J. Math. Kyoto Univ. (JMKYAZ)
29-3 (1989) 365–403

The imbedding theorems for weighted Sobolev spaces

By

Toshio HORIUCHI

§ 0. Introduction

Let F be a closed set of $\boldsymbol{R}^n$, and let $\Omega=\boldsymbol{R}^n\backslash F$. The purpose of this paper is to study imbedding theorems for weighted Sobolev spaces of three categories $W_\alpha^{k,p}(\Omega)$, $\mathcal{W}_\alpha^{k,p}(\Omega)$ and $H_\alpha^{k,p}(\boldsymbol{R}^n)$, where the weight functions considered here are powers of $\delta(x)$, where $\delta(x)$ is equivalent to the distance from x to the closed set F, see the § 1 for the precise definitions of those spaces.

In [37–39] S. L. Sobolev introduced a notion of generalized derivative and proved general integral inequalities for differentiable functions of several variables, which are usually lumped together in a single theorem as the so-called Sobolev imbedding theorem. Later the Sobolev theorem was generalized and refined variously (Kondrat'ev, Il'in, Gagliardo, Nirenberg etc.), and such theorems proved to be a usefull tool in functional analysis and in the theory of linear and nonlinear partial differential equations.

Weighted Sobolev spaces also have been studied intensively for more than twenty years, and the main field of application are the degenerated (elliptic) operators. This fact makes clear that a large part of papers concerned with Sobolev spaces with weights where the weight functions are powers of a distance to manifolds (see Grisvard [14], Kufner [20], Lizorkin [23–24], Uspenskii [45] etc.). For the further references, see a survey paper by A. Avantaggiati [8]. The author also studied in [16] the degenerated elliptic operators, and the present paper is strongly motivated by the author's research in this field.

Recently in [30], V. G. Maz'ja has proved a variant of Sobolev imbedding inequalities in the case of a weighted norm in the right-hand side with weights being powers of the distance to a linear subspace of $\boldsymbol{R}^n$, and refined both the Sobolev and Hardy inequalities. Here we present his result as our starting point.

Let us consider functions u of z, where $z=(x, y)\in\boldsymbol{R}^{n-s}\times\boldsymbol{R}^s$ $(1\leq s<n)$. Moreover let μ be a given positive measure on $\boldsymbol{R}^n$ such that the number

$$(0.1)\qquad K=\operatorname*{Sup}_{\rho,z}\,(\rho+|y|)^{-\alpha}\rho^{1-n}[\mu(B_\rho(z))]^{1/q}$$

Communicated by Prof. S. Mizohata October 6, 1987

is finite, where $1\leq q<+\infty$, $-s<\alpha<+\infty$ and $B_\rho(z)=\{\zeta\in\boldsymbol{R}^n;\ |\zeta-z|<\rho\}$.

Theorem 0.1 (V. G. Maz'ja [30]). *We have the following inequality*

$$(0.2)\qquad \Big(\int_{\boldsymbol{R}^n}|u(z)|^q d\mu\Big)^{1/q}\leq C\int_{\boldsymbol{R}^n}|y^\alpha|\,|\nabla_z u(z)|\,dz \qquad u\in C_0^\infty(\boldsymbol{R}^n)$$

with a positive constant C independent of u.

Let C be the best constant in this inequality. Then we have

$$(0.3)\qquad C_1K\leq C\leq C_2K$$

where K is the number defined by (0.1), *C_1 and C_2 are positive numbers independent of μ and depending only on n, s, q, α.*

Our main interest in this paper is to study imbeddings of weighted Sobolev spaces in parallel to this result, however we do not work with a general measure μ but the Lebesgue measure with weight for simplicity. In Maz'ja's theorem the subspace $\{z=(x, y)\in\boldsymbol{R}^{n-s}\times\boldsymbol{R}^s;\ y=0\}$ is considered as F in our notation. In order to treat more general F, we shall introduce in § 2 two regularity properties $P(s)$ and $SP(s)$ which a closed set F of $\boldsymbol{R}^n$ may possess, and roughly speaking, we shall generalize and refine Theorem 0.1 in the following ways.

In the first place, the sets F considered in this paper are not necessarily linear subspaces but arbitrary closed sets having the property $P(s)$. In Theorem 1, we shall prove the existence and compactness of imbedding operators of $H_\alpha^{1,p}(\boldsymbol{R}^n)$ into $L_\beta^q(\boldsymbol{R}^n)$ under the assumption $P(s)$ for F. Here the property $P(s)$ is defined by means of the behavior as $\eta\to 0$ of the n-dimensional Lebesgue measure of the tubular neighborhood of F defined by $F_\eta=\{x\in\boldsymbol{R}^n;\ dist(x, F)<\eta\}$.

What is essential for the proof of Theorem 1 is the equivalence of imbedding for Sobolev spaces with weights and isoperimetric inequality with weights (cf. Federer [11], Talenti [41] and Maz'ja [26–29]). The property $P(s)$ is a sufficient condition for the validity of isoperimetric inequalities with weights (see (2.5) in § 2). A fairly large class of sets F of $\boldsymbol{R}^n$ satisfies this property, for example, a set of finite points ($s=n$), a $(n-k)$-dimensional subspace of $\boldsymbol{R}^n$ ($s=k$), a finitely many union of $(n-k)$-dimensional Lipschitz manifolds of $\boldsymbol{R}^n$ ($s=k$), a Cantor set ($s=1-\log_3 2$) and so on (see §2, for the detailed).

Secondly we consider the case where F is an arbitrary closed set of $\boldsymbol{R}^n$ without any regularity assumption. It is already known that there exists in general no imbedding of the type (0.1) in Theorem 0.1 (cf. [4], [6], [12], [13], [25], see also [7]). However we can prove an analogous result Theorem 2 by the aid of suitable norms with weights and the covering lemma in § 4 (cf. Triebel [42], § 3).

Thirdly we shall assume that F satisfies the property $SP(s)$ which is a stronger assumption than $P(s)$ in general. Then we shall show the existence of imbeddings of $W_\alpha^{k+j,p}(\Omega)$ into $W_\beta^{j,q}(\Omega)$ using the previous theorems and the extension lemma in § 4. In particular if $p\geq n$, these spaces can be imbedded into Schauder spaces with weights denoted by $SC_\beta^j(\Omega)$ and $SC_\beta^{j,\lambda}(\Omega)$. For the proof we shall make use of suitable averaging functions in addition to potential estimate initiated by Sobolev (cf. [7], [18] and [32–34]). $SP(s)$ is also a regularity assumption for F. For instance, if F satisfies $SP(1)$, then the boundary of F has the strongly local Lipschitz property (for the precise definition, see § 2).

Our main imbedding results can be classified according to the following list: Let p satisfy that $1\leq p<+\infty$.
Case 1 (Theorem 1): F has the property $p(s)$ for some $s\in(0, n]$. Then

$$H_{\alpha}^{1,p}(\boldsymbol{R}^{n})\rightarrow L_{\beta}^{q}(\boldsymbol{R}^{n}). \tag{0.4}$$

Case 2 (Theorem 2): F is an arbitrary closed set of $\boldsymbol{R}^{n}$. Then

$$\mathcal{W}_{\alpha}^{k+j,p}(\Omega)\rightarrow\mathcal{W}_{\beta}^{j,q}(\Omega),\ \mathcal{SC}_{\beta}^{j}(\Omega)\quad\text{or}\quad\mathcal{SC}_{\beta}^{j,\lambda}(\Omega),\qquad 0\leq\lambda\leq 1. \tag{0.5}$$

Case 3 (Theorem 3): F has the property $SP(s)$ for some positive integer $s\leq n$. Then

$$W_{\alpha}^{k+j,p}(\Omega)\rightarrow W_{\beta}^{j,q}(\Omega),\ SC_{\beta}^{j}(\Omega)\quad\text{or}\quad SC_{\beta}^{j,\lambda}(\Omega),\qquad 0\leq\lambda\leq 1. \tag{0.6}$$

In asserting an imbedding $H_{\alpha}^{1,p}(\boldsymbol{R}^{n})\rightarrow L_{\beta}^{q}(\boldsymbol{R}^{n})$ for example, it is intended that there exists a positive constant C such that

$$\|u;\ L_{\beta}^{q}\|\leq C\|u;\ H_{\alpha}^{1,p}\|, \tag{0.7}$$

where C is independent of each $u\in H_{\alpha}^{1,p}(\boldsymbol{R}^{n})$.

We also construct counterexamples just after the statements of Theorems showing that they give in some sense best possible imbedding results for the spaces considered.

There are already many authors who have studied various aspect of weighted Sobolev spaces, see also [15], [31] and [35]. For the complete references, see the books by R. A. Adams [6], A. Kufner [21], V. G. Maz'ja [30] and H. Triebel [42] for example.

This paper is organized as follows:
In § 1, we define weighted Sobolev spaces and Schauder spaces. In § 2, we standardize some geometrical concepts and notations which are useful in this paper. In § 3, our main imbedding results will be stated, and most of counterexamples are also given there. The § 4 is devoted to prepare the lemmas concerned with covering, extension and isoperimetric inequalities. Theorem 1 will be proved in a chain of auxiliary lemmas through § § 5, 6 and 7. The proofs of Theorem 2 and Theorem 3 will be given in § 8 and § 9 respectively. The § 10 is devoted to establish techinical lemmas. To end this paper we shall give the proof of extension lemma stated in § 4 for the sake of self-containedness.

The author wishes to express his gratitude to Prof. S. Mizohata, Prof. N. Shimakura, Prof. K. Yabuta, and the staff in Institut Mittag Leffler, especially to Prof. L. Hörmander, for their kind help and a number of useful conversations.

§ 1. Definitions of weighted Sobolev spaces

Let F be a closed set in $\boldsymbol{R}^n$ and set $\Omega=\boldsymbol{R}^n\backslash F$.

Let $\delta(x)\in C^\infty(\Omega)$ be a nonnegative function satisfying

$$C^{-1}\leq\delta(x)/\mathrm{dist}\,(x, F)\leq C\,, \tag{1.1}$$

$$|\partial^\gamma\delta(x)|\leq C(\gamma)\,\mathrm{dist}\,(x, F)^{1-|\gamma|}\,, \qquad x\in\Omega=\boldsymbol{R}^n\backslash F\,, \tag{1.2}$$

where γ is an arbitrary milti-index, C and $C(\gamma)$ are positive numbers depending only on γ and n.

Here we note that for an arbitrary closed set F, there exist functions $\delta(x)$ with required properties (1.1) and (1.2), see [Triebel [42], P. 250, § 3], for example.

In this paper we shall deal with weighted Sobolev spaces denoted by

$$W^{k,p}_\alpha(\Omega),\ \mathcal{W}^{k,p}_\alpha(\Omega) \quad\text{and}\quad H^{k,p}_\alpha(\boldsymbol{R}^n)\,, \tag{1.3}$$

where k is a nonnegative integer, p is a real number ≥ 1 and α is a real number.

The spaces $W^{k,p}_\alpha(\Omega)$ and $\mathcal{W}^{k,p}_\alpha(\Omega)$ are the set of functions on $\Omega=\boldsymbol{R}^n\backslash F$, whose generalized derivatives $\partial^\gamma u$ of order $\leq k$ satisfy

$$\|u;\ W^{k,n}_\alpha\| = \sum_{|\gamma|\leq k}(\int_\Omega|\partial^\gamma u(x)|^p\delta(x)^{\alpha p}\,dx)^{1/p}<+\infty \tag{1.4}$$

and

$$\|u;\ \mathcal{W}^{k,p}_\alpha\| = \sum_{|\gamma|\leq k}(\int_\Omega|\partial^\gamma u(x)|^p\delta(x)^{(\alpha+|\gamma|-k)p}dx)^{1/p}<+\infty\,, \tag{1.5}$$

respectively.

These spaces are Banach spaces with the norms (1.4) and (1.5) respectively. Conventionally we set

$$L^p_\alpha(\Omega) = W^{0,p}_\alpha(\Omega) = \mathcal{W}^{0,p}_\alpha(\Omega) \tag{1.6}$$

and

$$\|u;\ L^p_\alpha\| = \|u;\ W^{0,p}_\alpha\| = \|u;\ \mathcal{W}^{0,p}_\alpha\|\,. \tag{1.7}$$

In order to define $H^{k,p}_\alpha(\boldsymbol{R}^n)$, we assume that $|F|$ (n-dimensional Lebesgue measure of F)$=0$. By $H^{k,p}_\alpha(\boldsymbol{R}^n)$, we mean the completion of $C^k_0(\boldsymbol{R}^n)$, the space of functions of the class C^k having compact support, with respect to the norm defined by

$$\|u;\ H^{k,p}_\alpha\| = \sum_{|\gamma|\leq k}(\int_{\boldsymbol{R}^n}|\partial^\gamma u(x)|^p\delta(x)^{\alpha p}dx)^{1/p}\,, \tag{1.8}$$

where the range of α will be specified later.

This space is also a Banach space with the norm (1.8) under additional assumptions on α and F (see the property $P(s)$ in § 2 and Theorem 1 in §3).

Here we note that these spaces $W^{k,p}_\alpha(\Omega)$, $\mathcal{W}^{k,p}_\alpha(\Omega)$ and $H^{k,p}_\alpha(\boldsymbol{R}^n)$ do not essentially depend on the choice of the function $\delta(x)$ having the properties (1.1) and (1.2).

We shall define Schauder spaces with weights as target of our imbeddings when p and k are large numbers.

Let j and k be nonnegative integers, and let λ satisfy $0<\lambda\leq 1$. Let us define for any $u\in C_0^\infty(\Omega)$,

$$|u|_{j,\alpha}=\sum_{|\gamma|=j}(\gamma !)^{-1}\operatorname*{Sup}_{\Omega}\delta(x)^{\operatorname{Max}(\alpha,0)}|\partial^\gamma u(x)|, \tag{1.8}$$

$$|u|^*_{j,\alpha}=\sum_{|\gamma|=j}(\gamma !)^{-1}\operatorname*{Sup}_{\Omega}\delta(x)^{\alpha}|\partial^\gamma u(x)|, \tag{1.9}$$

$$|u|_{j,\lambda,\alpha}=\sum_{|\gamma|=j}(\gamma !)^{-1}\operatorname*{Sup}_{\substack{(x,y)\in\Omega\times\Omega\\ x\neq y}}\operatorname{Min}[\delta(x)^\alpha,\delta(y)^\alpha]\frac{|\partial^\gamma u(x)-\partial^\gamma u(y)|}{|x-y|^\lambda}, \tag{1.10}$$

$$||u||_{k,\alpha}=\sum_{j=0}^{k}|u|_{j,\alpha-k+j}, \tag{1.11}$$

$$||u||^*_{k,\alpha}=\sum_{j=0}^{k}|u|^*_{j,\alpha-k+j}. \tag{1.12}$$

Then we set

$$SC_\alpha^k(\Omega)=\{u\in C^k(\Omega);\ ||u||_{k,\alpha}<+\infty\}, \tag{1.13}$$

$$\mathcal{S}C_\alpha^k(\Omega)=\{u\in C^k(\Omega);\ ||u||^*_{k,\alpha}<+\infty\}. \tag{1.14}$$

We also set

$$SC_\alpha^{k,\lambda}(\Omega)=\{u\in C^k(\Omega);\ ||u||_{k,\alpha-\lambda}+|u|_{k,\lambda,\alpha}<+\infty\},\qquad \alpha\geq 0, \tag{1.15}$$

and

$$\mathcal{S}C_\alpha^{k,\lambda}(\Omega)=\{u\in C^k(\Omega);\ ||u||^*_{k,\alpha-\lambda}+|u|_{k,\lambda,\alpha}<+\infty\}. \tag{1.16}$$

These are Banach spaces with the norms:

$$||u;\ SC_\alpha^k||=||u||_{k,\alpha},\quad ||u;\ \mathcal{S}C_\alpha^k||=||u||^*_{k,\alpha}, \tag{1.17}$$

$$||u;\ SC_\alpha^{k,\lambda}||=||u||_{k,\alpha-\lambda}+|u|_{k,\lambda,\alpha}, \tag{1.18}$$

$$||u;\ \mathcal{S}C_\alpha^{k,\lambda}||=||u||^*_{k,\alpha-\lambda}+|u|_{k,\lambda,\alpha}. \tag{1.19}$$

Here we remark that these spaces do not essentially depend on the choice of $\delta(x)$ as before.

§ 2. Sufficient conditions on F for the imbedding theorems

In this section we shall introduce two regularity properties which a closed set F of $\boldsymbol{R}^n$ may possess, and we also introduce weighted isoperimetric inequalities which are essentially equivalent to our imbedding results stated as Theorem 1 in § 3.

Let s be a positive number satisfying $0<s\leq n$. We first define the property $P(s)$ which concerns the $(n-s)$-dimensional Hausdorff measure of F and the n-dimensional Lebesgue measure of the tubular neighborhood of F denoted by F_η with $\eta>0$, that is:

Definition of $P(s)$. Let s be a positive number satisfying $0<s\leq n$ and $s^*=\mathrm{Min}(s, 1)$. A closed set F is said to have the property $P(s)$ if $|F|=0$ and there exist positive numbers C_s and A_0 such that

$$|B\cap(F_\eta\backslash F_{\eta'})|\leq C_s\eta^{s-s^*}(\eta-\eta')^{s^*}d(B)^{n-s}, \tag{2.1}$$

where $$F_\eta=\{x\in \boldsymbol{R}^n;\ \mathrm{dist}(x,F)<\eta\},$$

B is an arbitrary open ball with diameter $d(B)\leq A_0$, η and η' are arbitrary numbers satisfying

$$0\leq\eta'<\eta\leq d(B)\leq A_0. \tag{2.2}$$

Here $|*|$ stands for the n-dimensional Lebesgue measure.

In order to make clear in advance the role of this condition $P(s)$ as well as what it means, we shall describe here the weighted isoperimetric inequalities which will play an important role in the proof of Theorem 1 and will be established under this condition in the §§ 6 and 7. To do so, we need the definition of d-dimensional Hausdorff measure. Let S be a set in $\boldsymbol{R}^n$. Consider various coverings of S by balls of radii$\leq\varepsilon$. We put $H^d_\varepsilon(S)=v_d\,\mathrm{Inf}\sum_j r_j^d$, where r_j is the radius of the j-th ball, v_d is the volume of the unit ball in $\boldsymbol{R}^d$ and the infimum is taken over all such coverings. Then we define $\mathcal{H}^d(S)$ (the d-dimensional Hausdorff measure of S) by

$$\mathcal{H}^d(S)=\lim_{\varepsilon\to 0}H^d_\varepsilon(S). \tag{2.3}$$

Since H^d_ε is monotone, this limit, finite or infinite, obviously exists. We also define $d(S)$ (the diameter of S) by

$$d(S)=\mathrm{Sup}\,[\mathrm{dist}(x, y);\ x, y\in S]. \tag{2.4}$$

Proposition 2.1. *Suppose that F has the property $P(s)$ with $s\in(0, n]$. Let M be an arbitrary bounded open subset of $\boldsymbol{R}^n$ with smooth boundary. Moreover suppose that $d(M)\leq A_1A_0$. Then there exists a positive number C_1 such that*

$$\Big(\int_M \mathrm{dist}(x, F)^{\beta q}dx\Big)^{1/q}\leq C_1\int_{\partial M}\mathrm{dist}(x, F)^{\alpha}d\mathcal{H}^{n-1}(x), \tag{2.5}$$

where $$0\leq 1-1/q=(1-\alpha+\beta)/n \quad \text{and either}$$
$$-s/q<\beta\leq\alpha\leq 0,\ 0<s\leq 1 \quad \text{or} \quad -s/q<\beta\leq\alpha,\ 1<s\leq n.$$

Here C_1 is independent of M, A_1 is also a positive number defined in Proposition 6.2 in § 6 and $\mathcal{H}^{n-1}(x)$ is the $(n-1)$-dimensional Hausdorff measure.

For the sake of simplicity we assume that $A_0=+\infty$ for a moment. As will be seen later, this proposition leads us to the imbedding of the type $H^{1,1}_\alpha(\boldsymbol{R}^n)\to L^q_\beta(\boldsymbol{R}^n)$, and we also have

$$\|u;\ L^q_\beta\|\leq\mathrm{Const.}\||\nabla u|;\ L^1_\alpha\|, \quad \text{for any} \quad u\in \boldsymbol{C}^\infty_0(\mathbf{R}^n). \tag{2.6}$$

Conversely we have the following:

Proposition 2.2. *The inequality* (2.6) *implies* (2.5) *with* $A_0=+\infty$.

Proof. We take a bounded open subset M of $\boldsymbol{R}^n$ with smooth boundary and construct approximative characteristic functions u_ε's of M for sufficiently small $\varepsilon>0$ as follows.

Let us set $M_\varepsilon=\{x\in M; \operatorname{dist}(x, \partial M)>\varepsilon\}$ and

$$(2.7)\qquad u_\varepsilon(x)=\begin{cases} 1 & x\in M_\varepsilon \\ \operatorname{dist}(x, \partial M)/\varepsilon & x\in M\backslash M_\varepsilon \\ 0 & x\in M^c\,. \end{cases}$$

Since u_ε's belong to the space $H^{1,1}_\alpha(\boldsymbol{R}^n)$ with $\alpha>-s$, it follows from (2.6) that

$$(2.8)\qquad \|u_\varepsilon;\, L^q_\beta\|\leq \text{Const.}\, \||\nabla u_\varepsilon|;\, L^1_\alpha\|\,.$$

Then we have, letting $\varepsilon\to 0$,

$$(2.9)\qquad \lim_{\varepsilon\to 0}\|u_\varepsilon;\, L^q_\beta\| = \Big(\int_M \delta(x)^{\beta q}dx\Big)^{1/q}\,.$$

Since ∂M_ε is a smooth manifolds for sufficiently small $\varepsilon>0$, we also have

$$(2.10)\qquad \limsup_{\varepsilon\to 0}\int_{\boldsymbol{R}^n}|\nabla u_\varepsilon|\delta(x)^\alpha dx = \limsup_{\varepsilon\to 0}\varepsilon^{-1}\int_0^\varepsilon dt\int_{\partial M_t}\delta(x)^\alpha d\mathcal{H}^{n-1}(x)$$
$$= \int_{\partial M}\delta(x)^\alpha d\mathcal{H}^{n-1}(x)\,.$$

Noting that $\delta(x)$ is equivalent to $\operatorname{dist}(x, F)$, we have the desired inequality. Q.E.D.

Before defining another property $SP(s)$, we shall give some important remarks here.

Let F satisfy $P(s)$ with $s\in(0, n]$. Take and fix an arbitrary ball B of $\boldsymbol{R}^n$ with $d(B)\leq A_0$.
Let us set

$$(2.11)\qquad f_B(\eta) = |B\cap F_\eta|,\qquad 0\leq\eta\leq d(B)\leq A_0\,.$$

Since $|F|=|F_0|=0$, we get $f_B(\eta)\leq C_s d(B)^{n-s}\eta^s$. Moreover $f_B(\eta)$ is Hölder continuous with exponent $s^*=\operatorname{Min}(s, 1)$. If $s\geq 1$, then we have the following lemma which will be proved in § 6 (see Proposition 6.1).

Lemma 2.1. *Assume that* $s\geq 1$ *and* F *has the property* $P(s)$. *Then we have the followings:*
(1) *It holds that*

$$(2.12)\qquad \mathcal{H}^{n-1}(B\cap\partial F_\eta)\leq C\eta^{s-1}d(B)^{n-s}\,,$$

for any $\eta\in(0, d(B))$ *with* $d(B)\leq A_0$.
(2) *For any* $\eta\in(0, d(B))$ *with* $d(B)\leq A_0$, *there exists a sequence of smooth manifolds* $\{N_k\}_{k=1}^\infty$ *such that* $N_k\subset F^c_\eta$ $(k=1, 2, \cdots)$, N_k *converges* ∂F_η *as* $k\to+\infty$ *and*

(2.13) $$\limsup_{k\to+\infty} \mathcal{H}^{n-1}(B\cap N_k)\leq C\eta^{s-1}d(B)^{n-s}\,.$$

Here $\mathcal{H}^{n-1}(x)$ is the $(n-1)$-dimensional Hausdorff measure, and C is a positive number depending only on the dimension of the space (independent of B and η).

The followings are typical examples of the set F with the required property $P(s)$. First we assume that s is a positive integer $\leq n$.

Example 1. Assume that F consists of finitely many points of $\boldsymbol{R}^n$. Then F satisfies $P(s)$ with $s=n$.

Example 2. Let F be a $(n-s)$-dimensional linear subspace:

$$F=\{z=(x,y)\in \boldsymbol{R}^{n-s}\times \boldsymbol{R}^s;\ x\in \boldsymbol{R}^{n-s},\ y=0\}\,.$$

Then F satisfies $P(s)$ with $A_0=+\infty$ and $C_s=2s$.

Example 3. Let F be a $(n-s)$-dimensional compact Lipschitz manifolds in $\boldsymbol{R}^n$ with $s<n$. Then F satisfies $P(s)$ for some $A_0<+\infty$.

As is easily verified, if F^s and F^t satisfy $P(s)$ and $P(t)$ respectively, then $F^s\cup F^t$ satisfies $P(\min[s,t])$. Therefore the Lipschitz property and compactness of the set F are not necessary for the validity of $P(s)$. To illustrate this we also give the following example:

Example 4. Let $n=2$ and $s=1$. Let us set

$$F_m=\{(x,y)\in \boldsymbol{R}^2;\ (x-4m)^2+y^2=m^{-2}\}\quad\text{and}\quad F=\bigcup_{m=1}^{\infty}F_m\,.$$

Then F satisfies $P(1)$ with $A_0=1/2$, $C_1=2\pi$.

Proof. First note that $\operatorname{dist}(F_j, F_k)>5/2$ provided $k\neq j$. So that each ball B_r with radius r can meet only one component of the set F_η, provided $0<\eta\leq d(B_r)=2r\leq 1/2$. Then we immediately get

$$|(F_\eta\backslash F_{\eta'})\cap B_r|\leq 2\ (\textit{length of }\partial B_r)|\eta-\eta'|=2\pi d(B_r)|\eta-\eta'|\,,$$
$$\text{for}\quad 0\leq\eta'\leq\eta\leq d(B_r)\leq 1/2\,.$$

Q.E.D.

As for the case that $0<s<1$, the following is typical.

Example 5. Assume that $n=1$ and $s=1-\log_3 2$. Let F be the Cantor set in $H_0=[0,1]$. Then F satisfies $P(s)$ with $A_0=1$, $C_3=18$, and F does not satisfy $P(t)$ for $t>s$. (The proof will be given at the end of this subsection for the sake of self-containedness.).

We note that there exists a set F with null the d-dimensional Hausdorff measure $\mathcal{H}^d(F)$ for any nonnegative d such that F fails to satisfy $P(s)$ for any $s\in(0,n]$.

Counterexample 6. Assume that $n=2$ and $b>0$. Let t be a positive integer which will be specified later. Let us set

$$
\begin{aligned}
(2.14)\qquad & P^m_{j,k} = (m+jm^{b-1},\, km^{-b-t})\,, \\
& Q^m_{j,k} = (m+km^{-b-t},\, jm^{-b-1})\,, \\
& F_m = \{P^m_{j,k},\, Q^m_{j,k};\, j=0,\cdots,m, \text{ and } k=0,\cdots,m^t.\}\,, \\
& F = \bigcup_{m=1}^{\infty} F_m\,.
\end{aligned}
$$

Then F is a null set with respect to $\mathcal{H}^d(x)$ for any positive number d, nevertheless F does not have the property $P(s)$ for any $s\in(0, n]$.

Proof. Since F consists of a countable set of points, it holds that $\mathcal{H}^d(F)=0$, $0<d\leq n$. Let $\eta=2\eta'=m^{-b-t}/4$, $R_m=(m+m^{-b}/2,\, m^{-b}/2)$ and let $B(m)=B_{m^{-b}}(R_m)$ be the ball with center R_m and radius m^{-b}. Then there exists a positive number C independent of each m such that

$$|(F_\eta\backslash F_{\eta'})\cap B(m)|/\{|\eta-\eta'|^{s^*}\eta^{s-s^*}d(B(m))^{n-s}\}\geq Cm^{(s-1)t+1}\,.$$

with $s^*=\text{Min}\,[1, s]$. If we set $t=1$ for example, this inequality implies the assertion. Here we also note that Theorem 1 fails to hold for this set F, see § 3.

We proceed to define another regularity property of F, which is a variant of the well-known "strongly local Lipschitz property" (see Adams. [6] P. 66, for example).

Definition of $SP(s)$. Let s be a positive integer$\leq n$. A closed set F has the property $SP(s)$ provided there exist positive numbers K, L, M, a locally finite open cover $\{U_j\}$ of $\partial\Omega=\partial F=F\backslash\text{Int}\,F$, and if $s<n$, there exists for each U_j a system of real valued functions $G^j=\{g^j_1, \cdots, g^j_s\}$ of $n-s$ variables such that the following conditions (1), $\cdots$, (6) hold:

(1) If $s=n$, F consists at most of countable points, and every pair of distinct points $x, \zeta\in F$ satisfies $|x-\zeta|>M$.

(2) For $1\leq s<n$, every collection of $K+1$ of the set U_j has empty intersection.

(3) For every pair of points $x, \zeta\in\Omega_M=\{x\in\Omega;\, \delta(x)<M\}$ such that $|x-\zeta|<M$, there exists j such that

$$x, \zeta\in V_j = \{z\in U_j;\, \text{dist}\,(z, \partial U_j)>M\}\,.$$

(4) If $1<s<n$, for some Cartesian coordinate system

$$(x^j, y^j) = (x^j_1, \cdots, x^j_{n-s}, y^j_1, \cdots, y^j_s) \text{ depending on } j\,,$$

the set $F\cap U_j$ is represented by the equalities

$$y^j_l = g^j_l(x^j_1, \cdots, x^j_{n-s})\,, \qquad l=1, \cdots, s.$$

(5) If $s=1$, $\Omega\cap U_j=(\boldsymbol{R}^n\backslash F)\cap U_j$ is represented by the single inequality

$$y^j_1>g^j_1(x^j_1, \cdots, x^j_{n-1})\,.$$

for some Cartesian coordinate system depending on j.

(6) Each of functions g_l^j in (4) and (5) satisfies a Lipschitz condition with constant L,

$$|g_l^j(x_1, \cdots, x_{n-s})-g_l^j(\xi_1, \cdots, \xi_{n-s})| \leq L|(x_1-\xi_j, \cdots, x_{n-s}-\xi_{n-s})|,$$

for $l=1, \cdots, s$, $0<s<n$ and $j \in \mathrm{N}$.

Here we note that if $s=1$, then this condition is equivalent to the strongly local Lipschitz property of $\Omega=\boldsymbol{R}^n \backslash F$. If a closed set F of measure zero satisfies $SP(s)$, then it also satisfies $P(s)$, see example 2.

Proof of Example 5. Let F be the Cantor set in $\mathrm{H}_0=[0,1]$, more precisely

$$F=\bigcap_{j=0}^{\infty} \cup \mathrm{H}_j, \tag{2.15}$$

where

$$\mathrm{H}_j = \cup \{\Phi(I); I \in \mathrm{H}_{j-1}\}, \qquad j \geq 1,$$
$$\Phi(I) = \{\operatorname{Inf} I \leq x \leq \operatorname{Inf} I+(1/3) \operatorname{diam} I\} \cup$$
$$\cup \{\operatorname{Inf} I+(2/3) \operatorname{diam} I \leq x \leq \operatorname{Inf} I+\operatorname{diam} I\}.$$

Without loss of generality we assume that

$$3^{-k-1} \leq d(B) \leq 3^{-k}, \text{ for some nonnegative integer } k. \tag{2.16}$$

First we take η and η' such that

$$3^{-j-1} \leq \eta' < \eta \leq 3^{-j}, \text{ for some nonnegative integer } j. \tag{2.17}$$

From the definition of F and H_j, H_j consists of 2^j disjoint closed intervals with lengths 3^{-j}, and $\overline{F_\eta \backslash F_{\eta'}}$ consists of disjoint closed intervals with lengths $\leq 3^{-j}-3^{-j-1}=3^{-j-1}2$. Then it holds that

$$\overline{F_\eta \backslash F_{\eta'}} \subset \overline{\mathrm{H}_0 \backslash \mathrm{H}_j} \cup [-3^{-j}, 0] \cup [1, 1+3^{-j}]. \tag{2.18}$$

Here $\overline{\mathrm{H}_0 \backslash \mathrm{H}_j}$ consists of 2^j-1 disjoint closed intervals, and each connected component contains at most 2 components of $\overline{F_\eta \backslash F_{\eta'}}$. Hence the number of components of $\overline{F_\eta \backslash F_{\eta'}}$ is at most $2(2^j-1)+2=2^{j+1}$. It is easy to see that at most 2^{j+1-k} components of $F_\eta \backslash F_{\eta'}$ can be contained in B provided $d(B) \leq 3^{-k}$. So that we have

$$\begin{aligned} |B \cap (F_\eta \backslash F_{\eta'})| &\leq 2^{j+1-k}(\eta-\eta') \leq 2(\eta-\eta') 3^{(j-k)(1-s)} \\ &\leq 2(\eta-\eta')\eta^{s-1}[3d(B)]^{1-s} \leq 4(\eta-\eta')^s d(B)^{1-s}, \end{aligned} \tag{2.19}$$

if $s=1-\log_3 2$.

Secondly we assume that, for some positive integer m,

$$3^{-j-m-1} \leq \eta' < 3^{-j-m} \leq 3^{-j-1} < \eta \leq 3^{-j}. \tag{2.20}$$

If $m=1$, it follows by the repeated use of the previous step that

$$|B \cap (F_\eta \backslash F_{\eta'})| \leq 3(\eta-\eta')\eta^{s-1}[3d(B)]^{1-s}. \tag{2.21}$$

So we assume that $m \geq 2$. In this case we have

$$(2.22) \qquad F_\eta \setminus F_{\eta'} \subset \bigcup_{p=1}^{m+1} \{F_{\xi_{p-1}} \setminus F_{\xi_p}\}, \quad \text{where} \quad \xi_p = \eta 3^{-p}.$$

By (2.21) we have

$$(2.23) \qquad \begin{aligned} |B \cap (F_\eta \setminus F_{\eta'})| &\leq 2[3d(B)]^{1-s}\eta^s[1+3^{-s}+\cdots+3^{-sm}] \\ &\leq 2[3d(B)]^{1-s}\eta^s(1-3^{-s})^{-1} = 6[3d(B)]^{1-s}\eta^s \\ &\leq 18d(B)^{1-s}|\eta-\eta'|^s. \end{aligned}$$

Consequently, F satisfies $P(s)$ for $s=1-\log_3 2$.

On the contrary, assume that $B=\mathrm{H}_0$ and $3^{-j-1} \leq \eta' < \eta \leq 2^{-1}3^{-j}$. Since it holds that

$$(2.24) \qquad \mathrm{Min}\,[\mathrm{diam}\, I;\ I \in B \setminus \mathrm{H}_j] = 3^{-j},$$

we have

$$(2.25) \qquad \begin{aligned} |F_\eta \setminus F_{\eta'}| &\geq (2^j-1)(\eta-\eta') \geq 2^{j-1}(\eta-\eta') \\ &\geq 4^{-1}(\eta-\eta')\eta^{s-1}. \end{aligned}$$

So that F does not satisfy $P(t)$ provided $t>s$. Q.E.D.

§ 3. Results and Counterexamples

In this section we shall describe our imbedding theorems according to the list in § 0.

Let p satisfy $1 \leq p < +\infty$. Let α and β be real numbers, and let k and j be nonnegative integers. Let F be a closed set in $\boldsymbol{R}^n$, and set $\Omega = \boldsymbol{R}^n \setminus F$, $\partial\Omega = \partial F = F \setminus$ Interior of F. Note that $F = \partial F$ if either F satisfies $P(s)$ or $SP(s)$ with $s \neq 1$.

Theorem 1. *Let p satisfy $1 \leq p < +\infty$. Assume that a closed set F has the property $P(s)$ with $s \in (0, n]$. Let D be a bounded subdomain of $\boldsymbol{R}^n$. Then the following imbeddings are valid:*

Suppose $(1-\alpha+\beta)p<n$, $0 \leq 1/p-1/q \leq (1-\alpha+\beta)/n$ and either $-s/q < \beta \leq \alpha \leq -\beta q(1-1/p)$, $0<s\leq 1$, or $-s/q<\beta\leq\alpha$, $1<s\leq n$. Then

$$(3.1) \qquad H_\alpha^{1,p}(\boldsymbol{R}^n) \to L_\beta^q(\boldsymbol{R}^n), \qquad p \leq q \leq np/[n-p(1-\alpha+\beta)].$$

Moreover if $0 \leq 1/p-1/q < (1-\alpha+\beta)/n$, then the following restrictions of the mappings defined by (3.1) *are compact;*

$$(3.2) \qquad H_\alpha^{1,p}(\boldsymbol{R}^n) \to L_\beta^q(D), \qquad p \leq q < np/[n-p(1-\alpha+\beta)].$$

Theorem 2. *Let p satisfy $1 \leq p < +\infty$. Let F be an arbitrary closed set in $\boldsymbol{R}^n$, and set $\Omega = \boldsymbol{R}^n \setminus F$. Then the following imbeddings are valid:*

Case A Suppose $(k-\alpha+\beta)p<n$, $0 \leq 1/p-1/q \leq (k-\alpha+\beta)/n$ and $\beta \leq \alpha$. Then

$$(3.3) \qquad \mathcal{W}_\alpha^{k+j,p}(\Omega) \to \mathcal{W}_\beta^{j,q}(\Omega), \qquad p \leq q \leq np/[n-p(k-\alpha+\beta)].$$

Case B *Suppose* $(k-\alpha+\beta)p=n$ *and* $\beta\leq\alpha$. *Then* (3.3) *is valid for* $p\leq q<+\infty$. *Moreover if either* $p=1$ *or* $\beta<\alpha$, *then*

$$\mathcal{W}_{\alpha}^{k+j,p}(\Omega)\to\mathcal{S}C_{\beta}^{j}(\Omega)\,. \tag{3.4}$$

Case C *Suppose* $(k-1-\alpha+\beta)p<n<(k-\alpha+\beta)p$ *and* $\beta\leq\alpha$, *then*

$$\mathcal{W}_{\alpha}^{k+j,p}(\Omega)\to\mathcal{S}C_{\beta}^{j,\lambda}(\Omega)\,,\qquad 0<\lambda\leq k-\alpha+\beta-n/p\,. \tag{3.5}$$

Case D *Suppose* $(k-1-\alpha+\beta)p=n$ *and* $\beta\leq\alpha$. *Then* (3.5) *is valid for* $0<\lambda<1$. *Moreover if either* $p=1$ *or* $\beta<\alpha$, *then* (3.5) *is valid for* $\lambda=1$ *as well.*

Theorem 3. *Let* p *satisfy* $1\leq p<+\infty$. *Assume that a closed set* F *has the property* $SP(s)$ *with* s *nonnegative integer* $\leq n$. *Then the following imbeddings are valid:*

Case A *Suppose* $(k-\alpha+\beta)p<n$, $0\leq 1/p-1/q\leq(k-\alpha+\beta)/n$ *and* $-s/q<\beta\leq\alpha$, *then*

$$W_{\alpha}^{k+j,p}(\Omega)\to W_{\beta}^{j,q}(\Omega)\,,\qquad p\leq q\leq np/[n-p(k-\alpha+\beta)]\,. \tag{3.6}$$

Case B *Suppose* $(k-\alpha+\beta)p=n$ *and* $0\leq\beta\leq\alpha$, *then* (3.6) *is valid for* $p\leq q<+\infty$. *Moreover if either* $p=1$ *or* $0<\beta<\alpha$, *then it holds that*

$$W_{\alpha}^{k+j,p}(\Omega)\to SC_{\beta}^{j}(\Omega)\,. \tag{3.7}$$

Case C *Suppose* $(k-1-\alpha+\beta)p<n<(k-\alpha+\beta)p$ *and* $0\leq\beta\leq\alpha$, *then*

$$W_{\alpha}^{k+j,p}(\Omega)\to SC_{\beta}^{j,\lambda}(\Omega)\,,\qquad 0<\lambda<k-\alpha+\beta-n/p\,. \tag{3.8}$$

Moreover if $\alpha\neq k-n/p$, *then* (3.8) *holds for* $\lambda=k-\alpha+\beta-n/p$ *as well.*

Case D *Suppose* $(k-1-\alpha+\beta)p=n$ *and* $0\leq\beta\leq\alpha$, *then* (3.8) *is valid for* $0<\lambda<1$. *Moreover if either* $p=1$ *or* $0<\beta<\alpha$, *then* (3.8) *holds for* $\lambda=1$ *as well.*

Here we remark that: Since elements of weighted Sobolev spaces are not functions defined everywhere but rather equivalence classes of such functions defined and equal up to values on sets of measure zero, by an imbedding of $W_{\alpha}^{k+j,p}(\Omega)\to SC_{\beta}^{j,\lambda}(\Omega)$ for example we mean that the equivalence class $u\in W_{\alpha}^{k+j,p}(\Omega)$ should contain an element belonging to the target space.

In the rest of this subsection, we construct examples showing that in certain respects these theorems 1, 2 and 3 give best possible imbedding results for the spaces considered. For some questions it is helpful to consider spaces without weights as special cases of spaces with weights.

First, we give counterexamples to Theorem 1 assuming that s is a positive integer $\leq n$, $\delta(z)=|y|$ for simplicity and that

$$F=\{z=(x,y)\in\boldsymbol{R}^{n-s}\times\boldsymbol{R}^{s};\ x\in\boldsymbol{R}^{n-s},\ y=0\}\ . \tag{3.9}$$

Here we note that F satisfies $P(s)$. (See § 2).

Let $B_h^n=\{z=(x,y)\in\boldsymbol{R}^{n-s}\times\boldsymbol{R}^{s};\ |z|<h\}$, and choose a function $f\in C_0^{\infty}(B_1^n)$ so

that $f \geq 0$ and $f=1$ for $z \in B^n_{1/2}$.

Counterexample to Theorem 1 ($1/p-1/q>(1-\alpha+\beta)/n$). Suppose that

$$1/p-1/q>(1-\alpha+\beta)/n \quad \text{and} \quad -s/q<\beta\leq\alpha\,. \tag{3.10}$$

From this condition, we can choose t and τ so that

$$n+(t+\tau+\alpha-1)p>0,\ n+(t+\tau+\beta)q<0 \text{ and } t \text{ is sufficiently large}\,. \tag{3.11}$$

Let us set

$$u(z)=f(z)|y|^t|z|^\tau\,. \tag{3.12}$$

Then we have $u \in H^{1,p}_\alpha(\Omega)$ but $u \in L^q_\beta(\Omega)$, where $\Omega=\boldsymbol{R}^n\backslash F$. In fact, both of the following inequalities are valid:

$$\int_{B^n_1}|y|^{\alpha p}|\partial u(z)|^p dz \leq C\int_{B^n_1}|z|^{(t+\tau+\alpha-1)p}dz<+\infty\,, \tag{3.13}$$

$$\int_{B^n_1}|y|^{\beta q}|u(z)|^q dz \geq C\int_{|z|/2\leq|y|<1/2}|z|^{(t+\tau+\beta)q}dz \tag{3.14}$$
$$\geq C'\int_0^{1/3} r^{n-1+(t+\tau+\beta)q}dr=+\infty\,.$$

Here we used a trivial estimate $|\nabla u(z)|\leq C(|y|^{t-1}|z|^\tau+|y|^t|z|^{\tau-1})$. Hence no imbedding of the type $H^{1,p}_\alpha(\Omega)\to L^q_\beta(\Omega)$ is possible provided $q>np/\{n-p(1-\alpha+\beta)\}$.

If $0<s\leq 1$ and $\alpha>-\beta q(1-1/p)$, then the imbeddings in Theorem 1 do not hold in general. To see this we give the following example:

Counterexample to Theorem 1 ($s=1$, $p=1$). Suppose that $s=1$, $p=1$, $n=2$, $-1/q<\beta\leq\alpha$ and $0<\alpha$. Let F be the closed set defined in the example 4 in § 2. Then the imbedding (3.1) can not hold.

Proof. Let us set

$$u^\varepsilon_m(x)=\begin{cases} 0\,, & x\in(G_{m,\varepsilon})^c \\ \operatorname{dist}(x,\partial G_{m,\varepsilon})/\varepsilon\,, & x\in G_{m,\varepsilon} \text{ and } \operatorname{dist}(x,\partial G_{m,\varepsilon})\leq\varepsilon \\ 1\,, & x\in G_{m,\varepsilon} \text{ and } \operatorname{dist}(x,\partial G_{m,\varepsilon})\geq\varepsilon\,, \end{cases} \tag{3.15}$$

where $G_{m,\varepsilon}=\{(x,y)\in\boldsymbol{R}^2;\,(x-4m)^2+y^2\leq(m^{-1}+\varepsilon)^2\}$ and ε is a sufficiently small positive number.
Assume that $1-1/q=(1-\alpha+\beta)/2$. Then we have

$$\lim_{\varepsilon\to 0}\int|u^\varepsilon_m(x)|^q\delta(x)^{\beta q}dx=C(q,\beta)d(F_m)^{(\alpha+1)q}\,, \tag{3.16}$$

$$\lim_{\varepsilon\to 0}\int|u^\varepsilon_m(x)|\delta(x)^\alpha dx=C(1,\alpha)d(F_m)^{\alpha+2}\,, \tag{3.17}$$

$$\lim_{\varepsilon\to 0}\int|\nabla u^\varepsilon_m(x)|\delta(x)^\alpha dx\leq\lim_{\varepsilon\to 0}2\pi\varepsilon^{-1}\int_0^\varepsilon[r+d(F_m)/2]r^\alpha dr=0\,, \tag{3.18}$$

where $C(q, \beta)$ and $C(1, \alpha)$ are positive numbers independent of m and ε. Therefore the imbedding $H^{1,1}_{\alpha} \rightarrow L^{q}_{\beta}$ does not hold. Q.E.D.

Assume that F does not satisfy $P(s)$. Then Theorem 1 is not necessarily valid however the set F is "small". To see this, we give the following:

Counterexamples to Theorem 1 (F fails to have $P(s)$). Let F be the set defined in the counterexample 6 in the § 2. Then we can choose a parameter t so that Theorem 1 ($n=2$) does not hold for any $s \in (0, 2]$.

Proof. We note that Theorem 1 is essentially equivalent to Proposition 2.1 in § 2 (see also Propositions 2.2 and 5.1).

(1) Assume that $\alpha \geq \beta > -2/q$ and $\alpha > 0$. In this case we set $M = \{\text{Convex hull of } F_m\}$ in the isoperimetric inequality (2.5) with $1-1/q=(1-\alpha+\beta)/2$. Then we have

$$(3.19) \qquad (\int_M \delta(x)^{\beta q} dx)^{1/q} \geq C m^{-(\alpha+1)(b+1)+2/q},$$

and
$$\int_{\partial M} \delta(x)^{\alpha} d\mathcal{H}^{n-1}(x) \leq 4m^t \int_0^{m^{-d-t}} r^{\alpha} dr = C m^{-(b+t)(\alpha+1)+t}.$$

So that

$$(3.20) \qquad (\int_M \delta(x)^{\beta q} dx)^{1/q} (\int_{\partial M} \delta(x)^{\alpha} \mathcal{H}^{n-1}(x))^{-1} \geq C m^{\alpha(t-1)+2/q-1},$$

where C is a positive number independent of each m. Then we put $t=1$, if $\alpha > \beta$; $t=2$, if $\alpha = \beta > 0$. Anyway the right side of the above inequality can not remain bounded as $m \rightarrow +\infty$. Therefore the propositions 2.1 and 5.1 do not hold.

(2) Assume that $-2/q < \beta \leq \alpha < 0$. In this case we put $t=1$ and set

$$M = \{x \in \boldsymbol{R}^2;\ \mathrm{dist}\,(x, F_m) \leq m^{-b}\}.$$

Then we have in a similar way

$$(3.21) \qquad (\int_M \delta(x)^{\beta q} dx)^{1/q} \geq (\int_{\{\text{Convex bull of } F_m\}} \delta(x)^{\beta q} dx)^{1/q} = \\ = O(m^{-(b+1)(\alpha+1)+2/q})$$

and
$$\int_{\partial M} \delta(x)^{\alpha} d\mathcal{H}^{n-1}(x) = O(m^{-b(\alpha+1)}).$$

So that we have

$$(3.22) \qquad (\int_M \delta(x)^{\beta q} dx)^{1/q} (\int_{\partial M} \delta(x)^{\alpha} d\mathcal{H}^{n-1}(x))^{-1} \geq C m^{-\beta}.$$

Therefore the propositions 2.1 and 5.1 do not hold. Q.E.D.

Counterexamples to Theorem 3. We retain the notation introduced by (3.9). Moreover let us set $\Omega^* = \Omega$, if $s > 1$; $\Omega \cap \boldsymbol{R}^n_+$, if $s=1$. It suffices to give counterexamples in the cases B and C.

Case B. Suppose that $\beta=0$ and $p>1$. Then we have $\alpha=1-n/p\geq 0$. Let us set

$$(3.23)\qquad u(z)=f(z)\log\{\log(2|z|^{-1})\}\ .$$

Then clearly $u\notin L^\infty(\Omega^*)$. But we have $u\in W^{1,p}_\alpha(\Omega^*)$. In fact,

$$(3.24)\qquad \int_{B_1^n}|\partial u(z)|^p|y|^{\alpha p}dz\leq C\int_{B_1^n}|y|^{\alpha p}|z|^{-p}|\log(|z|/2)|^{-p}dz\leq$$
$$\leq C\int_{B_1^n}|z|^{-n}|\log(|z|/2)|^{-p}dz\leq C'\int_0^1 r^{-1}|\log(r/2)|^{-p}dr<+\infty\ .$$

Therefore the imbedding $W^{1,p}_\alpha(\Omega^*)\to SC^0_\beta(\Omega^*)\subset L^\infty_\beta(\Omega^*)$ does not hold in case $\beta=0$ and $p>1$. It is also clear from the Sobolev imbedding theorem that the space $W^{1,p}_\alpha$ can not be imbedded into L^∞_β in case $\alpha=\beta$ and $p>1$.

Case C. Suppose that $\mu=1-\alpha+\beta-n/p>0$. From the previous example it suffices to assume that $\alpha\neq 1-n/p$. In fact, if $\alpha=1-n/p$, then we have $\|u;\ SC^{0,\mu}_\beta\|\geq \|u;\ SC^0_{\beta-\mu}\|=\|u;\ L^\infty\|$. Since $p>1$, the function u defined by (3.23) becomes a counterexample again. We proceed to the case that $\alpha\neq 1-n/p$. Let λ satisfy $\lambda>\mu$. Then we can choose t and τ so that

$$(3.25)\qquad \mu=1-\alpha+\beta-n/p<t+\tau+\beta<\lambda,\ \tau+2t\neq 0 \text{ and is } \tau \text{ sufficiently large}\ .$$

Again we set $u(z)=f(z)|y|^t|z|^\tau$. Then we have $u\in H^{1,p}_\alpha(\boldsymbol{R}^{n-s}\times\boldsymbol{R}^s)$ from (3.13) and (3.25) But u fails to belong to $SC^{0,\lambda}_\beta(\boldsymbol{R}^{n-s}\times\boldsymbol{R}^s)$. To see this, we take any points (x, y) and $(0, 2y)$ with $0<|y|=|x|\leq 1/4$. Then

$$(3.26)\qquad \|u;\ SC^{0,\lambda}_\beta\|\geq|y|^\beta|u(x,y)-u(0,2y)|/|(x,y)-(0,2y)|^\lambda=$$
$$=2^{(\tau-\lambda)/2}(1-2^{t+\tau/2})|y|^{t+\tau+\beta-\lambda}\to+\infty\ ,$$

as $|y|$ tends to zero. Thus u has the required properties.

Counterexample to Theorem 2. Case A: In this case the assertion in Theorem 2 is equivalent to the isoperimetric inequality (8.1) in § 8, therefore it suffices to show the following is sharp up to constants C_1 and C_2.

$$(8.1)^*\qquad \left(\int_M\delta(x)^{\beta q}dx\right)^{1/q}\leq C_1\int_{\partial M}\delta(x)^\alpha d\mathcal{H}(x)^{n-1}+C_2\int_M\delta(x)^{\alpha-1}dx\ ,$$

where M is a compact domain $\subset\Omega$ with smooth boundary, $1-1/q=(1-\alpha+\beta)/n$ and $\beta\leq\alpha$.

Let us set

$$\Omega=\boldsymbol{R}^n_+=\{(x,y)\in\boldsymbol{R}^n;\ x\in\boldsymbol{R}^{n-1},\ y>0\}$$

and

$$M=\{(x,y)\in\boldsymbol{R}^n;\ x^2+(y-k^{-1})^2\leq(2k)^{-2}\}\ .$$

Then it follows by the dimension argument that the relation among α, β and q is sharp. Moreover the following example shows that if $\alpha>0$, then the second term in the right side of (8.1)* is needed. Let $n=2$ and $h_k=\exp(-k)$. Let us set

$$A_k = \{k<x<k+k^{-1},\ h_k<y<h_k+k^{-1}\}\ ,$$
$$B_k = \{k<x<k+h_k,\ 0\leq y\leq h_k\}\ ,$$
$$\Omega = \{(x, y)\in \boldsymbol{R}^2;\ y<0\} \cup \{\overset{\infty}{\cup} A_k\} \cup \{\overset{\infty}{\cup} B_k\}\ .$$

Since A_k can be approximated by smooth domain from inside, we can put $M=A_k$. Then as is ovbiously seen, the first term in the right side of (8.1)* vanishes rapidly when k tends to $+\infty$, so that the second term becomes the principal part.

In the cases B, C and D, we can show in a similar way that there exist no imbeddings of the type considered in the previous examples, except that those explicitly stated in the theorem 2. Q.E.D.

§ 4. A review of covering, extension and isoperimetric inequalities

In this section we shall prepare some lemmas which will be needed later. In the first place we give two covering lemmas (see [30] for the proof).

Lemma 4.1. *Let $l\geq 1$ and let Ω be a domain of $\boldsymbol{R}^n$. Then there exists an uniformly locally finite cover of Ω by open balls*

$$B_{r_j}(\xi^j) = \{x;\ |x-\xi^j|<r_j\}$$

such that:

(1) $\Omega\subset \cup B_{r_j}(\xi^j)$.

(2) $2^{-l-1}\operatorname{dist}(\xi^j, \partial\Omega)\leq r_j\leq 2^{-l}\operatorname{dist}(\xi^j, \partial\Omega)$.

(3) $\Omega\subset \cup (1-2^{-l})B_{r_j}(\xi^j)$.

(4) *For an arbitrary point $x\in\Omega$, the number of balls $B_{r_j}(\xi^j)$ which intersect $B_{2^{-l}\operatorname{dist}(x,\partial\Omega)}(x)$ is uniformly less than N, where $N=(1+2^{l+5}n)^n$.*

Lemma 4.2. *Let D be a bounded set in $\boldsymbol{R}^n$. With each point $x\in D$, we associate the ball $B_{d(x)}(x)$ whose center is x and radius is $d(x)>0$. Then we can select a sequence of balls $\{B_j\}$ in $\{(B_{r(x)}(x)\}_{x\in D}$ such that:*

(1) $D\subset \cup B_j$.

(2) *There exists a number K depending only on the dimension of the space, such that every collection of $K+1$ of the sets B_j has empty intersection.*

Secondly we give a lemma on the existence of extension operator which is due to E. M. Stein provided $\alpha=0$. The proof will be given in § 11.

Let δ^* be a modification of δ such that $\delta^*\in C^\infty(\boldsymbol{R}^n\backslash\partial F)$ and such that $\delta^*(x)=\delta(x)$ if $x\in\Omega=\boldsymbol{R}^n\backslash F$, $\delta^*(x)$ is equivalent to dist $(x, \partial F)$ if $x\in$ {Interior of F}.

Lemma 4.3. *Let F satisfy the property $SP(1)$. Let $\Omega=\boldsymbol{R}^n\backslash F$. Then there exists a continuous extension operator $E_{k,p,\alpha}$ mapping each element of $W_\alpha^{k,p}(\Omega)$ into $H_\alpha^{k,p}(\boldsymbol{R}^n)$ such that: $E_{k,p,\alpha}u=u$ on Ω and there exists a positive constant C such that we have*

$$||E_{k,p,\alpha}u;\ H_\alpha^{k,p}(\boldsymbol{R}^n)||\leq C||u;\ W_\alpha^{k,p}(\Omega)||\ , \tag{4.1}$$
$$\textit{for an arbitrary } u\in W_\alpha^{k,p}(\Omega)\ .$$

Here the space $H_{\alpha}^{k,p}(\boldsymbol{R}^n)$ *is defined in analogous way, except that we use* δ^* *in place of* δ.

Thirdly we prepare isoperimetric inequalities which are essential for the present paper (See [30], P. 163, § 3 for the proof).

Lemma 4.4. *Let* $n\geq 2$ *and let* B *be an open unit ball of* $\boldsymbol{R}^n$. *Let* M *be an open subset of* B *such that* $\partial M\cap B$ *is a manifold of class* $C^{0,1}$. *Then*

$$\operatorname{Min}(|M|,\ |B\backslash M|)\leq C_0 S(\partial M\cap B)^{n/(n-1)}, \quad \text{for } C_0=2^{-1}v_n v_{n-1}{}^{n/(n-1)}, \tag{4.2}$$

where $S(\partial M\cap B)=\int_{\partial\Omega\cap B}d\mathcal{H}^{n-1}(x)$, $\mathcal{H}^{n-1}(x)$ *denotes the* $(n-1)$*-dimensional Hausdorff measure and* v_n *denotes the Lebesgue measure of the n-dimensional unit ball.*

Lemma 4.5. *Let* $n\geq 2$ *nad let* B *be an open unit ball of* $\boldsymbol{R}^n$. *Let* M_1 *and* M_2 *be two open subsets of* B *whose boundary portions* $\partial M_i\cap B$, $i=1, 2$, *are of class* $C^{0,1}$. *Then*

$$\operatorname{Min}\{|M_1\cap M_2|,\ |B\backslash(M_1\cap M_2)|\}\leq C_0 S(\partial(M_1\cap M_2)\cap B)^{n/(n-1)}. \tag{4.3}$$

where C_0 *is the same number as the one in the lemma* 4.4.

For the proof of Lemma 4.5, it suffices to note that $M_1\cap M_2$ can be approximated from inside by a sequence of open subsets of B with smooth boundaries and with smaller volumes than $|M_1\cap M_2|$.

Lastly we prepare the following (The proof is omitted.).

Lemma 4.6. *If* $f\colon \boldsymbol{R}^n\to\boldsymbol{R}^1$ *is Lipschitzian, then*

$$\int_{\boldsymbol{R}^n} g(x)|\nabla f(x)|\,dx=\int_{\boldsymbol{R}^1}dt\int_{f^{-1}(t)}g(x)\,d\mathcal{H}^{n-1}(x), \tag{4.4}$$

for every Lebesgue integrable real-valued function g.
Here $\mathcal{H}^{n-1}(x)$ *denotes the* $(n-1)$*-dimensional Hausdorff measure and* $|\nabla f(x)|=(\sum_{j=1}^{n}|\partial_j f(x)|^2)^{1/2}$.

This formulation is due to Federer [11; Theorem 3.2.12.], see also Maz'ja [30; Theorem 1.24.]. We recall that if $n=1$ and $g(x)=1$, then this equality is known as Banach's theorem.

§ 5. Proof of Theorem 1 (First step)

In this section we shall reduce the statements of Theorem 1 to the following proposition 5.1 in terms of isoperimetric inequalities.

Definition. An open subset M of $\boldsymbol{R}^n$ is called admissible if M is bounded and ∂M is a manifold of class C^∞.

We note that if $u\in C_0^\infty(\boldsymbol{R})$, then for almost all t the sets

$$M_t = \{x\in \boldsymbol{R}^n;\ u(x) = t\} \tag{5.1}$$

are admissible. In fact Sard's lemma implies that the set of critical values of u has measure zero in $\boldsymbol{R}^1$.

Then Theorem 1 is reduced to the next proposition 5.1 which will be established in § 6.

Proposition 5.1. *Let F have the property $P(s)$ with $s\in(0, n]$. Let M be an arbitrary admissible set with $d(M)=$*diam *$M\leq A_1A_0$. Then there exists positive constant C_1 such that*

$$\left(\int_M \delta(x)^{\beta q}dx\right)^{1/q}\leq C_1 d(M)^{1-\alpha+\beta-n(1-1/q)}\int_{\partial M}\delta(x)^\alpha d\mathcal{H}^{n-1}(x)\,, \tag{5.2}$$

where $0\leq 1-1/q\leq(1-\alpha+\beta)/n$ and either

$$-s/q<\beta\leq\alpha\leq 0,\ 0<s\leq 1, \quad or \quad -s/q<\beta\leq\alpha,\ 1<s\leq n\,.$$

Here C_1 is independent of M, and A_1 is a positive number independent of M defined in Proposition 6.2 in *in* § 6.

Admitting this in the present, we shall establish Theorem 1.

Proof of Theorem 1. By virtue of a suitable partition of unity with radius less than Min $[A_1A_0, 1]$, it suffices to establish the following inequality for an arbitrary $u\in C_0^\infty(B_0)$, where B_0 is an arbitrary ball with $d(B_0)\leq$ Min $[A_1A_0, 1]$. Let p, q, α and β satisfy the hypotheses in Theorem 1. Then it holds that

$$\left(\int |u|^q\delta^{\beta q}dx\right)^{1/q}\leq C_1\left(\int|\nabla u|^p\delta^{\alpha p}dx\right)^{1/p}\,, \tag{5.3}$$

where C_1 is the same constant as the one in Proposition 5.1.

In the first place we have

$$\begin{aligned}\left(\int_{B_0}|u(x)|^q\delta(x)^{\beta q}dx\right)^{1/q} &= \left(\int_0^\infty d(t^q)\int_{|u(x)|\geq t}\delta(x)^{\beta q}dx\right)^{1/q}\leq \\ &\leq\int_0^\infty\left(\int_{|u(x)|\geq t}\delta(x)^{\beta q}dx\right)^{1/q}dt\,.\end{aligned} \tag{5.4}$$

Here we used the following elementary inequality which is valid for an arbitrary non-increasing function $g\geq 0$:

$$\left(\int_0^\infty g(t)^q d(t^q)\right)^{1/q}\leq\int_0^\infty g(t)dt\,, \tag{5.5}$$

which is a direct consequence of the inequality $g(t)\leq\int_0^t g(s)ds$. By the proposition 5.1 with $M=\{|u(x)|\geq t\}$ and the lemma 4.6 with $f=|u|$, $g=\delta^\alpha$, we have the desired estimate (5.3) provided $p=1$. If $p>1$, it suffices to replace q^{-1}, α, β in the inequality (5.3) with $p=1$ by $1-1/p+1/q$, $\alpha+\beta q(p-1)/p$, $\beta q(1-1/p+1/q)$ respectively and u by $|u|^{1+q(p-1)/p}$.

The assertion (3.2) concerned with compactness also follows from Proposition 5.1:

First we shall prove that the imbedding opreator of $H^{1,p}_{\alpha}(\boldsymbol{R}^n)$ into $L^p_{\alpha}(D)$ is compact, if D is a bounded subdomain of $\boldsymbol{R}^n$.

Let $F(\eta)=\{x\in\boldsymbol{R}^n;\ \delta(x)<\eta\}$, and let β satisfy that $\max[-s/p,\ \alpha-1]<\beta<\alpha$. Then it follows from the previous part that

$$(5.6)\qquad ||u;\ L^p_{\alpha}(D\cap F(\eta))||\leq\eta^{\alpha-\beta}||u;\ L^p_{\beta}(D\cap F(\eta))||\leq C\eta^{\alpha-\beta}||u;\ H^{1,p}_{\alpha}(\boldsymbol{R}^n)||\,.$$

Since $F(\eta)^c\cap D$ is an open subset of D with smooth boundary and compact closure for almost all $\eta>0$, the imbedding operator $H^{1,p}_{\alpha}(\boldsymbol{R}^p)$ into $L^p_{\alpha}(F(\eta)^c\cap D)$ is compact. So that the assertion follows.

Secondly we proceed to the general case. We need more notations. We construct a covering $\{B_j\}$ of $\bar{D}$ by balls with diam $B_j=\eta<\mathrm{Min}[A_1A_0,\ 1]$ the multiplicity of the covering being not more than a constant that depends only on n. Here we used the lemma 4.2. Let $\{\varphi_j\}$ be a partition of unity subordinate to this cover. Then it follows from Proposition 5.1 that for $\varepsilon=1-\alpha+\beta-n(1/p-1/q)$,

$$\begin{aligned}(5.7)\quad \int_D |u(x)|^q\delta(x)^{\beta q}dx &\leq \sum_{j=1}^{\infty}\int_D |u\varphi_j|^q\delta^{\beta q}dx\leq\\ &\leq C_1\eta^{\varepsilon q}\sum_{j=1}^{\infty}\Big(\int_D |\nabla(u\varphi_j)|^p\delta^{\alpha p}\,dx\Big)^{q/p}\leq\\ &\leq CC_1\eta^{\varepsilon q}||u;\ H^{1,p}_{\alpha}(\boldsymbol{R}^n)||^q+C_2(\eta)||u;\ L^p_{\alpha}(\boldsymbol{R}^n)||^q\,,\qquad \text{for any}\quad u\in C^{\infty}_0(\boldsymbol{R}^n)\,.\end{aligned}$$

Here we used the estimate $|\nabla\varphi_j|\leq$Const. η^{-1}. Since the imbedding $H^{1,p}_{\alpha}(\boldsymbol{R}^n)\to L^p_{\alpha}(D)$ is compact, it is clear that the imbedding (3.2) is compact provided $\varepsilon>0$.

Q.E.D.

§ 6. Proof of Theorem 1 (Second step: Proof of Proposition 5.1)

In this section we shall prove Proposition 5.1 using Proposition 6.3 which will be established in § 7.

First we state the following which is a direct consequence of the property $P(s)$.

Proposition 6.1. *Let F have the property $P(s)$. Then the following inequalities are valid for any η, η' and any ball B satisfying*

$$0\leq\eta'<\eta\leq d(B)\leq A_0\,.$$

(1) $\displaystyle\int_{B\cap(F_{\eta}\setminus F_{\eta'})}\delta(x)^{\alpha}dx\leq C\eta^{s-s^*+\alpha}|\eta-\eta'|^{s^*}d(B)^{n-s},\qquad -s<\alpha.$

(2) $\displaystyle\int_B \delta(x)^{\alpha}dx\leq Cd(B)^n[d(B)+\mathrm{dist}\,(B,\ F)]^{\alpha},\qquad -s<\alpha\leq 0.$

(3) $\int_B \delta(x)^\alpha dx \geq Cd(B)^n[d(B)+\mathrm{dist}(B, F)]^\alpha, \qquad \alpha\geq 0.$

(4) $\int_{B\cap\partial F_\eta} d\mathcal{H}^{n-1}(x)\leq C\eta^{s-1}d(B)^{n-s}, \qquad s\geq 1.$

Moreover if $s\geq 1$, then it holds that:

(5) *For any $\eta\in(0, d(B))$, there exists a sequence of smooth manifolds $\{N_k\}_{k=1}^\infty$ such that $N_k\subset F_\eta^c$, N_k converges ∂F_η as $k\to+\infty$ and*

$$\limsup_{k\to+\infty} \mathcal{H}^{n-1}(B\cap N_k)\leq C\eta^{s-1}d(B)^{n-s}.$$

Here C is a positive number independent of B, η and η'.

Proof. For the sake of simplicity we assume that $\delta(x)=\mathrm{dist}(x, F)$.

(1) In case $\alpha\geq 0$ the assertion is obvious, so we assume $-s<\alpha\leq 0$. Let $\xi_j=\eta'+(\eta-\eta')2^{-j}=\xi_{j+1}+(\eta-\eta')2^{-j-1}$ inductively on j. Then $F_\eta\backslash F_{\eta'}=\bigcup_{j=0}^\infty \omega_j$, $\omega_j=F_{\xi_j}\backslash F_{\xi_{j+1}}$, and in ω_j we have $\delta(x)\geq\xi_{j+1}$. Hence it follows from the property $P(s)$ that:

$$\int_{B\cap(F_\eta\backslash F_{\eta'})} \delta(x)^\alpha dx\leq C2^{s-s^*}d(B)^{n-s}|\eta-\eta'|^{s^*}\sum_{j\geq 0}\xi_{j+1}^{s-s^*+\alpha}2^{(-j-1)s^*}\leq$$
$$\leq C'\eta^{s-s^*+\alpha}|\eta-\eta'|^{s^*}d(B)^{n-s}.$$

Here we used $\xi_j\leq 2\xi_{j+1}$ and $\eta/2^{j+1}<\xi_{j+1}<\eta$.

(2) In case $d(B)<\mathrm{dist}(B, F)$, we get

$$(6.1) \qquad (1/2)[d(B)+\mathrm{dist}(B, F)]\leq\delta(x)\leq d(B)+\mathrm{dist}(B, F), \qquad x\in B.$$

Therefore the assertions (2) and (3) are clear. Now we assume $\mathrm{dist}(B, F)\leq d(B)\leq A_0/3$, then $3B\subset F_{A_0}$. From (1) we have, letting $\eta'=0$ and $\eta=3d(B)$,

$$(6.2) \qquad \int_B \delta(x)^\alpha dx\leq\int_{3B}\delta(x)^\alpha dx\leq Cd(B)^{n+\alpha}\leq$$
$$\leq C2^{-\alpha}d(B)^n[d(B)+\mathrm{dist}(B, F)]^\alpha.$$

We note that the excluded case that $A_0/3<d(B)<A_0$ can be treated in a similar way by dividing B into small balls with finite multiplicity.

(3) Again we assume $d(B)\geq\mathrm{dist}(B, F)$. For a small η we have

$$(6.3) \qquad \int_B \delta(x)^\alpha dx\geq\int_{B\cap F_\eta^c}\delta(x)^\alpha dx\geq\eta^\alpha(|B|-|F_\eta\cap B|)\geq$$
$$\geq\eta^\alpha\{1-C_s[\eta/d(B)]^s\}\,|B|.$$

Then putting $\eta=\mathrm{Min}[1, (2C_s)^{-1/s}]d(B)/2$, we get the desired estimate.

(4) Let ε and t be sufficiently small positive numbers, and set

$$B_t=\{x\in B;\ \mathrm{dist}(x, \partial B)>t\}, \qquad \text{for} \quad \varepsilon<t.$$

Then it follows from the property $P(s)$ and the lenmm 4.2 that there exists an uniformly locally finite cover of $B_t \cap \partial F_\eta$ by open balls B^j with $d(B^j)=\varepsilon$ and centers on ∂F_η such that:

$$\sum_j d(B^j)^{n-1} \leq C\,|B \cap (F_{\eta+\varepsilon} \backslash F_{\eta-\varepsilon})|/(2\varepsilon) \leq C'(\eta+2\varepsilon)^{s-1} d(B)^{n-s},$$

where C and C' are positive constants independent of η and ε. In view of the definition of the Hausdorff measure, we have, letting $\varepsilon \to 0$,

$$\mathcal{H}^{n-1}(B_t \cap \partial F_\eta) \leq C\eta^{s-1} d(B)^{n-s}.$$

Since t is arbitrary, the desired estimate follows.

Here we note the following: From the general measure theory it also holds that

$$\mathcal{H}^{n-1}(B \cap \partial F_\eta) \leq \limsup_{h \to 0} h^{-1} |(F_{\eta+h} \backslash F_\eta) \cap B| \leq C_s \eta^{s-1} d(B)^{n-s},$$

for allmost all η. For the detailed proof, see [11, Theorem 3.2.15, p. 252, Theorem 3.2.26, p. 261 and Theorem 3.2.39. p. 275] for example. Since $\bar{B} \cap \partial F_\eta$ may have a positive measure, the first inequality can not be replaced by the equality.

(5) For the sake of simplicity, we assume that $A_0=+\infty$. Take and fix an arbitrary $\eta \in (0, +\infty)$. Let $g(x)$ be a nonnegative smooth function satisfying

$$\begin{gathered} C^{-1} d(x, \partial F_\eta) \leq g(x) \leq C d(x, \partial F_\eta) \\ |\nabla g(x)| \leq C, \qquad x \in F_\eta^c \end{gathered} \tag{6.4}$$

for some positive number C independent of x and η.

Then we have for a sufficiently small $\varepsilon>0$

$$\begin{aligned} \operatorname{meas}\{x \in B;\ 0<g(x)<\varepsilon\} &\leq |(F_{\eta+C\varepsilon} \backslash F_\eta) \cap B| \leq \\ &\leq C'\eta^{s-1} \varepsilon d(B)^{n-s}. \end{aligned} \tag{6.5}$$

By the lemma 4.6, we have

$$\int_{(0<h(x)<\varepsilon) \cap B} |\nabla g(x)|\,dx = \int_0^\varepsilon \mathcal{H}^{n-1}(\{x \in B;\ g(x)=t\})\,dt. \tag{6.6}$$

Then by (6.4) and (6.5) we have

$$\int_0^\varepsilon \mathcal{H}^{n-1}(\{x \in B;\ g(x)=t\})\,dt \leq C\eta^{s-1} \varepsilon d(B)^{n-s}. \tag{6.7}$$

Since $g(x)$ is of class C^∞, we can choose a sequence $\{t_k\}_{k=1}^\infty$ so that $t_k \to 0$ as $k \to +\infty$, $N_k=\{x;\ g(x)=t_k\}$ is of class C^∞ and $\limsup_{k \to +\infty} \mathcal{H}^{n-1}(N_k \cap B) \leq C\eta^{s-1} d(B)^{n-s}$. Q.E.D.

The following is also a direct consequence of the property $P(s)$:

Proposition 6.2. *Assume that F has the property $P(s)$ and $\alpha>-s$. Then there exists a positive number A_1 such that:*

For any admissible set M satisfying $d(M) \leq A_1 A_0$ and for any $x \in M$, there exists an open ball B with center x and $d(B) \leq A_0$ such that

$$\int_{M \cap B} \delta(x)^{\alpha} dx = (1/2) \int_{B} \delta(x)^{\alpha} dx \tag{6.8}$$

holds, where A_1 is depending only on α and the dimension of the space.

Before the proof we remark that A_1 is uniformly bounded with respect to α if α is contained in a interval $[-\varepsilon, +\infty)$ for a given ε satisfying $\varepsilon < s$.

Proof. Take an arbitrary amdissible set M satisfying $d(M) \leq A_0/(4N)$, where N is a positive number ≥ 2 and will be specified later. Moreover we take an arbitrary ball B_0 with $d(B_0) = A_0/(2N)$ so that $M \subset B_0$ and the center of $B_0 \in M$. For simplicity we assume the center is the origin. Let B be an arbitrary ball concentric with B_0 and satisfying $d(B) \leq A_0$. Then we set

$$\gamma = 2 \int_{M \cap B} \delta(x)^{\alpha} dx \left(\int_{B} \delta(x)^{\alpha} dx \right)^{-1}. \tag{6.9}$$

As is obviously seen, γ equals 2 for sufficiently small B. Therefore it suffices to show that $\gamma \leq 1/2$ if $d(B) \geq A_0$. By Proposition 6.1, we have

$$\gamma \leq C_0 [d(B_0)/d(B)]^n \{[d(B_0) + \mathrm{dist}(B_0, F)]/[d(B) + \mathrm{dist}(B, F)]\}^{\alpha}, \tag{6.10}$$

where C_0 is a positive number depending only on n.
(1) Assume that $\mathrm{dist}(B_0, F) > (N-1) d(B_0)$. In this case we put $B = NB_0$. Noting that $\mathrm{dist}(B, F) = \mathrm{dist}(B_0, F) - (N-1) d(B_0)/2$ and $d(B) = Nd(B_0)$, we get

$$\begin{aligned} \gamma &\leq C_0 N^{-n} \{[d(B_0) + \mathrm{dist}(B_0, F)]/[Nd(B_0)/2 + \mathrm{dist}(B_0, F)]\}^{\alpha} \leq \\ &\leq C C_0 N^{-n} = \gamma_1, \qquad \text{for any} \quad \alpha > -s, \end{aligned} \tag{6.11}$$

where C is a positive number depending only on n.
(2) Assume that $\mathrm{dist}(B_0, F) \leq (N-1) d(B_0)$. Then we put $B = (2N) B_0$ so that $\mathrm{dist}(B, F) = 0$. Using another positive number C depending only on n, we have in a similar way

$$\gamma \leq C C_0 N^{-n} \mathrm{Max}[1, N^{-\alpha}] = \gamma_2, \qquad \text{for any} \quad \alpha > -s. \tag{6.12}$$

Here we choose N so that $\mathrm{Max}[\gamma_1, \gamma_2] \leq 1/2$, and we set $A_1 = [4N]^{-1}$. Then the assertion is now clear. Q.E.D.

Now we state the main proposition which will be proved in § 7.

Proposition 6.3. *Assume that F has the property $P(s)$. Let M be an admissible set with* diam $M < A_1 A_0$, *and let B be an ball with $d(B) < A_0$. Assume that*

$$\int_{M \cap B} \delta(x)^{\alpha} dx = (1/2) \int_{B} \delta(x)^{\alpha} dx. \tag{6.13}$$

Then it holds that

$$(6.14)\qquad \int_{M\cap B}\delta(x)^{\alpha}dx\leq C(\alpha)d(B)\int_{\partial M\cap B}\delta(x)^{\alpha}d\mathcal{H}^{n-1}(x),$$

where either $-s<\alpha\leq 0$, $0<s\leq 1$ *or* $-s<\alpha$, $1<s\leq n$, $C(\alpha)$ *is independent of each* M *and* B, *and* A_1 *is the positive number defined in Proposition* 6.2.

Admitting this in the rest of this section we shall establish Proposition 5.1.

Proof of Proposition 5.1. Let M be an arbitrary admissible set with diam $M\leq A_1A_0$, where A_0 is possibly permitted to be $+\infty$. By virtue of the lemma 4.2 we can construct an uniformly locally finite open cover by a sequence of balls $\{B_j\}_{j=1}^{\infty}$ with $d(B_j)<A_0$ such that the equality (6.13) holds for $B=B_j$ $(j=1,2,\cdots)$ and the constant C depends only on n. Then

$$\begin{aligned}(6.15)\quad (\int_M\delta(x)^{\beta q}dx)^{1/q}&\leq\sum_{j=1}^{\infty}(\int_{M\cap B_j}\delta^{\beta q}dx)^{1/q}\leq\\ &\leq\sum_{j=1}^{\infty}(\int_{B_j}\delta^{\beta q}dx)^{1/q}\leq C\sum_{j=1}^{\infty}[d(B_j)+\mathrm{dist}(B_j,F)]^{\beta}d(B_j)^{n/q}\leq\\ &\leq C'd(M)^{1-\alpha+\beta-n(1-1/q)}\sum_{j=1}^{\infty}\int_{B_j}\delta(x)^{\alpha}dx\,d(B_j)^{-1}\leq\\ &\leq C'C(\alpha)d(M)^{1-\alpha+\beta-n(1-1/q)}\sum_{j=1}^{\infty}\int_{\partial M\cap B_j}\delta(x)^{\alpha}d\mathcal{H}^{n-1}(x)=\\ &=C'C(\alpha)Kd(M)^{1-\alpha+\beta-n(1-1/q)}\int_{\partial M}\delta(x)^{\alpha}d\mathcal{H}^{n-1}(x).\end{aligned}$$

Thus Proposition 5.1 has been established. Q.E.D.

§ 7. Proof of Theorem 1 (Final step: Proof of Proposition 6.3)

In the first place we shall deal with the case that $-s<\alpha\leq 0$, which is rather simple. We prepare the following:

Lemma 7.1. *Let* F *have the property* $P(s)$ *with* $s\in(0,n]$. *Assume that* $-s<\alpha\leq 0$. *Let* B *be an arbitrary ball with* $d(B)<A_0$. *Then*

$$(7.1)\qquad \int_B|u(x)-u_B|\delta(x)^{\alpha}dx\leq C(\alpha)d(B)\int_B|\nabla u(x)|\delta(x)^{\alpha}dx,$$

for any $u\in C^1(\bar{B})$, *where* $u_B=|B|^{-1}\int_B u(x)dx$ *and* $C(\alpha)$ *is a positive constant independent of* B *and* u.

Proof of Proposition 6.3 *in the case* $\alpha\leq 0$. We can replace u in (7.1) by u_ε defined by (2.7), the modification of the characteristic function of the admissible set M, and letting ε to 0 we have the desired estimate. Q.E.D.

Proof of Lemma 7.1. The following inequality is familiar (see [30], p. 20 for example):

$$(7.2)\qquad |u(x)-u_B|\leq d(B)^n/(n|B|)\int_B|x-y|^{1-n}|\nabla u(y)|dy,$$

for any $u\in C^1(\bar{B})$.
Without loss of generality we assume that B is a unit ball $\{|x|<1\}$ and set

$$V(y, \rho)=\int_{|x|\leq\rho}\delta(x+y)^{\alpha}dx\,. \tag{7.3}$$

Then it follows from Proposition 6.1 that

$$V(y, \rho)\leq C(\alpha)\delta(y)^{\alpha}\rho^{n}\,,\qquad 0\leq\rho\leq 2\,. \tag{7.4}$$

By multiplying $\delta(x)^{\alpha}$ to the both sides of (7.2) and integrating over B with respect to the variable x we get

$$\begin{aligned}\int_B|u(x)-u_B|\delta(x)^{\alpha}dx&\leq C\int_B|\nabla u(y)|\,dy\int_B|x-y|^{1-n}\delta(x)^{\alpha}dx\\&\leq C\int_B|\nabla u(y)|\,dy\int_{2B}|x|^{1-n}\delta(x+y)^{\alpha}dx\\&\leq C\int_B|\nabla u(y)|\,dy\int_0^3\rho^{1-n}dV(y, \rho)\\&\leq CC(\alpha)\int_B|\nabla u(y)|\delta(y)^{\alpha}dy\,.\end{aligned} \tag{7.5}$$

Therefore the lemma 7.1 follows. Q.E.D.

Proof of Proposition 6.3 *in the case* $\alpha>0$. For the sake of simplicity we assume that $\delta(x)=\operatorname{dist}(x, F)$. The proof will be carried out in a chain of auxiliary lemmas. Though it is rather lengthy, the techiniques involved here are quite elementary, being based on isoperimetric inequalities. First we treat the case that $d(B)<\operatorname{dist}(B, F)$.

Lemma 7.2. *Suppose the same hypotheses as in Proposition* 6.3. *Further suppose that* $\alpha\geq 0$ *and* $d(B)<\operatorname{dist}(B, F)$. *Then the inequality* (6.11) *holds.*

Proof. By the lemma 4.4, (6.1) and Hölder's inequality we have

$$\begin{aligned}\int_{M\cap B}\delta(x)^{\alpha}dx&\leq\operatorname*{Sup}_B\delta(x)^{\alpha}d(B)\operatorname{Min}\{|M\cap B|,\ |M^c\cap B|\}^{(n-1)/n}\leq\\&\leq C_0^{(n-1)/n}d(B)\operatorname*{Sup}_B\delta(x)^{\alpha}S(\partial M\cap B)\leq\\&\leq 2^{\alpha}C_0^{(n-1)/n}d(B)\int_{\partial M\cap B}\delta(x)^{\alpha}d\mathcal{H}^{n-1}(x)\,.\end{aligned} \tag{7.6}$$

Q.E.D.

Lemma 7.3. *Suppose the same hypotheses as in Proposition* 6.3. *Further suppose that* $\alpha\geq 0$ *and* $d(B)<\mathit{dist}(B, F)$. *Then it holds that*

$$\operatorname{Min}\{|M\cap B|,\ |M^c\cap B|\}\geq\kappa|B|\,,\qquad 0<\kappa\leq 1/2\,, \tag{7.7}$$

where κ *is a constant independent of* M *nad* B.

Proof. From Proposition 6.1 we have for some constant κ with $0<\kappa\leq 1/2$,

$$\int_{M\cap B}\delta(x)^{\alpha}dx=(1/2)\int_{B}\delta(x)^{\alpha}dx\geq\kappa|B|(d(B)+\mathrm{dist}(B,F))^{\alpha}. \tag{7.8}$$

On the other hand, $\int_{M\cap B}\delta(x)^{\alpha}dx\leq|M\cap B|\operatorname{Sup}_{B}\delta(x)^{\alpha}$, hence we have

$$|M\cap B|\geq\kappa|B|(d(B)+\mathrm{dist}(B,F))^{\alpha}/\operatorname{Sup}_{B}\delta(x)^{\alpha}\geq\kappa|B|. \tag{7.9}$$

Q.E.D.

Let us set for small $\eta>0$

$$M_1=M \quad\text{and}\quad M_2=F_\eta^c=\{x\in \boldsymbol{R}^n;\ \mathrm{dist}(x,F)\geq\eta\}. \tag{7.10}$$

Here we remark that M_2 is not admissible in general. Nevertheless it can be apporximated from inside by a sequence of admissible sets with smaller volume than $|M_2|$ for all $\eta>0$ (Proposition 6.1).

Then from Lemma 7.3 and Lemma 4.5 we have

Lemma 7.4. *Suppose the same hypotheses as in Lemma 7.3. Then*

$$\begin{aligned}&|M_1\cap M_2\cap B|\leq C|(M_1\cap M_2)^c\cap B| \quad \textit{and}\\ &|M_1\cap M_2\cap B|\leq C'd(B)S(\partial(M_1\cap M_2)\cap B),\end{aligned} \tag{7.11}$$

for any $\eta>0$.
Here C and C' are positive constants independent of B and M.

Proof. First we have from Lemma 7.3 and the remark just after (7.10),

$$\begin{aligned}|(M_1\cap M_2)^c\cap B|&=|(M_1^c\cap B)\cup(M_2^c\cap B)|\geq|M_1^c\cap B|\geq\\ &\geq\kappa|B|\geq\kappa|M_1\cap M_2\cap B|.\end{aligned}$$

Also from Lemma 4.5, Proposition 6.1 (5) and Hölder's inequality we have the desired estimate. Q.E.D.

End of the proof of Proposition 6.3. Asumme that $\alpha\geq 0$, $0<s\leq n$ and $d(B)\geq \mathrm{dist}(B,F)$. By virtue of the property $P(s)$,

$$\begin{aligned}C|B|\operatorname{Sup}_{B}\delta(x)^{\alpha}&\leq\int_{M\cap B}\delta(x)^{\alpha}dx\leq\\ &\leq\operatorname{Sup}_{B}\delta(x)^{\alpha}|M_1\cap M_2\cap B|+C_s d(B)^{n-s}\eta^{s}\operatorname{Sup}_{B}\delta(x)^{\alpha}.\end{aligned} \tag{7.12}$$

Here M_1 and M_2 are defined by (7.10). Without loss of generality we assume $\partial M_2=\{x\in B;\ \mathrm{dist}(x,F)=\eta\}$ is adimissible as well. Also from Lemma 7.4

$$\begin{aligned}|M_1\cap M_2\cap B|&\leq C'd(B)S(\partial(M_1\cap M_2)\cap B)\leq\\ &\leq C'\eta^{-\alpha}d(B)\int_{\partial M_1\cap B}\delta(x)^{\alpha}d\mathcal{H}^{n-1}(x)+C_s d(B)^{n-s+1}\eta^{s-1}.\end{aligned} \tag{7.13}$$

Let us set $\eta=d(B)/N$, for $N>1$.
Then combining (7.12) with (7.13) we have

$$
\begin{aligned}
(7.14)\qquad C\operatorname*{Sup}_{B}\delta(x)^{\alpha}d(B)^{n}&\leq\int_{M_1\cap B}\delta(x)^{\alpha}dx\leq\\
&\leq C'\operatorname*{Sup}_{B}\delta(x)^{\alpha}d(B)^{-\alpha+1}N^{\alpha}\int_{\partial M_1\cap B}\delta(x)^{\alpha}d\mathcal{H}^{n-1}(x)+\\
&\quad+2C_s d(B)^n\operatorname*{Sup}_{B}\delta(x)^{\alpha}N^{1-s}\leq\\
&\leq C''(d(B)N^{\alpha}\int_{\partial M_1\cap B}\delta(x)^{\alpha}d\mathcal{H}^{n-1}(x)+d(B)^n N^{1-s}\operatorname*{Sup}_{B}\delta(x)^{\alpha}),
\end{aligned}
$$

where $C''=\mathrm{Max}\,[2^{\alpha}C', 2C_s]$.

Letting $N=N_0=\mathrm{Max}\,[2, (2C''/C)^{1/(s-1)}]$, we have

$$(7.15)\qquad \int_{M\cap B}\delta(x)^{\alpha}dx\leq 2C''N_0^{\alpha}d(B)\int_{\partial M\cap B}\delta(x)^{\alpha}d\mathcal{H}^{n-1}(x).$$

Therefore Propostion 6.3 is now established. Q.E.D.

§ 8. Proof of Theorem 2

Let F be an arbitrary closed set and set $\Omega=\boldsymbol{R}^n\backslash F$. Let $\{B_j\}_{j=0}^{\infty}=\{B_{r_j}(\xi_j)\}_{j=0}^{\infty}$ be a open cover of Ω so that the properties (1)~(4) in Lemma 4.1 are fulfilled. We assume for the sake of simplicity that $k=1, j=0$ and $r_j\leq 1$, because most of the proofs in the excluded cases follow from this inductively.

[Case A]. In this case Theorem 2 is proved in a quite similar way be using the following Propostion 8.1 in place of Proposition 5.1:

Proposition 8.1. *Let M be an arbitrary admissible set such that*

$$M\subset\Omega \quad \textit{and} \quad d(M)=\mathrm{diam}\,(M)<+\infty\,.$$

Then there exist positive constants C_1 and C_2 such that for $\lambda=1-\alpha+\beta-(1-1/q)\geq 0$

$$(8.1)\quad (\int_M\delta(x)^{\beta q}dx)^{1/q}\leq C_1 d(M)^{\lambda}\int_{\partial M}\delta(x)^{n}d\mathcal{H}^{\alpha-1}(x)+C_2 d(M)^{\lambda}\int_M\delta(x)^{\alpha-1}dx,$$

where $0\leq 1-1/q\leq(1-\alpha+\beta)/n$ and $\beta\leq\alpha$, C_1 and C_2 depend only on the dimension of the space.

Since $C_0^{\infty}(\Omega)$ is densely contained in $\mathcal{W}_{\alpha}^{k,p}(\Omega)$, it is easy to see that Proposition 8.1 implies Theorem 2 [case A] by the arguments in § 5. Now we prove this proposition. Again we assume that $\delta(x)=\mathrm{dist}\,(x, F)$ for simplicity. Let $\{\varphi_j\}_{j=1}^{\infty}$ be a partition of unity subordinate to the cover $\{B_j\}_{j=1}^{\infty}$ such that $|\nabla\varphi_j|\leq Cr_j^{-1}$, let M be an admissible set in Ω and let u_{ε}'s be approximative characteristic functions of M defined by (2.7). Then for any $\eta>0$, we have for a sufficiently small ε

(8.2) $$\Big(\int_M \delta(x)^{\beta q}dx\Big)^{1/q} \le \eta + \Big(\int_M (u_\varepsilon \delta^\beta)^q dx\Big)^{1/q} \le$$
$$\le \eta + \Big(\int \sum_{j=1}^\infty (\varphi_j u_\varepsilon)^q \delta^{\beta q} dx\Big)^{1/q} \le \eta + \sum_{j=1}^\infty \Big(\int (\varphi_j u_\varepsilon)^q \delta^{\beta q} dx\Big)^{1/q}.$$

Assume that $\lambda=0$. Then from Lemma 4.1 and Sobolev's imbedding theorem we have

(8.3) $$\Big(\int (\varphi_j u_\varepsilon)^q \delta^{\beta q} dx\Big)^{1/q} \le C r_j^\beta \Big(\int (\varphi_j u_\varepsilon)^q dx\Big)^{1/q} \le$$
$$\le C' r_j^{\beta+1-n(1-1/q)} \int |\nabla(\varphi_j u_\varepsilon)|\, dx \le C'' \int |\nabla(\varphi_j u_\varepsilon)|\, \delta^\alpha dx \le$$
$$\le C'' \int_{B_j} |\nabla u_\varepsilon| \delta^\alpha dx + C'' C \int_{B_j} |u_\varepsilon| \delta^{\alpha-1} dx.$$

Letting $\varepsilon \to 0$ we get from (8.2) and (8.3)

(8.4) $$\Big(\int_M \delta(x)^{\beta q} dx\Big)^{1/q} \le \eta + C_1 \int_{\partial M} \delta^\alpha d\mathcal{H}^{n-1}(x) + C_2 \int_M \delta^{\alpha-1} dx.$$

Since η is arbitrary (8.1) follows. In case $\lambda>0$, we can derive the desired estimate from (8.4) by virtue of Hölder's inequality.

Proof of Case B and Case C. First we assume that

$$0<\lambda = 1-\alpha+\beta-n/p<1 \quad \text{and} \quad \beta \le \alpha.$$

Then, for each $x\in\Omega_j$, there exist a ball B_j and a positive number C depending only on n such that

(8.5) $$|u(x)| \le C \int_{B_j} |x-y|^{1-n} |\nabla(\varphi_j u)|\, dy, \qquad \text{for any} \quad u\in C^1(\Omega).$$

Then noting $p>n\ge 1$, we get

(8.6) $$|u(x)| \le C r_j^{1-n/p} \{ r_j^{-1} \|u;\ L^p(B_j)\| + \|\,|\nabla u|\,;\ L^p(B_j)\| \}.$$

Therefore

(8.7) $$\delta(x)^{\alpha+n/p-1} |u(x)| \le C' \|u;\ \mathcal{W}_\alpha^{1,p}(\Omega)\|, \qquad \text{for any} \quad x\in\Omega.$$

On the other hand, from the inequality (7.2) it holds that for any ball B.

(8.8) $$|u(x)-u_B| \le C r^{1-n/p} \|\,|\nabla u|\,;\ L^p(B)\|, \qquad r = d(B).$$

Take an arbitrary point (x, y) from $\Omega\times\Omega$ and put $r=|x-y|$. First we assume that $r=|x-y| < \text{Min}\,\{\delta(x), \delta(y)\}/4$. So that we have

$$\text{dist}\,(B_r(x), \partial\Omega) \ge 3\delta(x)/4 \quad \text{and} \quad 4\delta(x)/5 \le \delta(y) \le 5\delta(x)/4.$$

Then we have from (8.7) and (8.8)

$$(8.9)\qquad \|u;\, \mathcal{S}C^{0,\lambda}_{\beta}\| = \|u\|^{*}_{0,\beta-\lambda} + |u|_{0,\lambda,\beta} = \\ = \delta(x)^{\beta-\lambda}|u(x)| + \mathrm{Min}\,\{\delta(x)^{\beta},\, \delta(y)^{\beta}\}|u(x)-u(y)|/|x-y|^{\lambda} \leq \\ \leq C\|u;\, \mathcal{W}^{1,p}_{\alpha}\| < +\infty\,.$$

If $|x-y| > \mathrm{Min}\,\{\delta(x), \delta(y)\}/4$, then we have

$$\mathrm{Max}\,\{\delta(x), \delta(y)\} \leq \mathrm{Min}\,\{\delta(x), \delta(y)\} + |x-y| \leq 5|x-y|\,.$$

So that the desired estimate (8.9) follows in a similar way. Thus the assertion in the case C has been established.

Secondly assume that

$$1-\alpha+\beta-n/p = 0 \quad \text{and} \quad \beta \leq \alpha\,.$$

Since the weight function δ does not vanish for any $x \in \Omega$, by Sobolev's imbedding theorem, the imbedding (3.4) can not be true in case that $\alpha=\beta$ and $p>1$. Hence it suffices to assume either $p=1$ or $\beta<\alpha$. If $\beta<\alpha$, then from (8.7) it holds that $\delta(x)^{\beta}|u(x)| \leq C\|u;\, \mathcal{W}^{1,p}_{\alpha}(\Omega)\|$. Hence this implies that u belongs to the class $\mathcal{S}C^{0}_{\beta}(\Omega)$. We proceed to the case $p=1$. We may also assume that $n=1$ and $\alpha=\beta$, otherwise p should be strictly larger than 1. For simplicity we assume that $0 \in F$ and $(0, 2T) \subset \Omega$, $T>0$. Then the assertion follows from the next elementary inequality:

$$(8.10)\qquad |u(x)|x^{\alpha} \leq \int_0^T |u'(t)|t^{\alpha}dt + \alpha\int_0^T |u(t)|t^{\alpha-1}dt \leq C\|u;\, \mathcal{W}^{1,1}_{\alpha}\|\,,$$

for any $u \in C^1(0, T) \cap \mathcal{W}^{1,1}_{\alpha}(\Omega)$ and any $x \in (0, T)$.

Proof of (8.10). Since $\lim_{x\to 0} u(x)x^{\alpha}=0$, this is ovbious. Q.E.D.

In a similar way we can prove the cases C and D using the previous result, so that Thoerem 2 has been just established. Q.E.D.

§ 9. Proof of Theorem 3

We shall begin with the case A.

Proof of the case A. For simplicity assume that $k=1$ and $j=0$ as before. Assume that F has the property $SP(s)$. So that ∂F has measure zero, and if $s \neq 1$, F satisfies $P(s)$ as well. Therefore the assertion in the case A follows from Theorem 1 and the extension lemma 4.3 ($s=1$), if either $-s/p<\alpha\leq 0$, $s=1$ or $-s/p<\alpha$, $1<s\leq n$. Hence it suffices to consider the excluded case $s=1$, $\alpha>0$. In this case $F=\partial F$ has the strongly local Lipschitz property. By Proposition 8.1 we have already established that:

$$(9.1)\qquad \Big(\int_{\Omega} |u|^{q}\delta^{\beta q}dx\Big)^{1/q} \leq C\Big(\int_{\Omega} |\nabla u|\delta^{\alpha}dx + \int_{\Omega} |u|\delta^{\alpha-1}dx\Big)\,,$$

for any $u \in C^{\infty}_{0}(\Omega)$. Here $1-1/q=(1-\alpha+\beta)/n$, $\beta\leq\alpha$ and C is a positive constant independent of u and Ω.

Assume that Ω satisfies the stronlgy local Lipschitz property. Then Hardy's inequality implies that the space $C_0^\infty(\Omega)$ is densely contained in $W_\alpha^{1,1}(\Omega)$ provided $\alpha>0$ or $\alpha\leq -1$. For the detailed proof, see [15], [21], [42].
Therefore the assertion follows easily from the following lemma.

Lemma 9.1. *Let Ω have the strongly local Lipschitz property. Assumue that $\alpha>0$. Then there exist positive numbers C_1 and C_2 such that we have*

$$\int_\Omega |u|\delta^{\alpha-1}dx\leq C_1\int_\Omega |\nabla u|\delta^\alpha dx+C_2\int_\Omega |u|\delta^\alpha dx, \tag{9.2}$$

for any $u\in C_0^\infty(\Omega)$. Here C_1 and C_2 are independent of each u.

Proof. In the first place we asuume that

$$\Omega = \{x = (x', x_n)\in \boldsymbol{R}^n;\ x_n>0\} \quad \text{and} \quad \delta(x) = x_n .$$

Then integration by parts gives

$$\int_\Omega |u|x_n^{\alpha-1}dx\leq \alpha^{-1}\int |\partial_n u|x_n^\alpha dx . \tag{9.3}$$

The proof in the general case follows from this elementary inequality using a partition of unity and diffeomorfism. Q.E.D.

We proceed to the proofs of the cases B, C and D. Here we note that the assertions in the case D follow from those in the cases B and C. Again by a partition of unity and diffeomorfism, we can reduce the assertions to its simplest form as follows:

We denote by $z=(x, y)$ and $\zeta=(\xi, \eta)$ points in $\boldsymbol{R}^{n-s}\times \boldsymbol{R}^s$ with $x, \xi\in \boldsymbol{R}^{n-s}$, $y, \eta\in \boldsymbol{R}^s$, where s is a positive integer $\leq n$. Let $F=\{z=(x, y)\in \boldsymbol{R}^{n-s}\times \boldsymbol{R}^s;\ x\in \boldsymbol{R}^{n-s}, y=0\}$ if $s>1$, and let $F=\{z=(x, y)\in \boldsymbol{R}^{n-1}\times \boldsymbol{R};\ x\in \boldsymbol{R}^{n-1}, y\leq 0\}$ if $s=1$. We adopt $|y|$ as $\delta(z)$ for simplicity (See Example 2 in § 2.). Lastly we set $\Omega=\boldsymbol{R}^n\backslash F$ (If $s=1$, $\Omega=\boldsymbol{R}_+^n$.). Then

Proposition 9.1. *Let p satisfy $p\geq 1$, and let α and β be real numbers.*
Case B. Suppose that $p(1-\alpha+\beta)=n$ and $0\leq\beta\leq\alpha$. Then the following imbeddings are valid:

$$W_\alpha^{1,p}(\Omega)\rightarrow L_\beta^q(\Omega), \qquad p\leq q<+\infty . \tag{9.4}$$

Moreover if either $p=1$ or $0<\beta<\alpha$, then it holds that

$$W_\alpha^{1,p}(\Omega)\rightarrow SC_\beta^0(\Omega). \tag{9.5}$$

Case C. Suppose that $1-\alpha+\beta-n/p>0$ and $0\leq\beta\leq\alpha$. Then the following imbeddings are valid:

$$W_\alpha^{1,p}(\Omega)\rightarrow SC_\beta^{0,\lambda}(\Omega), \qquad 0<\lambda<1-\alpha+\beta-n/p . \tag{9.6}$$

Moreover if $\alpha\neq 1-n/p$, then it holds that

$$W_{\alpha}^{1,p}(\Omega) \to SC_{\beta}^{0,\lambda}(\Omega)\,, \qquad 0<\lambda\leq 1-\alpha+\beta-n/p\,. \tag{9.7}$$

In order to prove this proposition we prepare more notations. Let $B_h^s(a)$ denote the s-dimensional ball with center $a\in \boldsymbol{R}^s$ and radius h, and let S^{s-1} equal $\partial B_1^s(0)$. Let us set

$$C_{h,e}^s = \{y=(y_1, \cdots, y_s)\in \boldsymbol{R}^s;\ 0\leq (y_1^2+\cdots+y_{s-1}^2)^{1/2}<y_s<h\}\,, \tag{9.8}$$

where $e=(0, \cdots, 0, 1)\in S^{s-1}$ denotes the direction of the cone. By $C_{h,\omega}^s$ with $\omega\in S^{s-1}$, we mean the congruent cone to $C_{h,e}^s$ with the same vertex, where ω denotes the direction of the cone. We also set for $s>1$.

$$\Gamma_{h,e}^s = \{y=(y_1, \cdots, y_s)\in \boldsymbol{R}^s;\ 0\leq (y_1^2+\cdots+y_{s-1}^2)^{1/2}<y_s=h\}\,. \tag{9.9}$$

In a similar way let $\Gamma_{h,\omega}^s$ denote the boundary portion of $C_{h,\omega}^s$ which is perpendicular to ω and congruent to $\Gamma_{h,e}^s$. We introduce averaging functions as follows:

$$B_h^{n-s}[u_1] = |B_h^{n-s}|^{-1}\int_{B_h^{n-s}} u_1(x)dx = \fint_{B_h^{n-s}} u_1(x)dx\,, \tag{9.10}$$

$$\text{for}\quad u_1\in C^0(\boldsymbol{S}^{n-s})\,.$$

$$\Gamma_{h,\omega}^s[u_2] = \begin{cases} |\Gamma_{h,\omega}^s|^{-1}\displaystyle\int_{\Gamma_{h,\omega}^s} u_2(y)\,dS_y = \fint_{\Gamma_{h,\omega}^s} u_2(y)dS_y\,, & s>1\,,\\ u_2(h\omega)\,, & s=1\,,\end{cases}$$

$$\text{for}\quad u_2\in C^0(\boldsymbol{R}^s)\,.$$

$$M_h^m[u_3] = mh^{-m}\int_0^h t^{m-1}u_3(t)dt\,,\qquad \text{for}\quad u_3\in C^0(\boldsymbol{R})\ \text{ and }\ m>0\,.$$

$$A_{h,\omega}^{m+s}[u] = \underset{t\to h}{M_h^{m+s}}[\underset{y\to t}{\Gamma_{t,\omega}^s}[B_h^{n-s}[u(*, y)]]] =$$

$$= \begin{cases} (m+s)^{-1}h^{-m-s}\displaystyle\int_0^h t^{m+s-1}dt\fint_{\Gamma_{t,\omega}^s} dS_y\fint_{B_h^{n-s}} u(x, y)dx, & s>1\,,\\ (m+1)^{-1}h^{-m-1}\displaystyle\int_0^h t^m dt\fint_{B_h^{n-1}} u(x, t)dx, & s=1\,,\end{cases}$$

for $u\in C^0(\boldsymbol{R}^{n-s}\times \boldsymbol{R}^s)$ and $\omega\in S^{s-1}$. Here S_y is the $(s-1)$-dimensional Lebesgue measure.

The proof of Proposition 9.1 will be carried out in a chain of auxiliary lemmas being based on potential estimates. The next lemma 9.2 will be established in § 10.

Lemma 9.2. *Let $u\in C_0^1(\boldsymbol{R}^n)$, and let m be an arbitrary positive integer. Then for an arbitrary point $z=(x, y)$ with $x\in B_h^{n-s}(0)$ and $0<|y|<h$, the following inequalities are valid:*

(1) *Suppose that $s=1$. Then*

$$|u(z)-A_{h,\omega}^{m+1}[u]|\leq C\int_{B_h^{n-1}} d\xi\int_{C_{h,\omega}^1} \frac{|\nabla u(\xi, \eta)|}{|z-\zeta|^{n-1}}\,\frac{|\eta|^m}{|z-\zeta^*|^m}\,d\eta\,, \tag{9.11}$$

where $\omega=y/|y|$ and $\xi^*=(\xi, -\eta)$.
(2) Suppose that $1<s\le n$. Then

(9.12) $$|u(z)-A^{m+s}_{h,\omega}[u]|\le C\int_{B^{n-s}_h}d\xi\int_{C^s_{h,\omega}}\frac{|\nabla u(\xi,\eta)|}{|z-\zeta|^{n-1}}\frac{|\eta|^m}{(|y|+|\eta|)^m}d\eta+ +C\int_{B^{n-s}_h}d\xi\int_{C^s_{h,\omega}}\frac{|\nabla u(\xi,\eta)|\,|\eta|^m}{(|y|+|\eta|)^{m+s-1}}d\eta,$$

where ω is an arbitrary element of S^{s-1} satisfying $|\omega-\omega_0|<\sqrt{2}/4$ with $\omega_0=y/|y|$, and C is a positive constant depending only on n and m.

The following lemmas are easy corollaries of this lemma and (7.2) in § 7.

Lemma 9.3. *Let p satisfy $p>n$ and let $u\in C^1_0(\boldsymbol{R}^n)$. Suppose that $1-\alpha+\beta-n/p>0$ and $0\le\beta\le\alpha$. Then for any $z=(x, y)\in\boldsymbol{R}^n$ with $x\in B^{n-s}_h(0)$ and $0<|y|<h$, it holds that*

(9.13) $$|y|^\beta|u(z)-A^{m+s}_{h,\omega}[u]|\le Ch^{1-\alpha+\beta-n/p}\||\nabla u|;\, L^p_\alpha\|,$$

where ω is an arbitrary element of S^{s-1} satisfying $|\omega-\omega_0|<\sqrt{2}/4$ with $\omega_0=y/|y|$, and C is a positive number independent of each u and h.

Lemma 9.4. *Let p satisfy $p>n$ and let $u\in C^1_0(\boldsymbol{R}^n)$. Suppose that $0<\alpha<1-n/p$. Then for any point $a=(a', a_n)\in\boldsymbol{R}^{n-1}\times\boldsymbol{R}$ and any positive number h, it holds that*

(9.14) $$|u(z)-B^n_h[u(*+a)]|\le C(h+|a_n|)^{-\alpha}h^{1-n/p}\||\nabla u|;\, L^p_\alpha\|,$$
$$|B^n_h[u(*+a)]|\le C(h+|a_n|)^{-\alpha}h^{-n/p}\|u;\, L^p_\alpha\|,\qquad \text{for any}\quad z\in B^n_h(a).$$

Here C is a positive number depending only on n and α.

Proof of Proposition 9.1. Since the weight function $|y|$ does not vanish in Ω, the assertion (9.4) follows from Theorem 1 by making use of partition of unity. So we proceed to the assertion (9.5). First we assume that $p=1$. Hence we have $n=s=1$ and $0\le\alpha=\beta$. From (9.2) with $B=(y, y+1)$, $y\ge0$, we have

(9.15) $$y^\beta|u(y)|\le\int_y^{y+1}\eta^\alpha|u(\eta)|d\eta+C\int_y^{y+1}\eta^\alpha|\partial u(\eta)|d\eta,\qquad u\in C^1_0(\boldsymbol{R}).$$

So that we have

(9.16) $$y^\beta|u(y)|\le C\|u;\, W^{1,1}_\alpha\|.$$

Secondly we assume that $p>1$ and $0<\beta<\alpha$. Again from (7.2) with $\Omega=C^n_{h,\omega}(z)=z+C^n_{h,\omega}$, $z\in\boldsymbol{R}^{n-s}\times\boldsymbol{R}^s$, $y\ne0$ and $\omega=(0, y/|y|)$, it holds that for an arbitrary $u\in C^1_0(\boldsymbol{R}^n)$

(9.17) $$|u(z)|\le h^{-n}\int_{C^n_{h,\omega}(z)}|u(\zeta)|d\zeta+C\int_{C^n_{h,\omega}(z)}|\nabla u(\zeta)|\,|z-\zeta|^{1-n}d\zeta= =I+J.$$

where $\zeta=(\xi, \eta)\in\boldsymbol{R}^{n-s}\times\boldsymbol{R}^s$.

By Theorem 1 and Hölder's inequality we have $I\times|y|^{\beta}\leq Ch^{-n/p}\|u;\ W_{\alpha}^{1,p}\|$. As for J we further divide it into two terms. Let us set $D_1=C_{h,\omega}^{n}(z)\backslash B_{|y|}^{n}(z)$ and $D_2=C_{h,\omega}^{n}(z)\backslash D_1$, and set $J=C\int_{D_1}[\cdots]d\zeta+C\int_{D_2}[\cdots]d\zeta=J_1+J_2$. Then by Hölder's inequality we have

$$(9.18)\qquad \begin{aligned} J_1\times|y|^{\beta}&\leq C|y|^{\beta}\Big(\int_{D_1}|z-\zeta|^{(1-\alpha-n)p'}d\zeta\Big)^{1/p'}\|u;\ W_{\alpha}^{1,p}\|\leq\\ &\leq C'|y|^{\beta}\Big(\int_{|y|}^{\infty}r^{(1-\alpha-n)p'+n-1}dr\Big)^{1/p'}\|u;\ W_{\alpha}^{1,p}\|\leq\\ &\leq C''\|u;\ W_{\alpha}^{1,p}\|\,. \end{aligned}$$

Let ε satisfy $0<\varepsilon<\alpha-\beta$, then we have

$$(9.19)\qquad \begin{aligned} J_2\times|y|^{\beta}&\leq C|y|^{-\varepsilon}\Big(\int_{D_2}|z-\zeta|^{(1-\alpha+\beta-n+\varepsilon)p'}d\zeta\Big)^{1/p'}\|u;\ W_{\alpha}^{1,p}\|\leq\\ &\leq C'|y|^{-\varepsilon}\Big(\int_{0}^{|y|}r^{(1-n-\alpha+\beta+\varepsilon)p'+n-1}dr\Big)^{1/p'}\|u;\ W_{\alpha}^{1,p}\|\leq\\ &\leq C''\|u;\ W_{\alpha}^{1,p}\|\,,\ \text{where } p'=p(p-1)^{-1}\,. \end{aligned}$$

Therefore we have the desired estimates.

Lastly we prove the assertion in the case c.

Suppose that $\mu=1-\alpha+\beta-n/p>0$ and $0\leq\beta\leq\alpha$. Then it follows from the previous part and the lemma 9.4 that:

(a) If $\alpha=1-n/p$, $\|u;\ SC_{\beta-\lambda}^{0}\|\leq C\|u;\ W_{\alpha}^{1,n}\|$, $\quad 0<\lambda<\mu$.

(b) If $\alpha\neq1-n/p$, $\|u;\ SC_{\beta-\lambda}^{0}\|\leq C\|u;\ W_{\alpha}^{1,p}\|$, $\quad 0<\lambda\leq\mu$.

Here C is a positive constant independent of u.

Proof. If $\alpha=1-n/p$, then we have $0=\beta-\mu<\beta-\lambda<\alpha$ and $1-\alpha+(\beta-\lambda)-n/p>0$. Hence one can choose r so that $n<r<p$ and $1-\alpha+(\beta-\lambda)-n/r=0$. Then the assertion (a) follows from (9.5) and $W_{\alpha}^{1,p}(B)\to W_{\alpha}^{1,r}(B)$ by virtue of a partition of unity. If $\alpha>1-n/p$, then we have $0<\beta-\lambda<\beta\leq\alpha$, so that the assertion (b) follows from (9.5) replacing β by $\beta-\mu$. In a similar way, if $0<\alpha<1-n/p$, the assertion follows from the lemma 9.4. Q.E.D.

We proceed to the estimates of Hölder norms of $u\in W_{\alpha}^{1,p}$. Let $\mu=1-\alpha+\beta-n/p>0$. We shall establish the following:

(c) $|u|_{0,\lambda,\beta}\leq C\|u;\ W_{\alpha}^{1,p}\|$, $\quad 0<\lambda\leq\mu$.

Here $|*|_{0,\lambda,\beta}$ is the semi-norm defined by (1.10), and C is a positive constant independent of each u.

Proof. Let $z=(x, y)$ and $\zeta=(\xi, \eta)$ be arbitrary points in Ω, and put $\rho=2|z-\zeta|$. Without loss of generality we assume that $|y|\geq|\eta|$ and $\rho\geq1$. Now we shall classify the case by the values of ρ. In the first place, assume that $\rho=2|z-\zeta|<|y|/2$. Then we have $z, \zeta\in B_{\rho}^{n}(z)$ and $|\eta|>3\rho/2$. By the inequality (7.2) we have for $z'=(x', y')\in\Omega$

$$(9.20)\qquad |u(z)-B_\rho^n[u(*+z)]|\leq C\int_{B_\rho^n(z)}|\nabla u(z')|\,|z-z'|^{1-n}dz'.$$

Since $0\leq\beta\leq\alpha$ and $3\rho/2<|\eta|\leq|y|$, we have

$$(9.21)\qquad |y|^\beta|u(z)-B_\rho^n[u(*+z)]|\leq$$
$$\leq C|y|^\beta(|y|-\rho)^{-\alpha}\int_{B_\rho^n(z)}|y|^\alpha|\nabla u(z')|\,|z-z'|^{1-n}dz'\leq C\rho^\mu\|u;\ W_\alpha^{1,p}\|.$$

In a similar way,

$$(9.22)\qquad |\eta|^\beta|u(\zeta)-B_\rho^n[u(*+z)]|\leq C\rho^\mu\|u;\ W_\alpha^{1,p}\|.$$

So that

$$(9.23)\qquad \mathrm{Min}[|y|^\beta,\ |\eta|^\beta]|u(z)-u(\zeta)|\leq 2C\rho^\mu\|u;\ W_\alpha^{1,p}\|.$$

Secondly we assume that $\rho=2|z-\zeta|\geq|y|/2\geq|\eta|/2$. Without loss of generality, we also assume that $z=(0, y)$ with $y\neq 0$ and $\eta\neq 0$. Then $z,\ \zeta\in B_{3\rho}^n(0)$ $(s>1)$, and $z,\ \zeta\in B_{3\rho}^n(0)\cap \boldsymbol{R}_+^n$ $(s=1)$. If $0\leq\alpha<1-n/p$, the desired estimate (9.23) follows from the lemma 9.4. So we assume that $\alpha\geq 1-n/p$. Moreover if $s=1$, then the assertion follows directly from the lemma 9.3 in place of the lemma 9.4. If $s>1$, one can choose a finite number of points ω_i, $i=1, \cdots, n(s)$, so that $\omega_1=y/|y|$, $\omega_{n(s)}=\eta/|\eta|$, $\omega_i\in S^{s-1}$ and $C_{1,\omega_i}^s\cap C_{1,\omega_{i+1}}^s\neq\phi$, where the number $n(s)$ depends only on n and s. Then it holds that $y\in C_{3\rho,\omega_1}^s$, $\eta\in C_{3\rho,\omega_{n(s)}}^s$ and $B_{3\rho}^{n-s}(0)\cap B_{3\rho}^{n-s}(\xi)\neq 0$. By an obvious inequality we have $|u(z)-u(\zeta)|\leq|u(0, y)-u(0, \eta)|+|u(0, \eta)-u(\xi, \eta)|$. After multiplying the factor $\mathrm{Min}[|y|^\beta,\ |\eta|^\beta]$ to the both sides, the repeated use of the lemma 9.3 gives the desired estimate (c). Q.E.D.

§ 10. Proof of Lemma 9.2

We have postponed to this section the proof of Lemma 9.2 which will be established in a chain of lemmas being based on Potential estimates. We retain the notations introduced in § 9. Without loss of generality, we assume that $\omega=\omega_0=y/|y|=(0, \cdots, 0, 1)\in S^{s-1}$. We begin with the following lemma which implies the assertion (1) of the lemma 9.2 in the case $n=s=1$.

Lemma 10.1. *Let $v\in C^1(\overline{\boldsymbol{R}_+})$. Let m and h be nonnegative numbers. Then*

$$(10.1)\qquad |v(r)-M_h^m[v]|\leq 2^{m+1}\int_0^h|v'(\tau)|\tau^m/(r+\tau)^m d\tau\,,\qquad \textit{for any}\quad r\in[0, h].$$

Proof. By virtue of integration by parts, we have

$$(10.2)\qquad |v(r)-M_h^m[v]|=|v(r)-mh^{-m}\int_0^h\tau^{m-1}v(\tau)d\tau|\leq$$
$$\leq 2\int_r^h|v'(\tau)|d\tau+h^{-m}\int_0^r\tau^m|v'(\tau)|d\tau\,,$$

and hence

$$\int_r^h |v'(\tau)|d\tau \leq 2^m \int_r^h |v'(\tau)|\tau^m/(r+\tau)^m d\tau\,, \tag{10.3}$$

$$\int_0^r (\tau/h)^m |v'(\tau)|d\tau \leq 2^m \int_0^r |v'(\tau)|\tau^m/(r+\tau)^m d\tau\,.$$

So that we have the desired estimate. Q.E.D.

From now on we assume that $n \geq 2$. First we deal with the case $s=1$. Let $t=(t_1, \cdots, t_{m+1})$ and $\sigma=(\sigma_1, \cdots, \sigma_{m+1})$ denote points in $\boldsymbol{R}^{m+1}$ with m positive integer. Take an arbitrary function $u \in C_0^1(\boldsymbol{R}^n)$, and set

$$U(x, t) = u(x, |t|)\,, \tag{10.4}$$

where $x=(x_1, \cdots, x_{n-1}) \in \boldsymbol{R}^{n-1}$ and $|t|=(t_1^2+\cdots+t_{m+1}^2)^{1/2}$. Then setting

$$B_h^{n-1,m+1}[u] = (|B_h^{n-1}|\,|B_h^{m+1}|)^{-1} \int_{B_h^{n-1}} dx \int_{B_h^{m+1}} U(x, t)dt\,, \tag{10.5}$$

we have $B_h^{n-1,m+1}[u]=A_{h,1}^{m+1}[u]$. Since $B_h^{n-1} \times B_h^{m+1}$ is convex, we also have by the inequality (7.2), for any $(x, t) \in B_h^{n-1} \times B_h^{m+1}$,

$$|U(x, t)-A_{h,1}^{m+1}[u]| \leq C \int_{B_h^{n-1} \times B_h^{m+1}} |(x, t)-(\xi, \sigma)|^{1-n-m} |\nabla u(\xi, |\sigma|)|d\xi d\sigma\,. \tag{10.6}$$

In order to estimate the right-hand side of this, we prepare the following:

Lemma 18.2. *Let v and w be of the class $C_0^1(\overline{\boldsymbol{R}_+})$. Let m be a positive integer. Then we have*

$$\int_{B_h^{m+1}} v(|t-\sigma|)w(|\sigma|)d\sigma = \tag{10.7}$$

$$= C_m \int_0^h w(r)r^m dr \int_0^1 [\theta(1-\theta)]^{-1+m/2} v(f(|t|, r, \theta))d\theta\,,$$

where $C_m=2^{m-1}|S^{m-1}|$, $f(|t|, r, \theta)=[(|t|-r)^2(1-\theta)+(|t|+r)^2\theta]^{1/2}$.

Admitting this for a moment, we establish the assertion (1) of the lemma 9.2. Let $v(r)=[|z-\zeta|^2+r^2]^{(1-n-m)/2}$ with $z \neq \zeta$, and let $w(r)=|\nabla u(\xi, r)|$. Then we have for any $z=(x, y) \in B_h^{n-1} \times C_{h,1}^1$

$$|u(z)-A_{h,1}^{m+1}[u]| \leq C \iint_{B_h^{n-1} \times C_{h,1}^1} \eta^m |\nabla u(\xi, \eta)|d\xi d\eta \times \tag{10.8}$$

$$\times \int_0^1 [\theta(1-\theta)]^{-1-m/2}[|z-\zeta|^2(1-\theta)+|z-\zeta^*|^2\theta]^{(1-n-m)/2}d\theta\,,$$

where $\zeta^*=(\xi, -\eta)$.
Thus with somewhat more calculations we have

$$(10.9) \qquad |u(z)-A^{m+1}_{h,1}[u]| \leq C \int_{B^{n-1}_h \times C^1_{h,1}} |\nabla u(\zeta)|\, |z-\zeta|^{1-n} \eta^m |z-\zeta^*|^{-m} d\zeta ,$$

for any $z=(x, y) \in B^{n-1}_n \times C^1_{h,1}$, which is the assertion (1) in the lemma 9.2.

Proof of Lemma 10.2. We make use of the polar coordinate system defined by

$$(10.10) \qquad t = r\tau,\ \sigma = \rho\tau' \quad \text{with} \quad \tau, \tau' \in S^m ,$$
$$dS^m_s = (\rho \sin \psi)^{m-1} dS^{m-1}_{\tau''} d\psi \quad \text{with} \quad 0<\psi<\pi, \quad \text{and} \quad \tau'' \in S^{m-1} .$$

Then we have

$$(10.11) \qquad \int_{B^{m+1}_h} v(|t-\sigma|) w(|\sigma|) d\sigma =$$
$$= \int_0^h \rho^m w(\rho) d\rho \int_{S^{m-1}} dS^{m-1}_{\tau''} \int_0^\pi v((r^2+\rho^2+2r\rho \cos \psi)^{1/2}) (\sin \psi)^{m-1} d\psi .$$

Carrying out the change of variables defined by

$$(10.12) \qquad \sin \psi = 2[\theta(1-\theta)]^{1/2} , \quad \text{that is, } \cos \psi = 2\theta - 1 ,$$
$$d\psi = (-2/\sin \psi) d\theta ,$$

the desired estimate follows. Q.E.D.

We proceed to the case $s \geq 2$. For the sake of simplicity we assume that $\omega = \omega_0 = e = (0, \cdots, 0, 1) \in S^{s-1}$ as before. Letting $y = \rho e$ with $0<\rho \leq h$, we have

$$(10.13) \qquad u(z) - A^{m+s}_{h,e}[u] = \mathrm{I} + \mathrm{J} + \mathrm{K} ,$$

where

$$\mathrm{I} = u(z) - 2/\rho \int_{\rho/2}^{\rho} \Gamma^s_{r,e}[B^{n-s}_h[u(*, \eta)]] dr ,$$
$$\mathrm{J} = 2/\rho \int_{\rho/2}^{\rho} \Gamma^s_{r,e}[B^{n-s}_h[u(*, \eta)]] dr - \Gamma^s_{\rho,e}[B^{n-s}_h[u(*, \eta)]] ,$$
$$\mathrm{K} = \Gamma^s_{\rho,e}[B^{n-s}_h[u(*, \eta)]] - A^{m+s}_{h,e}[u] ,$$
$$\Gamma^s_{\rho,e}[B^{n-s}_h[u(*, \eta)]] = \fint_{\Gamma^s_{\rho,e}} dS_\eta \fint_{B^{n-s}_h} u(\xi, \eta) d\xi .$$

In the first place we can rewrite I to obtain

$$(10.14) \qquad \mathrm{I} = u(z) - |\Omega|^{-1} \int_\Omega u(\zeta) d\zeta ,$$

where $\qquad \Omega = B^{n-s}_h \times \{\eta \in C^s_{h,e};\ \rho/2 < \eta_s < \rho\}$.

Since Ω is convex, we have by an easy variant of the inequality (7.2)

$$(10.15) \qquad \mathrm{I} \leq C \int_\Omega |\nabla u(\zeta)|\, |z-\zeta|^{1-n} d\zeta \leq$$
$$\leq C' \int_\Omega |\nabla u(\zeta)|\, |z-\zeta|^{1-n} |\eta|^m / (|\eta|+\rho)^m d\zeta .$$

As for J and K, we have

$$\text{(10.16)} \qquad \mathrm{J} = 2/\rho \int_{\rho/2}^{\rho} v(r)dr - v(\rho) \quad \text{and} \quad \mathrm{K} = v(\rho) - M_h^{m+s}[v]\,,$$

where $\qquad v(\rho) = \Gamma_{\rho,e}^{s}[B_h^{n-s}[u(*, \eta)]]\,.$

By the lemma 10.1 and its easy variant,

$$\text{(10.17)} \qquad \mathrm{K} \leq 2^{m+n+1} \int_0^h |v'(r)|\, r^{m+s}/(r+\rho)^{m+s}\, dr\,,$$

$$\mathrm{J} \leq \int_{\rho/2}^{\rho} |v'(r)|\, dr \leq C \int_0^h |v'(r)|\, r^{m+s}/(r+\rho)^{m+s} dr\,.$$

Since it holds that

$$\text{(10.18)} \qquad |v'(r)| \leq C[|B|_h^{n-s}|\,|\Gamma_{r,e}^{s}|]^{-1} \iint_{B_h^{n-s}\times\Gamma_{r,e}^{s}} |\nabla_\eta u(\xi, \eta)|\, d\xi\, dS_\eta\,,$$

we have the desired estimates. Q.E.D.

§ 11. Proof of Lemma 4.1

We shall prove the extension lemma 4.3 in § 3 which is estentially due to E. M. Stein. He treated the case $\alpha = 0$. Following his arguement we shall show the existence of extension operator $E_{k,p,\alpha}$: $W_\alpha^{k,p}(\Omega) \to H_\alpha^{k,p}(\boldsymbol{R}^n)$ satisfying $\|E_{k,p,\alpha}u;\ H_\alpha^{k,p}(\boldsymbol{R}^n)\| \leq C\|u;\ W_\alpha^{k,p}(\Omega)\|$. For the sake of simplicity we assume that $\Omega = \{(x, y) = (x_1, \cdots, x_{n-1}, y) \in \boldsymbol{R}^n;\ \varphi(x) < y$, where $\varphi;\ \boldsymbol{R}^{n-1} \to \boldsymbol{R}$ is a Lipschitz function with Lipschitz constant 1. In fact the main part of his proof is based on the existence of functions defined in neighborhood of a simple domain as above. Then by virtue of a partition of unity he constructs a global extension in general case. Choose a smooth function $\psi(\lambda)$ so that

$$\text{(11.1)} \qquad \int_1^\infty \psi(\lambda)d\lambda = 1\,, \quad \int_1^\infty \psi(\lambda)\lambda^l d\lambda = 0\,, \qquad l = 1, 2, \cdots\,.$$

For the sake of simplicity we assume that δ introduced in § 1 satisfies

$$\text{(11.2)} \qquad \begin{aligned} &\delta(x, y) = 0 && \text{for} \quad y \geq \varphi(x)\,,\\ &2(\varphi(x)-y) \leq \delta(x, y) \leq 3(\varphi(x)-y) && \text{for} \quad y \leq \varphi(x)\,,\end{aligned}$$

and $\qquad |\partial^\gamma \delta(x, y)| \leq C(r)|\varphi(x)-y|^{1-|\gamma|}\,.$

Let us set

$$\text{(11.3)} \qquad (E_{k,p,\alpha}u)(x, y) = \begin{cases} u(x, y)\,, & y \geq \varphi(x) \\ \displaystyle\int_1^\infty u(x, y+\lambda\delta(x, y))\psi(\lambda)d\lambda\,, & y \leq \varphi(x)\,. \end{cases}$$

In the first place we shall see that $E_{k,p,\alpha}u \in H_\alpha^{1,p}(\boldsymbol{R}^n)$ provided $u \in W_\alpha^{1,p}(\Omega)$. Since $|\psi(\lambda)|$ is rapidly decreasing, it is majorated by Const. $1/\lambda^2$. So we have for

$y<\varphi(x)$

(11.4) $\delta(x, y)^{\alpha}(E_{k,p,\alpha}u)(x, y)\leq C\int_{y+\delta(x,y)}^{\infty}|\varphi(x)-y|^{\alpha+1}|u(x, t)|/|t-y|^{2}dt$.

Raising both sides to the p-th power and integrating for $y\in(-\infty, \varphi(x)]$,

$$\text{(11.5)}\qquad \mathrm{I}(x)\equiv\int_{-\infty}^{\varphi(x)}\delta(x, y)^{\alpha p}|(E_{k,p,\alpha}u)(x, y)|^{p}dy\leq$$
$$\leq C\int_{-\infty}^{\varphi(x)}|\int_{y+\delta(x,y)}^{\infty}|y-\varphi(x)|^{\alpha+1}|u(x, t)|/|t-y|^{2}dt|^{p}dy\,.$$

Using the fact $t-y\geq t-\varphi(x)\geq y+\delta(x, y)-\varphi(x)\geq\varphi(x)-y\geq 0$, for $y<\varphi(x)$, we have

$$\text{(11.6)}\qquad \mathrm{I}(x)\leq C\int_{-\infty}^{\varphi(x)}|\int_{\varphi(x)-y}^{\infty}(\varphi(x)-y)^{\alpha+1}|u(x, \varphi(x)+t)|/t^{2}dt|^{p}dy\leq$$
$$\leq C\int_{0}^{\infty}|\int_{z}^{\infty}z^{\alpha+1}|u(x, \varphi(x)+t)|/t^{2}dt|^{p}dz\,.$$

The following inequality is familiar. For the proof, see [15].

Lemma 11.1 (Hardy). *Let p satisfy $p\geq 1$ and $p(\gamma+1)+1>0$. Then*

$$\text{(11.7)}\qquad \int_{0}^{\infty}(\int_{t}^{\infty}|f(s)|ds)^{p}t^{p(\gamma+1)}dt\leq\{p/[p(\gamma+1)+1]\}^{p}\int_{0}^{\infty}|sf(s)|^{p}s^{p(\gamma+1)}ds\,,$$

for any $f\in C_{0}^{\infty}(\overline{\boldsymbol{R}_{+}})$.

From this we get

$$\text{(11.8)}\qquad \mathrm{I}(x)\leq C\int_{0}^{\infty}|u(x, \varphi(x)+t)|^{p}t^{p\alpha}dt=C\int_{\varphi(x)}^{\infty}|u(x, y)|^{p}|y-\varphi(x)|^{p\alpha}dy\,.$$

Hence we have, by integrating with respect to x over $\boldsymbol{R}^{n-1}$,

$$\text{(11.9)}\qquad \int_{\boldsymbol{R}^{n}}|(E_{k,p,\alpha}u)(x, y)|^{p}\delta(x, y)^{\alpha p}dxdy\leq C||u;\ L_{\alpha}^{p}(\Omega)||^{p}\,.$$

Secondly we proceed to the estimate of derivatives $\partial^{\gamma}(E_{k,p,\alpha}u)$ with $|\gamma|=1$. Let ∂_{j} and δ_{j} denote ∂_{x_j} and $\partial_{x_j}\delta$ respectively.

$$\text{(11.10)}\qquad \partial_{j}(E_{k,p,\alpha}u)=\int_{1}^{\infty}(\partial_{j}u)(x, y+\lambda\delta)\psi(\lambda)d\lambda+\int_{1}^{\infty}(\partial_{y}u)(x, y+\lambda\delta)\lambda\delta_{j}\psi(\lambda)d\lambda\,,$$
$$\partial_{y}(E_{k,p,\alpha}u)=\int_{1}^{\infty}(\partial_{y}u)(x, y+\lambda\delta)(1+\lambda\delta_{y})\psi(\lambda)d\lambda\,.$$

Since the first derivatives of δ are bounded, we have in a similar way that $||\nabla(E_{k,p,\alpha}u);\ L_{\alpha}^{p}(\boldsymbol{R}^{n})||\leq C||u;\ W_{\alpha}^{1,p}||$. The proof for $k>0$ is similar. Consider for example $k=2$. We shall handle typical term $\partial_{1}^{2}(E_{k,p,\alpha})$ only.
Then it is easy to see that only the following term

$$\text{(11.11)}\qquad \int_{1}^{\infty}(\partial_{y}u)(x, y+\lambda\delta)\lambda\delta_{1,1}\psi(\lambda)d\lambda$$

needs to be dealt with separately. We write

$$\partial_y u(x, y+\lambda\delta) = \partial_y u(x, y+\delta)+\int_{y+\delta}^{y+\lambda\delta} u_{yy}(x, t)dt . \tag{11.12}$$

Since ψ is rapidly decreasing, it suffices to estimate

$$J \equiv |y-\varphi(x)|^{-1}\int_1^\infty \lambda^{-3}d\lambda \int_{y+\delta}^{y+\lambda\delta} u_{yy}(x, t)dt . \tag{11.13}$$

By interchange of the order of integration we have

$$\mathrm{J} \equiv 1/2(y-\varphi(x))^{-1}\int_{y+\delta}^\infty u_{yy}(x, t)\delta(x, y)^2/|t-y|^2 dt . \tag{11.14}$$

So that it follows from the earlier case that the desired estimate holds. Q.E.D.

DEPARTMENT OF MATHEMATICS,
FACULTY OF SCIENCE,
IBARAKI UNIVERSITY

Bibliography

[1] R. A. Adams, Compact imbeddings for pepper sets, J. Math. Anal. Appl., **27** (1969), 405–408.
[2] R. A. Adams, Capacity and compact imbeddings, J. Math. Mech., **19**)(1970), 923–929.
[3] R. A. Adams, Traces of potentials arising from transration invariant operators, Ann. Scuola Norm. Super. Pisa. **25** (1971), 203–217.
[4] R. A. Adams, Some integral inequalities with applications to the imbedding of Sobolev spaces defined over irregular domains, Trans. Amer. Math. Soc., **178** (1973), 401–429.
[5] R. A. Adams, A trace inequality for generalized potential, Studia Math., **48** (1973), 99–105.
[6] R. A. Adams, Sobolev spaces, Academic press, 1975, New York-San Francisco-London.
[7] R. Andersson, The type set of a generalized Sobolev operator, Meddelanden fran Lunds Universitets Matem. Semin., **19** (1972), 1–101.
[8] A. Avantaggiati, Spaci di Sobolev con peso ed alcune applicazioni, Boll. Unione Mat. Ital., **13**–A (1976), 1–52.
[9] O. V. Besov, V. P. Il' in, S. M. Nikol' skii, Integral representations of functions and imbedding theorems, V. H. Winston and Sons, Washington, D. C., Vol. 1, 1978; Vol. 2 1979.
[10] S. Campanato, Proprieta di Hölderianta di alcune classi di funzioni, Ann. Scuola Norm. Sup. Pisa, **18** (1964), 137–160.
[11] H. Federer, Geometric measure theory, Springer, 1969, Berlin-Heidelberg-New York.
[12] I. G. Globenko, Embedding theorems for a region with null salient points, Soviet Maht. Dokl., **1** (1960), 517–519.
[13] I. G. Globenko, Some questions in the theory of imbedding for domains with singularities on the boundary, Mat. Sb., **57** **(99)** (1962), 201–224.
[14] P. Grisvard, Espaces intermediaires entre espaces de Sobolev avec poids, Ann. Scuola Norm. Sup. Pisa, **17** (1963), 255–296.
[15] G. H. Hardy, J. E. Littlewood, G. Polya, Inequalities, Camblidge Univ. Press, 1943, London and New York.
[16] T. Horiuchi, Existence and uniqueness of classical solutions for certain degenerated elliptic equations of second order, J. Math. Kyoto Univ. ,**24**–3 (1984), 557–576.

[17] V. P. Il'in, On an imbedding theorem for a limiting exponent, Dokl. Acad. Nauk SSSR, **96** (1954), 905–908 (Russian).

[18] V. P. Il'in, Integral representations of differentiable functions and their application to questions of the extension of functions of the class $W_p^l(G)$, Siberian Math. J., **8** (1967), 421–432.

[19] A. Jonsson, H. Wallin, A Whitney extension theorem in L^p and Besov spaces, Ann. Inst. Fourier, **28** (1978), 139–192.

[20] A. Kufner, Imbedding theorems for general Sobolev weight spsces, Ann. Scuola Norm. Sup. Pisa, **21** (1969), 373–386.

[21] A. Kufner, Weighted Sobolev spaces, Leipzig, Teubner-Texte zur Math. 1980.

[22] J. L. Lions, B. Magenes, Non-homogeneous limit problems and applications, Springer, 1972, Berlin-Heidelberg-York.

[23] P. I. Lizorkin, Boundary properties of functions from "weight classes", Soviet Math. Dokl., **1** (1960), 589–593.

[24] P. I. Lizorkin, M. Otelbaev, Imbedding theorems and compactness for the spaces of Sobolev type with weights, Math. USSR-Sb., **16** (1980), 331–349; **40** (1981), 51–77.

[25] V. G. Maz' ja, Classes of regions and imbedding theorems for function spaces, Soviet Math. Dokl., **1** (1960), 882–885.

[26] V. G. Maz' ja, The p-conductivity and theorems on imbedding certain function spaces into a C-spaces, Soviet Math. Dokl., **2** (1961), 1200–1203.

[27] V. G. Maz' ja, The Dirichert proble n for elliptic equations of arbitrary order in unbounded regions, Soviet Math. Dokl., **4** (1963), 860–863.

[28] V. G. Maz' ja, On Neumann's problem in domains with nonregular boundaries, Siberian Math. J., **9** (1968), 990–1012.

[29] V. G. Maz' ja, On weak solutions of the Dirichret and Neumann probrems. Trans. Moscow Math. Soc., **20** (1969), 135–172.

[30] V. G. Maz' ja, Sobolev spaces, Berlin-Heidelberg-New York, Springer, 1985.

[31] B. Mucknehoupt, Hardy's inequality with weights, Studia Math., **44** (1972), 31–38.

[32] T. Muramatu, On imbedding theorems for Sobolev spaces and some of their generalizations, Publ. Res. Inst. Math. Sci. Kyoto Univ., Ser. A, **1** (1968), 393–416.

[33] T. Muramatu, On imbedding theorems for Besov spaces of functions defined in general regions, Publ. Res. Inst. Math. Sci. Kyoto Univ., **7** (1971/1972), 262–285.

[34] T. Muramatu, On Besov spaces and Sobolev spaces of generalized functions defined on a general region, Publ. Res. Inst. Math. Kyoto Univ., **9** (1974), 325–396.

[35] J. Nečas, Sur une méthode pour résoudre les équations aux dérivées Partielles du type elliptique, voisine de la variationnelle, Ann. Scuora Norm. Sup. Pisa, ser. 3, **14–4** (1962), 305–326.

[36] L. Nirenberg, On elliptic partial differential equations (Lecutre II), Ann. Scuola Norm. Sup. Pisa, ser. 3, **11** (1959), 115–162.

[37] S. L. Sobolev, On some estimates relating to families of functions having derivatives that are square integrable, Dokl. Akad. Nauk SSSR. (1936), 267–270.

[38] S. L. Sobolev, On a theorem in functional analysis, Am. Math. Soc. Translations, vol. (2) 34 (1963), 39–68.

[39] S. L. Sobolev, Applications of functional analysis in mathematical physics, Am. Math. Soc. Translations, vol. 7 (1963).

[40] E.M. Stein, Singular integrals and differentiability properties of functions, Princeton Univ. Press, 1970, Princeton, N. J.

[41] G. Talenti, Best constant in Sobolev inequality, Ann. Mat. pura Appl. IV. Ser., **118** (1976), 353–372.

[42] H. Triebel, Interporation theory, function spaces, differential operators, VEB Deutscher Verlag der Wissenschaften, 1978, Berlin.

[43] S. V. Uspenskii, An imbedding theorem for S. L. Sobolev' classes of fractional order, Soviet Math. Dokl., **1** (1960), 132–133.

[44] S. V. Uspenskii, Properites of the W_p^r classes with a fractional derivative on differentiable manifolds, Soviet Math. Dokl., **1** (1960), 495–497.

[45] S. V. Uspenskii, Imbedding theorems for classes with weights, Amer. Math. Soc. Transl., **87** (1970), 121–145.