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Hyperbolic operators with symplectic multiple characteristics

By

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1. Introduction

Let U be an open set in $\boldsymbol{R}^d$ with coordinates $x'=(x_1, \cdots, x_d)$. Denote by T^*U the contangent bundle on U and by $(x', \xi')=(x_1, \cdots, x_d, \xi_1, \cdots, \xi_d)$ standard coordinates in T^*U. Let I be an open interval containing the origin and set $\Omega=I\times U$. We denote by $(x, \xi)=(x_0, x', \xi_0, \xi')$ standard coordinates in $T^*\Omega$ and

$$D_j = -i\partial/\partial x_j, j = 0, \cdots, d,\ D = (D_0, D'),\ D' = (D_1, \cdots, D_d)\,.$$

Let

$$P(x, D) = D_0^m + \sum_{j=1}^{m} A_j(x, D')\, D_0^{m-j} \tag{1.1}$$

be a differential operator in D_0 of order m with coefficients $A_j(x, D')$ which are classical pseudodifferential operators of order j defined near $(\hat{x}, \hat{\xi}')=(0, \hat{x}', \hat{\xi}')\in I\times (T^*U\backslash 0)$. We denote by $p(x, \xi)$ the principal symbol of P and we assume that $p(x, \cdot)$ is hyperbolic with respect to dx_0 near $(\hat{x}, \hat{\xi}')$ that is the zeros ξ_0 of $p(x, \xi_0, \xi_0')$ are all real near $(\hat{x}, \hat{\xi}')$. We shall study the microlocal and local Cauchy problem for $P(x, D)$ with data on $x_0=0$.

Denote by Σ the set of real characteristics of order m of P;

$$\Sigma = \{(x, \xi)\in T^*\Omega\backslash 0;\ p(x, \xi) = dp(x, \xi) = \cdots = d^{m-1}p(x, \xi)=0\}\,. \tag{1.2}$$

We assume that

$$\begin{aligned}&\Sigma \text{ is a } C^\infty \text{ manifold near } \rho = (\hat{x}, \hat{\xi}) = (\hat{x}, \hat{\xi}_0, \hat{\xi}') \text{ with the}\\ &\text{tangent space } T_\rho\Sigma \text{ at } \rho \text{ such that } T_\rho S\supset T_\rho^\sigma\Sigma\cap T_\rho\Sigma\end{aligned} \tag{1.3}$$

where $S=\{x_0=0\}$ is a initial surface and $T_\rho^\sigma\Sigma$ is the σ orthogonal space of $T_\rho\Sigma$. Here σ is a natural 2 form on $T^*\Omega$ given in any standard coordinates (x, ξ) by

$$\sigma = \sum_{j=0}^{d} d\xi_j\wedge dx_j\,.$$

Note that if $T_\rho\Sigma$ is a symplectic subspace, that is $T_\rho^\sigma\Sigma\cap T_\rho\Sigma=\{0\}$, this condition

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is verified obviously. As another example we consider the case $T_\rho S \cap T_\rho^\sigma \Sigma$ is involutive which was treated in [13]. Since $T_\rho S \cap T_\rho^\sigma \Sigma \supset T_\rho^\sigma \Sigma \cap T_\rho \Sigma$ (1.3) is also verified.

We introduce the localization $p_\rho(x, \xi)$ of $p(x, \xi)$ at ρ;

$$p_\rho(x, \xi) = \lim_{s \to 0} s^{-m} p(\rho + s(x, \xi)) \quad \text{(cf. [3], [4])}\,. \tag{1.4}$$

It is well known that $p_\rho(x, \xi)$ is a hyperbolic polynomial in $T_\rho(T^*\Omega)$ with respect to H_{x_0}, the Hamilton field of x_0 (see [5], [7]). Then we can define the hyperbolic cone $\Gamma(p_\rho, H_{x_0})$ as the component of H_{x_0} in $\{X \in T_\rho(T^*\Omega);\, p_\rho(X) \neq 0\}$ and the propagation cone $C(p_\rho, H_{x_0})$;

$$C(p_\rho, H_{x_0}) = \{X \in T_\rho(T^*\Omega);\, \sigma(X, Y) \geqq 0 \quad \text{for any} \quad Y \in \Gamma(p_\rho, H_{x_0})\}\,.$$

Note that (1.3) implies

$$C(p_\rho, H_{x_0}) \cap T_\rho \Sigma = \{0\} \tag{1.5}$$

(but the converse is not true in general). Let Σ be defined by $f_0(x, \xi) = \cdots = f_k(x, \xi) = 0$ for instance, where $df_j(\rho)$ are linearly independent, then $p(x, \xi)$ can be written as

$$p(x, \xi) = \sum_{|\alpha| = m} c_\alpha(x, \xi)\, F(x, \xi)^\alpha$$

near ρ with $F = (f_0, \cdots, f_k)$. This gives that $d^m p(\rho)(X_1, \cdots, X_m) = 0$ if some X_j belongs to $T_\rho \Sigma$. Hence $p_\rho(X) = d^m p(\rho)(X, \cdots, X)$ is well defined as a polynomial in $N_\Sigma(T^*\Omega)_\rho = T_\rho(T^*\Omega)/T_\rho \Sigma$. We assume that

(1.6) $\quad p_\rho(x, \xi)$ is strictly hyperbolic with respect to $j(H_{x_0})$ in $N_\Sigma(T^*\Omega)_\rho$

where j is a natural projection from $T_\rho(T^*\Omega)$ onto $N_\Sigma(T^*\Omega)_\rho$ ($T_\rho \Sigma$ is the linearity space of $p_\rho(x, \xi)$, cf. [3], [4]). When $m = 2$ and Σ is a C^∞ manifold near ρ (1.6) is always verified except for a special case $\dim N_\Sigma(T^*\Omega) = 1$ (that is the case of characteristic of constant multiplicity). We note that these conditions are invariant under a change of homogeneous symplectic coordinates preserving $x_0 = \text{const}$.

Let $P(x, \xi)$ be the full ymbol of P;

$$P(x, \xi) = p(x, \xi) + p_{m-j}(x, \xi) + \cdots + p_i(x, \xi) + \cdots$$

where $p_i(x, \xi)$ is the homogeneous part of degree i of $P(x, \xi)$. We assume that

(1.7) $\quad p_{m-j}(x, \xi)$ vanishes of order $m-2j$ on Σ near ρ whenever $m-2j > 0$.

Clearly (1.7) is invariant under conjugation by elliptic Fourier integral operators. When P is a differential operator, Theorem 4.1 in [7] asserts that for the Cauchy problem for P to be C^∞ well posed it is necessary that p_{m-j} vanishes of order $m-2j$ at ρ. This necessary condition is independent of any geometric character of Σ and $C(p_\rho, H_{x_0})$. In this sense the condition (1.7) is the weakest one on lower order terms to expect the C^∞ correctness of the Cauchy problem when $p(x, \xi)$ has a characteristic of order m.

Under these assumptions we have the microlocal correctness in C^∞ of the Cauchy problem for P;

Theorem 1.1. *Assume that* (1.3), (1.6) *and* (1.7). *Then there is a parametrix of P at* $(0, \hat{x}', \hat{\xi}')$ *with finite propagation speed of wave front sets.*

We shall give the definition of a parametrix of P at $(0, \hat{x}', \hat{\xi}')$ with finite propagation speed of wave front sets in Appendix. When $m=2$ (1.5) and (1.6) imply that $p(x, \xi)$ is effectively hyperbolic at ρ and hence more general results were obtained (see [6], [8], [10], [11]). We give two simple examples which also motivate our hypotheses (1.3) and (1.6).

Example 1.1. Consider the following operator in $\boldsymbol{R}^2$ with $\rho=(0, 0, 0, 1)\in T^*\boldsymbol{R}^2\backslash 0$

$$P(x, D) = (D_0-x_0 D_1)\{(D_0+x_0 D_1)^2+\alpha D_1\}, \quad \alpha\neq 0 .$$

This verifies (1.3) and (1.7) but (1.6). It is clear that the Cauchy problem for this P is not C^∞ well posed.

Example 1.2. Let $P(x, D)$ be

$$(D_0-ax_1 D_1)(D_0-bx_1 D_1)(D_0-cx_1 D_1)+\alpha D_1, \quad \alpha\neq 0$$

which is also considered in $\boldsymbol{R}^2$ with $\rho=(0, 0, 0, 1)$ where a, b, c are mutually different real constants. This verifies (1.6) and (1.7) but (1.3). The Cauchy problem for this P is not C^∞ well posed in view of Theorem 4.1 in [7].

Now we study the propagation of wave front sets in a slightly more general setting. Let P be a classical pseudodifferential operator of order m in an open set $\Omega\subset\boldsymbol{R}^{d+1}$ with the real principal symbol $p(x, \xi)\in C^\infty(T^*\Omega\backslash 0)$. Let $\rho\in T^*\Omega\backslash 0$ be a characteristic of p of order r. Denote by Σ_r the set of real characteristics of order r of p defined by (1.2) with $m=r$;

$$\Sigma_r = \{(x, \xi)\in T^*\Omega\backslash 0;\, p(x, \xi) = dp(x, \xi) = \cdots = d^{r-1}p(x, \xi) = 0\} .$$

We assume that there is a conic neighborhood of V of ρ such that

$$\text{(1.8)} \qquad \Sigma_r\cap V \text{ is a } C^\infty \text{ manifold near } \rho .$$

We introduce the localization $p_\rho(x, \xi)$ of $p(x, \xi)$ at ρ by (1.4) with $m=r$. As noted before, $p_\rho(x, \xi)$ is a well defined polynomial in $N_{\Sigma_r}(T^*\Omega)_\rho$. In what follows we regard $N_{\Sigma_r}(T^*\Omega)_\rho$ as a subspace of $T_\rho(T^*\Omega)$ and denote by $[X]\in N_{\Sigma_r}(T^*\Omega)_\rho$ the residue classe of $X\in T_\rho(T^*\Omega)$. We suppose that

(1.9) p_ρ is strictly hyperbolic in $N_{\Sigma_r}(T^*\Omega)_\rho$ with respect to some $[\theta]\in N_{\Sigma_r}(T^*\Omega)_\rho$.

By $\Gamma(p_\rho, [\theta]) \subset N_{\Sigma_r}(T^*\Omega)_\rho$ we denote the hyperbolic cone of p_ρ regarded as a polynomial in $N_{\Sigma_r}(T^*\Omega)_\rho$. Let $P(x, \xi)$ be the full symbol of P and $p_i(x, \xi)$ be the homogeneous part of degree i of $P(x, \xi)$. Now we assume that

(1.10) $p_{m-j}(x, \xi)$ vanishes of order $r-2j$ on $\Sigma_r \cap V$ near ρ whenever $r-2j>0$.

As mentioned above we regard $T_\rho^\sigma \Sigma_r / T_\rho^\sigma \Sigma_r \cap T_\rho \Sigma_r$ as a subspace of $N_{\Sigma_r}(T^*\Omega)_\rho$ and hence is equal to $\{[X] \in N_{\Sigma_r}(T^*\Omega)_\rho ;\ X \in T_\rho^\sigma \Sigma_r\}$.

Theorem 1.2. *Suppose* (1.8)–(1.10). *Let $\varphi(x, \xi)$ be real, homogeneous of degree* 0 *in ξ, C^∞ in a conic neighborhood of ρ such that*

$$\varphi(\rho) = 0\,, \quad [H_\varphi(\rho)] \in \Gamma(p_\rho, [\theta]) \cap (T_\rho^\sigma \Sigma_r / T_\rho^\sigma \Sigma_r \cap T_\rho \Sigma_r)\,.$$

Let ω be a sufficiently small conic neighborhood of ρ. Then it follows from

$$\omega \cap \{\varphi<0\} \cap WF(u) = \phi,\ \rho \notin WF(Pu)$$

that

$$\rho \notin WF(u)$$

for any distribution $u \in \mathcal{D}'(\Omega)$.

Note that if $T_\rho \Sigma_r$ is symplectic then $T_\rho^\sigma \Sigma_r$ is identified with $N_{\Sigma_r}(T^*\Omega)_\rho$ and hence the hypothesis in Theorem 1.2 is reduced to

$$[H_\varphi(\rho)] \in \Gamma(p_\rho, [\theta])\,. \tag{1.11}$$

Note that the hypothesis in this theorem is equivalent to

$$H_\varphi(\rho) \in \Gamma(p_\rho, \theta) \cap T_\rho^\sigma \Sigma_r + T_\rho \Sigma_r\,. \tag{1.12}$$

Since $\Gamma(p_\rho, \theta) \cap T_\rho^\sigma \Sigma_r \neq \phi$ is equivalent to $C(p_\rho, \theta) \cap T_\rho \Sigma_r = \{0\}$, then this theorem gives a rough estimate of wave front sets when

$$C(p_\rho, \theta) \cap T_\rho \Sigma_r = \{0\} \tag{1.5$'$}$$

and p satisfies (1.9), (1.10). As noted before when $r=2$ (1.5)$'$ and (1.9) imply that $p(x, \xi)$ is effectively hyperboic at ρ and then Theorem 1.2 holds under (1.11) (see [10], [12]). If we assume further that $\operatorname{codim} \Sigma_2 = 2$, detailed discussions were given in [1], [2], [9].

We turn to the local Cauchy problem. To simplify notation, we say $p(x, \xi) \in \Sigma^{p,q}$ near ρ if $p(x, \xi)$ is homogeneous of degree p, C^∞ in a conic neighborhood of ρ which is a polynomial in ξ_0 such that $p(x, \xi)$ vanishes of order q on Σ near ρ. Then (1.7) is equivalent to say

$$p_{m-j} \in \Sigma^{m-j, m-2j} \quad \text{near } \rho \text{ whenever } m-2j>0\,. \tag{1.7$'$}$$

We assume that $A_j(x, D')$ in (1.1) are classical pseudodifferential operators of order

j in Ω. We also assume that $p(x, \cdot)$ is hyperbolic with respect to dx_0 near $\hat{x}$, that is the zeros ξ_0 of $p(x, \xi_0, \xi')$ are all real for any $(x, \xi') \in \tilde{\Omega} \times (\boldsymbol{R}^d \backslash 0)$, where $\tilde{\Omega}$ is an open neighborhood of $\hat{x}$. Let $\kappa \in T^*_{\hat{x}}\Omega \backslash 0$ be a multiple characteristic of p. We denote by $m(\kappa)$ its multiplicity and by $\Sigma_{(\kappa)}$ the component of κ in the real characteristic set of order $m(\kappa)$ of p. We recall our hypotheses (1.3), (1.6) and (1.7) at κ;

$(1.3)_\kappa$ $\quad\Sigma_{(\kappa)}$ is a C^∞ manifold near κ with the tangent space $T_\kappa \Sigma_{(\kappa)}$ at κ such that $T_\kappa S \supset T^\sigma_\kappa \Sigma_{(\kappa)} \cap T_\kappa \Sigma_{(\kappa)}$

$(1.6)_\kappa$ $\quad p_\kappa(x, \xi)$ is strictly hyperbolic with respect to $[H_{x_0}]$ in $N_{\Sigma_{(\kappa)}}(T^*\Omega)_\kappa$

$(1.7)_\kappa$ $\quad p_{m-j}(x, \xi) \in \Sigma^{m-j, m(\kappa)-2j}_{(\kappa)}$ near κ whenever $m(\kappa)-2j>0$.

Theorem 1.3. *Let $p(x, \cdot)$ be hyperbolic with respect to dx_0 near $\hat{x}$. Assume that $(1.3)_\kappa$, $(1.6)_\kappa$ and $(1.7)_\kappa$ are verified for every multiple characteristic $\kappa \in T^*_{\hat{x}}\Omega \backslash 0$. Then the Cauchy problem for P is locally solvable near $\hat{x}$ in C^∞ with initial data on $x_0=0$.*

Proof. By Proposition A.4, it will suffice to show that P has a parametrix with finite propagation speed of wave front sets at $(0, \hat{x}', \xi')$ for any $|\xi'|=1$. Fix $\tilde{\xi}'$ with $|\tilde{\xi}'|=1$ arbitrarily and show that P has such a parametrix at $(0, \hat{x}', \tilde{\xi}')$. Let $\kappa_j \in T^*_{\hat{x}}\Omega \backslash 0$ $(j=1, \cdots, r)$ be multiple characteristics of p such that their projection off ξ_0 coordinate are $(\hat{x}, \tilde{\xi}')$, that is $\kappa_j = (\hat{x}, \tilde{\xi}_0^{(j)}, \tilde{\xi}')$, where $\tilde{\xi}_0^{(j)}$ are different zeros of $p(\hat{x}, \xi_0, \tilde{\xi}')$. Let $m(\kappa_j)=m_j$ then it is clear that

$$p(x, \xi) = \prod_{j=1}^r p^{(j)}(x, \xi)$$

where $p^{(j)}(x, \xi)$ are homogeneous of degree m_j, C^∞ in a conic neighborhood of $(\hat{x}, \tilde{\xi}')$ which are polynomials in ξ_0 and κ_j are characteristics of order m_j of $p^{(j)}(x, \xi)$. We note that

$$p_{\kappa_j}(x, \xi) = p^{(j)}_{\kappa_j}(x, \xi) \{ \prod_{k \neq j} p^{(k)}(\kappa_j) \} .$$

It is clear from hypothesis that

$$p^{(j)}(x, \xi) \in \Sigma^{m_j, m_j}_{(\kappa_j)} \quad \text{near} \quad \kappa_j . \tag{1.13}$$

The zeros ξ_0 of $p^{(j)}(x, \xi)$ $(j=1, \cdots, r)$ are different each other when (x, ξ') is near $(\hat{x}, \tilde{\xi}')$ and then we can write

$$P(x, D) = P^{(1)}(x, D) \cdots P^{(r)}(x, D) + \sum_{j=1}^m B_j(x, D') D_0^{m-j}$$

where $B_j(x, \xi') \in S^{-\infty}$ near $(\hat{x}, \tilde{\xi}')$ uniformly when $|x_0|$ is small and $P^{(j)}(x, \xi)$ has an asymptotic expansion;

$$P^{(j)}(x, \xi) \sim p^{(j)}(x, \xi) + p^{(j)}_{m_j-1}(x, \xi) + \cdots + p^{(j)}_i(x, \xi) + \cdots$$

with $p_i^{(j)}(x, \xi)$ which are homogeneous of degree i, C^∞ in a conic neighborhood of $(\hat{x}, \tilde{\xi}')$, polynomials in ξ_0. Comparing the homogeneous part of degree $m-i$ of the symbols of $P(x, D)$ and $P^{(1)}(x, D)\cdots P^{(r)}(x, D)$ it follows that

$$\{\prod_{k\neq j} p^{(k)}(x, \xi)\}\, p^{(j)}_{m_j-i}(x, \xi)+R_{j,i}(x, \xi) = p_{m-i}(x, \xi)\,.$$

By induction on i, it follows easily that $R_{j,i}(x, \xi)$ vanishes of order m_j-2i on $\Sigma_{(\kappa_j)}$ near κ_j. Since $p^{(k)}(x, \xi)$ never vanish on $\Sigma_{(\kappa_j)}$ if $k\neq j$ it follows that

$$p^{(j)}_{m_j-i}(x, \xi)\in\Sigma^{m_j, m_j-2i}_{(\kappa_j)} \quad \text{whenever} \quad m_j-2i>0\,. \tag{1.14}$$

From (1.13), (1.14) we can apply Theorem 1.1 with $P=P^{(j)}$, $\Sigma=\Sigma_{(\kappa_j)}$ and we conclude that $P^{(j)}(x, D)$ has such a parametrix at $(0, \hat{x}', \tilde{\xi}')$ $(j=1, \cdots, r)$. On the other hand, by Corollary A.1, to prove the existence of a parametrix with finite propagation speed of wave front sets of P at $(0, \hat{x}', \tilde{\xi}')$ it suffices to show that each $P^{(j)}(x, D)$ has such a parametrix at $(0, \hat{x}', \tilde{\xi}')$. This remark completes the proof.

Proof of Theorem 1.2. Let $\Sigma_r\cap V$ be given by

$$b_1(x, \xi) = \cdots = b_k(x, \xi) = 0$$

where $b_j(x, \xi)$ are homogeneous of degree 1 in ξ and $db_j(\rho)$ are linearly independent. Choose $c_j(x, \xi)$ $(1\leqq j\leqq l=2(d+1)-k)$ with $c_j(\rho)=0$, homogeneous of degree 1 in ξ, C^∞ in a conic neighborhood of ρ so that $H_{c_j}(\rho)$ form a basis for $T_\rho\Sigma_r$. Let

$$[H_\varphi(\rho)]\in\hat{\Gamma}(p_\rho, [\theta])\cap(T^\sigma_\rho\Sigma_r/T^\sigma_\rho\Sigma_r\cap T_\rho\Sigma_r)\,.$$

Then we can write $H_\varphi(\rho)=\sum_{j=1}^{k}\alpha_j H_{b_j}(\rho)+\sum_{j=1}^{l}\beta_j H_{c_j}(\rho)$ with real constants α_j, β_j. Set $\psi(x, \xi)=\sum\alpha_j b_j(x, \xi)+\sum\beta_j c_j(x, \xi)+M|(x, \xi|\xi|^{-1})-\rho|^2|\xi|$ and we may assume that $\{\varphi\leqq 0\}\supset\{\psi\leqq 0\}$ near ρ taking M sufficiently large. Since $[H_\psi(\rho)]\in\hat{\Gamma}(p_\rho, [\theta])$ it follows that

$$p_\rho(H_\psi(\rho))\neq 0$$

and this implies that $(H_\psi(\rho))^r p_\rho(0)\neq 0$. From the definition of localization we have

$$(H_\psi(\rho)^r p)(\rho)\neq 0\,. \tag{1.15}$$

Put $X_0=\psi(x, \xi)$ and note that $H_\psi(\rho)$ and the radial vector field at ρ (which is in $T_\rho\Sigma_r$) are linearly independent because $\sum\alpha_j H_{b_j}(\rho)\neq 0$. Then we can extend X_0 to a full homogeneous symplectic coordinates $\{X, \Xi\}$ such that $X(\rho)=0$, $\Xi(\rho)=e_d$. We write (x, ξ) instead of (X, Ξ) then (1.15) implies that $H^r_{x_0}p(\rho)\neq 0$. Since $H^j_{x_0}p(\rho)=0$ for $0\leqq j\leqq r-1$, Malgrange's preparation theorem gives that

$$p(x, \xi) = q(x, \xi)\,\{\xi_0^r+a_1(x, \xi')\,\xi_0^{r-1}+\cdots+a_r(x, \xi')\}$$

with $q(\rho)\neq 0$ where $a_j(x, \xi')$ are real, homogeneous of degree j in ξ', C^∞ in a conic neighborhood of $\rho'=(0, e_d')$, $e_d'=(0, \cdots, 0, 1)\in\boldsymbol{R}^d$ and $a_j(\rho')=0$, $\xi'=(\xi_1, \cdots, \xi_d)$.

A pseudodifferential operator analogue of the Malgrange division theorem shows that

$$P(x, D) \equiv Q(x, D)\ \{D_0^r + A_1(x, D')\, D_0^{r-1} + \cdots + A_r(x, D')\}$$

modulo a smoothing operator near ρ where Q is non characteristic at ρ. Snice our result is invariant under multiplication by Q it will suffice to consider

$$P(x, D) = D_0^r + A_1(x, D')\, D_0^{r-1} + \cdots + A_r(x, D')\,.$$

Denote by $P(x, \xi)$ the full symbols of $P(x, D)$;

$$P(x, \xi) = p(x, \xi) + p_{r-1}(x, \xi) + \cdots + p_i(x, \xi) + \cdots\,.$$

It follows from the assumption (1.10) that $p_{r-j}(x, \xi)$ vanishes of order $r-2j$ whenever $r-2j>0$. Since p_ρ is strictly hyperbolic with respect to $[H_{x_0}]$ in $N_{\Sigma_r}(T^*\Omega)_\rho$ and clearly

$$T_\rho\{x_0=0\} \supset T_\rho^\sigma \Sigma_r \cap T_\rho \Sigma_r$$

we can apply Proposition 6.2 to $P(x, D)$ (after reducing to a second order system following §7) to conclude that; if

$$(\omega' \times \boldsymbol{R}) \cap \{x_0<0\} \cap WF(u) = \phi\,, \quad \rho \notin WF(Pu) \tag{1.16}$$

then one has $\rho \notin WF(u)$ where ω' is a sufficiently small conic neighborhood of $\rho'=(0, e_d')$. If $p(x, \xi)=0$ then $|\xi_0|$ is bounded by $B|(x, \xi'|\xi'|^{-1})-\rho'|^2|\xi'|$ near ρ with a positive constant B and hence we can replace $\omega' \times \boldsymbol{R}$ in (1.16) by a small conic neighborhood of ρ in $T^*\Omega\backslash 0$. This proves the theorem.

In §§2 and 3 we reduce our study on P to first order operators using a blow up like process along $T_\rho\Sigma$. We shall also estimate the derivatives of symbols of such reduced first order operators. In §4 we recall some properties of pseudodifferential operators with symbols defined in §3 which are found in [11] with different notation. In §5, we shall therefore study such first order operators using calculus in §4. We follow [11] and a basic estimate is proved by energy integral method. A modified version of such estimate will be applied to study the propagation of wave front sets of solutions. In §6 we shall extend estimates obtained in §5 to a product of two first order operators. §7 is devoted to a reduction of the Cauchy problem for P to that of a second order system with diagonal principapl part to which we can apply our results in §6. We complete the proof of Theorems 1.1 here. In Appendix we shall give the definition and some properties of parametrices with finite propagation speed of wave front sets which are found in §3 in [14] in a slightly different formulation.

2. Blow up of principal symbol

We can choose a homogeneous symplectic coordinates (x, ξ) at $(\hat{x}, \hat{\xi})$ preserving $x_0=$const., such that $\rho''=(\hat{x}', \hat{\xi}')=(0, e_d')$, $e_d'=(0, \cdots, 0, 1)\in\boldsymbol{R}^d$ and

$$p(x,\xi)=\xi_0^m+\sum_{j=2}^{m}\hat{a}_j(x,\xi')\xi_0^{m-j}$$

where $\hat{a}_j(x,\xi')$ are homogeneous of degree j in ξ', C^∞ in a conic neighborhood of $\rho'=(0,0,e_d')$. Since $\Sigma\subset\{\xi_0=0\}$ in this coordinates, Σ is given by

$$\Sigma=\{\xi_0=0,\ b_j(x,\xi')=0,\ j=1,\cdots,k\}$$

where $db_j(\rho')$ are linearly independent. Here we have assumed that the codimension of Σ is $k+1$.

Lemma 2.1. *Suppose* (1.3). *Then we can choose a homogeneous symplectic coordinates at* $(\hat{x}',\hat{\xi}')$ *so that* $\rho''=(0,e_d')$ *and*

$$\Sigma=\{\xi_0=0,\ b_j(x,\xi')=0,\ j=1,\cdots,k\}$$

where $b_j(x,\xi')$ *are homogeneous of degree* 1 *in* ξ', C^∞ *in a conic neighborhood of* ρ' *and*

$$db_1(\rho')=dx_0+bdx_{d-1}+adx_d$$
$$db_{2j}(\rho')=dx_j+a_{2j}\,dx_d,\ db_{2j+1}(\rho')=d\xi_j+a_{2j+1}\,dx_d,\ 1\leqq j\leqq q$$
$$db_{2q+1+j}(\rho')=dx_{q+j}+a_{2q+1+j}\,dx_d,\ 1\leqq j\leqq r,\ db_k(\rho')=b_k\,dx_{q+r+1}+a_k\,dx_d$$

with $r=k-2q-2$, $q\leqq d-1$ *and* $b=0$ *if* $q=d-1$.

Remark 2.1. In the case $H_{b_j}(\rho')$ $(1\leqq j\leqq k)$ and $\partial/\partial\xi_d$ are linearly dependent we have $b_k=0$ in this lemma.

For later use, renumbering the coordinates, we may assume that $\rho'=(0,0,e_p')$ and Σ is given by

$$\Sigma=\{\xi_0=0,\ b_j(x,\xi')=0,\ j=1,\cdots,k\}$$

where $b_j(x,\xi')$ are homogeneous of degree 1 in ξ', C^∞ near ρ' and

$$db_1(\rho')=dx_0+bdx_{p-1}+adx_p$$
$$db_{2j}(p')=dx_{p+j}+a_{2j}\,dx_p,\ db_{2j+1}(\rho')=d\xi_{p+j}+a_{2j+1}\,dx_p\quad(1\leqq j\leqq q,\ p+q=d)$$
$$db_j(\rho')=dx_{j-1}+a_j\,dx_p(2\leqq j\leqq r=k-2q-2),\ db_{r+1}(\rho')=b_{r+1}\,dx_r+a_{r+1}\,dx_p\,.$$

Note that $b_1(x,\xi')$ can be written as

(2.1) $$b_1(x,\xi')=(x_0+bx_{p-1}+ax_p+f_1(x',\xi'))\,e_1(x,\xi')$$

near ρ' with $df_1(\rho')=0$ where $f_1(x',\xi')$, $e_1(x,\xi')$ are homogeneous of degree 0, 1 in ξ' respectively and $e_1(\rho')=1$.

Put $b_0(x,\xi)=\xi_0$ then we can write $p(x,\xi)$ as

(2.2) $$p(x,\xi)=b_0(x,\xi)^m+\sum_{|\alpha|=m,\alpha_0\leqq m-2}\tilde{a}_\alpha(x,\xi')\,b(x,\xi)^\alpha$$

where $\alpha=(\alpha_0,\alpha_1,\cdots,\alpha_k)\in\boldsymbol{N}^{k+1}$, $b(x,\xi)=(b_0(x,\xi),b_1(x,\xi'),\cdots,b_k(x,\xi'))$ and $\tilde{a}_\alpha(x,\xi')$

are homogeneous of degree 0 in ξ'. Choose $Y_j \in T_\rho(T^*\Omega)$ $(0 \leqq j \leqq k)$ so that

$$db_i(\rho)(Y_j) = \delta_{ij}$$

whose residue classes form a basis to $N_\Sigma(T^*\Omega)_\rho$. With this basis we may assume that $N_\Sigma(T^*\Omega)_\rho = \boldsymbol{R}^{k+1}$, $[H_{x_0}] = (1, 0, \cdots, 0) \in \boldsymbol{R}^{k+1}$. Then p_ρ in $N_\Sigma(T^*\Omega)_\rho = \boldsymbol{R}^{k+1}$ is given by

$$q(\zeta) = \zeta_0^m + \sum_{|\alpha|=m, \alpha_0 \leqq m-2} \tilde{a}_\alpha(\rho')\zeta^\alpha, \ \zeta = (\zeta_0, \zeta_1, \cdots, \zeta_k)$$

and hence

$$p_\rho(x, \xi) = q(db(x, \xi)) .$$

Assumption (1.3) implies that

(2.3) $q(\zeta)$ is a strictly hyperbolic polynomial in ζ with respect to $(1, 0, \cdots, 0) \in \boldsymbol{R}^{k+1}$.

Using this q we can write $p(x, \xi)$ as

$$p(x, \xi) = q(b(x, \xi)) + \sum_{|\alpha|=m, \alpha_0 \leqq m-2} a_\alpha(x, \xi')\, b(x, \xi)^\alpha \tag{2.4}$$

where $a_\alpha(x, \xi') = \tilde{a}_\alpha(x, \xi') - a_\alpha(\rho')$ and hence $a_\alpha(\rho') = 0$. We make a blow up like process to $p(x, \xi)$ along Σ using the expression (2.4) and Nuij approximation [15]. Let

$$\hat{q}(\zeta; x, \xi') = q(\zeta) + \sum_{|\alpha|=m, \alpha_0 \leqq m-2} a_\alpha(x, \xi')\zeta^\alpha$$

$$\hat{q}(\zeta, \sigma; x, \xi') = (1 - \sigma^2 \partial^2/\partial\zeta_0^2)^{[m/2]}\, \hat{q}(\zeta; x, \xi')$$

where $[k]$ denotes the integer part of k. Then we can write of course

$$\hat{q}(\zeta; x, \xi') = \tilde{q}(\zeta, \sigma; x, \xi') + \sum_{j=1}^{[m/2]} \sigma^{2j}\, r_{m-2j}(\zeta; x, \xi') \tag{2.5}$$

where $r_{m-2j}(\zeta; x, \xi')$ are polynomials in ζ of degree $m-2j$ with coefficients which are homogeneous of degree 0 in ξ', C^∞ in a conic neighborhood of ρ'. We note that

$$\hat{q}(\xi; \rho') = q(\zeta), \tilde{q}(\zeta, 0; x, \xi') = \hat{q}(\zeta; x, \xi') .$$

Proposition 2.1. *The equation* $\tilde{q}(\zeta, \sigma; \rho') = 0$ *in* ζ_0 *has* m *real distinct roots for any* $(\zeta', \sigma) \neq (0, 0)$.

Since $\tilde{q}$ is homogeneous of degree m, 0 in (ζ, ρ), ξ' respectively, it follows from Proposition 2.1 and Rouché's theorem that there are m functions $\tilde{\lambda}_j(\zeta', \sigma; x, \xi') \in C^\infty((\boldsymbol{R}^{k+1}\backslash 0) \times W)$ such that

$$\tilde{q}(\zeta, \sigma; x, \xi') = \prod_{j=1}^{m} (\zeta_0 - \tilde{\lambda}_j(\zeta', \sigma; x, \xi')) \tag{2.6}$$

when $(\zeta', \sigma) \neq (0, 0)$ where W is a conic neighborhood of ρ'. Here we note that $\tilde{\lambda}_j(\zeta', \sigma; x, \xi')$ is homogeneous of degree 1, 0 in (ζ', σ), ξ' respectively and

$$|\tilde{\lambda}_i(\zeta', \sigma; x, \xi') - \tilde{\lambda}_j(\zeta', \sigma; x, \xi')| \geqq c(|\zeta'|^2 + \sigma^2)^{1/2} \tag{2.7}$$

for any $(x, \xi') \in W$, $(\zeta', \sigma) \in \boldsymbol{R}^{k+1} \backslash 0$, $i \neq j$, with a positive constant c. By (2.5) and (2.6) we have an expression of $p(x, \xi)$;

$$p(x, \xi) = \prod_{j=1}^{m} (\xi_0 - \tilde{\lambda}_j(b'(x, \xi'), \sigma; x, \xi')) + \sum_{j=1}^{[m/2]} \sigma^{2j}\, r_{m-2j}(b(x, \xi); x, \xi') \tag{2.8}$$

with $b'(x, \xi') = (b_1(x, \xi'), \cdots, b_k(x, \xi'))$.

3. Estimate of blown up symbol

Let $\chi_1(s) \in C_0^\infty(\boldsymbol{R})$ be equal to 1 in $|s| \leqq 1$ vanish in $|s| \geqq 2$ such that $0 \leqq \chi_1(s) \leqq 1$. We define $y_j = y_j(x, \mu)$, $\eta_j = \eta_j(\xi, \mu)$ following [11] by

$$y_0 = \mu x_0,\ y_j = \mu \chi_1(x_j)\, x_j (1 \leqq j \leqq p),\ y_j = \mu^{1/2} \chi_1(\mu^{-1/2} x_j)\, x_j \quad (p+1 \leqq j)$$
$$\eta_0 = \mu^{-1} \xi_0,\ \eta_j = \mu^{-1/2} \chi_1(\mu^{-1/2} \xi_j \langle \xi' \rangle^{-1})\, \xi_j \quad (p+1 \leqq j)$$
$$\eta_j = \mu^{-1} \chi_1(\mu^{-1}(\xi_j \langle \xi' \rangle^{-1} - \delta_{jp}))\, (\xi_j - \delta_{jp} \langle \xi' \rangle) + \mu^{-1} \delta_{jp} \langle \xi' \rangle \quad (1 \leqq j \leqq p)$$

where $0 < \mu \leqq 1$ and δ_{ij} is Kronecker's delta. It is easy to check that

$$\begin{aligned} &y_j \in S(\mu, dx_0^2 + \tilde{G}'_\mu) \text{ for any } j,\ \mu \eta_j \in S(\mu \langle \xi' \rangle, dx_0^2 + \tilde{G}'_\mu) \quad (p+1 \leqq j) \\ &\mu \eta_j - \delta_{jp} \langle \xi' \rangle \in S(\mu \langle \xi' \rangle, dx_0^2 + \tilde{G}_\mu) \quad (1 \leqq j \leqq p) \end{aligned} \tag{3.1}$$

uniformly when $0 < \mu \leqq 1$ with $\tilde{G}'_\mu = |dx''|^2 + \mu^{-1} |dx'''|^2 + \mu^{-1} \langle \xi' \rangle^{-2} |d\xi'|^2$, $\tilde{G}_\mu = |dx''|^2 + \mu^{-1} |dx'''|^2 + \mu^{-2} \langle \xi' \rangle^{-2} |d\xi'|^2$ where $x'' = (x_1, \cdots, x_p)$, $x''' = (x_{p+1}, \cdots, x_d)$. Put $W_c = \{(x', \xi');\ |x_j| \leqq c,\ |\xi_j |\xi'|^{-1} - \delta_{jp}| \leqq c\}$ and note that

$$|\mu \eta_j - \delta_{jp} \langle \xi' \rangle| \leqq 2\mu \langle \xi' \rangle,\ (1 - C\mu) \langle \xi' \rangle \leqq \mu |\eta'| \leqq (1 + C\mu) \langle \xi' \rangle \tag{3.2}$$

with a positive constant C independent of μ. Then there is a positive constant $\hat{c}$ such that $(y', \eta') \in W_{\hat{c}\mu}$. Moreover if $|x_j| \leqq \mu^{1/2}$, $|\xi_j \langle \xi' \rangle^{-1} - \delta_{jp}| \leqq \mu$ it follows that

$$(y, \eta) = (\mu x_0, \mu x'', \mu^{1/2} x''', \mu^{-1} \xi_0, \mu^{-1} \xi'', \mu^{-1/2} \xi''') = M_\mu(x, \xi) \tag{3.3}$$

with $\xi'' = (\xi_1, \cdots, \xi_p)$, $\xi''' = (\xi_{p+1}, \cdots, \xi_d)$. Let I be an open interval containing 0 and $b(x, \xi') \in C^\infty(I \times W_c)$ be homogeneous of degree 1 in ξ' such that

$$b(0, e_p) = 0\ , \quad (\partial/\partial \xi_j)\, b(0, e_p) = 0 \quad j = 1, \cdots, p-1\ . \tag{3.4}$$

For such $b(x, \xi')$ we define $B(x, \xi', \mu)$ by

$$B(x, \xi', \mu) = \mu b(y, \eta') = b(y, \mu \eta'),\ 0 < \mu \leqq \hat{\mu} \tag{3.5}$$

with $\hat{c}\hat{\mu} < c$. Remark that $B(x, \xi', \mu)$ is defined for all $(x', \xi') \in \boldsymbol{R}^d \times \boldsymbol{R}^d$. First we estimate the derivatives of B. By the Taylor expansion of B at $(0, \langle \xi' \rangle e_p)$ we can write

$$B(y, \mu\eta') = \sum_{|\alpha+\beta|=1} (\alpha!\,\beta!)^{-1} y^{\beta}(\mu\eta')^{\alpha}\, b_{(\beta)}^{(\alpha)}(0, \langle\xi'\rangle e_p)+$$
$$+2 \sum_{|\alpha+\beta|=2} (\alpha!\,\beta!)^{-1} y^{\beta}(\mu\eta'-\langle\xi'\rangle e_p)^{\alpha} \int_0^1 (1-\theta)\, b_{(\beta)}^{(\alpha)}(\theta y, \theta(\mu\eta'-\langle\xi'\rangle e_p)+\langle\xi'\rangle e_p)\, d\theta\,.$$

Here we have used $(\partial/\partial\xi_p)\, b(0, e_p)=0$ which follows from Euler's identity and (3.4). We note that the integral belongs to $S(\langle\xi'\rangle^{1-|\alpha|}, dx_0^2+\tilde{G}_\mu)$ by (3.1) and (3.2). Hence the second term of the right hand side belongs to $S(\mu^2\langle\xi'\rangle, dx_0^2+\tilde{G}_\mu)$. From (3.4) the first term of the right-hand side contains no $\mu\eta_j (1\leqq j\leqq p)$ and hence belongs to $S(\mu\langle\xi'\rangle, dx_0^2+\tilde{G}'_\mu)$ in view of (3.1). These two facts give that

$$\begin{aligned} &|B_{(\beta)}^{(\alpha)}(x, \xi', \mu)| \leqq c_{\alpha\beta}\, \mu^{1-|\alpha'''+\beta'''|/2}\langle\xi'\rangle^{1-|\alpha|} \quad \text{for} \quad |\alpha|\leqq 1 \\ &|B_{(\beta)}^{(\alpha)}(x, \xi', \mu)| \leqq c_{\alpha\beta}\, \mu^{2-|\alpha|-|\beta'''|/2}\langle\xi'\rangle^{1-|\alpha|} \quad \text{for} \quad |\alpha|\geqq 2 \end{aligned} \tag{3.6}$$

where $\beta=(\beta_0, \beta'', \beta''')\in \boldsymbol{N}^{d+1}$, $\alpha=(\alpha'', \alpha''')\in \boldsymbol{N}^d$. Let $f(x, \xi')\in C^\infty(I\times W_c)$ be homogeneous of degree m in ξ'. We set

$$F(x, \xi', \mu) = \mu^m f(y, \eta')\,.$$

Then the same argument as to prove (3.6) shows that

Lemma 3.1. $F(x, \xi', \mu) = f(0, e_p)\langle\xi'\rangle^m+\tilde{F}(x, \xi', \mu)$ *with* $\tilde{F}(x, \xi', \mu)\in S(\mu\langle\xi'\rangle^m, dx_0^2+\tilde{G}_\mu)$. *If* $f_{(\beta)}^{(\alpha)}(0, e_p)=0$ *for* $|\alpha+\beta|<r$ *then*

$$F(x, \xi', \mu)\in S(\mu^r\langle\xi'\rangle^m, dx_0^2+\tilde{G}_\mu)$$

uniformly when $0<\mu\leqq\hat{\mu}$.

Let $B_1(x, \xi', \mu)$ be defined by (3.5) with $b(x, \xi')=b_1(x, \xi')$. From (2.1) we have

$$\begin{aligned} B_1(x, \xi', \mu) &= (x_0+b\chi_1(x_{p-1})\, x_{p-1}+a\chi_1(x_p)\, x_p+\mu^{-1}f_1(y', \eta'))\, \mu e_1(y, \mu\eta') \\ &= \varphi(x, \xi', \mu)\, E_1(x, \xi', \mu) \end{aligned} \tag{3.7}$$

where $E_1=\mu\langle\xi'\rangle+\tilde{E}_1$, $\tilde{E}_1\in S(\mu^2\langle\xi'\rangle, dx_0^2+\tilde{G}_\mu)$ in view of Lemma 3.1 and hence $E_1\geqq c\mu\langle\xi'\rangle$ with a positive constant c. Since $f_{1(\beta)}^{(\alpha)}(0, e_p)=0$ for $|\alpha+\beta|<2$ it follows from Lemma 3.1 again that

$$\mu^{-1}f_1(y', \eta')\in S(\mu, dx_0^2+\tilde{G}_\mu)$$

hence

$$\varphi(x, \xi', \mu),\ \varphi_{(\beta)}^{(\alpha)}(x, \xi', \mu)\in S(1, dx_0^2+\tilde{G}_\mu) \quad \text{for} \quad |\alpha+\beta| = 1\,. \tag{3.8}$$

We return to estimate the derivatives of B. Let $\chi(s)\in C^\infty(\boldsymbol{R})$ be equal to zero in $|s|\leqq 1$ and equal to 1 in $|s|\geqq 2$. According to the remark preceding to Remark 4.1, we may consider $B\chi(\mu^4\langle\xi'\rangle)$ instead of B hence we may suppose that

$$\mu^4\langle\xi'\rangle\geqq 1 \tag{3.9}$$

on the support of B. We use this abbreviation without refering.

Lemma 3.2. *Let* $|\beta''|\geqq s$, $|\beta'''|\geqq t$. *Then*

$$|B_{(\beta)}^{(\alpha)}|\leqq c_{\alpha\beta}\,\mu^{1-|\alpha'''|/2-t/2}\langle\xi'\rangle^{1-|\alpha|}\langle\mu\xi'\rangle^{(|\beta''|-s)/2}\langle\xi'\rangle^{(|\beta'''|-t)/2}$$

if $|\alpha|\leqq 1$ *and if* $|\alpha|\geqq r\geqq 2$ *then*

$$|B_{(\beta)}^{(\alpha)}|\leqq c_{\alpha\beta}\,\mu^{2+|\alpha|-2r}\langle\xi'\rangle^{1-r}\langle\xi'\rangle^{-(|\alpha|-r)/2}\langle\mu\xi'\rangle^{(|\beta''|-s)/2}\langle\xi'\rangle^{(|\beta'''|-t)/2}.$$

Corollary 3.1. *Let* $|\beta''|\geqq s$, $|\beta'''|\geqq t$. *Then*

$$|B_{(\beta)}^{(\alpha)}|\leqq c_{\alpha\beta}\,\mu^{1-t/2}\langle\xi'\rangle^{1-|\alpha|/2}\langle\mu\xi'\rangle^{(|\beta''|-s)/2}\langle\xi'\rangle^{(|\beta'''|-t)/2}$$

for any α,

$$\begin{aligned}|B_{(\beta)}^{(\alpha)}|&\leqq c_{\alpha\beta}\,\mu^{1-t/2+(|\alpha''|-1)/2}\langle\xi'\rangle^{-(|\alpha|-1)/2}\langle\mu\xi'\rangle^{-(|\beta''|-s)/2}\\&\quad\times\langle\xi'\rangle^{(|\beta'''|-t)/2}\quad \textit{for}\quad|\alpha|\geqq 1,\\|B_{(\beta)}^{(\alpha)}|&\leqq c_{\alpha\beta}\langle\xi'\rangle^{-1}\langle\xi'\rangle^{-(|\alpha|-2)/2}\langle\mu\xi'\rangle^{(|\beta''|-s)/2}\langle\xi'\rangle^{(|\beta'''|-t)/2}\end{aligned}$$

if $|\alpha|\geqq 2$.

Lemma 3.3. *For* $|\alpha+\beta|\geqq 1$ *we have*

$$|B_{(\beta)}^{(\alpha)}|\leqq c_{\alpha\beta}\langle\mu\xi'\rangle^{1/2}\langle\xi'\rangle^{-|\alpha|/2}\langle\mu\xi'\rangle^{|\beta''|/2}\langle\xi'\rangle^{|\beta'''|/2}.$$

Let $\tilde{\lambda}(\zeta',\sigma;x,\xi')\in C^\infty((\boldsymbol{R}^{k+1}\backslash 0)\times W)$ be homogeneous of degree n, m in (ζ',σ), ξ' respectively where W is a conic neighborhood of $(0,e_p)$. The homogeneity shows that

$$(3.10)\qquad |\partial_{\xi'}^{\alpha}\partial_x^{\beta}\partial_{(\xi',\sigma)}^{\gamma}\tilde{\lambda}(\zeta',\sigma;x,\xi')|\leqq c_{\alpha\beta\gamma}(|\zeta'|^2+\sigma^2)^{(n-|\gamma|)/2}|\xi'|^{m-|\alpha|}.$$

Put

$$\begin{aligned}a(x,\xi',\mu)&=\tilde{\lambda}(B_1(x,\xi',\mu),\cdots,B_k(x,\xi',\mu),\langle\mu\xi'\rangle^{1/2};y,\mu\eta')\\m(B')&=\{\sum_{j=1}^{k}B_j(x,\xi',\mu)^2\langle\mu\xi'\rangle^{-2}+\langle\mu\xi'\rangle^{-1}\}^{1/2},\end{aligned}$$

where $B_j(x,\xi',\mu)$ is given by (3.5) with $b=b_j(x,\xi')$ which was defined after Remark 2.1. Note that when $\tilde{\lambda}$ is homogeneous of degree 1 and 0 in (ζ',σ), ξ' respectively then in view of (3.3) and (3.5) we have

$$(3.11)\quad a(x,\xi',\mu)=\mu\tilde{\lambda}(b_1(M_\mu(x,\xi')),\cdots,b_k(M_\mu(x,\xi')),\langle\mu\xi'\rangle^{1/2};M_\mu(x,\xi'))$$

when $|x_j|\leqq\mu^{1/2}$, $|\xi_j\langle\xi'\rangle^{-1}-\delta_{jp}|\leqq\mu$.

Our aim in this section is to prove the following proposition.

Proposition 3.1.

$$a_{(\beta)}^{(\alpha)}\in S(\mu^{-|\alpha'''+\beta'''|/2}\langle\mu\xi'\rangle^{n}\langle\xi'\rangle^{m}\,m(B')^{n-|\alpha+\beta|}\langle\xi'\rangle^{-|\alpha|},\tilde{g}_\mu+\langle\mu\xi'\rangle\,dx_0^2)$$

if $|\alpha+\beta|\leqq 1$,

$$a_{(\beta)}^{(\alpha)}\in S(\mu^{-|\alpha+\beta'''|/2}\langle\mu\xi'\rangle^{n}\langle\xi'\rangle^{m}\,m(B')^{n-|\alpha+\beta|}\langle\xi'\rangle^{-|\alpha|},\tilde{g}_\mu+\langle\mu\xi'\rangle\,dx_0^2)$$

if $|\alpha+\beta|=2$ *uniformly when* $0<\mu\leqq\hat{\mu}$ *where*

$$\tilde{g}_\mu=\langle\mu\xi'\rangle|dx''|^2+\langle\xi'\rangle|dx'''|^2+\langle\xi'\rangle^{-1}|d\xi'|^2 .$$

Remark 3.1. Taking into account (3.9) we may suppose that $\mu^{-1}\leqq\langle\xi'\rangle$, $\mu^{-2}\langle\xi'\rangle^{-2}\leqq\langle\xi'\rangle^{-1}$ and hence we may assume that

$$dx_0^2+\tilde{G}_\mu\leqq\tilde{g}_\mu+dx_0^2 .$$

Proof. First we prove this proposition when $\tilde{\lambda}$ is independent of (x,ξ'). After this we shall reduce the proof of the general case to this. Put $\tilde{B}=(B_1(x,\xi',\mu),\cdots,B_k(x,\xi',\mu),\langle\mu\xi'\rangle^{1/2})$ and consider $\tilde{\lambda}(\tilde{B})$ where $\tilde{\lambda}(\zeta',\sigma)$ satisfies

$$|\partial^\gamma_{(\zeta',\sigma)}\tilde{\lambda}(\zeta',\sigma)|\leqq c_\gamma(|\zeta'|^2+\sigma^2)^{(n-|\gamma|)/2} .$$

Since $|(\langle\mu\xi'\rangle^{1/2})^{(\alpha)}|\leqq c_\alpha\{\langle\mu\xi'\rangle^{1/2}\langle\xi'\rangle^{-1}\}\langle\xi'\rangle^{1-|\alpha|}$ and we may assume $\langle\mu\xi'\rangle^{1/2}\langle\xi'\rangle^{-1}\leqq c\mu$ modulo $S_\mu^{-\infty}$ hence $B_{k+1}=\langle\mu\xi'\rangle^{1/2}$ verifies (3.6). We start with

$$\begin{aligned}\partial^{\hat{\alpha}}_{\xi'}\partial^{\hat{\beta}}_x(a^{(\tilde{\alpha})}_{(\tilde{\beta})}) &= \Sigma C(\gamma,\alpha,\beta)\,\partial^\gamma_{(\zeta',\sigma)}\tilde{\lambda}(B)\,\tilde{B}^{(\alpha_1)}_{i_1(\beta_1)}\cdots B^{(\alpha_s)}_{i_s(\beta_s)}= \\ &=\sum_{s=1}+\sum_{s\geqq2}=\Sigma'+\Sigma''\end{aligned}\tag{3.12}$$

where $|\gamma|\geqq1$, $\beta=\hat{\beta}+\tilde{\beta}$, $\alpha=\hat{\alpha}+\tilde{\alpha}$, $|\alpha_i+\beta_i|\geqq1$, $\alpha=\alpha_1+\cdots+\alpha_s$, $\beta=\beta_1+\cdots+\beta_s$. We study the case $|\tilde{\alpha}+\tilde{\beta}|=1$. Noting that $|\tilde{B}|=\langle\mu\xi'\rangle m(B')$ it follows that

$$|\Sigma'|\leqq c|\tilde{B}|^{n-1}|B^{(\alpha)}_{j_1(\beta)}|\leqq c\langle\mu\xi'\rangle^{n-1}m(B')^{n-1}|B^{(\alpha)}_{j_1(\beta)}| .$$

Since $\alpha=\hat{\alpha}+\tilde{\alpha}$, $\beta=\hat{\beta}+\tilde{\beta}$, from Corollary 3.1 $|B^{(\alpha)}_{i_1(\beta)}|$ is estimated by $c_{\alpha\beta}\,\mu^{1-|\tilde{\alpha}'''+\tilde{\beta}'''|/2}\langle\xi'\rangle^{1-|\tilde{\alpha}|}\langle\xi'\rangle^{-|\hat{\alpha}|/2}\langle\mu\xi'\rangle^{-|\hat{\beta}''|/2}\langle\xi'\rangle^{|\hat{\beta}'''|/2}$ and hence

$$\begin{aligned}|\Sigma'|\leqq c_{\hat{\alpha}\hat{\beta}}\,\mu^{-|\tilde{\alpha}'''+\tilde{\beta}'''|/2}\,m(B')^{n-1}\langle\mu\xi'\rangle^{n}\langle\xi'\rangle^{-|\tilde{\alpha}|} \\ \times\langle\xi'\rangle^{-|\hat{\alpha}|/2}\langle\mu\xi'\rangle^{|\hat{\beta}''|/2}\langle\xi'\rangle^{|\hat{\beta}'''|/2} .\end{aligned}\tag{3.13}$$

We turn to Σ''. When $|\tilde{\alpha}|=1$ there is k such that $\alpha_k\geqq\tilde{\alpha}$ and hence we may assume that $\alpha_1\geqq\tilde{\alpha}$. Then as in the preceding argument $|\tilde{B}|^{n-1}|B^{(\alpha_1)}_{i_1(\beta_1)}|$ is estimated by

$$\begin{aligned}c_{\alpha_1\beta_1}\,\mu^{-|\tilde{\alpha}'''/2|}\,m(B')^{n-1}\langle\mu\xi'\rangle^{n}\langle\xi'\rangle^{-1}\langle\xi'\rangle^{-(|\alpha_1|-1)/2}\langle\mu\xi'\rangle^{|\beta_1''|/2}\times \\ \times\langle\xi'\rangle^{|\beta_1'''|/2} .\end{aligned}\tag{3.14}$$

On the other hand by Lemma 3.2 $|\tilde{B}|^{1-s}\prod_{j=2}^{s}|B^{(\alpha_j)}_{i_j(\beta_j)}|$ is bounded by

$$c_{\alpha\beta}\langle\xi'\rangle^{-(|\alpha|-|\alpha_1|)/2}\langle\mu\xi'\rangle^{(|\beta''|-|\beta_1''|)/2}\langle\xi'\rangle^{(|\beta'''|-|\beta_1'''|)/2}\tag{3.15}$$

for $|\tilde{B}|^{-1}\leqq\langle\mu\xi'\rangle^{-1/2}$. Now (3.14) and (3.15) imply that $|\Sigma''|$ is estimated by

$$\begin{aligned}c_{\hat{\alpha}\hat{\beta}}\,\mu^{-|\tilde{\alpha}'''|/2}\,m(B')^{n-1}\langle\mu\xi'\rangle^{n}\langle\xi'\rangle^{-1}\langle\xi'\rangle^{-|\hat{\alpha}|/2} \\ \times\langle\mu\xi'\rangle^{|\hat{\beta}''|/2}\langle\xi'\rangle^{|\hat{\beta}'''|/2} .\end{aligned}\tag{3.16}$$

(3.13) and (3.16) show that

$$\partial_{\xi'}^{\alpha}\partial_x^{\beta}\tilde{\lambda}(\tilde{B})\in S(\mu^{-|\alpha'''+\beta'''|/2}\,m(B')^{n-|\alpha+\beta|}\langle\mu\xi'\rangle^n\langle\xi'\rangle^{-|\alpha|},\ \tilde{g}_\mu+\langle\mu\xi'\rangle\,dx_0^2)$$

if $|\alpha+\beta|\leqq 1$. In the case $|\tilde{\beta}|=1$ the proof is similar.

Now we shall study the case $|\tilde{\alpha}+\tilde{\beta}|=2$. Let $\alpha=\hat{\alpha}+\tilde{\alpha}$, $\beta=\hat{\beta}+\tilde{\beta}$. Then Corollary 3.1 gives that

$$|B_{i_1(\beta)}^{(\alpha)}|\leqq c_{\alpha\beta}\,\mu^{1-|\alpha+\beta'''|/2}\langle\xi'\rangle^{1-|\tilde{\alpha}|}\langle\xi'\rangle^{-|\hat{\alpha}|/2}\langle\mu\xi'\rangle^{|\hat{\beta}''|/2}\langle\xi'\rangle^{|\hat{\beta}'''|/2}$$

and hence $|\tilde{B}|^{n-1}|B_{i_1(\beta)}^{(\alpha)}|$ and $|\Sigma'|$ are bounded by

$$\begin{aligned}&c_{\hat{\alpha}\hat{\beta}}\,\mu^{-|\alpha+\beta'''|/2}\,m(B')^{n-2}\langle\mu\xi'\rangle^n\langle\xi'\rangle^{-|\tilde{\alpha}|}\langle\xi'\rangle^{-|\hat{\alpha}|/2}\\&\times\langle\mu\xi'\rangle^{|\hat{\beta}''|/2}\langle\xi'\rangle^{|\hat{\beta}'''|/2}\end{aligned}\tag{3.17}$$

for $m(B')^{n-1}\leqq Cm(B')^{n-2}$. We estimate Σ''. Let $|\tilde{\beta}|=2$. Then there are j, k such that $\beta_j+\beta_k\geqq\tilde{\beta}$ hence we can assume that $\beta_1+\beta_2\geqq\tilde{\beta}$. Corollary 3.1 shows that $\prod_{l=1}^{2}|B_{i_l(\beta_l)}^{(\alpha_l)}|$ is estimated by

$$\begin{aligned}&c_{\alpha\beta}\,\mu^{2-(t_1+t_2)/2}\langle\xi'\rangle^{2-|\alpha_1+\alpha_2|/2}\langle\mu\xi'\rangle^{(|\beta_1''+\beta_2''|-(s_1+s_2))/2}\\&\times\langle\xi'\rangle^{(|\beta_2'''+\beta_2'''|-(t_1+t_2))/2}\end{aligned}$$

for $\beta_i''\geqq s_i$, $\beta_i'''\geqq t_i$. Since $\beta_1'''+\beta_2'''\geqq\tilde{\beta}'''$ it follows that $|\tilde{B}|^{n-2}\times\prod_{l=1}^{2}|B_{i_l((\beta_l)}^{(\alpha_l)}|$ is bounded by

$$\begin{aligned}&c_{\hat{\alpha}\hat{\beta}}\,\mu^{-|\tilde{\beta}'''|/2}\,m(B')^{n-2}\langle\mu\xi'\rangle^n\langle\xi'\rangle^{-|\alpha_1+\alpha_2|/2}\langle\mu\xi'\rangle^{(|\beta_1''+\beta_2''|-|\tilde{\beta}''|)/2}\\&\times\langle\xi'\rangle^{(|\beta_1'''+\beta_2'''|-|\tilde{\beta}'''|)/2}.\end{aligned}$$

On the other hand Lemma 3.3 shows that $|\tilde{B}|^{2-s}\prod_{l=3}^{s}|B_{i_l(\beta_l)}^{(\alpha_l)}|$ is estimated by

$$\begin{aligned}&c_{\alpha\beta}\langle\xi'\rangle^{-(|\alpha|-|\alpha_1+\alpha_2|)/2}\langle\mu\xi'\rangle^{(|\beta''|-|\beta_1''+\beta_2''|)/2}\\&\times\langle\xi'\rangle^{(|\beta'''|-|\beta_1'''+\beta_2'''|)/2}.\end{aligned}\tag{3.18}$$

From these estimates it follows that $|\Sigma''|$ is bounded by

$$c_{\hat{\alpha}\hat{\beta}}\,\mu^{-|\tilde{\beta}'''|/2}\,m(B')^{n-2}\langle\mu\xi'\rangle^n\langle\xi'\rangle^{-|\hat{\alpha}|/2}\langle\mu\xi'\rangle^{|\hat{\beta}''|/2}\langle\xi'\rangle^{|\hat{\beta}'''|/2}.\tag{3.19}$$

Let $|\tilde{\alpha}|=2$. As in the preceding argument we may suppose that $\alpha_1+\alpha_2\geqq\tilde{\alpha}$. In view of Corollary 3.1 $\prod_{l=1}^{2}|B_{i_l(\beta_l))}^{(\alpha_l)}|$ is estimated by

$$c_{\alpha\beta}\,\mu\langle\xi'\rangle^{-(|\alpha_1+\alpha_2|-2)/2}\langle\mu\xi'\rangle^{|\beta_2''+\beta_2''|/2}\langle\xi'\rangle^{|\beta_1'''+\beta_2'''|/2}.$$

Since $|\tilde{B}|^{n-2}\mu\leqq C\mu^{-1}\,m(B')^{n-2}\langle\mu\xi'\rangle^n\langle\xi'\rangle^{-2}$ using (3.18) we can estimate $|\Sigma''|$ by

$$c_{\hat{\alpha}\hat{\beta}}\,\mu^{-1}\,m(B')^{n-2}\langle\mu\xi'\rangle^n\langle\xi'\rangle^{-2}\langle\xi'\rangle^{-|\hat{\alpha}|/2}\langle\mu\xi'\rangle^{|\hat{\beta}''|/2}\langle\xi'\rangle^{|\hat{\beta}'''|/2}.\tag{3.20}$$

Let $|\tilde{\alpha}|=|\tilde{\beta}|=1$. We may assume that $\alpha_1\geqq\tilde{\alpha}_1$ and $\beta_1+\beta_2\geqq\tilde{\beta}$. By Corollary 3.1 we can give an estimate of $\prod_{l=1}^{2}|B_{i_l(\beta_l)}^{(\alpha_l)}|$ by

$$\begin{aligned}&c_{\alpha\beta}\,\mu^{2-|\tilde{\beta}'''|/2-|\tilde{\alpha}'''|/2}\langle\xi'\rangle^{1-(|\alpha_1+\alpha_2|-1)/2}\langle\mu\xi'\rangle^{(|\beta_1''+\beta_2''|-|\tilde{\beta}''|)/2}\\&\times\langle\xi'\rangle^{(|\beta_1'''+\beta_2'''|-|\tilde{\beta}'''|)/2}.\end{aligned}$$

Taking this estimate into account the same argument as above gives an estimate of $|\Sigma''|$ by

$$c_{\hat{\alpha}\hat{\beta}}\, m(B')^{n-2}\langle\mu\xi'\rangle^{n}\langle\xi'\rangle^{-1}\langle\xi'\rangle^{-|\hat{\alpha}|/2}\langle\mu\xi'\rangle^{|\hat{\beta}''|/2}\langle\xi'\rangle^{|\hat{\beta}'''|/2}\,. \tag{3.21}$$

These estimates prove that

$$\partial_{\xi'}^{\alpha}\,\partial_{x}^{\beta}\,\tilde{\lambda}(\tilde{B})\in S(\mu^{-|\alpha+\beta'''|/2}\, m(B')^{n-|\alpha+\beta|}\langle\mu\xi'\rangle^{n}\langle\xi'\rangle^{-|\alpha|},\ \tilde{g}_{\mu}+\langle\mu\xi'\rangle\, dx_0^2)$$

if $|\alpha+\beta|=2$. Now we shall give a proof for the general case reducing it to that we have proved. First note that

$$\begin{aligned}\partial_{\xi'}^{\alpha}\,\partial_{x}^{\beta}\, a(x,\xi',\mu) &= \Sigma C(\nu,\delta,\gamma,\hat{\alpha},\tilde{\alpha},\hat{\beta},\tilde{\beta})\,\partial_{\xi'}^{\nu}\,\partial_{x}^{\delta}\,\partial_{(\zeta',\alpha)}^{\gamma}\,\tilde{\lambda}(\tilde{B};y,\mu\eta')\\ &\times\prod_{l=1}^{s}B_{i_l(\hat{\beta}_l)}^{(\hat{\alpha}_l)}\prod_{l=1}^{t}y_{j_l(\tilde{\beta}_l)}\prod_{l=1}^{u}(\mu\eta_{k_l})^{(\tilde{\alpha}_l)}\end{aligned}$$

where $|\gamma|=s$, $|\delta|=t$, $|\nu|=u$, $|\hat{\alpha}_i+\hat{\beta}_i|$, $|\tilde{\beta}_i|$, $|\tilde{\alpha}_i|\geqq 1$, $\alpha=\hat{\alpha}+\tilde{\alpha}$ $\beta=\hat{\beta}+\tilde{\beta}$. We note that

$$|\langle\xi'\rangle y_{j(\beta)}|\leqq c_{\beta}\,\mu^{1-|\beta'''|/2}\langle\xi'\rangle,\ \mu|(\mu\eta_j)^{(\alpha)}|\leq c_{\alpha}\,\mu^{2-|\alpha|}\langle\xi'\rangle^{1-|\alpha|}\,. \tag{3.22}$$

That is $\langle\xi'\rangle y_j$, $\mu(\mu\eta_j)$ satisfy the smae estimate (3.6). On the other hand it follows from (3.2) and (3.10) that

$$\begin{aligned}|\partial_{\xi'}^{\nu}\,\partial_{x}^{\delta}\,\partial_{(\zeta',\sigma)}^{\gamma}\,\tilde{\lambda}(\tilde{B};y,\mu\eta')| &\leqq c_{\nu\delta\gamma}(|B'|^2+\langle\mu\xi'\rangle)^{(n-s)/2}\langle\mu\eta'\rangle^{m-u}\leqq\\ &\leqq c_{\nu\delta\gamma}(|B'|^2+\langle\mu\xi'\rangle)^{(n-s)/2}\langle\xi'\rangle^{m-u}\,.\end{aligned}$$

Since $\langle\xi'\rangle^{-u}\langle\xi'\rangle^{-t}\,\mu^{-u}\leqq C(|B'|^2+\langle\mu\xi'\rangle)^{-(u+t)/2}$ (modulo $S_{\mu}^{-\infty}$) one can estimate $|\partial_{\xi'}^{\alpha}\,\partial_{x}^{\beta}\, a|$ by

$$\begin{aligned}&\Sigma C(\nu,\delta,\gamma,\hat{\alpha},\tilde{\alpha},\hat{\beta},\tilde{\beta})\,(|B'|^2+\langle\mu\xi'\rangle)^{(n-(s+u+t))/2}\langle\xi'\rangle^{m}\\ &\times\prod_{l=1}^{s}|\hat{B}_{i_l(\hat{\beta}_l)}^{(\alpha_l)}|\prod_{l=1}^{t}|\langle\xi'\rangle y_{j_l(\tilde{\beta}_l)}|\prod_{l=1}^{u}|\mu(\mu\eta_{k_l})^{(\tilde{\alpha}_l)}|\,.\end{aligned}$$

Since to prove the proposition for $\tilde{\lambda}(\tilde{B})$ we have used only the formula (3.21) and the estimate (3.6) then noting (3.22) the proof can be reduced to the previous case.

4. Some properties of pseudodifferential operators

We use notation and calculus in [5] (Chapter 18). We shall use the following metrics;

$$\begin{aligned}g_{\mu}(dx',d\xi') &= \langle\mu\xi'\rangle|dx'|^2+\langle\xi'\rangle^{-2}\langle\mu\xi'\rangle|d\xi'|^2\,,\\ \tilde{g}_{\mu}(dx',d\xi') &= \langle\mu\xi'\rangle|dx''|^2+\langle\xi'\rangle|dx'''|^2+\langle\xi'\rangle^{-1}|d\xi'|^2\quad\text{at}\quad(x',\xi')\,.\end{aligned}$$

These metrics are slowly varying and σ temperate uniformly when $0<\mu\leqq 1$. We denote $g(dx',d\xi')=g_1(dx',d\xi')=\tilde{g}_1(dx',d\xi')$. Remark that $g^{\sigma}=g$, $g_{\mu}\leqq\tilde{g}_{\mu}\leqq g$. We say that a positive function $m(x,\xi',\mu)\in C^{\infty}(I\times\boldsymbol{R}^{2d}\times(0,\mu(m)])$ is a weight function if $m(x,\xi',\mu)$ is σ temperate with respect to the metric g uniformly when $0<\mu\leqq\mu(m)$

and satisfies

$$C^{-1}\mu^{N_2}\langle\xi'\rangle^{-N_1}\leqq m(x,\xi',\mu)\leqq C\mu^{-N_2}\langle\xi'\rangle^{N_1},\ 0<\mu\leqq\mu(m) \tag{4.1}$$

with constants C, N_i independent of μ. We denote by $\mathcal{K}$ the set of all weight functions. It is clear that if $m_i\in\mathcal{K}(i=1,2)$ then $m_1m_2\in\mathcal{K}$ and if $m\in\mathcal{K}$ so does $m^s\in\mathcal{K}$ for any real $s\in\boldsymbol{R}$.

We define $S(m,G_\mu)$ with $m\in\mathcal{K}$, $G_\mu=g_\mu$ or $\tilde{g}_\mu$ the set of all $a(x,\xi',\mu)=a(x_0,x',\xi',\mu)\in C^\infty(I\times\boldsymbol{R}^{2d}\times(0,\mu(a)])$ such that

$$a(x,\xi',\mu)\in S(m,\langle\mu\xi'\rangle dx_0^2+G_\mu)$$

uniformly when $0<\mu\leqq\mu(a)$. We also define $S_\mu^{-\infty}$ the set of all $a(x,\xi',\mu)\in C^\infty(I\times\boldsymbol{R}^{2d}\times(0,\mu(a)])$ such that for any $l\in\boldsymbol{N}$ there is $k(l)\in\boldsymbol{R}$ with

$$\mu^{k(l)}a(x,\xi',\mu)\in S(\langle\xi'\rangle^{-l},g)$$

uniformly when $0<\mu\leqq\mu(a)$. Let $\chi(s)\in C^\infty(\boldsymbol{R})$ be $\chi(s)=0$ in $|s|\leqq1$ and equal to 1 in $|s|\geqq2$ and set $\tilde{\chi}(\xi',\mu)=\chi(2^{-1}\mu\langle\xi'\rangle)$. For $a(x,\xi',\mu)\in S(m,G_\mu)$ it is obvious that $a\tilde{\chi}\in S(m,G_\mu)$. On the other hand it is clear that $a(1-\tilde{\chi})\in S_\mu^{-\infty}$ in view of (4.1) and $G_\mu\leqq g$. Since the operator with symbol in $S_\mu^{-\infty}$ is bounded from $L^2(\boldsymbol{R}^d_{x'})$ to a Sobolev space of any order on $\boldsymbol{R}^d_{x'}$ (although the operator norm depends possibly on μ) and hence is quite harmless in our arguments. Then we shall usually work with $S(m,G_\mu)/S_\mu^{-\infty}$ instead of $S(m,G_\mu)$. According to this note we shall ofen identify $a\in S(m,G_\mu)$ with $a\tilde{\chi}$.

Remark 4.1. Since we have $(2\mu\langle\xi'\rangle\langle\mu\xi'\rangle^{-1})^s\geqq1$ on the support of $\tilde{\chi}$ if $s\geqq0$ so it follows that

$$S(m,G_\mu)\subset S(\mu^s m\langle\mu\xi'\rangle^{-s}\langle\xi'\rangle^s,G_\mu)\,,\quad S(\mu^{-s}m\langle\xi'\rangle^{-s},G_\mu)\subset S(m\langle\mu\xi'\rangle^{-s},G_\mu)\,.$$

Lemma 4.1 *Let $\varphi_j(x,\xi',\mu)\in C^\infty(I\times\boldsymbol{R}^{2d}\times(0,\bar{\mu}])$ $(j=1,2,\cdots,k)$. Assume*

$$|\varphi_{j(\beta)}^{(\alpha)}(x,\xi',\mu)|\leqq C\mu^{-|\alpha+\beta|/2}\langle\xi'\rangle^{-|\alpha|}\quad\text{for}\quad|\alpha+\beta|\leqq1$$

with positive constnat C independent of μ. Then

$$m=\{\sum_{j=1}^k\varphi_j(x,\xi',\mu)^2+\langle\mu\xi'\rangle^{-1}\}^{1/2}\in\mathcal{K}\,. \tag{4.2}$$

For $a(x,\xi',\mu)\in S(m,G_\mu)$ we denote by $a(x,D',\mu)$ (or Opa) the operator with symbol $a(x,\xi',\mu)$. By $\sigma(a(x,D',\mu))$ we denote the symbol of $a(x,D',\mu)$. But sometimes we do not distinguish the operator and its symbol if there will be no confusion. $a(x,D',\mu)^*$ is the adjoint of a with respect to the scalar product in $L^2(\boldsymbol{R}^d_{x'})$. Noting that $g_\mu/g_\mu^\sigma\leqq\langle\mu\xi'\rangle\langle\xi'\rangle^{-1}$, $\tilde{g}_\mu/\tilde{g}_\mu^\sigma\leqq1$ we set $h(G_\mu)=\mu$, 1 according to $G_\mu=g_\mu$, $\tilde{g}_\mu$.

Lemma 4.2. *Let $m_i(x,\xi',\mu)\in\mathcal{K}$ and $a_i(x,\xi',\mu)\in S(m_i,G_\mu)$ $(i=1,2)$. Then $a_1(x,D',\mu)\,a_2(x,D',\mu)\in S(m_1m_2,G_\mu)$ and*

$$\sigma(a_1 a_2)-\sum_{|\alpha|<N}(\alpha!)^{-1} a_1^{(\alpha)}(x, \xi', \mu)\, a_{2(\alpha)}(x, \xi', \mu)\in S(h(G_\mu)^N, G_\mu)\,.$$

Lemma 4.3. *Let $m(x, \xi', \mu)\in\mathcal{K}$ and $a(x, \xi', \mu)\in S(m, G_\mu)$. Then $a(x, D', \mu)^*\in S(m, G_\mu)$ and*

$$\sigma(a^*)-\sum_{|\alpha|<N}(-1)^{|\alpha|}(\alpha!)^{-1}\,\overline{a_{(\alpha)}^{(\alpha)}(x, \xi', \mu)}\in S(h(G_\mu)^N, G_\mu)$$

where $\bar{a}$ denotes the complex conjugate of a.

Lemma 4.4. *Let $a(x, \xi', \mu)\in S(m, g_\mu)$ and $a(x, \xi', \mu)\geqq cm(x, \xi', \mu)$ with positive constant c independent of μ. Then there are $b(x, \xi', \mu)$, $\tilde{b}(x, \xi', \mu)\in S(m^{-1}, g)$ such that*

$$a(x, D', \mu)\, b(x, D', \mu)\equiv b(x, D', \mu)\, a(x, D', \mu)\equiv 1\,,$$
$$a(x, D', \mu)^*\, \tilde{b}(x, D', \mu)\equiv \tilde{b}(x, D', \mu)\, a(x, D', \mu)^*\equiv 1$$

modulo $S_\mu^{-\infty}$.

Proof. Put $b_0(x, \xi', \mu)=a(x, \xi', \mu)^{-1}\in S(m^{-1}, g_\mu)$ then $\mathrm{Op}\, a\, \mathrm{Op}\, b_0-1=r(x, D', \mu)\in S(\mu, g_\mu)\subset S(\mu, g)$. Since

$$q(x, D', \mu)=\sum_{j=0}^{\infty} r(x, D', \mu)^j\in S(1, g)$$

we get $a(x, D', \mu)\, b(x, D', \mu)\equiv 1$ with $b(x, D', \mu)=b_0(x, D', \mu)\, q(x, D', \mu)\in S(m^{-1}, g)$. To prove the existence of $\tilde{b}(x, D', \mu)$ we note that $a(x, D', \mu)^*=a(x, D', \mu)+T$ with $T\in S(\mu m, g_\mu)$. Using the first part in Lemma 4.5 below one can write

$$a(x, D', \mu)^*\equiv a(x, D', \mu)\,(1+r) \quad \text{with} \quad r\in S(\mu, g)\,.$$

Denote by $\tilde{q}\in S(1, g)$, $q\in S(m^{-1}, g)$ parametrices of $(1+r)$ and a that we have just constructed above it follows that $\tilde{b}=\tilde{q}\; q\in S(m^{-1}, g)$ is the desired one.

Lemma 4.5. *Assume that $a_i(x, \xi', \mu)\in S(m_i, g_\mu)$ and $a_i(x, \xi', \mu)\geqq c_i\, m_i(x, \xi', \mu)$ with positive constants c_i independent of $\mu(i=1, 2)$. Let $b(x, \xi', \mu)\in S(m, g_\mu)$ then we have*

$$\mathrm{Op}\, b\equiv \mathrm{Op}\, a_1\{\mathrm{Op}\, c+r\}\ \mathrm{Op}\, a_2,\ \mathrm{Op}\, b\equiv(\mathrm{Op}\, a_1)^*\{\mathrm{Op}\, c+\tilde{r}\}\ \mathrm{Op}\, a_2$$

with $\sigma(c)=\sigma(b)\,\sigma(a_1)^{-1}\,\sigma(a_2)^{-1}$ *and* r, $\tilde{r}\in S(\mu m m_1^{-1}\, m_2^{-1}, g)$.

Remark 4.2. The same argument shows that

$$\mathrm{Op}\, b\equiv\{\mathrm{Op}\, c+r\}\ \mathrm{Op}\, a_1\, \mathrm{Op}\, a_2\,, \quad \mathrm{Op}\, b\equiv \mathrm{Op}\, a_1\, \mathrm{Op}\, a_2\, \{\mathrm{Op}\, c+r\} \quad \text{etc.}\,,$$

with possibly different $r\in S(\mu m m_1^{-1}\, m_2^{-1}, g)$. If $b(x, \xi', \mu)\in S(m, g)$ then we can write

$$\mathrm{Op}\, b\equiv \mathrm{Op}\, a_1\, c(x, D', \mu)\ \mathrm{Op}\, a_2\,, \quad \mathrm{Op}\, b\equiv(\mathrm{Op}\, a_1)^*\, c(x, D'\, \mu)\ \mathrm{Op}\, a_2 \quad \text{etc.},$$

with possibly difierent $c(x, D', \mu)\in S(m m_1^{-1}\, m_2^{-1}, g)$.

Let $\Lambda(x, \xi')=(\mu^{1/2} x, \mu^{-1/2} \xi')$ and put

$$\bar{g}_\mu(dx', d\xi') = \langle\mu^{1/2} \xi'\rangle |dx'|^2+\langle\mu^{1/2} \xi'\rangle^{-1} |d\xi'|^2 .$$

Then it is easy to see that $a\in S(1, g_\mu)$ if and only if $\Lambda^*a\in S(1, \mu\bar{g}_\mu)$, where Λ^*a denotes the pull back of a by Λ (here we have identified a with $a\tilde{\chi}$).

Lemma 4.6. *Let $a(x, \xi', \mu)\in S(1, g_\mu)$ and $\sup|a(x, \xi', \mu)|=c$, with a constant c independent of μ. Then*

$$||a(x, D', \mu) u||\leqq(c+c(a) \mu)||u||$$

where $||\cdot||$ is the $L^2(\boldsymbol{R}^d)$ norm.

Proof. Recall hat $\Lambda^*a\in S(1, \mu\bar{g}_\mu)\subset S(1, \bar{g}_\mu)$. Hence Op Λ^*a is L^2 bounded. Moreover $\Lambda^*a\in S(1, \mu\bar{g}_\mu)$ implies that

$$|\Lambda^*a|_k^{\bar{g}_\mu}\leqq(c+c_k \mu) \quad \text{for any} \quad k\in\boldsymbol{N} .$$

Then it follows that $||(\mathrm{Op}\ \Lambda^*a) u||\leqq(c+c(a) \mu) ||u||$ which proves the lemma.

Now we observe some special symbols. Let $\varphi_j(x, \xi', \mu)\in C^\infty(I\times\boldsymbol{R}^{2d}\times(0, \bar{\mu}])$ $(j=1, \cdots, k)$ and assume that

$$\begin{aligned} &|\varphi_{j(\beta)}^{(\alpha)}(x, \xi', \mu)|\leqq c_{\alpha\beta}\langle\xi'\rangle^{-|\alpha|} \quad \text{for} \quad |\alpha+\beta|\leqq 1 \\ &|\varphi_{j(\beta)}^{(\alpha)}(x, \xi', \mu)|\leqq c_{\alpha\beta}\, \mu^{-|\alpha+\beta|}\langle\xi'\rangle^{-|\alpha|} \quad \text{for} \quad |\alpha+\beta|\geqq 2 \end{aligned} \tag{4.3}$$

with positive constants $c_{\alpha\beta}$ independent of μ. We define $m(\Phi)$ by (4.2).

Lemma 4.7. *Assume* (4.3). *Then*

$$m(\Phi)_{(\beta)}^{(\alpha)}\in S(m(\Phi)^{1-|\alpha+\beta|}\langle\xi'\rangle^{-|\alpha|}, g_\mu) \quad \text{for} \quad |\alpha+\beta|\leqq 1 .$$

Remark 4.3. Assume that $\varphi_j(x, \xi', \mu)\in C^\infty(I\times\boldsymbol{R}^{2d}\times(0, \bar{\mu}])$ $(j=1, \cdots, k)$ satisfy

$$\begin{aligned} &|\varphi_{j(\beta)}^{(\alpha)}(x, \xi', \mu)|\leqq c_{\alpha\beta}\, \mu^{-|\alpha+\beta|/2}\langle\xi'\rangle^{-|\alpha|} \quad \text{for} \quad |\alpha+\beta|\leqq 1 \\ &|\varphi_{j(\beta)}^{(\alpha)}(x, \xi', \mu)|\leqq c_{\alpha\beta}\, \mu^{-|\alpha+\beta|}\langle\xi'\rangle^{-|\alpha|} \quad \text{for} \quad |\alpha+\beta|\geqq 2 \end{aligned} \tag{4.4}$$

uniformly in μ. Then the same argument gives that

$$m(\Phi)_{(\beta)}^{(\alpha)}\in S(\mu^{-|\alpha+\beta|/2}\, m(\Phi)^{1-|\alpha+\beta|}\langle\xi'\rangle^{-|\alpha|}, g) \quad \text{if} \quad |\alpha+\beta|\leqq 1 .$$

Suppose that $\varphi_j(x, \xi', \mu)\in C^\infty(I\times\boldsymbol{R}^{2d}\times(0, \bar{\mu}])$ verify the hypothesis in Lemma 4.7. Let $a(x, \xi', \mu)\in S(m, g_\mu)$ satisfy;

$$a_{(\beta)}^{(\alpha)}\in S(mm(\Phi)^{-|\alpha+\beta|}\langle\xi'\rangle^{-|\alpha|}, g_\mu) \quad \text{for all} \quad \alpha, \beta \tag{4.5}$$

$$C^{-1}\langle\mu\xi'\rangle^{-1/2}\leqq m(\Phi)\leqq C\langle\mu\xi'\rangle^{-1/2} \quad \text{on} \quad \operatorname{supp} a_{(\beta)}^{(\alpha)} \quad \text{with} \quad |\alpha+\beta|\geqq 1 \tag{4.6}$$

with a positive constant C independent of μ. As an example choose $\chi(s)\in C^\infty(\boldsymbol{R})$

such that $\chi(s)=1$ for $|s|\geqq 1$. Then $a(x, \xi', \mu)=\chi(m(\Phi)\langle\mu\xi'\rangle^{1/2})$ satisfy the conditions (4.5) and (4.6) since $\langle\mu\xi'\rangle^{1/2}m(\Phi)\in S(m(\Phi)\langle\mu\xi'\rangle^{1/2}, g_\mu)$ and $m(\Phi)\langle\mu\xi'\rangle^{1/2}$ is bounded on the support of $\chi^{(k)}(m(\Phi)\langle\mu\xi'\rangle^{1/2})$ $(k\geqq 1)$.

Lemma 4.8. *Assume* (4.5) *and* (4.6). *Let* $b(x, \xi', \mu)\in S(\tilde{m}, g_\mu)$ *then*

$$a(x, D', \mu)\,b(x, D', \mu)-\mathrm{Op}\,(ab)\in S(m\tilde{m}\langle\xi'\rangle^{-1}\langle\mu\xi'\rangle^{1+r/2}\,m(\Phi)^r, g_\mu)$$

for any $r\in\boldsymbol{R}$.

Proof. By (4.5) and (4.6), $a^{(\alpha)}$ belongs to $S(m\langle\xi'\rangle^{-|\alpha|}\langle\mu\xi'\rangle^{i(\alpha)/2}\,m(\Phi)^{-|\alpha|+i(\alpha)}$, $g_\mu)$ for any real $i(\alpha)$. On the other hand $b_{(\alpha)}\in S(\tilde{m}\langle\mu\xi'\rangle^{|\alpha|/2}, g_\mu)$ and hence

$$a^{(\alpha)}\,b_{(\alpha)}\in S(m\tilde{m}\langle\xi'\rangle^{-|\alpha|}\langle\mu\xi'\rangle^{|\alpha|/2+i(\alpha)/2}\,m(\Phi)^{-|\alpha|+i(\alpha)}, g_\mu), \quad |\alpha|\geqq 1\,.$$

With $i(\alpha)=|\alpha|+r$ this proves the statement.

Lemma 4.9. *Assume that* $a(x, \xi', \mu)\in S(m, g_\mu)$, $b(x, \xi', \mu)\in S(\tilde{m}, g_\mu)$ *satisfy* (4.5) *and* (4.6) *with* $m(\Phi)$, $m(\Psi)$ *respectively. Then*

$$[a, b]\in S(m\tilde{m}\langle\xi'\rangle^{-1}\langle\mu\xi'\rangle^{1+(r+s)/2}\,m(\Phi)^r\,m(\Psi)^s, g_\mu)$$

for any real r, $s\in\boldsymbol{R}$ *where* $[a, b]$ *is the commutator of* a *and* b.

5. Energy estimate for first order operators

Energy estimate for the first order operator

$$L(x, D, \mu) = D_0-i\theta-a(x, D', \mu)\,, \quad \theta\gg 1$$

will be proved by energy integral method, where $a(x, \xi', \mu)$ is real and satisfies

$$\text{(5.1)}\qquad \begin{aligned} &a^{(\alpha)}_{(\beta)}(x, \xi', \mu)\in S(\mu^{-|\alpha''+\beta''|/2}\,m(B')^{1-|\alpha+\beta|}\langle\mu\xi'\rangle\langle\xi'\rangle^{-|\alpha|}, g)\\ &a^{(\alpha)}_{(\beta)}(x, \xi', \mu)\in S(\mu^{-|\alpha+\beta''|/2}\,m(B')^{1-|\alpha+\beta|}\langle\mu\xi'\rangle\langle\xi'\rangle^{-|\alpha|}, g)\end{aligned}$$

when $|\alpha+\beta|\leqq 1$ and $|\alpha+\beta|=2$ respectively where $B_j(x, \xi', \mu)$ and $m(B')$ are defined in section 3. Note that $B_j(x, \xi', \mu)\langle\mu\xi'\rangle^{-1}$ verifies the conditions in Lemma 4.1 for $B_j(x, \xi', \mu)$ satisfy (3.6) and hence $m(B')\in\mathcal{K}$. Let $\chi_2(s)$, $\chi_3(s)\in C^\infty(\boldsymbol{R})$ such that

$$\chi_2(s) = 0 \text{ in } s\leqq -1/2,\ \chi_2(s) = 1 \text{ in } s\geqq -1/4,\ 0\leqq\chi_2(s)\leqq 1 \quad \text{for} \quad s\in\boldsymbol{R}$$
$$\chi_3(s) = 0 \text{ in } s\leqq -1,\ \chi_3(s) = 1 \text{ in } s\geqq 1,\ 0\leqq\chi_3(s)\leqq 1,\ \chi_3(s)+\chi_3(-s) = 1 \quad \text{for} \quad s\in\boldsymbol{R}\,.$$

We introduce following symbols;

$$\begin{aligned} \alpha_\varepsilon(x, \xi', \mu) &= \chi_3(\varepsilon\, n^{1/2}\,\varphi(x, \xi', \mu)\langle\mu\xi'\rangle^{1/2})\\ J_\varepsilon(x, \xi', \mu) &= \varepsilon\{2\chi_2(\varepsilon\varphi(x, \xi', \mu)\langle\mu\xi'\rangle^{1/2})-1\}\,\varphi(x, \xi', \mu)+\langle\mu\xi'\rangle^{-1/2}\\ I_\varepsilon(r)\,(x, \xi', \mu) &= \langle\mu\xi'\rangle^{n\tilde{\varepsilon}}\,J_\varepsilon(x, \xi', \mu)^{-n\varepsilon-r}\end{aligned}$$

with $\varphi(x, \xi', \mu)$ defined by (3.7) where $\varepsilon=\pm 1$, $\tilde{\varepsilon}=\max\,(0, -\varepsilon)$, $r\in\boldsymbol{R}$, $n\in\boldsymbol{R}^+$. Note that $\varphi(x, \xi', \mu)$ satisfies (4.3) by (3.8). We define $m(\varphi)$ by (4.2) with $\varphi_1=\varphi$ and $k=1$ then $m(\varphi)\in\mathcal{K}$ in view of Lemma 4.1.

Lemma 5.1. *With positive constants c_i, one has*

$$c_1\, m(\varphi)\leqq J_\varepsilon(x, \xi', \mu)\leqq c_2\, m(\varphi)$$

uniformly when $0<\mu\leqq\hat{\mu}$.

Lemma 5.2. $J^{(\alpha)}_{\varepsilon(\beta)}\in S(m(\varphi)^{1-|\alpha+\beta|}\langle\xi'\rangle^{-|\alpha|}, g_\mu)$ *for* $|\alpha+\beta|\leqq 1$.

From Lemma 5.1 we have $J^r_\varepsilon\in S(m(\varphi)^r, g_\mu)$ for any $r\in\boldsymbol{R}$ and by Lemma 5.2 it follows that

$$I_\varepsilon(r)^{(\alpha)}_{(\beta)}\in S(\langle\mu\xi'\rangle^{n\varepsilon}\, m(\varphi)^{-n\tilde{\varepsilon}-r-|\alpha+\beta|}\langle\xi'\rangle^{-|\alpha|}, g_\mu) \tag{5.2}$$

for $|\alpha+\beta|\leqq 1$. Notice that Lemma 5.1 gives that

$$I_\varepsilon(r)\geqq c\langle\mu\xi'\rangle^{n\tilde{\varepsilon}}\, m(\varphi)^{-n\varepsilon-r} \tag{5.3}$$

with a positive constant c independent of μ. We start with the following identity;

$$\begin{aligned}
&-2\,\mathrm{Im}(I_\varepsilon(r)\, L\alpha_\varepsilon\, u, I_\varepsilon(r)\,\alpha_\varepsilon\, u) = -2\,\mathrm{Im}([I_\varepsilon(r), L]\,\alpha_\varepsilon\, u, I_\varepsilon(r)\,\alpha_\varepsilon\, u)\,,\\
&-2\,\mathrm{Im}(LI_\varepsilon(r)\,\alpha_\varepsilon\, u, I_\varepsilon(r)\,\alpha_\varepsilon\, u) = \partial_0\|I_\varepsilon(r)\,\alpha_\varepsilon\, u\|^2+2\theta\|I_\varepsilon(r)\,\alpha_\varepsilon\, u\|^2\\
&-2\,\mathrm{Im}(aI_\varepsilon(r)\,\alpha_\varepsilon\, u, I_\varepsilon(r)\,\alpha_\varepsilon\, u)-2\,\mathrm{Im}([I_\varepsilon(r), L]\,\alpha_\varepsilon\, u, I_\varepsilon(r)\,\alpha_\varepsilon\, u)
\end{aligned} \tag{5.4}$$

where $\partial_0=\partial/\partial x_0$, Im A=the imaginary part of A and $(\cdot, \cdot)$ denotes the scalar product in $L^2(\boldsymbol{R}^d)$. Take $r=1/2$ in (5.4) and estimate the third term in the right-hand side of (5.4). From Lemma 4.3 and (5.1) it follows that $a^*-a\in S(\mu^{-1}\langle\mu\xi'\rangle\langle\xi'\rangle^{-1} m(B')^{-1}, g)\subset S(m(B')^{-1}, g)$. Noting that $Cm(B')\geqq m(\varphi)$ one obtains

$$a^*-a\in S(m(\varphi)^{-1}, g)\,. \tag{5.5}$$

(5.2), (5.3) and Remark 4.2 show that with $J_\varepsilon(r)=\mathrm{Op}(J^{-r}_\varepsilon)$

$$I_\varepsilon(1/2)^*\equiv I_\varepsilon(1)^*\, J_\varepsilon(-1/2)\,(1+r)\,,\quad I_\varepsilon(1/2)\equiv(1+\tilde{r})\, J_\varepsilon(-1/2)\, I_\varepsilon(1) \tag{5.6}$$

with $r, \tilde{r}\in S(\mu, g)$. On the other hand one has

$$J_\varepsilon(-1/2)\,(1+r)\,(a^*-a)\,(1+\tilde{r})\, J_\varepsilon(-1/2)\equiv A+R \tag{5.7}$$

with $A=J_\varepsilon(-1/2)\,(a^*-a)\, J_\varepsilon(-1/2)\in S(1, g)$ in view of (5.5) and $R\in S(\mu, g)$. We note that A does not depend on n whereas R depends possibly on n. From (5.6) and (5.7) it follows that $I_\varepsilon(1/2)^*(a^*-a)\, I_\varepsilon(1/2)\equiv I_\varepsilon(1)^*(A+R)\, I_\varepsilon(1)$ and this proves that

$$|\mathrm{Im}(aI_\varepsilon(1/2)\,\alpha_\varepsilon\, u, I_\varepsilon(1/2)\,\alpha_\varepsilon\, u)|\leqq(c+c(n)\,\mu)\|I_\varepsilon(1)\,\alpha_\varepsilon\, u\|^2\,. \tag{5.8}$$

Next we estimate the last term of the right-hand side of (5.4), $[I_\varepsilon(1/2), L]=i\partial_0\, I_\varepsilon(1/2)$-$[I_\varepsilon(1/2), a]$. Since $a^{(\alpha)}_{(\beta)}$ belongs to $S(\mu^{-|\alpha''|+\beta''|/2}\langle\mu\xi'\rangle\, m(B')^{1-|\alpha+\beta|}\langle\xi'\rangle^{-|\alpha|}, g)\subset S(\mu^{-1/2}\langle\mu\xi'\rangle\langle\xi'\rangle^{-|\alpha|}, g)$ for $|\alpha+\beta|=1$, Lemma 4.2 and (5.2) give that

$$
\begin{aligned}
(5.9)\qquad &[I_\varepsilon(r), a] \in S(\mu^{-1/2}\langle\mu\xi'\rangle^{n\tilde{\varepsilon}+1}\, m(\varphi)^{n\varepsilon-r-1}\langle\xi'\rangle^{-1}, g) \subset \\
&\subset S(\mu^{1/2}\langle\mu\xi'\rangle^{n\tilde{\varepsilon}}\, m(\varphi)^{-n\varepsilon-r-1}, g)\,.
\end{aligned}
$$

Noting $I_\varepsilon(1/2)^*[I_\varepsilon(1/2), a] \in S(\mu^{1/2}\langle\mu\xi'\rangle^{2n\tilde{\varepsilon}}\, m(\varphi)^{-2n\varepsilon-2}, g)$, it follows from Remark 4.2 that $I_\varepsilon(1/2)^*[I_\varepsilon(1/2), a] \equiv I_\varepsilon(1)^* A I_\varepsilon(1)$ with $A \in S(\mu^{1/2}, g)$. This gives that

$$
(5.10)\qquad |\operatorname{Im}([I_\varepsilon(1/2), a]\,\alpha_\varepsilon u, I_\varepsilon(1/2)\,\alpha_\varepsilon u)| \leqq c(n)\,\mu^{1/2}\|I_\varepsilon(1)\,\alpha_\varepsilon u\|^2\,.
$$

Now we consider $-2\operatorname{Im}(i\partial_0 I_\varepsilon(1/2)\,\alpha_\varepsilon u, I_\varepsilon(1/2)\,\alpha_\varepsilon u) = -2\operatorname{Re}(\partial_0 I_\varepsilon(1/2)\,\alpha_\varepsilon u, I_\varepsilon(1/2)\,\alpha_\varepsilon u)$. Let us write down $\partial_0 I_\varepsilon(1/2)$ explicitly.

$$
\begin{aligned}
(5.11)\qquad \partial_0 I_\varepsilon(1/2) &= -(n\varepsilon+1/2)\, I_\varepsilon(3/2)\,\{2\langle\mu\xi'\rangle^{1/2}\,\chi_2^{(1)}(\varepsilon\varphi\langle\mu\xi'\rangle^{1/2})\,\varphi \\
&\quad + \varepsilon(2\chi_2(\varepsilon\varphi\langle\mu\xi'\rangle^{1/2})-1)\} \\
&= -(n\varepsilon+1/2)\,\varepsilon I_\varepsilon(3/2) + (n\varepsilon+1/2)\, I_\varepsilon(3/2)\,(\varepsilon-K)
\end{aligned}
$$

with $K = 2\langle\mu\xi'\rangle^{1/2}\,\chi_2^{(1)}(\varepsilon\varphi\langle\mu\xi'\rangle^{1/2})\,\varphi + \varepsilon\{2\chi_2(\varepsilon\varphi\langle\mu\xi'\rangle^{1/2})-1\}$, here we have used $\partial_0\varphi = 1$. Note that $(\varepsilon-K)\,\alpha_\varepsilon(x, \xi', \mu) = 0$ for $n \geqq 16$ and apply Lemma 4.8 to $(\varepsilon-K)$ to get

$$
(\varepsilon-K)\,\alpha_\varepsilon \in S(\langle\xi'\rangle^{-1}\langle\mu\xi'\rangle^{1+k/2}\, m(\varphi)^k, g_\mu) \subset S(\mu\langle\mu\xi'\rangle^{k/2}\, m(\varphi)^k, g_\mu)
$$

for any $k \in \boldsymbol{R}$, $n \geqq 16$. Then it follows from (5.11) that

$$
\partial_0 I_\varepsilon(1/2)\,\alpha_\varepsilon + (n+\varepsilon/2)\, I_\varepsilon(3/2)\,\alpha_\varepsilon \in S(\mu\langle\mu\xi'\rangle^{n\tilde{\varepsilon}+k/2}\, m(\varphi)^{-n\varepsilon-3/2+k}, g_\mu)\,.
$$

Put $T = I_\varepsilon(1/2)^*\{\partial_0 I_\varepsilon(1/2)\,\alpha_\varepsilon + (n+\varepsilon/2)\, I_\varepsilon(3/2)\,\alpha_\varepsilon\}$ then T is in $S(\mu\langle\mu\xi'\rangle^{2n\tilde{\varepsilon}+k/2}$ $m(\varphi)^{-2n\varepsilon-2+k}, g_\mu)$. Choose $k = -2n\tilde{\varepsilon}$ hence $2n\tilde{\varepsilon}+k/2 = n\tilde{\varepsilon}$, $-2n\varepsilon-2+k = -n\varepsilon-n-2$. Then by Lemma 4.5, T is written

$$
T \equiv I_\varepsilon(1)^*\, r I_1(1) \quad \text{with} \quad r \in S(\mu, g)\,.
$$

Choose $k = -2n\tilde{\varepsilon}+2n$ then $2n\tilde{\varepsilon}+k/2 = n\tilde{\varepsilon}+n$, $-2n\varepsilon-2+k = -n\varepsilon+n-2$. Hence Lemma 4.5 shows again that

$$
T \equiv I_\varepsilon(1)^*\tilde{r}\, I_{-1}(1) \quad \text{with} \quad \tilde{r} \in S(\mu, g)\,.
$$

On the other hand since $\alpha_1+\alpha_{-1} = 1$ we can write

$$
(5.12)\qquad T \equiv T(\alpha_1+\alpha_{-1}) \equiv I_\varepsilon(1)^* r\, I_1(1)\,\alpha_1 + I_\varepsilon(1)^*\tilde{r}\, I_{-1}(1)\,\alpha_{-1} \equiv \sum_\delta I_\varepsilon(1)^* r_\delta\, I_\delta(1)\,\alpha_\delta
$$

Again Lemma 4.5 gives $I_\varepsilon(1/2)^* I_\varepsilon(3/2) \equiv I_\varepsilon(1)^*(1+r)\, I_\varepsilon(1)$ with $r \in S(\mu, g)$ then combining this and (5.12) one has

$$
I_\varepsilon(1/2)^*\partial_0 I_\varepsilon(1/2)\,\alpha_\varepsilon \equiv -(n+\varepsilon/2)\, I_\varepsilon(1)^* I_\varepsilon(1)\,\alpha_\varepsilon + \sum_\delta I_\varepsilon(1)^* r_\delta\, I_\delta(1)\,\alpha_\delta
$$

with $r_\delta \in S(\mu, g)$. This implies that

$$
\begin{aligned}
(5.13)\qquad &-2\operatorname{Im}(i\partial_0 I_\varepsilon(1/2)\,\alpha_\varepsilon u, I_\varepsilon(1/2)\,\alpha_\varepsilon u) \geqq (2n+\varepsilon)\|I_\varepsilon(1)\,\alpha_\varepsilon u\|^2 \\
&\quad - c(n)\,\mu \sum_\delta \|I_\delta(1)\,\alpha_\delta u\|^2\,.
\end{aligned}
$$

From (5.10) and (5.13), $-2\,\mathrm{Im}([I_\varepsilon(1/2), L]\,\alpha_\varepsilon u, I_\varepsilon(1/2)\,\alpha_\varepsilon u)$ is bounded from below by $(2n+\varepsilon)\|I_\varepsilon(1)\,\alpha_\varepsilon u\|^2 - c(n)\,\mu^{1/2}\sum_\delta \|I_\delta\,\alpha_\delta u\|^2$. Set $H^{-\infty}=\bigcap_s H^s(\boldsymbol{R}^d)$ where $H^s(\boldsymbol{R}^d)$ is the usual Sobolev space of order s and we summarize above estimates.

Lemma 5.3.

$$-2\,\mathrm{Im}(I_\varepsilon(1/2)\,L\alpha_\varepsilon u, I_\varepsilon(1/2)\,\alpha_\varepsilon u) \geqq \partial_0\|I_\varepsilon(1/2)\,\alpha_\varepsilon u\|^2 + 2\theta\|I_\varepsilon(1/2)\,\alpha_\varepsilon u\|^2$$
$$+(2n+\varepsilon)\|I_\varepsilon(1)\,\alpha_\varepsilon u\|^2 - (c+c(n)\,\mu^{1/2})\sum_\delta \|I_\delta(1)\,\alpha_\delta u\|^2$$

for any $u\in C^\infty(I, H^{-\infty})$.

Our next task is to estimate $(I_\varepsilon(1/2)\,D_0\,\alpha_\varepsilon u, I_\varepsilon(1/2)\,\alpha_\varepsilon u)$, $(I_\varepsilon(1/2)\,[\alpha_\varepsilon, a]\,u, I_\varepsilon(1/2)\,\alpha_\varepsilon u)$ which come from the commutator $[L, \alpha_\varepsilon]=[D_0, \alpha_\varepsilon]+[\alpha_\varepsilon, a]$. We recall that for any $k\in\boldsymbol{R}$, $n\geqq 16$ we have $D_0\,\alpha_\varepsilon = -i\varepsilon n^{1/2}\langle\mu\xi'\rangle^{1/2}\,\chi_3^{(1)}(\varepsilon n^{1/2}\,\varphi\langle\mu\xi'\rangle^{1/2})\in S(\langle\mu\xi'\rangle^{k/2}\,m(\varphi)^{-1+k}, g_\mu)$ for $|\varphi\langle\mu\xi'\rangle^{1/2}|\leqq 1/4$ on the support of $\chi_3^{(1)}(\varepsilon n^{1/2}\,\varphi\langle\mu\xi'\rangle^{1/2})$ when $n\geqq 16$. The same argument as in the proof of (5.12) gives

$$I_\varepsilon(1/2)^*I_\varepsilon(1/2)\,D_0\,\alpha_\varepsilon \equiv \sum_\delta I_\varepsilon(1)^*(R_\delta+r_\delta)\,I_\delta(1)$$

with $r_\delta\in S(\mu, g)$ where $R_\delta(x, \xi', \mu)=I_\delta(1)^{-1}\,I_\varepsilon(1)^{-1}\,I_\varepsilon(1/2)\,I_\varepsilon(1/2)\,D_0\,\alpha_\varepsilon = \langle\mu\xi'\rangle^{-n\tilde{\delta}+n\tilde{\varepsilon}}\,J_\delta^{n\delta+1}\,J_\varepsilon^{-n\varepsilon}\,D_0\,\alpha_\varepsilon$ which is equal to

$$\{\langle\mu\xi'\rangle^{-\tilde{\delta}+\tilde{\varepsilon}}\,J_\delta^\delta\,J_\varepsilon^{-\varepsilon}\}^n\,J_\delta\,D_0\,\alpha_\varepsilon\,.$$

We shall examine that $R_\delta\in S(1, g_\mu)$ and that the maximum of the symbol $R_\delta(x, \xi', \mu)$ has a bound independent of n and μ. Note that $J_\delta=\delta\varphi+\langle\mu\xi'\rangle^{-1/2}$ when $|\varphi\langle\mu\xi'\rangle^{1/2}|\leqq 1/4$. If $\delta\varepsilon=1$ then it is obvious $\langle\mu\xi'\rangle^{-\tilde{\delta}+\tilde{\varepsilon}}\,J_\delta^\delta\,J_\varepsilon^{-\varepsilon}=1$. If $\delta\varepsilon=-1$ then it follows that

$$\langle\mu\xi'\rangle^{-\tilde{\delta}+\tilde{\varepsilon}}\,J_\delta^\delta\,J_\varepsilon^{-\varepsilon} = (\langle\mu\xi'\rangle\,J_1\,J_{-1})^{\pm 1} = (1-\varphi^2\langle\mu\xi'\rangle)^{\pm 1}$$

when $|\varphi\langle\mu\xi'\rangle^{1/2}|\leqq 1/4$. This implies that, on the support of $D_0\,\alpha_\varepsilon$ $(n\geq 16)$ where $|\varphi\langle\mu\xi'\rangle^{1/2}|\leqq n^{-1/2}$, we have

$$|\langle\mu\xi'\rangle^{-\tilde{\delta}+\tilde{\varepsilon}}\,J_\delta^\delta\,J_\varepsilon^{-\varepsilon}|^n \leqq (1\pm n^{-1})^{\pm n}\leqq c \tag{5.15}$$

with a constant c independent of n. Recall that $J_\delta\,D_0\,\alpha_\varepsilon = -i\varepsilon n^{1/2}\langle\mu\xi'\rangle^{1/2}\,J_\delta\,\chi_3^{(1)}(\varepsilon n^{1/2}\,\varphi\langle\mu\xi'\rangle^{1/2})$. Since $\langle\mu\xi'\rangle^{1/2}\,J_\delta=\delta\langle\mu\xi'\rangle^{1/2}\,\varphi+1$ on the support of $D_0\,\alpha_\varepsilon$ $(n\geqq 16)$ we have $|J_\delta\,D_0\,\alpha_\varepsilon|\leqq cn^{1/2}$ with c independent of n. (5.15) and this show that

$$|R_\delta(x, \xi', \mu)|\leqq cn^{1/2} \tag{5.16}$$

with c independent of n. Then Lemma 4.6 implies that

$$|(I_\varepsilon(1/2)\,D_0\,\alpha_\varepsilon u, I_\varepsilon(1/2)\,\alpha_\varepsilon u)|\leqq (cn^{1/2}+c(n)\,\mu)\sum_\delta \|I_\delta(1)\,\alpha_\delta u\|^2\,. \tag{5.17}$$

To estimate $[\alpha_\varepsilon, a]$ we note that $\alpha_{\varepsilon(\beta)}^{(\alpha)}\in S(m(\varphi)^{-|\alpha+\beta|+k}\langle\mu\xi'\rangle^{k/2}\langle\xi'\rangle^{-|\alpha|}, g_\mu)$, for any $|\alpha+\beta|\geqq 1$, $k\in\boldsymbol{R}$ which follows from the note preceding to Lemma 4.8 and the fact that $a_{(\beta)}^{(\alpha)}\in S(\mu^{-|\alpha''+\beta''|/2}\langle\mu\xi'\rangle\,m(B')^{1-|\alpha+\beta|}\langle\xi'\rangle^{-|\alpha|}, g)$ for $|\alpha+\beta|\leqq 1$. Taking these notes into account Lemma 4.2 shows $[\alpha_\varepsilon, a]$ belongs to $S(\mu^{1/2}\, m(\varphi)^{-1+k} \langle\mu\xi'\rangle^{k/2}, g)$ for any $k\in \boldsymbol{R}$. We apply the same argument to obtain (5.12) to get

$$I_\varepsilon(1/2)^*\, I_\varepsilon(1/2)\,[\alpha_\varepsilon, a] \equiv \sum_\delta I_\varepsilon(1)^*\, R_\delta\, I_\delta(1)\, \alpha_\delta \quad \text{with} \quad R_\delta \in S(\mu^{1/2}, g)$$

and consequently one has

$$(5.18)\qquad |(I_\varepsilon(1/2)\,[\alpha_\varepsilon, a]\, u,\, I_\varepsilon(1/2)\, \alpha_\varepsilon\, u)| \leqq c(n)\, \mu^{1/2} \sum_\delta ||I_\delta(1)\, \alpha_\delta\, u||^2 .$$

Now (5.17) and (5.18) imply

$$(5.19)\qquad |(I_\varepsilon(1/2)\,[L, \alpha_\varepsilon]\, u,\, I_\varepsilon(1/2)\, \alpha_\varepsilon\, u)| \leqq (cn^{1/2}+c(n)\, \mu^{1/2}) \sum_\delta ||I_\delta(1)\, \alpha_\delta\, u||^2 .$$

Summing the inequality in Lemma 5.3 for $\varepsilon=\pm 1$ and using the above estimate (5.19) we get

Lemma 5.4.

$$\begin{aligned} -2\,\mathrm{Im} \sum_\varepsilon (I_\varepsilon(1/2)\, \alpha_\varepsilon\, Lu,\, I_\varepsilon(1/2)\, \alpha_\varepsilon\, u) &\geqq \partial_0 \sum_\varepsilon ||I_\varepsilon(1/2)\, \alpha_\varepsilon\, u||^2 \\ &+2\theta \sum_\varepsilon ||I_\varepsilon(1/2)\, \alpha_\varepsilon\, u||^2 + (2n-1-cn^{1/2}-c(n)\, \mu^{1/2}) \sum_\varepsilon ||I_\varepsilon(1)\, \alpha_\varepsilon\, u||^2 \end{aligned}$$

for any $u\in C^\infty(I, H^{-\infty})$.

Proposition 5.1.

$$\begin{aligned} \sum_\varepsilon ||I_\varepsilon(0)\, \alpha_\varepsilon\, Lu||^2 &\geqq n\partial_0 \sum_\varepsilon ||I_\varepsilon(1/2)\, \alpha_\varepsilon\, u||^2 + 2n\theta \sum_\varepsilon ||I_\varepsilon(1/2)\, \alpha_\varepsilon\, u||^2 \\ &+cn^2 \sum_\varepsilon ||I_\varepsilon(1)\, \alpha_\varepsilon\, u||^2 \end{aligned}$$

with a positive constant c independent of n and μ for any $\hat{n}\leqq n$, $0<\mu\leqq\hat{\mu}(n)$, $u\in C^\infty(I, H^{-\infty})$.

Remark 5.1. If we start with $r=1$, $3/2$ in (5.4) we shall obtain the following estimates instead of that in Proposition 5.1,

$$\begin{aligned} \sum_\varepsilon ||I_\varepsilon(r-1/2)\, \alpha_\varepsilon\, Lu||^2 &\geqq n\partial_0 \sum_\varepsilon ||I_\varepsilon(r)\, \alpha_\varepsilon\, u||^2 + 2n\theta \sum_\varepsilon ||I_\varepsilon(r)\, \alpha_\varepsilon\, u||^2 \\ &+cn^2 \sum_\varepsilon ||I_\varepsilon(r+1/2)\, \alpha_\varepsilon\, u||^2, \quad r = 1,\, 3/2 . \end{aligned}$$

Corollary 5.1.

$$c_1 \int^t ||m(\varphi)^n \langle\mu D'\rangle^n\, Lu(\tau, \cdot)||^2\, d\tau \geqq c_2\, n^2 \int^t ||m(\varphi)^{-n-1}\, u(\tau, \cdot)||^2\, d\tau$$

with positive constants c_i independent of n and μ for any $\hat{n}\leqq n$, $0<\mu\leqq\hat{\mu}(n)$, $u\in C^\infty(I, H^{-\infty})$ vanishing in $x_0<0$.

Remark 5.2. Note that $m(\varphi)\,(x, \xi', \mu)=\langle\mu\xi'\rangle^{-1/2}$ when $\varphi(x, \xi', \mu)=0$.

We shall prove a variant of Proposition 5.1. We put $(u, v)_{(s)}=(\langle\mu D'\rangle^s u, \langle\mu D'\rangle^s v)$, $||u||_{(s)}=||\langle\mu D'\rangle^s u||$.

Proposition 5.2. *Fix $0<\nu<1$. Then*

$$c_1 \sum_\varepsilon \|I_\varepsilon(0)\, \alpha_\varepsilon\, Lu\|_{(s)}^2 \geqq n\partial_0 \sum_\varepsilon \|I_\varepsilon(1/2)\, \alpha_\varepsilon \langle \mu D'\rangle^s u\|^2$$
$$+c_2\, n\theta^\nu \sum_\varepsilon \|I_\varepsilon(1/2)\, \alpha_\varepsilon u\|_{(s)}^2 + c_2\, n^2 \sum_\varepsilon \|I_\varepsilon(1)\, \alpha_\varepsilon\, u\|_{(s)}^2$$

for any $\hat{n}\leqq n$, $0<\mu\leqq\hat{\mu}(n)$, $\theta(n, \mu, s)\leqq\theta$, $s\in \boldsymbol{R}$, $u\in C^\infty(I, H^{-\infty})$ *where* c_i *are positive constants independent of* n, μ, θ, s.

Lemma 5.5. *Let* $T\in S(m, g)$. *Then*

$$I_\varepsilon(r)\, \alpha_\varepsilon T \equiv T\, I_\varepsilon(r)\, \alpha_\varepsilon + \sum_\delta R_\delta \langle \mu D'\rangle^{(k-1)/2}\, I_\delta(r+1-k)\, \alpha_\delta\,, \quad R_\delta\in S(\mu^{1/2}\, m, g)$$

for any $k\geqq 0$. *In particular if* $T=\langle \mu D'\rangle^s$ *we have*

$$I_\varepsilon(r)\, \alpha_\varepsilon\langle \mu D'\rangle^s - \langle \mu D'\rangle^s\, I_\varepsilon(r)\, \alpha_\varepsilon \equiv \sum_\delta R_\delta \langle \mu D'\rangle^{s-1+k/2}\, I_\delta(r+1-k)\, \alpha_\delta$$
$$\equiv \sum_\delta \tilde{R}_\delta \langle \mu D'\rangle^{-1+k/2}\, I_\delta(r+1-k)\, \alpha_\delta \langle \mu D'\rangle^s$$

for any $k\geqq 0$ *where* R_δ, $\tilde{R}_\delta \in S(\mu, g)$.

Proof of Proposition 5.2. First we note that Lemma 5.4 can be stated as;

$$-2\,\mathrm{Im} \sum_\varepsilon (I_\varepsilon(0)\, \alpha_\varepsilon\, Lu, I_\varepsilon(1)\, \alpha_\varepsilon\, u) \geqq \partial_0 \sum_\varepsilon \|I_\varepsilon(1/2)\, \alpha_\varepsilon\, u\|^2$$
$$+2\theta \sum_\varepsilon \|I_\varepsilon(1/2)\, \alpha_\varepsilon\, u\|^2 + cn \sum_\varepsilon \|I_\varepsilon(1)\, \alpha_\varepsilon\, u\|^2$$

for any $\hat{n}\leqq n$, $0<\mu\leqq\hat{\mu}(n)$. We replace u by $\langle \mu D'\rangle^s u$ in this estimate. Since $[L, \langle \mu D'\rangle^s]=[\langle \mu D'\rangle^s, a]\in S(\mu^{1/2}\langle \mu\xi'\rangle^s, g)$ it follows from Lemma 5.5 that

$$I_\varepsilon(0)\, \alpha_\varepsilon\, L\langle \mu D'\rangle^s \equiv \langle \mu D'\rangle^s\, I_\varepsilon(0)\, \alpha_\varepsilon\, L + \sum_\delta R_\delta \langle \mu D'\rangle^{s-1/2}\, I_\delta(0)\, \alpha_\delta\, L$$
$$+ \sum_\delta \tilde{R}_\delta \langle \mu D'\rangle^s\, I_\delta(0)\, \alpha_\delta\,,$$
$$I_\varepsilon(1)\, \alpha_\varepsilon \langle \mu D'\rangle^s \equiv \langle \mu D'\rangle^s\, I_\varepsilon(1)\, \alpha_\varepsilon + \sum_\delta \hat{R}_\delta \langle \mu D'\rangle^{s-1/4}\, I_\delta(1/2)\, \alpha_\delta$$

with R_δ, $\tilde{R}_\delta$, $\hat{R}_\delta \in S(\mu^{1/2}, g)$. Then these imply that

$$2\,|(I_\varepsilon(0)\, \alpha_\varepsilon\, L\langle \mu D'\rangle^s u, I_\varepsilon(1)\, \alpha_\varepsilon \langle \mu D'\rangle^s u)| \leqq 2\,|(I_\varepsilon(0)\, \alpha_\varepsilon\, Lu, I_\varepsilon(1)\, \alpha_\varepsilon\, u)_{(s)}|$$
$$+n^{-1}\|I_\varepsilon(0)\, \alpha_\varepsilon\, Lu\|_{(s-1/4)}^2 + c(n, \mu, s) \sum_\delta \|I_\delta(1/2)\, \alpha_\delta\, u\|_{(s)}^2\,,$$

and this is also estimated by

$$(c_1+1)\, n^{-1}\|I_\varepsilon(0)\, \alpha_\varepsilon\, Lu\|_{(s)}^2 + c_1^{-1} n\|I_\varepsilon(1)\, \alpha_\varepsilon\, u\|_{(s)}^2 + c_1(n, \mu, s) \sum_\delta \|I_\delta(1/2)\, \alpha_\delta\, u\|_{(s)}^2\,. \tag{5.20}$$

On the other hand one has from Lemma 5.5 that $I_\varepsilon(1)\, \alpha_\varepsilon\langle \mu D'\rangle^s \equiv \langle \mu D'\rangle^s I_\varepsilon(1)\, \alpha_\varepsilon + \sum_\delta R_\delta \langle \mu D'\rangle^{s-1/4}\, I_\delta(1/2)\, \alpha_\delta$ hence

$$2\|I_\varepsilon(1)\, \alpha_\varepsilon \langle \mu D'\rangle^s u\|^2 \geqq \|I_\varepsilon(1)\, \alpha_\varepsilon\, u\|_{(s)}^2 - c_2(n, \mu, s) \sum_\delta \|I_\delta(1/2)\, \alpha_\delta\, u\|_{(s-1/4)}^2\,. \tag{5.21}$$

Another application of Lemma 5.5 shows that

$$\hat{c}(n, \mu, s)\|I_\varepsilon(1/2)\, \alpha_\varepsilon \langle \mu D'\rangle^s u\|^2 \geqq \|I_\varepsilon(1/2)\, \alpha_\varepsilon\, u\|_{(s)}^2\,. \tag{5.22}$$

Taking $c_1^{-1}<c/2$, $\theta-\hat{c}(n,\mu,s)\,\theta^{1/2}\geqq\theta^{\nu}$, we get from (5.20)–(5.22)

$$(c_1+1)\,n^{-1}\|I_\varepsilon(0)\,\alpha_\varepsilon\,Lu\|_{(s)}^2\geqq\partial_0\sum_\varepsilon\|I_\varepsilon(1/2)\langle\mu D'\rangle^s u\|^2+(\theta^\nu-c_1(n,\mu,s)$$
$$-c_2(n,\mu,s)\,n)\sum_\varepsilon\|I_\varepsilon(1/2)\,\alpha_\varepsilon\,u\|_{(s)}^2+c_2\,n\sum_\varepsilon\|I_\varepsilon(1)\,\alpha_\varepsilon\,u\|_{(s)}^2\,.$$

This proves the proposition.

Remark 5.1. If we start with $r=1$, $3/2$ then we shall obtain

$$c_1\sum_\varepsilon\|I_\varepsilon(r-1/2)\,\alpha_\varepsilon\,Lu\|_{(s)}^2\geqq n\partial_0\sum_\varepsilon\|I_\varepsilon(r)\,\alpha_\varepsilon\langle\mu D'\rangle^s\,u\|^2$$
$$+c_2\,n\theta^\nu\sum_\varepsilon\|I_\varepsilon(r)\,\alpha_\varepsilon\,u\|_{(s)}^2+c_2\,n^2\sum_\varepsilon\|I_\varepsilon(r+1/2)\,\alpha_\varepsilon\,u\|_{(s)}^2$$

with $r=1$, $3/2$ for any $\hat{n}\leqq n$, $0<\mu\leqq\hat{\mu}(n)$, $\theta(n,\mu,s)\leqq\theta$.

Corollary 5.2.

$$\sum_\varepsilon\|I_\varepsilon(0)\,\alpha_\varepsilon\,Lu\|_{(s-1/4)}^2-2\,\mathrm{Im}\sum_\varepsilon(I_\varepsilon(1/2)\,\alpha_\varepsilon\,Lu,\,I_\varepsilon(1/2)\,\alpha_\varepsilon\,u)_{(s)}$$
$$\geqq\partial_0\sum_\varepsilon\|I_\varepsilon(1/2)\,\alpha_\varepsilon\langle\mu D'\rangle^s\,u\|^2+c\theta^\nu\sum_\varepsilon\|I_\varepsilon(1/2)\,\alpha_\varepsilon\,u\|_{(s)}^2+cn\sum_\varepsilon\|I_\varepsilon(1)\,\alpha_\varepsilon\,u\|_{(s)}^2$$

for any $\hat{n}\leqq n$, $0<\mu\leqq\hat{\mu}(n)$, $\theta(n,\mu,s)\leqq\theta$, $u\in C^\infty(I,H^{-\infty})$ where c is a positive constant independent of n, μ, θ, s.

The rest of this section is devoted to obtain an estimate of wave front sets. Let

$$f(x',\xi',\mu)\in S(\mu,\tilde{G}_\mu)\,.$$

Set $\psi(x,\xi',\mu)=x_0-f(x',\xi',\mu)$ and define $\Psi(x,\xi',\mu)$ by

$$\Psi(x,\xi',\mu)=\begin{cases}\exp\,(1/\psi(x,\xi',\mu) & \text{if}\quad\psi(x,\xi',\mu)\leqq0\\ 0 & \text{if}\quad\psi(x,\xi',\mu)>0\end{cases}$$

(cf. [9]). Then it is clear that $\Psi(x,\xi',\mu)\in S(1,\tilde{G}_\mu)$. We give two examples of such $f(x,\xi')$. Let $\chi(x')\in C_0^\infty(\boldsymbol{R}^d)$ be equal to 1 near $x'=0$ vanish in $|x'|\geqq1$. Let $(\tilde{x}',\tilde{\xi}')\in T^*\boldsymbol{R}^d\backslash0$ and set

$$d_\varepsilon(x'\,\xi')=\{\chi(x'-\tilde{x}')\,|x'-\tilde{x}'|^2+|\xi'\langle\xi'\rangle^{-1}-\tilde{\xi}'|^2+\varepsilon^2\}^{1/2}\,.$$

Then it is easy to see that $\mu d_\varepsilon(M_\mu(x',\xi'))\in S(\mu,\tilde{G}_\mu)$. As another example we take $f(x',\xi')\in C^\infty(W)$ which is homogeneous of degree 0 in ξ' such that $f(0,e_p)=0$ where W is a conic neighborhood of $(0,e_p)$. It follows from Lemma 3.1 that $f(y',\eta')+\mu\varepsilon\in S(\mu,\tilde{G}_\mu)$.

Put

$$\tilde{\Psi}=\psi^{-1}\,\Psi\in S(1,\tilde{G}_\mu)\,.$$

Lemma 5.6.

$$c(n,\mu,s)\sum_\varepsilon\|I_\varepsilon(0)\,\alpha_\varepsilon\,u\|_{(s-1/4)}^2-2\,\mathrm{Im}\sum_\varepsilon(I_\varepsilon(1/2)\,\alpha_\varepsilon[\Psi,L]\,u,\,I_\varepsilon(1/2)\,\alpha_\varepsilon\,\Psi u)_{(s)}$$
$$\geqq(2-c(n,f)\,\mu^{1/2})\sum_\varepsilon\|I_\varepsilon(1/2)\,\alpha_\varepsilon\,\Psi u\|_{(s)}^2$$

for any $\hat{n}\leqq n$, $0<\mu\leqq\hat{\mu}(n)$, $u\in C^{\infty}(I, H^{-\infty})$.

Proof. Note that $\psi\tilde{\Psi}=\Psi-R$, $R\in S(\langle\mu\xi'\rangle^{-1}, g)$. Since $[\Psi, L]=-D_0\Psi+[a, \Psi]$ $\in S(1, g)$ Lemma 5.5 gives that

$$I=\mathrm{Im}(I_\varepsilon(1/2)\,\alpha_\varepsilon[\Psi, L]\,u, I_\varepsilon(1/2)\,\alpha_\varepsilon\,\Psi u)_{(s)}$$
$$\sim\mathrm{Im}(I_\varepsilon(1/2)\,\alpha_\varepsilon[\Psi, L]\,u, I_\varepsilon(1/2)\,\alpha_\varepsilon\,\psi\tilde{\Psi}u)_{(s)}\,.$$

Here and in the following $\sim$ denotes the equality modulo a term which is bounded by

$$\mu\sum_\varepsilon\|I_\varepsilon(1/2)\,\alpha_\varepsilon\,\Psi u\|_{(s)}^2+c(n, \mu, s)\sum_\delta\|I_\varepsilon(0)\,\alpha_\varepsilon\,u\|_{(s-1/4)}^2\,.$$

The argument as in the proof of Lemma 5.5 shows

$$\langle\mu D'\rangle^s I_\varepsilon(r)\,\alpha_\varepsilon\,\psi\equiv\psi\langle\mu D'\rangle^s I_\varepsilon(r)\,\alpha_\varepsilon+\sum_\delta R_\delta\langle\mu D'\rangle^{s-1/2} I_\delta(r)\,\alpha_\delta \tag{5.23}$$

with $R_\delta\in S(\mu, g)$. Using (5.23) it follows that

$$I\sim\mathrm{Im}(\langle\mu D'\rangle^s I_\varepsilon(1/2)\,\alpha_\varepsilon[\Psi, L]\,u, \psi\langle\mu D'\rangle^s I_\varepsilon(1/2)\,\alpha_\varepsilon\,\tilde{\Psi}u)$$
$$=\mathrm{Im}(\psi^*\langle\mu D'\rangle^s I_\varepsilon(1/2)\,\alpha_\varepsilon[\Psi, L]\,u, \langle\mu D'\rangle^s I_\varepsilon(1/2)\,\alpha_\varepsilon\,\tilde{\Psi}u)\,.$$

Remarking $\psi^*-\psi\in S(\langle\mu\xi'\rangle^{-1}, \tilde{G}_\mu)$ (5.23) shows that

$$I\sim\mathrm{Im}(I_\varepsilon(1/2)\,\alpha_\varepsilon\,\psi[\Psi, L]\,u, I_\varepsilon(1/2)\,\alpha_\varepsilon\,\tilde{\Psi}u)_{(s)}\,. \tag{5.24}$$

Set $[\Psi, L]=-iK+T$ with $K=\mathrm{Op}(\{\xi_0-a, \psi\}\,\psi^{-2}\,\Psi)\in S(1, g)$. From (5.1) it is clear that $T\in S(\mu^{-1}\langle\mu\xi'\rangle^{-1}\,m(B')^{-1}, g)\subset S(\mu^{-1}\langle\mu\xi'\rangle^{-1/2}, g)$. Substituting this expression into (5.24) we have

$$-I\sim\mathrm{Re}(I_\varepsilon(1/2)\,\alpha_\varepsilon\,\psi Ku, I_\varepsilon(1/2)\,\alpha_\varepsilon\,\tilde{\Psi}u)_{(s)}\,.$$

Noting $\psi K=\mathrm{Op}(\{\xi_0-a, \psi\}\,\tilde{\Psi})+\tilde{T}$, $\tilde{T}\in S(\mu^{1/2}\langle\mu\xi'\rangle^{-1/2}, g)$ it follows that $-I\sim\mathrm{Re}$ $(I_\varepsilon(1/2)\,\alpha_\varepsilon\,\mathrm{Op}(\{\xi_0-a, \psi\}\,\tilde{\Psi})\,u, I_\varepsilon(1/2)\,\alpha_\varepsilon\,\tilde{\Psi}u)_{(s)}$. Set

$$M=\mathrm{Op}\{\xi_0-a, \psi\}=\mathrm{Op}(1+\{a, f\})\quad\text{with}\quad\{a, f\}\in S(\mu^{1/2}, g) \tag{5.25}$$

then it follows that $M\tilde{\Psi}=\mathrm{Op}(\{\xi_0-a, \psi\}\,\tilde{\Psi})-\hat{T}$, $\hat{T}\in S(\langle\mu\xi'\rangle^{-1/2}, g)$ and hence

$$-I\sim\mathrm{Re}(I_\varepsilon(1/2)\,\alpha_\varepsilon\,M\tilde{\Psi}u, I_\varepsilon(1/2)\,\alpha_\varepsilon\,\tilde{\Psi}u)_{(s)}\,.$$

Here we apply Lemma 5.5 and we get by (5.25) that $I_\varepsilon(1/2)\,\alpha_\varepsilon M=MI_\varepsilon(1/2)\,\alpha_\varepsilon+\sum_\delta R_\delta\,I_\delta(1/2)\,\alpha_\delta$, $R_\delta\in S(\mu, g)$. Since $[\langle\mu D'\rangle^s, M]$, $[\langle\mu D'\rangle^s, R_\delta]$ are in $S(\mu\langle\mu\xi'\rangle^{s-1/2}, g)$ this gives that

$$\langle\mu D'\rangle^s I_\varepsilon(1/2)\,\alpha_\varepsilon\,M\equiv M\langle\mu D'\rangle^s I_\varepsilon(1/2)\,\alpha_\varepsilon+\sum_\delta R_\delta\langle\mu D'\rangle^s I_\delta(1/2)\,\alpha_\delta$$
$$+\sum_\delta T_\delta\langle\mu D'\rangle^{s-1/2} I_\delta(1/2)\,\alpha_\delta,\ T_\delta\in S(\mu, g)\,.$$

Here we note that M and R_δ are independent of s. Using above expression we have

$$-I\sim \mathrm{Re}(M\langle\mu D'\rangle^s I_\varepsilon(1/2)\,\alpha_\varepsilon\,\tilde{\Psi}u,\ \langle\mu D'\rangle^s I_\varepsilon(1/2)\,\alpha_\varepsilon\,\tilde{\Psi}u)$$
$$+\sum_\delta \mathrm{Re}(R_\delta\langle\mu D'\rangle^s I_\delta(1/2)\,\alpha_\delta\,\tilde{\Psi}u,\ \langle\mu D'\rangle^s I_\varepsilon(1/2)\,\alpha_\varepsilon\,\tilde{\Psi}u)\,.$$

Notice that the second term in the right-hand side is estimated by

$$c(n,f)\,\mu\sum_\delta ||I_\delta(1/2)\,\alpha_\delta\,\tilde{\Psi}u||^2_{(s)}\,.$$

Recalling (5.25) the first term of the right-hand side is estimated from below by

$$(1-c(f)\,\mu^{1/2})||I_\varepsilon(1/2)\,\alpha_\varepsilon\,\tilde{\Psi}u||^2_{(s)}\,.$$

These complete the proof.

Proposition 5.3. *Fix* $0<\nu<1$. *Then*

$$c(n,\mu,s,\theta)\sum_\varepsilon ||I_\varepsilon(0)\,\alpha_\varepsilon\,u||^2_{(s-1/4)}+c_1\sum_\varepsilon ||I_\varepsilon(0)\,\alpha_\varepsilon\Psi\,Lu||^2_{(s)}$$
$$\geqq n\partial_0\sum_\varepsilon ||I_\varepsilon(1/2)\,\alpha_\varepsilon\langle\mu D'\rangle^s\,\Psi u||^2+c_2\,n\theta^\nu\sum_\varepsilon ||I_\varepsilon(1/2)\,\alpha_\varepsilon\Psi u||^2_{(s)}$$
$$+c_2\,n^2\sum_\varepsilon ||I_\varepsilon(1)\,\alpha_\varepsilon\Psi u||^2_{(s)}$$

for any $\hat{n}\leqq n$, $0<\mu\leqq\hat{\mu}(n)$, $\theta(n,\mu,s)\leqq\theta$, $u\leqq C^\infty(I,H^{-\infty})$ *where* c_i *are positive constants independent of* n, μ, θ, s.

Remark 5.4. If we start with $r=1$, $3/2$ then we shall get

$$c(n,\mu,s,\theta)\sum_\varepsilon ||I_\varepsilon(r-1/2)\,\alpha_\varepsilon\,u||^2_{(s-1/4)}+c_1\sum_\varepsilon ||I_\varepsilon(r-1/2)\,\alpha_\varepsilon\Psi\,Lu||^2_{(s)}$$
$$\geqq n\partial_0\sum_\varepsilon ||I_\varepsilon(r)\,\alpha_\varepsilon\langle\mu D'\rangle^s\,\Psi u||^2+c_2\,n\theta^\nu\sum_\varepsilon ||I_\varepsilon(r)\,\alpha_\varepsilon\Psi u||^2_{(s)}$$
$$+c_2\,n^2\sum_\varepsilon ||I_\varepsilon(r+1/2)\,\alpha_\varepsilon\Psi u||^2_{(s)}\,,\quad r=1,\,3/2\,.$$

6. Energy estimate for second order operators

We shall extend Propositions 5.2 and 5.3 to operators of the form

$$L=q_1\,q_2,\ q_j(x,D,\mu)=D_0-i\theta-a_j(x,D',\mu)$$

where $a_j(x,\xi',\mu)$ are real and satisfy (5.1). We assume moreover

$$|a_1(x,\xi',\mu)-a_2(x,\xi',\mu)|\geqq c\langle\mu\xi'\rangle\,m(B') \tag{6.1}$$

with a positive constant c independent of μ. Put

$$H_{(s)}=\{\sum_\varepsilon ||I_\varepsilon(0)\,\alpha_\varepsilon\,q_1\,q_2\,u||^2_{(s)}+\sum_\varepsilon ||I_\varepsilon(0)\,\alpha_\varepsilon\,q_2\,q_1\,u||^2_{(s)}\}$$

then Proposition 5.2 gives ($\nu=2/3$)

$$\hat{c}H_{(s)}\geqq n\partial_0\sum_\varepsilon\sum_i ||I_\varepsilon(1/2)\,\alpha_\varepsilon\langle\mu D'\rangle^s\,q_i\,u||^2+cn\theta^{2/3}\sum_\varepsilon\sum_i ||I_\varepsilon(1/2)\,\alpha_\varepsilon\,q_i\,u||^2_{(s)}$$
$$+cn^2\sum_\varepsilon\sum_i ||I_\varepsilon(1)\,\alpha_\varepsilon\,q_i\,u||^2_{(s)} \tag{6.2}$$

with a positive constnat c independent of n, μ, θ, s. The same argument to obtain (5.12) shows that

(6.3) $\quad I_\varepsilon(1)\,\alpha_\varepsilon(q_1-q_2)\equiv I_\varepsilon(1)\,(q_1-q_2)\,\alpha_\varepsilon+\sum_\delta R_\delta\,I_\delta(2)\,\alpha_\delta\,,\quad R_\delta\in S(\mu^{1/2},g)\,.$

Since $q_2-q_1=a_1-a_2\in S(\langle\mu\xi'\rangle\,m(B'),g)$ and $|a_1-a_2|\geqq c\langle\mu\xi'\rangle\,m(B')$ by assumption (6.1) we can estimate $||\langle\mu D'\rangle\,I_\varepsilon(0)\,\alpha_\varepsilon\,u||$ by $||I_\varepsilon(1)\,(a_1-a_2)\,\alpha_\varepsilon\,u||$. Indeed

Lemma 6.1. *With a positive constant c independent of n, μ we have*

$$c||I_\varepsilon(1)\,(a_1-a_2)\,w||^2+(c+c(n)\,\mu)||I_\varepsilon(2)\,w||^2\geqq||\langle\mu D'\rangle\,I_\varepsilon(0)\,w||^2+\sum_i||I_\varepsilon(1)\,a_i\,w||^2\,.$$

Proof. By (5.1) and (5.2) it follows that $[I_\varepsilon(1),a_i]$ are in $S(\mu^{1/2}\langle\mu\xi'\rangle^{\tilde{n_\varepsilon}}\,m(\varphi)^{-n_\varepsilon-2},$ $g)$ then it suffices to prove that

$$c||(a_1-a_2)\,I_\varepsilon(1)\,w||^2+(c+c(n)\,\mu)||I_\varepsilon(2)\,w||^2\geqq||\langle\mu D'\rangle\,I_\varepsilon(0)\,w||^2+\sum_i||a_i\,I_\varepsilon(1)\,w||^2\,.$$

Put $T(x,\xi',\mu)=\langle\mu\xi'\rangle\,J_\varepsilon(x,\xi',\mu)\,(a_1-a_2)^{-1}$ then $T\in S(m(\varphi)\,m(B')^{-1},g)\subset S(1,g)$ for $(a_1-a_2)^{-1}\in S(\langle\mu\xi'\rangle^{-1}\,m(B')^{-1},g)$ and $m(\varphi)\leqq Cm(B')$. Taking this into account, it is easy to see that

(6.4) $\quad T^{(\alpha)}_{(\beta)}\in S(\mu^{-|\alpha+\beta|/2}\,m(B')^{-|\alpha+\beta|}\langle\xi'\rangle^{-|\alpha|},g)\quad\text{for}\quad|\alpha+\beta|=1\,.$

Then it follows that

(6.5) $\quad T(a_1-a_2)=\mathrm{Op}(\langle\mu\xi'\rangle\,J_\varepsilon)+R,\ R\in S(m(B')^{-1},g)\subset S(m(\varphi)^{-1},g)\,.$

From Lemma 4.2 we have $\mathrm{Op}(\langle\mu\xi'\rangle\,J_\varepsilon)\,I_\varepsilon(1)-\langle\mu D'\rangle\,I_\varepsilon(0)$ belongs to $S(\mu\langle\mu\xi'\rangle^{\tilde{n_\varepsilon}}$ $m(\varphi)^{-n_\varepsilon-2},g)$ then Lemma 4.5 snows that

(6.6) $\quad \mathrm{Op}(\langle\mu\xi'\rangle\,J_\varepsilon)\,I_\varepsilon(1)\equiv\langle\mu D'\rangle\,I_\varepsilon(0)+r_1\,I_\varepsilon(2),\ r_1\in S(\mu,g)\,.$

On the other hand from Lemma 4.5 again one has $I_\varepsilon(1)\equiv(1+r_2)\,J_\varepsilon\,I_\varepsilon(2)$ with $r_2\in S(\mu,g)$ then it follows that

(6.7) $\quad RI_\varepsilon(1)\equiv(\tilde{R}+\tilde{r}_2)\,I_\varepsilon(2)$

with $\tilde{R}=RJ_\varepsilon\in S(1,g)$ which is independent of n and $\tilde{r}_2=Rr_2\,J_\varepsilon\in S(\mu,g)$. Now (6.5)–(6.7) give that $T(a_1-a_2)\,I_\varepsilon(1)\equiv\langle\mu D'\rangle\,I_\varepsilon(0)+(\tilde{R}+\tilde{r})\,I_\varepsilon(2)$ with $\tilde{r}\in S(\mu,g)$ and this proves

$$||\langle\mu D'\rangle\,I_\varepsilon(0)\,w||^2\leqq c||(a_1-a_2)\,I_\varepsilon(1)\,w||^2+(c+c(n)\,\mu)||I_\varepsilon(2)\,w||^2$$

since $T\in S(1,g)$ and T is independent of n.

Next set $T_i(x,\xi',\mu)=a_i(x,\xi',\mu)\,(a_1(x,\xi',\mu)-a_2(x,\xi',\mu))^{-1}\in S(1,g)$ then T_i satisfy the estimate (6.4) and hence $T_i(a_1-a_2)\equiv a_i+R_i$ with $R_i\in S(m(B')^{-1},g)\subset S$ $(m(\varphi)^{-1},g)$. Then by (6.7) one obtains

(6.8) $\quad T_i(a_1-a_2)\,I_\varepsilon(1)\equiv a_i\,I_\varepsilon(1)+(\tilde{R}_i+\tilde{r}_i)\,I_\varepsilon(2),\ \tilde{r}_i\in S(\mu,g)$

with $\tilde{R}_i\in S(1,g)$ which are independent of n. From (6.8) $||a_i\,I_\varepsilon(1)\,w||^2$ is bounded by $c\,||(a_1-a_2)\,I_\varepsilon(1)\,w||^2+(c+c(n)\,\mu)||I_\varepsilon(2)\,w||^2$ and the proof of the lemma is complete.

Note that from this lemma and (6.3) it follows with a positive constant c that (to replace $a_i\,\alpha_\varepsilon$ by $\alpha_\varepsilon\,a_i$, see the proof of (5.18))

$$
\begin{aligned}
(6.9)\qquad & c\sum_i ||I_\varepsilon(1)\,\alpha_\varepsilon\,q_i\,u||^2+(c+c(n)\,\mu)\sum_\delta ||I_\delta(2)\,\alpha_\delta\,u||^2 \\
& \geqq ||\langle\mu D'\rangle\,I_\varepsilon(0)\,\alpha_\varepsilon\,u||^2+\sum_i ||I_\varepsilon(1)\,\alpha_\varepsilon\,a_i\,u||^2\,.
\end{aligned}
$$

Lemma 6.2. *With a positive constant c independent of n, μ, θ, s we have*

$$
\begin{aligned}
& c\sum_i ||I_\varepsilon(r-1/2)\,\alpha_\varepsilon\,q_i\,u||^2_{(s)}+(c+c(n)\,\mu)\sum_\delta ||I_\delta(r+1/2)\,\alpha_\delta\,u||^2_{(s)} \\
& \quad +c(n,\mu,s)\,\{\sum_i\sum_\delta ||I_\delta(r-1)\,\alpha_\delta\,q_i\,u||^2_{(s-1/4)}+\sum_\delta ||I_\delta(r-1/2)\,\alpha_\delta\,u||^2_{(s)}\} \\
& \geqq ||I_\varepsilon(r-3/2)\,\alpha_\varepsilon\,u||^2_{(s+1)}+||I_\varepsilon(r-1/2)\,\alpha_\varepsilon(D_0-i\theta)\,u||^2_{(s)}
\end{aligned}
$$

where $r=1$, $3/2$.

Proof. It will suffice to show the case $r=3/2$. In (6.9) we replace u by $\langle\mu D'\rangle^s u$. Noting that $[a_i, \langle\mu D'\rangle^s]\in S(\mu^{1/2}\langle\mu\xi'\rangle^s, g)$ and applying Lemma 5.5 it follows that

$$
\begin{aligned}
& c\sum_i ||I_\varepsilon(1)\,\alpha_\varepsilon\,q_i\,u||^2_{(s)}+(c+c(n)\,\mu)\sum_\delta ||I_\delta(2)\,\alpha_\delta\,u||^2_{(s)} \\
(6.10)\qquad & \quad +c(n,\mu,s)\,\{\sum_\delta ||I_\delta(1)\,\alpha_\delta\,u||^2_{(s)}+\sum_\delta\sum_i ||I_\delta(1/2)\,\alpha_\delta\,q_i\,u||^2_{(s-1/4)}\} \\
& \geqq ||I_\varepsilon(0)\,\alpha_\varepsilon\,u||^2_{(s+1)}+\sum_i ||I_\varepsilon(1)\,\alpha_\varepsilon\langle\mu D'\rangle^s\,a_i\,u||^2\,.
\end{aligned}
$$

Denoting by F the left-hand side of the inequality of the lemma, this implies that F is bounded from below by the right-hand side of (6.10). Note that $||I_\varepsilon(1)\,\alpha_\varepsilon(D_0-i\theta)\,u||^2_{(s)}\leqq 2\,||I_\varepsilon(1)\,\alpha_\varepsilon\langle\mu D'\rangle^s(D_0-i\theta)\,u||^2+c(n,\mu,s)\sum_\delta ||I_\delta(1/2)\,\alpha_\delta(D_0-i\theta)\,u||^2_{(s-1/4)}$ where the right-hand side is estimated by

$$
\begin{aligned}
& 4\{||I_\varepsilon(1)\,\alpha_\varepsilon\langle\mu D'\rangle^s\,q_i\,u||^2+||I_\varepsilon(1)\,\alpha_\varepsilon\langle\mu D'\rangle^s\,a_i\,u||^2\} \\
& \quad +c(n,\mu,s)\,\{\sum_\delta ||I_\delta(1/2)\,\alpha_\delta\,q_i\,u||^2_{(s-1/4)}+\sum_\delta ||I_\delta(1/2)\,\alpha_\delta\,a_i\,u||^2_{(s-1/4)}\}
\end{aligned}
$$

for $D_0-i\theta=q_i+a_i$. Further this term is estimated by $4F+c(n,\mu,s)\sum_\delta ||I_\delta(1/2)\,\alpha_\delta\,a_i\,u||^2_{(s-1/4)}$. It is clear that the argument to show (6.10) also gives that

$$
\begin{aligned}
& c(n,\mu,s)\,\{\sum_\delta\sum_i ||I_\delta(1/2)\,\alpha_\delta\,q_i\,u||^2_{(s-1/4)}+\sum_\delta ||I_\delta(1)\,\alpha_\delta\,u||^2_{(s)}\} \\
& \geqq \sum_\delta\sum_i ||I_\delta(1/2)\,\alpha_\delta\,a_i\,u||^2_{(s-1/4)}
\end{aligned}
$$

and hence we have proved

$$
4F\geqq ||I_\varepsilon(1)\,\alpha_\varepsilon(D_0-i\theta)\,u||^2_{(s)}\,.
$$

This completes the proof.

We return to estimate $H_{(s)}$. Using Lemma 6.2 and Remark 5.3 ($\nu=1/2$) we have

$$
(6.11)\quad
\begin{aligned}
c\sum_{\varepsilon}\sum_{i}\|I_\varepsilon(r-1/2)\,\alpha_\varepsilon\, q_i\, u\|^2_{(s)} \geq\ & n\partial_0\sum_{\varepsilon}\|I_\varepsilon(r)\,\alpha_\varepsilon\langle\mu D'\rangle^s u\|^2\\
&+\tilde{c}_1\sum_{\varepsilon}\|I_\varepsilon(r-3/2)\,\alpha_\varepsilon\, u\|^2_{(s+1)}+\tilde{c}_1\sum_{\varepsilon}\|I_\varepsilon(r-1/2)\,\alpha_\varepsilon(D_0-i\theta)\,u\|^2_{(s)}\\
&+\tilde{c}_2\, n^2\sum_{\varepsilon}\|I_\varepsilon(r+1/2)\,\alpha_\varepsilon\, u\|^2_{(s)}+\tilde{c}_2\, n\theta^{1/2}\sum_{\varepsilon}\|I_\varepsilon(r)\,\alpha_\varepsilon\, u\|^2_{(s)}\\
&-c(n,\mu,s)\sum_{i}\sum_{\varepsilon}\|I_\varepsilon(r-1)\,\alpha_\varepsilon\, q_i\, u\|^2_{(s-1/4)}
\end{aligned}
$$

for any $\hat{n}\leqq n$, $0<\mu\leqq\hat{\mu}(n)$, $\theta(n,\mu,s)\leqq\theta$ where $r=1$, $3/2$. The sum of the second and third term of the right-hand side of (6.2) is estimated from below obviously by

$$
\begin{aligned}
&2^{-1}cn\theta^{2/3}\sum_{\varepsilon}\sum_{i}\|I_\varepsilon(1/2)\,\alpha_\varepsilon\, q_i\, u\|^2_{(s)}+2^{-1}cn^2\sum_{\varepsilon}\sum_{i}\|I_\varepsilon(1)\,\alpha_\varepsilon\, q_i\, u\|^2_{(s)}\\
&\quad+2^{-1}cn\theta^{1/2}\sum_{\varepsilon}\sum_{i}\|I_\varepsilon(1/2)\,\alpha_\varepsilon\, q_i\, u\|^2_{(s)}+2^{-1}cn^2\sum_{\varepsilon}\sum_{i}\|I_\varepsilon(1)\,\alpha_\varepsilon\, q_i\, u\|^2_{(s)}\,.
\end{aligned}
$$

We substitute the estimate (6.11) into the last two terms of the above expression. To simplify notation we set

$$
\begin{aligned}
&Q(x,\xi,\mu)=(Q_1,Q_2,Q_3)=(D_0-i\theta,q_1,q_2)\,,\\
&e_{(s)}(u)=n\sum_{\varepsilon}\sum_{i}\|I_\varepsilon(1/2)\,\alpha_\varepsilon\langle\mu D'\rangle^s q_i\, u\|^2+\hat{c}_1\, n^2\theta^{1/2}\sum_{\varepsilon}\|I_\varepsilon(1)\,\alpha_\varepsilon\langle\mu D'\rangle^s u\|^2\\
&\quad+\hat{c}_1\, n^3\sum_{\varepsilon}\|I_\varepsilon(3/2)\,\alpha_\varepsilon\langle\mu D'\rangle^s u\|^2\,.
\end{aligned}
$$

Then by (6.11) with $r=1$ and $3/2$, $\hat{c}H_{(s)}$ is estimated from below by

$$
(6.12)\quad
\begin{aligned}
&\partial_0\, e_{(s)}(u)+\hat{c}_2\sum_{\varepsilon}\sum_{|\gamma|+j=2,j\geqq1}\sum_{k=0}^{[j/2]}\sum_{s=j-k}^{2j-2k}\theta^{j-k-s/2}\, n^s\\
&\quad\times\|I_\varepsilon(s/2-k)\,\alpha_\varepsilon\, Q^\gamma u\|^2_{(s+k)}
\end{aligned}
$$

where $Q^\gamma=Q_1^{\gamma_1}Q_2^{\gamma_2}Q_3^{\gamma_3}$, $\gamma=(\gamma_1,\gamma_2,\gamma_3)$.

Finally we estimate $q_1\, q_2-q_2\, q_1=[q_1,q_2]$. Note that $T=[q_1,q_2]$ is in $S(\langle\mu\xi'\rangle,g)$ for $(\partial/\partial\xi_0)\, q_i=1$ and $(\partial/\partial x_0)\, q_i\in S(\langle\mu\xi'\rangle,g)$. Then using Lemma 5.5 it follows that $I_\varepsilon(0)\,\alpha_\varepsilon\, T\equiv T\, I_\varepsilon(0)\,\alpha_\varepsilon+\sum_{\delta}R_\delta\langle\mu D'\rangle\, I_\delta(0)\,\alpha_\delta$ with $R_\delta\in S(\mu^{1/2},g)$. Since $[\langle\mu D'\rangle^s, T]\in S(\mu^{1/2}\langle\mu\xi'\rangle^{s+1/2},g)$ we have

$$
\begin{aligned}
&\langle\mu D'\rangle^s I_\varepsilon(0)\,\alpha_\varepsilon\, T\equiv\tilde{T}\langle\mu D'\rangle^{s+1} I_\varepsilon(0)\,\alpha_\varepsilon+\sum_{\delta}R_\delta\langle\mu D'\rangle^{s+1} I_\delta(0)\,\alpha_\delta\\
&\quad+\sum_{\delta}\tilde{R}_\delta\langle\mu D'\rangle^{s+1/2} I_\delta(0)\,\alpha_\delta\quad\text{with}\quad\tilde{R}_\delta\in S(\mu^{1/2},g)\,.
\end{aligned}
$$

We note that $\tilde{T}=T\langle\mu D'\rangle^{-1}$ is independent of n, s and R_δ do not depend on s. Then one obtains

$$
(6.13)\quad
\begin{aligned}
\|I_\varepsilon(0)\,\alpha_\varepsilon\, Tu\|^2_{(s)}\leqq\ &(c+c(n)\,\mu)\sum_{\delta}\|I_\delta(0)\,\alpha_\delta\, u\|^2_{(s+1)}\\
&+c(n,\mu,s)\sum_{\delta}\|I_\delta(0)\,\alpha_\delta\, u\|^2_{(s+1/2)}\,.
\end{aligned}
$$

Remarking that the second term in the right-hand side of (6.13) is estimated by $c(n,\mu,s)\sum_{\delta}\|I_\delta(-1/2)\,\alpha_\delta\, u\|^2_{(s+3/4)}$, we have from (6.12)

Proposition 6.1.

$$
\begin{aligned}
&c_3\sum_{\varepsilon}\|I_\varepsilon(0)\,\alpha_\varepsilon\, q_1\, q_2\, u\|^2_{(s)}\geqq\partial_0\, e_{(s)}(u)\\
&\quad+c_4\sum_{\varepsilon}\sum_{|\gamma|+j=2,j\geqq1}\sum_{k=0}^{[j/2]}\sum_{s=j-k}^{2j-2k}\theta^{j-k-s/2}\, n^s\|I_\varepsilon(s/2-k)\,\alpha_\varepsilon\, Q^\gamma u\|^2_{(s+k)}
\end{aligned}
$$

for any $\hat{n}\leqq n$, $0<\mu\leqq\hat{\mu}(n)$, $\theta(n, \mu, s)\leqq\theta$, $u\in C^\infty(I, H^{-\infty})$ *where* c_i *are positive constants independent of* n, μ, θ, s.

For later use we restate Proposition 6.1 in a slightly less precise form

Corollary 6.1.

$$c_3 \sum_\varepsilon \|I_\varepsilon(0)\,\alpha_\varepsilon\, q_1\, q_2\, u\|_{(s)}^2 \geqq \partial_0\, e_{(s)}(u)+c_4\, n^2 \sum_\varepsilon \{\sum_i \|I_\varepsilon(1)\,\alpha_\varepsilon\, q_i\, u\|_{(s)}^2 + \|I_\varepsilon(1)\,\alpha_\varepsilon(D_0-i\theta)\, u\|_{(s)}^2+\|I_\varepsilon(0)\,\alpha_\varepsilon\, u\|_{(s+1)}^2\}$$

for any $\hat{n}\leqq n$, $0<\mu\leqq\hat{\mu}(n)$, $\hat{\theta}(n, \mu, s)\leqq\theta$, $u\in C^\infty(I, H^{-\infty})$.

Corollary 6.2.

$$c\int^t \|m(\varphi)^n\langle\mu D'\rangle^n\, q_1\, q_2\, u(\tau, \cdot)\|_{(s)}^2\, d\tau\geqq n^2\int^t \|m(\varphi)^{-n}\, u(\tau, \cdot)\|_{(s+1)}^2\, d\tau$$

for any $\hat{n}\leqq n$, $0<\mu\leqq\hat{\mu}(n)$, $\hat{\theta}(n, \mu, s)\leqq\theta$, $u\in C^\infty(I, H^{-\infty})$ *vanishing in* $x_0<0$.

Now we extend Proposition 5.3. Put

$$E_{(s)}(u) = \sum_\varepsilon\sum_i \|I_\varepsilon(1)\,\alpha_\varepsilon\, q_i\, u\|_{(s-1)}^2+\sum_\varepsilon \|I_\varepsilon(1)\,\alpha_\varepsilon(D_0-i\theta)\, u\|_{(s-1)}^2+\sum_\varepsilon \|I_\varepsilon(0)\,\alpha_\varepsilon\, u\|_{(s)}^2\,,$$

$$H_{(s)}(\Psi, u) = \sum_\varepsilon \{\|I_\varepsilon(0)\,\alpha_\varepsilon\,\Psi q_1\, q_2\, u\|_{(s)}^2+\|I_\varepsilon(0)\,\alpha_\varepsilon\,\Psi q_2\, q_1\, u\|_{(s)}^2\}\,.$$

Then from Proposition 5.3 ($\nu=1/2$) it follows that

$$c(n, \mu, \theta, s)\, E_{(s+3/4)}(u)+\hat{c}H_{(s)}(\Psi, u)\geqq n\partial_0 \sum_\varepsilon\sum_i \|I_\varepsilon(1/2)\,\alpha_\varepsilon\langle\mu D'\rangle^s\,\Psi q_i\, u\|^2 +cn^2\sum_\varepsilon\sum_i \|I_\varepsilon(1)\,\alpha_\varepsilon\,\Psi q_i\, u\|_{(s)}^2+cn\,\theta^{1/2}\sum_\varepsilon\sum_i \|I_\varepsilon(1/2)\,\alpha_\varepsilon\,\Psi q_i\, u\|_{(s)}^2$$

with a positive constant c independent of n, μ, θ, s. Substituting the estimate of Remark 5.4 with $r=3/2$ ($\nu=1/2$) into the second term of the right-hand side we get

$$\begin{aligned}(6.14)\qquad & c(n, \mu, \theta, s)\, E_{(s+3/4)}(u)+\hat{c}H_{(s)}(\Psi, u)\geqq \partial_0\,\tilde{e}_{(s)}(\Psi, u)\\ & +c_5\, n^2\sum_\varepsilon\sum_i \|I_\varepsilon(1)\,\alpha_\varepsilon\,\Psi q_i\, u\|_{(s)}^2+c_5\, n^4\sum_\varepsilon \|I_\varepsilon(2)\,\alpha_\varepsilon\,\Psi u\|_{(s)}^2\\ & +c_5\, n\theta^{1/2}\sum_\varepsilon\sum_i \|I_\varepsilon(1/2)\,\alpha_\varepsilon\,\Psi q_i\, u\|_{(s)}^2+c_5\, n^3\,\theta^{1/2}\sum_\varepsilon \|I_\varepsilon(3/2)\,\alpha_\varepsilon\,\Psi u\|_{(s)}^2\end{aligned}$$

where $\tilde{e}_{(s)}(\Psi, u)=n\sum_\varepsilon\sum_i \|I_\varepsilon(1)\,\alpha_\varepsilon\langle\mu D'\rangle^s\,\Psi q_i\, u\|^2+n^3\sum_\varepsilon \|I_\varepsilon(3/2)\,\alpha_\varepsilon\langle\mu D'\rangle^s\,\Psi u\|^2$.
Noting that $[\Psi, q_i]\in S(1, g)$ we can replace Ψq_i by $q_i\Psi$ in the last four terms of the right-hand side of (6.14). From Lemma 6.2 the sum of these terms so obtained is estimated from below by

$$(6.15)\qquad \hat{c}_5\, n^2\, E_{(s+1)}(\Psi u)-c(n, \mu, \theta, s)\, E_{(s+3/4)}(u)\,.$$

We turn to $\|I_\varepsilon(0)\,\alpha_\varepsilon\,\Psi q_2\, q_1\, u\|_{(s)}^2$. Since $T=[q_2, q_1]\in S(\langle\mu\xi'\rangle, g)$ and $[\Psi, T]\in S(\mu^{-1/2}\langle\mu\xi'\rangle^{1/2}, g)$ it follows from Lemma 5.5 that $\|I_\varepsilon(0)\,\alpha_\varepsilon\,\Psi q_2\, q_1\, u\|_{(s)}^2$ is bounded by

$$c\|I_\varepsilon(0)\,\alpha_\varepsilon\,\Psi q_1\, q_2\, u\|_{(s)}^2+c\|I_\varepsilon(0)\,\alpha_\varepsilon\, T\Psi u\|_{(s)}^2+c(n, \mu, s)\|I_\varepsilon(0)\,\alpha_\varepsilon\, u\|_{(s+1/2)}^2\,.$$

From this estimate and (6.13) $\|I_\varepsilon(0)\,\alpha_\varepsilon\,\Psi q_2\, q_1\, u\|_{(2)}^2$ is estimated by

(6.16)
$$c\,||I_\varepsilon(0)\,\alpha_\varepsilon\,\Psi q_1\,q_2\,u||^2_{(s)}+(c+c(n)\,\mu)\sum_\delta ||I_\delta(0)\,\alpha_\delta\,\Psi u||^2_{(s+1)}$$
$$+c(n,\mu,s)\sum_\delta ||I_\delta(0)\,\alpha_\delta\,u||^2_{(s+1/2)}\,.$$

Now (6.15) and (6.16) show that

Proposition 6.2.

$$c(n,\mu,\theta,s)\,E_{(s+3/4)}(u)+c_6\sum_\varepsilon ||I_\varepsilon(0)\,\alpha_\varepsilon\,\Psi q_1\,q_2\,u||^2_{(s)}$$
$$\geqq \partial_0\,\tilde{e}_{(s)}(\Psi,u)+c_7\,n^2\,E_{(s+1)}(\Psi u)$$

for any $\hat{n}\leqq n$, $0<\mu\leqq\hat{\mu}(n)$, $\hat{\theta}(n,\mu,s)\leqq\theta$, $u\in C^\infty(I,H^{-\infty})$ *where* c_i *are positive constants independent of* n,μ,θ,s.

7. Reduction to a second order system

Put

$$q_j(x,\xi,\mu)=\xi_0-i\theta-a_j(x,\xi',\mu),\; a_j(x,\xi',\mu)=\tilde{\lambda}_j(B'(x,\xi',\mu),\langle\mu\xi'\rangle^{1/2};y,\mu\eta')$$

for $1\leqq j\leqq m$ where $\tilde{\lambda}_j(\zeta',\sigma;x,\xi')$ are defined in §2, (2.6). Let K be a set of indices $K=(i_1,i_2,\cdots,i_k)$, $1\leqq i_1<i_2<\cdots<i_k\leqq m$ and $|K|=k=$the number of indices. We denote

$$q_K(x,\xi,\mu)=\prod_{j\in K} q_j(x,\xi,\mu)\,.$$

For σ, a permutation on K, we put

$$Q_K^\sigma=\mathrm{Op}\,q_{\sigma(i_1)}\,\mathrm{Op}\,q_{\sigma(i_2)}\cdots\mathrm{Op}\,q_{\sigma(i_k)}\,.$$

When $K=(1,2,\cdots,m)$ we often write q, Q^σ instead of q_K, Q_K^σ.

Lemma 7.1.
$$q_{K(\beta)}=\sum_{j=1}^{|\beta|}\sum_{|M|=|K|-j}T_M^\beta\,q_M\quad with\quad T_M^\beta\in S(\mu^{j/2}\langle\xi'\rangle^{(j+|\beta|)/2},\tilde{g}_\mu)\,.$$

Put $\psi=q_i^{(\gamma)}\,q_{j(\gamma)}$ with $|\gamma|=1$. Then it is clear that

$$\psi^{(\alpha)}_{(\beta)}\in S(\mu^{-|\alpha+\beta''|/2}\langle\mu\xi'\rangle\,m(B')^{-|\alpha+\beta|}\langle\xi'\rangle^{-|\alpha|},\tilde{g}_\mu)\text{ for }|\alpha+\beta|\leqq 1.$$

Lemma 7.2.
$$\sigma(\mathrm{Op}\,q_i\,\mathrm{Op}\,q_K)=q_i\,q_K+\sum_{|L|=|K|-1}\psi_L\,q_L+\sum_{j=2}^{|K|}\sum_{|L|=|K|-j}T_L\,q_L$$

where $\psi^{(\alpha)}_{L(\beta)}\in S(\mu^{-|\alpha+\beta''|/2}\langle\mu\xi'\rangle\langle\xi'\rangle^{-|\alpha|}\,m(B')^{-|\alpha+\beta|},\tilde{g}_\mu)$ *for* $|\alpha+\beta|\leqq 1$ *and* $T_L\in S(\langle\mu\xi'\rangle^{j/2}\,m(B')^{-1},\tilde{g}_\mu)$.

Lemma 7.3. *Let* $\psi^{(\alpha)}_{(\beta)}\in S(\mu^{-|\alpha+\beta''|/2}\langle\mu\xi'\rangle^j\langle\xi'\rangle^{-|\alpha|}\,m(B')^{-|\alpha+\beta|},\tilde{g}_\mu)$ *for* $|\alpha+\beta|\leqq 2$. *Then for* $|L|=|K|-2j$ *we have*

$$\mathrm{Op}(\psi q_L)=\psi\,\mathrm{Op}\,q_L+\mathrm{Op}(\sum_{i=2j+1}^{|K|}\sum_{|M|=|K|-i}T_{M,i}\,q_M)$$

with $T_{M,i}\in S(\langle\mu\xi'\rangle^{(i-1)/2}\, m(B')^{-1}, \tilde{g}_\mu)$.

Lemma 7.4. *Assume that* $A\in S(\langle\mu\xi'\rangle^{j/2}\, m(B')^{-1}, \tilde{g}_\mu)$ *and* $|M|=|K|-j$. *Then*

$$\mathrm{Op}(Aq_M)=\sum_{i=j}^{|K|}\sum_{|S|=|K|-i} T_{S,i}\,\mathrm{Op}\, q_S \quad \text{with} \quad T_{S,i}\in S(\langle\mu\xi'\rangle^{i/2}\, m(B')^{-1}, \tilde{g}_\mu)\,.$$

Remark 7.1. If $A\in S(\langle\mu\xi'\rangle^{(j-1)/2}\, m(B')^{-1}, \tilde{g}_\mu)$ and $|M|=|K|-j$ then Lemma 7.4 implies that

$$\mathrm{Op}(Aq_M)=\sum_{i=j}^{|K|}\sum_{|S|=|K|-i} T_{S,i-1}\,\mathrm{Op}\, q_S\,.$$

Lemma 7.5.

$$\mathrm{Op}(q_i\, q_K)=\mathrm{Op}\, q_i\,\mathrm{Op}\, q_K+\sum_{|L|=|K|-1}\psi_L\,\mathrm{Op}\, q_L+\sum_{j=2}^{|K|}\sum_{|M|=|K|-j} T_{M,j}\,\mathrm{Op}\, q_M$$

with $T_{M,j}\in S(\langle\mu\xi'\rangle^{1/2}\, m(B')^{-1}, \tilde{g}_\mu)$ *where* $\psi_{L(\beta)}^{(\alpha)}$ $(|\alpha+\beta|\leqq 1)$ *belong to* $S(\mu^{-|\alpha+\beta''|/2}\langle\mu\xi'\rangle\langle\xi'\rangle^{-|\alpha|}\, m(B')^{-1}, \tilde{g}_\mu)$.

Proposition 7.1. *Let σ be a permutation on K. Then*

$$\begin{aligned}\mathrm{Op}\, q_K-Q_K^\sigma &=\sum_{j=1}^{[|K|/2]}\sum_{|L|=|K|-2j,\tau,L\subset K}\psi_{L,j}^{\sigma,\tau}\, Q_L^\tau \\ &\quad+\sum_{j=3}^{|K|}\sum_{|M|=|K|-j,\tau,M\subset L} C_{M,j}^{\sigma,\tau}\, Q_M^\tau\end{aligned}\tag{7.1}$$

with $C_{M,j}^{\sigma,\tau}\in S(\langle\mu\xi'\rangle^{(j-1)/2}\, m(B')^{-1}, \tilde{g}_\mu)$ *where for* $|\alpha+\beta|\leqq 1$ *one has*

$$\psi_{L,j(\beta)}^{\sigma,\tau(\alpha)}\in S(\mu^{-|\alpha+\beta''|/2}\langle\mu\xi'\rangle^j\langle\xi'\rangle^{-|\alpha|}\, m(B')^{-|\alpha+\beta|}, \tilde{g}_\mu)\,.\tag{7.2}$$

Proof. We shall proceed by induction on $|K|$. When $|K|=2$ one has clearly

$$\mathrm{Op}\, q_K=Q_K^\sigma+\tilde{\psi}_{0,1}^\sigma+T^\sigma,\ T^\sigma\in S(\langle\mu\xi'\rangle^{1/2}\, m(B')^{-1}, \tilde{g}_\mu)$$

where $\tilde{\psi}_{0,1}^\sigma$ satisfies (7.2). Hence $\psi_{0,1}^\sigma=\tilde{\psi}_{0,1}^\sigma+T^\sigma$ verifies the desired estimate (7.2) and (7.1) holds when $|K|=2$. We assume that (7.1) holds with K and let $T=K\cup\{\nu\}$, $1\leqq\nu\leqq m$. From Lemma 7.4 it follows that

$$\mathrm{Op}(q_\nu\, q_K)=\mathrm{Op}\, q_\nu\,\mathrm{Op}\, q_K+\sum_{|L|=|K|-1}\psi_L\,\mathrm{Op}\, q_L+\sum_{j=2}^{|K|}\sum_{|M|=|K|-j} T_{M,j}\,\mathrm{Op}\, q_M$$

with $T_{M,j}\in S(\langle\mu\xi'\rangle^{j/2}\, m(B')^{-1}, \tilde{g}_\mu)$. We substitute the expression (7.1) into $\mathrm{Op}\, q_K$. To do so we note that

$$\begin{aligned}&[\mathrm{Op}\, q_\nu, \psi_{L,j}^{\cdot,\cdot}]\in S(\langle\mu\xi'\rangle^{(2j+1-1)/2}\, m(B')^{-1}, \tilde{g}_\mu)\,,\\ &[\mathrm{Op}\, q_\nu, C_{M,j}^{\cdot,\cdot}]\in S(\langle\mu\xi'\rangle^{(j+1-1)/2}\, m(B')^{-1}, \tilde{g}_\mu)\,.\end{aligned}$$

These imply that $\mathrm{Op}\, q_\nu\,\mathrm{Op}\, q_K-\mathrm{Op}\, q_\nu\, Q_K^{\sigma'}$ is of the same form as the right-hand side of (7.1). We turn to the term $\psi_1\,\mathrm{Op}\, q_L$. Substituting the expression of $\mathrm{Op}\, q_L$, it follows that $\psi_1\,\mathrm{Op}\, q_L-\psi_L\, Q_L^{\sigma'}$ is equal to

$$\sum_{j=1}^{[|L|/2]}\sum_{|S|=|L|-2j}(\psi_1\,\psi_{M,j}^{\sigma',\tau'})\, Q_S^{\tau'}+\sum_{j=3}^{|L|}\sum_{|M|=|L|-j} T_{M,j}^{\sigma',\tau'}\, Q_M^{\tau'}$$

where $T^{\sigma',\tau'}_{M,j}\in S(\langle\mu\xi'\rangle^{(j+1)/2}\,m(B')^{-1},\tilde g_\mu)$. We examine that $\psi_L\,\psi^{\sigma',\tau'}_{S,j}=\psi^{\sigma',\tau'}_{S,j+1}$ verify the desired estimates. Since ψ_L satisfies (7.2) noting that $m(B')^{-1}\leqq C\langle\mu\xi'\rangle^{1/2}$ it follows that

$$\sigma(\psi_L\,\psi^{\sigma',\tau'}_{S,j})=\psi_L\,\psi^{\sigma',\tau'}_{S,j}+R \quad\text{with}\quad R\in S(\langle\mu\xi'\rangle^{(2j+2-1)/2}\,m(B')^{-1},\tilde g_\mu)$$

and this asserts that $\psi^{\sigma',\tau'}_{S,j+1}$ verify the desired estimates (7.2). Then $\psi_L\operatorname{Op}q_L-\psi_1\,Q^{\sigma'}_L$ is of the same form of the right-hand side of (7.1). Finally we treat $T_{M,j}\operatorname{Op}q_M$. Remark that

$$T_{M,j}\,\psi^{\cdot,\cdot}_{M,j}\in S(\langle\mu\xi'\rangle^{(j+2i+1-1)/2}\,m(B')^{-1},\tilde g_\mu)\quad(|M|=|K|-j,\ |L|=|M|-2i)$$
$$T_{M,j}\,C^{\cdot,\cdot}_{L,i}\in S(\langle\mu\xi'\rangle^{(i+j+1-1)/2}\,m(B')^{-1},\tilde g_\mu)\quad(|M|=|K|-j,\ |L|=|M|-i)\,.$$

Then it is clear that $T_{M,j}\operatorname{Op}q_M$ has the same form as of the right-hand side of (7.1). Now we have proved that (7.1) is valid when K is replaced by $K\cup\{\nu\}$.

For later reference we restate Proposition 7.1 with $|K|=m$ in a slightly different form. If j is even then with $j=2i$ we have

$$C^{\cdot,\cdot(\alpha)}_{M,2i(\beta)}\in S(\langle\mu\xi'\rangle^{(2j-1)/2}\,m(B')^{-1}\langle\mu\xi'\rangle^{|\beta''|/2}\langle\xi'\rangle^{|\beta'''|/2}\langle\xi'\rangle^{-|\alpha|/2},\tilde g_\mu)$$
$$\subset S(\mu^{-|\alpha+\beta'''|/2}\langle\mu\xi'\rangle^{i}\,\langle\xi'\rangle^{-|\alpha|}\,m(B')^{-|\alpha+\beta|},\tilde g_\mu)$$

for $|\alpha+\beta|\leqq 1$. Thus Proposition 7.1 stated as

$$\text{(7.3)}\qquad \operatorname{Op}q-Q^{\sigma}=\sum_{j=1}^{[m/2]}\ \sum_{|L|=m-2j,\tau,L}\psi^{\sigma,\tau}_{L,j}\,Q^{\tau}_L+\sum_{j=3,\text{odd}}^{m}\ \sum_{|M|=m-j,\tau,M}C^{\sigma,\tau}_{M,j}\,Q^{\tau}_M\,.$$

We proceed to the second step of our reduction. For a permutation σ on $(1,2,\cdots,m)$ we define

$$w^{\sigma}_j=\langle\mu D'\rangle^{j-1}\operatorname{Op}q_{\sigma(2j+1)}\operatorname{Op}q_{\sigma(2j+2)}\cdots\operatorname{Op}q_{\sigma(m)}\,u\quad\text{for}\quad 1\leqq j\leqq[m/2]=\hat m\,.$$

If m is odd we add further

$$w_{\hat m+1}=m(\varphi)^{-1}\langle\mu D'\rangle^{\hat m-1}\,u\,,\quad\text{if } m \text{ is odd.}$$

Proposition 7.2.

$$\operatorname{Op}q_{\sigma(1)}\operatorname{Op}q_{\sigma(2)}\,u=Q^{\sigma}u=(\operatorname{Op}q)\,u+\sum_{j=1}^{[(m+1)/2]}\sum_{\tau}A^{\sigma,\tau}_j\,w^{\tau}_j+\sum_{j=2}^{[m/2]}\sum_{\tau}\tilde A^{\sigma,\tau}_j\operatorname{Op}q_{\tau(2j)}\,w^{\tau}_j\,,$$
$$\operatorname{Op}q_{\sigma(2j-1)}\operatorname{Op}q_{\sigma(2j)}\,w^{\sigma}_j=\langle\mu D'\rangle\,w^{\sigma}_{j-1}+C^{\sigma}_{1j}\,w^{\sigma}_j+C^{\sigma}_{0j}(D_0-i\theta)\,w^{\sigma}_j\,,\quad 2\leqq j\leqq\hat m$$

where $A^{\sigma,\tau}_j\in S(\langle\mu\xi'\rangle,g)$, $\tilde A^{\sigma,\tau}_j\in S(m(\varphi)^{-1},\tilde g_\mu)$, $C^{\sigma}_{1j}\in S(\mu^{1/2}\langle\mu\xi'\rangle,\tilde g_\mu)$, $C^{\sigma}_{0j}\in S(\mu^{1/2},\tilde g_\mu)$. *If m is odd then we have further*

$$\operatorname{Op}q_{\sigma(m-1)}\operatorname{Op}q_{\sigma(m)}\,w_{\hat m+1}=A^{\sigma}\operatorname{Op}q_{\sigma(m-1)}\,w^{\sigma}_{\hat m}+\tilde A^{\sigma}\operatorname{Op}q_{\sigma(m-1)}\,w_{\hat m+1}+C^{\sigma}\,w^{\sigma}_{\hat m}+\tilde C^{\sigma}\,w_{\hat m+1}$$

where $A^{\sigma}\in S(m(\varphi)^{-1},\tilde g_\mu)$, $\tilde A^{\sigma}\in S(m(\varphi)^{-1},g)$, $C^{\sigma}\in S(\langle\mu\xi'\rangle,\tilde g_\mu)$, $\tilde C^{\sigma}\in S(\langle\mu\xi'\rangle,g)$.

Proof. For any $L(|L|=m-2j)$, τ (a permutation on L) we choose $\delta=\delta(L,\tau)$ (a permutation on $(1,2,\cdots,m)$) so that

$$\langle\mu D'\rangle^{j-1}\, Q_L^{\tau}\, u = w_j^{\delta}\,.$$

Setting

$$\sum_{\delta=\delta(L,\tau)} \psi_{L,j}^{\sigma,\tau}\langle\mu D'\rangle^{-(j-1)} = A_j^{\sigma,\delta}\in S(\langle\mu\xi'\rangle, \tilde{g}_\mu)$$

it is clear that the first term of the right-hand side of (7.3), operated on u, can be written as

$$\sum_{j=1}^{\hat{m}} \sum_{\delta} A_j^{\sigma,\delta}\, w_j^{\delta}\,.$$

Consider the second term of the right-hand side of (7.3) Let $M=(i_1, \cdots, i_{m-j})$. When $m-j\geqq 1$ (if m is even this is the case) we choose δ, a permutation on $(1, 2, \cdots, m)$ such that

$$\delta(2k) = \tau(i_1), \cdots, \delta(m) = \tau(i_{m-j})\,, \quad j = 2k-1\,.$$

Then it follows that

$$\begin{aligned} C_{M,j}^{\sigma,\tau}\, Q_M^{\tau}\, u &= C_{M,j}^{\sigma,\tau}\, \mathrm{Op}\, q_{\tau(i_1)}\langle\mu D'\rangle^{-(k-1)}\, w_k^{\delta} = C_{M,j}^{\sigma,\tau}\langle\mu D'\rangle^{-(j-1)/2}\, \mathrm{Op}\, q_{\delta(2k)}\, w_k^{\delta} \\ &\quad + C_{M,j}^{\sigma,\tau}[\mathrm{Op}\, q_{\delta(2k)}, \langle\mu D'\rangle^{-(j-1)/2}]\, w_k^{\delta} \end{aligned}$$

where $C_{M,j}^{\sigma,\tau}[\mathrm{Op}\, q_{\delta(2k)}, \langle\mu D'\rangle^{-(j-1)/2}]\in S(\mu^{1/2}\, m(B')^{-1}, \tilde{g}_\mu)$. Thus setting

$$\tilde{A}_j^{\sigma,\delta} = \sum_{\delta=\delta(M,\tau)} C_{M,j}^{\sigma,\tau}\langle\mu D'\rangle^{-(j-1)/2}\in S(m(B')^{-1}, \tilde{g}_\mu)$$

the second term of the right-hand side of (7.3), after operated on u, is written

$$\sum_{j=2}^{\hat{m}} \sum_{\delta} \tilde{A}_j^{\sigma,\delta}\, \mathrm{Op}\, q_{\tau(2j)}\, w_j^{\delta} + \sum_{j=1}^{[(m+1)/2]} \sum_{\delta} A_j^{\delta,\tau}\, w_j^{\delta}\,.$$

When $m=j=2k-1$ (hence in particular m is odd) note that from Lemma 4.4 there is $\tilde{m}\in S(m(\varphi), g)$ such that

$$\tilde{m}m(\varphi)^{-1} \equiv 1\,.$$

Then we can write $C_{M,m}^{\sigma}\, u\equiv A_{\hat{m}+1}^{\sigma}\, w_{\hat{m}-1}$ with $A_{\hat{m}+1}^{\sigma}=C_{M,m}^{\sigma}\langle\mu D'\rangle^{-(\hat{m}-1)}\, \tilde{m}$ which is in $S(\langle\mu\xi'\rangle, g)$. This proves the first part.

We shall prove the second part. Note that

$$\begin{aligned} \text{(7.4)} \qquad \mathrm{Op}\, q_{\sigma(2j-1)}\, \mathrm{Op}\, q_{\sigma(2j)}\, w_j^{\sigma} &= \langle\mu D'\rangle\, w_{j-1}^{\sigma} + \mathrm{Op}\, q_{\sigma(2j-1)}\, \psi w_j^{\sigma} \\ &\quad + \tilde{\psi}\, \mathrm{Op}\, q_{\sigma(2j)} \cdots \mathrm{Op}\, q_{\sigma(m)}\, u \end{aligned}$$

where $\psi=[\mathrm{Op}\, q_{\sigma(2j)}, \langle\mu D'\rangle^{j-1}]\,\langle\mu D'\rangle^{-(j-1)}$, $\tilde{\psi}=[\mathrm{Op}\, q_{\sigma(2j-1)}, \langle\mu D'\rangle^{j-1}]$. It is easy to see that $\psi\in S(\mu^{1/2}, \tilde{g}_\mu)$, $\tilde{\psi}\in S(\mu^{1/2}\langle\mu\xi'\rangle^{j-1}, \tilde{g}_\mu)$. It is obvious that $\mathrm{Op}\, q_{\sigma(2j-1)}\, \psi=\psi\,(D_0-i\theta)+(D_0\psi-a_{\sigma(2j-1)}\, \psi)$ and the second term of the right-hand side is in $S(\mu^{1/2}\langle\mu\xi'\rangle, \tilde{g}_\mu)$. On the other hand writing

$$\tilde{\psi}\, \mathrm{Op}\, q_{\sigma(2j)} = \tilde{\psi}\langle\mu D'\rangle^{-(j-1)}\,(D_0-i\theta)\,\langle\mu D'\rangle^{j-1} - \tilde{\psi}\, a_{\sigma(2j)}\langle\mu D'\rangle^{-(j-1)}\langle\mu D'\rangle^{j-1}$$

it is clear that the second term of the right-hand side of (7.4) is

$$C_{1j}^{\sigma}\, w_j^{\sigma}+C_{0j}^{\sigma}(D_0-i\theta)\, w_j^{\sigma}$$

with $C_{1j}^{\sigma}\in S(\mu^{1/2}\langle\mu\xi'\rangle, \tilde{g}_\mu)$, $C_{0j}^{\sigma}\in S(\mu^{1/2}, \tilde{g}_\mu)$. This proves the second statement. We turn to the last statement. Remark that $\mathrm{Op}\, q_{\sigma(m)}\, w_{\hat{m}+1}=m(\varphi)^{-1}\, w_{\hat{m}}^{\sigma}+\psi u$ with $\psi=[\mathrm{Op}\, q_{\sigma(m)}, m(\varphi)^{-1}\langle\mu D'\rangle^{\hat{m}-1}]\in S(m(\varphi)^{-2}\langle\mu\xi'\rangle^{\hat{m}-1}, \tilde{g}_\mu)$ and

$$\begin{aligned}\mathrm{Op}\, q_{\sigma(m-1)}\, \mathrm{Op}\, q_{\sigma(m)} w_{\hat{m}+1}&=m(\varphi)^{-1}\, \mathrm{Op}\, q_{\sigma(m-1)}\, w_{\hat{m}}^{\sigma}+[\mathrm{Op}\, q_{\sigma(m-1)}, m(\varphi)^{-1}]\, w_{\hat{m}}^{\sigma}\\ &\quad+\mathrm{Op}\, q_{\sigma(m-1)}\, \psi\langle\mu D'\rangle^{-(\hat{m}-1)}\, \tilde{m} w_{\hat{m}+1}\,.\end{aligned}$$

Since $\hat{\psi}=\psi\langle\mu D'\rangle^{-(\hat{m}-1)}\, \tilde{m}\in S(m(\varphi)^{-1}, g)$ it follows that $[\mathrm{Op}\, q_{\sigma(m-1)}, \hat{\psi}]$ is in $S(\langle\mu\xi'\rangle, g)$. It is also clear that $[\mathrm{Op}\, q_{\sigma(m-1)}, m(\varphi)^{-1}]$ is in $S(m(\varphi)^{-2}, \tilde{g}_\mu)\subset S(\langle\mu\xi'\rangle, \tilde{g}_\mu)$. These notes show the last statement.

Next we study the difference of the principal symbol and its blown up one. Recall that

$$\begin{aligned}p(x, \xi) &= \tilde{q}\,(b(x, \xi), \sigma; x, \xi')+\sum_{j=1}^{[m/2]}\sigma^{2j}\, r_{m-2j}(b(x, \xi); x, \xi')\,,\\ \tilde{q}\,(\zeta, \sigma; x, \xi') &= \prod_{j=1}^{m}(\zeta_0-\tilde{\lambda}_j(\zeta', \sigma; x, \xi'))\,.\end{aligned}$$

Put

$$\begin{aligned}\tilde{P}(x, \xi, \mu) &= \mu^m P(y, \eta),\ P_{m-j}(x, \xi, \mu) = \mu^{m-j}\, p_{m-j}(y, \eta)\,,\\ R_{m-2j}(x, \xi, \mu) &= \mu^{m-2j}\, r_{m-2j}(b(y, \eta); y, \eta)\,.\end{aligned}$$

Since $\mu b(y, \eta)=B(x, \xi, \mu)$ we have

$$\begin{aligned}\tilde{P}(x, \xi, \mu) &= \tilde{q}\,(B(x, \xi, \mu), \mu\sigma; y, \eta')+\sum_{j=1}^{[m/2]}(\mu\sigma)^{2j}\, R_{m-2j}(x, \xi, \mu)\\ &\quad+\sum_{j=1}\mu^j\, P_{m-j}(x, \xi, \mu)\,.\end{aligned}\tag{7.5}$$

Here we take $\mu\sigma=\langle\mu\xi'\rangle^{1/2}$ that is $\sigma=\mu^{-1}\langle\mu\xi'\rangle^{1/2}$ then it follows that

$$\begin{aligned}\tilde{P}(x, \xi_0-i\theta, \xi', \mu) &= \prod_{j=1}^{m} q_j(x, \xi, \mu)+\sum_{j=1}^{[m/2]}\langle\mu\xi'\rangle^j\, R_{m-2j}(x, \xi_0-i\theta, \xi', \mu)\\ &\quad+\sum_{j=1}\mu^j\, P_{m-j}(x, \xi_0-i\theta, \xi', \mu)\end{aligned}\tag{7.6}$$

where $q_j(x, \xi, \mu)=\xi_0-i\theta-\tilde{\lambda}_j(B'(x, \xi', \mu), \langle\mu\xi'\rangle^{1/2}; y, \mu\eta')$ which was studied in sections 5 and 6. Set

$$\tilde{q}_j(\zeta, \sigma; x, \xi') = \zeta_0-\tilde{\lambda}_j(\zeta', \sigma; x, \xi'),\ \tilde{q}_k(\zeta, \sigma; x, \xi') = \prod_{j\in K}\tilde{q}_j(\zeta, \sigma; x, \xi')\,.$$

Let $r(\zeta; x, \xi')$ be polynomial in ζ of degree $k\,(k\leqq m-1)$ with coefficients which are homogeneous of degree 0 in ξ', C^∞ in a conic neighborhood of $(0, e_p)$. Since $\tilde{\lambda}_j(\zeta', \sigma; x, \xi')$ are different for any $(\zeta', \sigma)\neq(0, 0)$, $(x, \xi')\in W$, a conic neighborhood of $(0, e_p)$, we can write

$$r(\zeta; x, \xi') = \sum_{|K|=k}\tilde{C}_K(\zeta', \sigma; x, \xi')\, \tilde{q}_K(\zeta, \sigma; x, \xi')\tag{7.7}$$

where $\tilde{C}_K(\zeta', \sigma; x, \xi')$ are homogeneous of degree 0 in (ζ', σ) and ξ' respectively. Set

$$C_K(x, \xi', \mu) = \tilde{C}_K(B'(x, \xi', \mu); \langle\mu\xi'\rangle^{1/2}; y, \mu\eta')$$

then Proposition 3.1 and (7.7) give

Lemma 7.6. *Let $R_{m-2j}(x, \xi, \mu)$ be as above. Then*

$$\langle\mu\xi'\rangle^j R_{m-2j}(x, \xi_0-i\theta, \xi', \mu) = \sum_{|K|=m-2j} C_{K,j}(x, \xi', \mu)\, q_K(x, \xi, \mu)$$

where $C_{K,j(\beta)}^{(\alpha)} \in S(\mu^{-|\alpha'''+\beta''|/2}\langle\mu\xi'\rangle^j \langle\xi'\rangle^{-|\alpha|} m(B')^{-|\alpha+\beta|}, \tilde{g}_\mu)$ for $|\alpha+\beta| \leqq 1$.

Lemma 7.3 and Remark 7.1 show that $\mathrm{Op}(C_{K,j}\, q_K)$ can be written as in the same form of the right-hand side of (7.3). Then we can apply Proposition 7.2 or rather its proof to conclude that

Proposition 7.3. *Let $R_{m-2j}(x, \xi, \mu)$ be as above. Then*

$$\mathrm{Op}(\langle\mu\xi'\rangle^j R_{m-2j}(x, \xi_0-i\theta, \xi', \mu))\, u = \sum_{j=1}^{[(m+1)/2]} \sum_{\tau} A_j^{\sigma,\tau} w_j^{\tau} + \sum_{j=2}^{[m/2]} \sum_{\tau} \tilde{A}_j^{\sigma,\tau} \mathrm{Op}\, q_{\tau(2j)}\, w_j^{\tau}$$

where $A_j^{\sigma,\tau}$ and $\tilde{A}_j^{\sigma,\tau}$ have the same properties as in Proposition 7.2.

Finally we study lower order terms satisfying (1.7). Assume that $p_{m-j}(x, \xi)$ vanishes of order $m-2j$ whenever $m-2j>0$ on Σ near $(0, e_p)$. Since $p_{m-j}(x, \xi)$ are polynomials in ξ_0 we can represent p_{m-j} as follows when $m-2j>0$;

$$p_{m-j}(x, \xi) = \sum_{|\alpha|=m-2j} d_\alpha(x, \xi')\, b(x, \xi)^\alpha + \sum_{i=j}^{2j-1} e_{i-j}(x, \xi')\, \xi_0^{m-i}$$

where $d_\alpha(x, \xi')$, $e_k(x, \xi')$ are homogeneous of degree j, k respectively. By the definition of $P_{m-j}(x, \xi, \mu)$ it follows that

$$\mu^j P_{m-j}(x, \xi, \mu) = \sum_{|\alpha|=m-2j} \mu^j d_\alpha(y, \mu\eta')\, B(x, \xi, \mu)^\alpha + \sum_{i=j}^{2j-1} \mu^{2j-i}(\mu^{i-j} e_{i-j}(y, \mu\eta'))\, \xi_0^{m-i}\,.$$

From (7.7) it follows that

$$B(x, \xi, \mu)^\alpha = \sum_{|K|=m-2j} C_K(x, \xi', \mu)\, q_K(x, \xi, \mu) \quad (|\alpha| = m-2j)\,,$$
$$\xi_0^{m-i} = \sum_{|L|=m-i} C_L(x, \xi', \mu)\, q_L(x, \xi, \mu)$$

where $C_{S(\beta)}^{(\alpha)}$ belong to $S(\mu^{-|\alpha''+\beta''|/2}\langle\xi'\rangle^{-|\alpha|} m(B')^{-|\alpha+\beta|}, \tilde{g}_\mu)$ if $|\alpha+\beta| \leqq 1$. Note that Lemma 3.1 and Remark 3.1 show that

$$(7.8)\qquad \begin{aligned} &(\mu^j d_\alpha(y, \mu\eta'))_{(\delta)}^{(\gamma)} \in S(\langle\mu\xi'\rangle^j \langle\xi'\rangle^{-|\gamma|}, \tilde{g}_\mu)\,, \quad |\gamma+\delta| \leqq 1\,,\\ &(\mu^{i-j} e_{i-j}(y, \mu\eta'))_{(\delta)}^{(\gamma)} \in S(\langle\mu\xi'\rangle^{i-j}\langle\xi'\rangle^{-|\gamma|}, \tilde{g}_\mu)\,, \quad |\gamma+\delta| \leqq 1\,. \end{aligned}$$

Since $j \leqq i \leqq 2j-1$ and hence $i-j \leqq (i-1)/2$ we see that $\mathrm{Op}(\mu^j P_{m-j}(x, \xi_0-i\theta, \xi', \mu))$ has the same form as in the right-hand side of (7.3).

We turn to the case $m-2j \leqq 0$. Let

$$p_{m-j}(x, \xi') = \sum_{i=j}^{m} e_{i-j}(x, \xi')\, \xi_0^{m-i}$$

where $e_{i-j}(x, \xi')$ are homogeneous of degree $i-j$. Write

$$\mu^j P_{m-j}(x, \xi, \mu) = \sum_{i=j}^{m} \mu^{2j-i}(\mu^{i-j} e_{i-j}(y, \mu\eta'))\, \xi_0^{m-i}$$

where $\mu^{i-j} e_{i-j}(y, \mu\eta')$ satisfies (7.8). If $m-2j<0$ then $(i+1)/2 \leqq (m+1)/2 \leqq j$ and hence we have

$$(i-j) \leqq (i-1)/2 . \tag{7.9}$$

This is also true if $m=2j$ and i is odd for $(i+1)/2 \leqq m/2 = j$. On the other hand if $m=2j$ and i is even we get $i-j=i-m/2 \leqq i/2$. In any case $\mathrm{Op}(\mu^j P_{m-j}(x, \xi_0, \xi', \mu))$ has the same form as in the right-hand side of (7.3).

Proposition 7.4. *Assume that $p_{m-j}(x, \xi)$ satisfy (1.7). Let $P_{m-j}(x, \xi, \mu)$ be defined as above. Then*

$$\mathrm{Op}(\mu^j P_{m-j}(x, \xi_0, \xi', \mu))\, u = \sum_{j=1}^{[(m+1)/2]} \sum_{\tau} A_j^{\sigma,\tau} w_j^{\tau} + \sum_{j=2}^{[m/2]} \sum_{\tau} \tilde{A}_j^{\sigma,\tau}\, \mathrm{Op}\, q_{\tau(2j)}\, w_j^{\tau}$$

where $A_j^{\sigma,\tau}$ and $\tilde{A}_j^{\sigma,\tau}$ have the same properties as in Proposition 7.2.

Propositions 7.2, 7.3 and 7.4 show that the equation

$$\tilde{P}(x, D_0 - i\theta, D', \mu)\, u = f$$

can be reduced to a second order system with diagonal principal part to which we can apply Corollary 6.1. Then we can conclude that $\tilde{P}(x, D, \mu)$ has a parametrix verifying (A.3) and (A.4) without modulo term. To prove that this parametrix satisfies (A.5) we apply Proposition 6.2 with suitable $\psi(x, \xi', \mu)$, for example $\psi_\varepsilon(x, \xi', \mu) = x_0 - \mu d_\varepsilon(M_\mu(x', \xi'))$. Remarking that

$$\tilde{P}(x, \xi, \mu) = \mu^m P(M_\mu(x, \xi))$$

when $|x_j| \leqq \mu^{1/2}$, $|\xi_j \langle \xi' \rangle^{-1} - \delta_{jp}| \leqq \mu$ it follows that $P(M_\mu(x, \xi))$ has a parametrix at $(0, e_p)$ with finite propagation speed of wave front sets. Hence $P(x, \xi)$ has such a parametrix at $(0, e_p)$.

Appendix

In this appendix, we shall give the definition of parametrices with finite propagation speed of wave front sets and give some properties of such parametrices. Consider operators of the form

$$P(x, D) = \sum_{j=0}^{m} A_j(x, D')\, D_0^{m-j} \tag{A.1}$$

where $A_j(x, D')$ are $N \times N$ matrix valued pseudodifferential operators with symbols in $S^j(\boldsymbol{R}^{d+1} \times \boldsymbol{R}^d)$ and $A_0(x, D') = I_N$, the identity matrix of order N. We call m the order of P. Let I be an open interval containing s and we denote by $C^k(I, H^p)$ the set of k times continuously differentiable functions from I to the usual Sobolev space $H^p = H^p(\boldsymbol{R}^d)$ of order p and by $||\cdot||_p$ the norm in H^p and set

$$||f||_p^2 = \sum_{j=1}^{N} ||f_j||_p^2 \quad \text{for} \quad f=(f_1, \cdots, f_N) \in (H^p)^N .$$

By $C^k(I, H^p)_s^+$ we denote the set of $f \in C^k(I, H^p)$ vanishing in $x_0 < s$. We shall say that $V \in \mathcal{V}_s$ if there is a positive constant $\delta(V)$ such that

$$\text{(A.2)} \qquad ||D_0^k Vf(t, \cdot)||_q^2 \leqq c_{p,q,k} \sum_{j=0}^{k} \int^t ||D_0^j f(\tau, \cdot)||_p^2 \, d\tau, \ t \leqq s+\delta(V)$$

for any $k \in \boldsymbol{N}$, $p, q \in \boldsymbol{R}$ and $f \in (C^k(I, H^p)_s^+)^N$.

To simplify notation we shall in what follows write simply $h \in S^j$ when $h=(h_{ik})$ is a $N \times N$ matrix valued pseudodifferential operator (or symbol) with $h_{ik} \in S^j(\boldsymbol{R}^n)$ where n will be clear in the context. We fix $(s, \hat{x}, \hat{\xi}') = (s, \kappa) \in I \times (T^* \boldsymbol{R}^d \backslash 0)$ and observe an operator G which satisfies the following conditions;

(A.3) $\quad PGh \equiv h$ modulo an operator in $C^\infty(I, S^{-\infty}) + \mathcal{V}_s$ for any $h = h(x', D') \in S^0(\boldsymbol{R}^d \times \boldsymbol{R}^d)$ supported near κ,

with a constant β we have

(A.4) $\quad ||D_0^j Gf(t, \cdot)||_p^2 \leqq c_{p,j} \int^t ||f(\tau, \cdot)||_{p+j+\beta}^2 \, d\tau , \quad 0 \leqq j \leqq m-1$
for any $p \in \boldsymbol{R}, f \in (C^0(I, H^{p+m-1+\beta})_s^+)^N$,

for any $h_1(x', D') \in S^\infty(\boldsymbol{R}^d \times \boldsymbol{R}^d)$ supported near κ and for any

(A.5) $\quad h_2(x', D') \in S^\infty(\boldsymbol{R}^d \times \boldsymbol{R}^d)$ with $\operatorname{supp} h_2 \subset\subset T^*\boldsymbol{R}^d \backslash (\operatorname{supp} h_1)$,
one has $D_0^j h_2 G h_1 \in \mathcal{V}_s$, $\ 0 \leqq j \leqq m-1$.

Note that (A.4) means that Gf loses β derivatives. From the dfinition of $\mathcal{V}_s$ it follows in particular that there is a positive constant $\delta(h_1, h_2)$ such that

$$WF(D_0^j G h_1 f(t, \cdot)) \cap \operatorname{supp} h_2 = \phi , \quad 0 \leqq j \leqq m-1$$

when $t \leqq s+\delta(h_1, h_2)$ for any $f \in (C^0(I, H^p)_s^+)^N$.

For $l \in \boldsymbol{N}$ we can write

$$\text{(A.6)} \qquad D_0^l = QP+R , \quad R = \sum_{j=1}^{m} B_j D_0^{m-j} , \quad B_j \in S^{l-m+j}$$

where Q is an operator of the form (A.1) of order $l-m$. Hence it follows that $D_0^l Gh = Q(h+S+V) + RGh$ with $S \in C^\infty(I, S^{-\infty})$, $V \in \mathcal{V}_s$. Then it is claer that for sufficiently small $|t-s|$,

$$\text{(A.7)} \qquad ||D_0^l Ghf(t, \cdot)||_p^2 \leqq c_p \sum_{k=0}^{l-m+1} \int^t ||D_0^k f(\tau, \cdot)||_{p+l+\beta-k}^2 \, d\tau ,$$

$$\int^t ||D_0^l Ghf(\tau, \cdot)||_p^2 \, d\tau \leqq c_p \sum_{k=0}^{l-m} \int^t ||D_0^k f(\tau, \cdot)||_{p+l+\beta-k}^2 \, d\tau$$

for any $f \in (C^{l-m+1}(I, H^{p+l+\beta})_s^+)^N$. Assume that $a(x, D') \in S^\infty(\boldsymbol{R}^{d-1} \times \boldsymbol{R}^d)$ and ah is in $S^{-\infty}$ near κ uniformly when $|x_0-s|$ is small. Then it is also clear that (from (A.4))

(A.8) $$\|aD_0^l\, Ghf(t,\cdot)\|_p^2 \leqq c_{p,q} \sum_{k=0}^{l-m+1} \int^t \|D_0^k f(\tau,\cdot)\|_q^2\, d\tau\,,$$

$$\int^t \|aD_0^l\, Ghf(\tau,\cdot)\|_p^2\, d\tau \leqq c_{p,q} \sum_{k=0}^{l-m} \int^t \|D_0^k f(\tau,\cdot)\|_q^2\, d\tau$$

for any $f\in(C^{l-m+1}(I, H^q)_s^+)^N$ and $p, q\in \boldsymbol{R}$, for small $|t-s|$. Let $\tilde{P}$ be another operator of the form (A.1) of order m such that

(A.9) $$P-\tilde{P} = \sum_{j=1}^{m} B_j(x, D')\, D_0^{m-j}$$

with $B_j\in S^j$ which are in $S^{-\infty}$ near κ uniformly when $|x_0-s|$ is small. In the following we write $P\equiv\tilde{P}$ near κ when P and $\tilde{P}$ satisfy (A.9). If G verifies (A.3)–(A.5) for P at (s, κ) then it follows that

(A.10) $$(P-\tilde{P})\, Gh\in \mathcal{V}_s$$

where $h(x', D')\in S^0$ has support in a sufficiently small conic neighborhood of κ. This implies that G satisfies (A.3)–(A.5) for $\tilde{P}$ also at (s, κ). This allows us to microlocalize our definition of parametrices; *we shall say that P has a parametrix with finite propagation speed of wave front sets at (s, κ) if there exist $\tilde{P}$, G with $P\equiv\tilde{P}$ near κ wihch satisfy* (A.3)–(A.5). *We call G a parametrix of P at (s, κ) with finite propagation speed of wave front sets.* For brevity we say it parametrix in this Appendix. In what follows we denote by J a small open interval containing s which may differ in each context. Now we give some properties of parametrices.

Proposition A.1. *Let P_i $(i=1, 2)$ be operators of the form* (A.1) *of order m_i. If each P_i has a parametrix at (s, κ) then $P_1 P_2$ so does at (s, κ). If $P_1 P_2$ has a parametrix at (s, κ) then so does P_1 at (s, κ).*

Corollary A.1. *Let $P_i(i=1, 2, \cdots, n)$ be operators of the form* (A.1) *of order m_i. If each P_i has a parametrix at (s, κ) then $P_1 P_2 \cdots P_n$ has a parametrix at (s, κ).*

Let $T(x, D')$ be $N\times N$ matrix valued pseudodifferential operator in $S^0(\boldsymbol{R}^{d+1}\times \boldsymbol{R}^d)$ which is elliptic near (s, κ) uniformly when $|x_0-s|$ is small.

Proposition A.2. *Let P, $\tilde{P}$ be operators of the form* (A.1) *of order m. Assume that $PT\equiv T\tilde{P}$ near κ. Then if $\tilde{P}$ has a parametrix at (s, κ) then so does P at (s, κ).*

Next we shall examine the invariance of existence of a parametrix by conjugation with a Fourier integral operator associated to a local homogeneous canonical transformation preserving the planes x_0=const. Let χ be a local homogeneous canonical transformation from a neighborhood of $(\hat{y}, \hat{\eta})=(\hat{y}_0, \hat{y}', \hat{\eta}_0, \hat{\eta}')$ to a neighborhood of $(\hat{x}, \hat{\xi})=(\hat{x}_0, \hat{x}', \hat{\xi}_0, \hat{\xi}')$ such that $y_0=x_0$. Since χ preserves y_0=const., a generating function of this canonical transformation has the form

$$x_0\, \eta_0+g(x, \eta')\,.$$

We work with a Fourier integral operator F associated with χ which is elliptic near $(\hat{x}, \hat{\xi}, \hat{y}, \hat{\eta})$, is represented as

$$Fu(x) = \int e^{ih(x,\eta')} a(x, \eta')\, \hat{u}(x_0, \eta')\, d\eta'$$

(in a convenient y' coordinates) in which x_0 can be regarded as a parameter. We also assume that F is bounded from $H^k(\boldsymbol{R}^d_{y'})$ to $H^k(\boldsymbol{R}^d_{x'})$ for every $k \in \boldsymbol{R}$ uniformly with respect to a parameter x_0 when $|x_0 - s|$ is small.

Proposition A.3. *Let χ, F be as above and $P(x, D)$, $P^{\chi}(y, D)$ be operators of the form* (A.1) *of order m. Assume that*

$$PF \equiv FP^{\chi} \quad near \quad (\hat{y}', \hat{\eta}')\,.$$

Then if P^{χ} has a parametrix at $(s, \hat{y}', \hat{\eta}')$ then so does P at $(s, \hat{x}', \hat{\xi}')$.

Proposition A.4. *Let P be an operator of the form* (A.1) *of order m. Assume that P has a parametrix at $(s, \hat{x}', \xi')$ for every ξ' with $|\xi'| = 1$. Then the Cauchy problem for P is locally solvable near $(s, \hat{x}')$ in C^∞ with initial data on $x_0 = s$.*

Proof. Denote by $G_{\xi'}$ a parametrix of P at $(s, \hat{x}', \xi')$. By hypothesis there are operators $P_{\xi'}$ of the form (A.1) of order m such that $P_{\xi'} \equiv P$ near $(\hat{x}', \xi')$ and $P_{\xi'}$, $G_{\xi'}$ verify (A.3)–(A.5). Then there are finite open conic neighborhood W_i of $(\hat{x}', \xi'_i)$ such that $\cup_i W_i \supset \Omega \times (\boldsymbol{R}^d \backslash 0)$ where Ω is a neighborhood of $\hat{x}'$. We may assume that

$$P_{\xi'_i} = P_i \equiv P \quad \text{in} \quad W_i$$

uniformly when $|x_0 - s|$ is small and (A.3), (A.5) hold for any $h, h_1 \in S^0$ with supports in W_i with $P = P_i$, $G = G_i = G_{\xi'_i}$. Now we take another open conic covering $\{V_i\}$ of $\Omega \times (\boldsymbol{R}^d \backslash 0)$, $V_i \subset\subset W_i$ and a partition of unity $\{\alpha_i(x', \xi')\}$ subordinate to $\{V_i\}$ so that

$$\sum_i \alpha_i(x', \xi') = \alpha(x')$$

where $\alpha(x')$ is equal to 1 in a small neighborhood of $\hat{x}'$. Put

$$G = \sum_i G_i\, \alpha_i$$

then we have from (A.3) and (A.8) that

$$PGf = \sum_i (P - P_i)\, G_i\, \alpha_i f + \sum_i P_i\, G_i\, \alpha_i f = \alpha(x') f + (V + S) f$$

with $S \in C^\infty(J, S^{-\infty})$, $V \in \mathcal{V}_s$. Set $T = -(V+S)$ and let $\beta(x'), \gamma(x') \in C_0^\infty(\boldsymbol{R}^d)$ be equal to 1 near $\hat{x}'$ such that supp $\gamma \subset\subset \{\beta = 1\}$, supp $\beta \subset\subset \{\alpha = 1\}$. By the definition of $\mathcal{V}_s$ it is clear that

$$\text{(A.11)} \qquad \int^t \|\beta V f(\tau, \cdot)\|^2\, d\tau \leqq c_p\, t \int^t \|f(\tau, \cdot)\|^2\, d\tau \quad \text{for} \quad t \leqq s + \delta(V)$$

for any $f \in (C^0(J, L^2)^+_s)^N$. It is also easy to see that

$$\|\beta S f(t, \cdot)\|^2 \leqq 4^{-1} \|f(t, \cdot)\|^2 \quad \text{for} \quad t \leqq s + \delta_1$$

if $||\beta||$ is sufficiently small. This implies that

(A.12) $$\int^t ||\beta Tf(\tau, \cdot)||^2 d\tau \leqq 2^{-1} \int^t ||f(\tau, \cdot)||^2 d\tau \quad \text{for} \quad t \leqq s+\delta_2$$

for any $f \in (C^0(J, L^2)_s^+)^N$. Define

$$U = \sum_{k=0}^{\infty} (\beta T)^k$$

then it follows from (A.12) that

(A.13) $$\int^t ||Uf(\tau, \cdot)||^2 d\tau \leqq c \int^t ||f(\tau, \cdot)||^2 d\tau \quad \text{for} \quad t \leqq s+\delta_2$$

and $(1-\beta T)\, U=1$. Since $\gamma(\alpha-T)\, U=\gamma$ it follows that

(A.14) $\gamma(x')\, PGUf = \gamma(x')f, \quad t \leq s+\delta_2 \quad$ for any $\quad f \in (C^0(J, L^2)_s^+)^N$.

Noting that

$$\int^t ||Tf(\tau, \cdot)||_p^2 d\tau \leqq c_p \int^t ||f(\tau, \cdot)||^2 d\tau, \quad t \leqq s+\delta_3$$

for any $p \in \boldsymbol{R}, f \in (C^0(J, L^2)_s^+)^N$ and

$$Uf = \sum_{k=0}^{\infty} (\beta T)^k f = f + \beta TUf$$

one has from (A.4)

$$||D_0^j\, GUf(t, \cdot)||_p^2 \leqq c_p \{||D_0^j\, Gf(t, \cdot)||_p^2 + \int^t ||Uf(\tau, \cdot)||^2 d\tau\}$$
$$\leqq \tilde{c}_p \{\int^t ||f(\tau, \cdot)||_{p+j+\beta}^2 d\tau + \int^t ||f(\tau, \cdot)||^2 d\tau\}$$

where $\beta = \max_i \beta_i$, $0 \leqq j \leqq m-1$. This gives that

(A.15) $$||D_0^j\, GUf(t, \cdot)||_p^2 \leqq 2c_p \int^t ||f(\tau, \cdot)||_{p+j+\beta}^2 d\tau, \; t \leqq s+\delta_4$$

for any $p \in \boldsymbol{R}$ with $p+\beta+m-1 \geqq 0$, $0 \leqq j \leqq m-1$, $f \in (C^0(J, H^{p+\beta+m-1})_s^+)^N$. Now (A.14) shows that GUf is a local solution near $\hat{x}'$ to the Cauchy problem

$$Pu = f, \quad f \in (C^0(J, H^{p+\beta+m-1})_s^+)^N.$$

From (A.15) it follows that $D_0^j\, GUf\, (0 \leqq j \leqq m-1)$ belong to $(L^2([0, \delta_4], H^p))^N$ and vanish in $x_0 < s$. This completes the proof.

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