J. Math. Kyoto Univ. (JMKYAZ)
29-4 (1989) 601-608

On a Hasse principle for σ-conjugacy

Dedicated to Professor Ichiro Satake on his sixtieth birthday

By

Hiroshi SAITO

§ 0. Introduction

Let M/K be a cyclic extension of finite algebraic number fields of degree l, and σ a generator of the Galois group $\mathrm{Gal}(M/K)$, which will be fixed. For an algebraic group G defined over K, we denote by $G(M)$ the set of all points of G with coordinates in M. The action of σ on $G(M)$ can be defined naturally. We denote it by ${}^{\sigma}g$ for $g \in G(M)$. In $G(M)$, we define an equivalence relation $\underset{\sigma}{\sim}$ by $g \underset{\sigma}{\sim} g'$ if and only if $g = h^{-1}g'{}^{\sigma}h$ for some $h \in G(M)$. This will be called σ-conjugacy. It was introduced for $GL(2)$ in the study of the twisted trace formula ([3], [4]). The purpose of this paper is to determine σ-conjugacy classes for G such that $G(K) = A^{\times}$, where A is a semi-simple algebra over K.

The σ-conjugacy has a close relation with the usual conjugacy, which will be denoted by $\sim$. For $g \in G(M)$, we define the "norm" of $g \in G(M)$ by $Ng = g^{\sigma}g^{\sigma^2}g \cdots {}^{\sigma^{l-1}}g$. Then the conjugacy class of Ng depends only on the σ-conjugacy class of g. We denote by $G(M)/\underset{\sigma}{\sim}$, $G(M)/\sim$ the sets of σ-conjugacy classes, and usual conjugacy classes in $G(M)$ respectively. Then N defines a map of $G(M)/\underset{\sigma}{\sim}$ to $G(M)/\sim$. This map is fundamental in our study of σ-conjugacy. In fact, for $G = A^{\times}$, this map is injective, and to determine $G(M)/\underset{\sigma}{\sim}$, it is sufficient to determine the image of $G(M)/\underset{\sigma}{\sim}$ by N. It is easy to see this image is contained in the set $(G(M)/\sim)^{\sigma}$ consisting of conjugacy classes invariant under σ. To describe the image, we consider the norm at each place of K. For each place v of K, let K_v be the completion of K at v and let $M_v = M \otimes_K K_v$. Then the action of σ can be extended to M_v and $G(M_v)$. We can define in $G(M_v)$ σ-conjugacy and the norm in the same way as above. Our main result asserts that for $G = A^{\times}$ a conjugacy class in $(G(M)/)^{\sigma}$ is contained in the image of $G(M)$ by N if and only if it is contained in image of $G(M_v)$ by N for all v (cf. Th. 2.1).

In § 1, we give preliminary results on σ-conjugacy. In § 2, we state our main result and reduce the proof to the cases of semi-simple and unipotent elements. The proofs of these two cases are given in § 3 and § 4 separately.

§ 1. σ-conjugacy

In this section, we prove some elementary properties of σ-conjugacy. Let K be a field of characteristic 0 and G a linear algebraic group defined over K. We define σ-conjugacy for M more general than that in the Introduction. Let M be a commutative

Received February 12, 1988.

semi-simple algebra over K of dimension l, and σ a K-automorphism of M such that $M^{\sigma}=K$. Here for a set X on which an automorphism σ is defined, we denote by X^{σ} the set of elements invariant under σ. Then we may assume M is the m-fold product M_1^m of a cyclic extension M_1 of K of degree l/m with a generator τ of $\mathrm{Gal}(M_1/K)$ for a divisor m of l and the action of σ on M is given by ${}^{\sigma}(x_1, \cdots, x_m)=(x_2, \cdots, x_m, {}^{\tau}x_1)$.

Let $G(M)$ the set of M-valued points of G. Then the action of σ on $G(M)$ can be defined naturally. We denote it by ${}^{\sigma}x$ for $x\in G(M)$. In $G(M)$, we define an equivalence relation $\underset{\sigma}{\sim}$, which will be called σ-conjugacy, by

$$X\underset{\sigma}{\sim}Y \Longleftrightarrow X=g^{-1}Y^{\sigma}g \qquad \text{for} \quad g\in G(M).$$

W denote by $\sim$ the usual conjugacy, that is, $x\sim y$ for x, $y\in G(M)$ if and only if $x=g^{-1}yg$ for $g\in G(M)$. For $X\in G(M)$, we define a "norm" N by

$$NX=X^{\sigma}X\cdots {}^{\sigma^{l-1}}X.$$

For a divisor n of l, we define N_1, N_2 by

$$N_1X=X^{\sigma}X\cdots {}^{\sigma^{n-1}}X.$$

$$N_2X=X^{\eta}X\cdots {}^{\eta^{l/n-1}}X.$$

with $\eta=\sigma^n$. Then we have

Proposition 1.1. *Let X, Y, $g\in G(M)$. Then*

(1) $NX=N_2(N_1(X))$;

(2) ${}^{\sigma}N_1X\underset{\eta}{\sim}N_1X$, *in particular for* $n=l$, ${}^{\sigma}NX\sim NX$;

(3) $N_1(g^{-1}X^{\sigma}g)=g^{-1}NX^{\eta}g$, *in particular for* $n=l$, $N(g^{-1}X^{\sigma}g)=g^{-1}(NX)g$.

Proof. The assertion (1) can be checked directly, and (2) and (3) follow from $X^{\sigma}(N_1X)^{\eta}X^{-1}=N_1X$ and $N_1(g^{-1}X^{\sigma}g)=g^{-1}X^{\sigma}g^{\sigma}(g^{-1}X^{\sigma}g)\cdots {}^{\sigma^{n-1}}(g^{-1}X^{\sigma}g)=g^{-1}(N_1X)^{\eta}g$.

From this, we see easily

Corollary 1.2. (1) *Let $G(M)/\underset{\sigma}{\sim}$ and $G(M)/\sim$ be the sets of equivalence classes with respect to $\underset{\sigma}{\sim}$ and $\sim$ respectively. Then N_1 and N induce maps*

$$N_1\colon G(M)/\underset{\sigma}{\sim} \longrightarrow (G(M)\underset{\eta}{\sim})^{\sigma},$$

$$N\colon G(M)/\underset{\sigma}{\sim} \longrightarrow (G(M)/\sim)^{\sigma}.$$

(2) *For $X\in G(M)$, let $C_{\sigma}(X)$, $C_{\eta}(X)$ be the σ and η-conjugacy classes containing X and $C(x)$ the conjugacy class containing x. Then N_1 and N induce surjective maps*

$$N_1\colon C_{\sigma}(X) \longrightarrow C_{\eta}(N_1X),$$

$$N\colon C_{\sigma}(X) \longrightarrow C(NX).$$

For $h\in G(M)$, we define the maps σ_h, N_h of $G(M)$ to $G(M)$ by

$${}^{\sigma_h}X=h^{\sigma}Xh^{-1},$$

$$N_h(X)=X^{\sigma_h}X\cdots {}^{\sigma_h^{l-1}}X.$$

Then we have ${}^{\sigma_h}(Nh)=Nh$. For $x\in G(M)$, let G_x denote the centralizer of x in G. Then we have

Proposition 1.3. *Let $X\in G(M)$. Then*
(1) $N_h(Xh^{-1})=NX(Nh)^{-1}$;
(2) *if* ${}^{\sigma_h}(NX)=NX$, Nh^{-1} *belongs to* $G_{NX}(M)$.

Proof. The assertion (1) can be verified directly, and (2) follows from $NX=h^{\sigma}(NX)h^{-1}=hX^{-1}(NX)Xh^{-1}$.

The formula (1) was remarked by Hijikata. If $h^{\sigma}xh^{-1}=x$, $Nh\in G_x(M)$ and σ_h gives rise to an automorphism of $G_x(M)$. Furthermore, if Nh belongs to the center of G_x, in particular, if $Nh=x$, the order of the automorphism is finite, and when M is a field, G_x has a K-structure so that x, $Nh\in G_x(K)$. The automorphism σ_h is nothing but the action of $\sigma\in\mathrm{Gal}(M/K)$ with respect to this K-structure, and N_h is the norm for this action.

Proposition 1.4. *Assume $x=NX$, and let $\mathfrak{g}$ be the group of automorphisms of $G_x(M)$ generated by σ_X. Then the set of σ-conjugacy classes in $N^{-1}(C(x))$ is in one to one correspondence with $H^1(\mathfrak{g}, G_x(M))$.*

Proof. If $NX\sim NY$, that is, $NX=g^{-1}(NY)g$ for $g\in G(M)$, then $NX=N(g^{-1}Y{}^{\sigma}g)$. Hence the inclusion induces a bijection $\{Y\mid YN=x\}/\underset{\sigma}{\sim}\to N^{-1}(C(x))/\underset{\sigma}{\sim}$. Since ${}^{\sigma_X}x=x$, $NY=x$ if and only if $YX^{-1}\in G_x(M)$ and $N_X(YX^{-1})=1$ by Prop. 1.3. For Y_1, Y_2 in $\{Y\mid NY=x\}$, we see if $Y_1=g^{-1}Y_2{}^{\sigma}g$ for $g\in G(M)$, then $g\in G_x(M)$, and $Y_1\underset{\sigma}{\sim}Y_2$ if and only if $Y_1X^{-1}=g^{-1}(Y_2X^{-1})^{\sigma_X}g$ for $g\in G_x(M)$. This proves the proposition.

Corollary 1.5. *If $H^1(\mathfrak{g}, G_x(M))=1$ for all $x\in G(M)$, then the map $N: G(M)/\underset{\sigma}{\sim}\to(G(M)/\sim)^{\sigma}$ is injective.*

Proposition 1.6. *Let M_1, m, τ be as above, and let $n=m$ in the definition of N_1, N_2, and η. Then the following assertions hold.*
(1) *The norm N_1 induces a bijection*

$$N_1: G(M)/\underset{\sigma}{\sim}\longrightarrow(G(M)/\underset{\eta}{\sim})^{\sigma},$$

and if $m=l$, N induces a bijection

$$N: G(M)/\underset{\sigma}{\sim}\longrightarrow(G(M)/)^{\sigma}.$$

(2) *The diagonal embedding $\Delta: x\to(x, \cdots, x)$ of M_1 into M induces bijections*

$$\Delta: (G(M_1)/\underset{\tau}{\sim})\longrightarrow(G(M)/\underset{\eta}{\sim})^{\sigma},$$
$$\Delta: (G(M_1)/\sim)^{\tau}\longrightarrow(G(M)/\sim)^{\sigma}.$$

Proof. (1) We note that if $n=m$, ${}^{\eta}(x_1, x_2, \cdots, x_m)=({}^{\tau}x_1, {}^{\tau}x_2, \cdots, {}^{\tau}x_m)$ for $(x_1, x_2, \cdots, x_m)\in G(M)=G(M_1)^m$. First we prove the surjectivity. For $x=(x_1, x_2, \cdots, x_m)\in G(M)$, assume ${}^{\sigma}x=g^{-1}x{}^{\tau}g$ for $g=(g_1, g_2, \cdots, g_m)\in G(M)$. Then we have $x_2=g_1^{-1}x_1{}^{\tau}g_1$,

$x_3=g_2^{-1}x_2{}^{\tau}g_2, \cdots, x_m=g_{m-1}^{-1}x_{m-1}{}^{\tau}g_{m-1}$, and $x\underset{\eta}{\sim}(x_1, x_1, \cdots, x_1)$. Since $N_1((1, 1, \cdots, x_1))=(x_1, x_1, \cdots, x_1)$, N_1 is surjective. For $X=(X_1, X_2, \cdots, X_m)$, let $g_1=1$, $g_2=X_1^{-1}$, $g_3=(X_1X_2)^{-1}, \cdots, g_m=(X_1X_2\cdots X_{m-1})^{-1}$ and $g=(g_1, g_2, \cdots, g_m)$. Then we have $g^{-1}X^{\sigma}g=(1, \cdots, 1, X_1X_2\cdots X_m)$. To prove the injectivity, it is enough to show $N_1X=N_1Y$ implies $X\underset{\sigma}{\sim}Y$ for $X=(1, \cdots, 1, \cdots, X_1)$ and $Y=(1, 1, \cdots, Y_1)$. But this is obvious, because $N_1X=(X_1, \cdots, X_1)$ and $N_1Y=(Y_1, \cdots, Y_1)$.

(2) The injectivity is obvious. The surjectivity of the first map follows from the proof of (1). Assume ${}^{\sigma}X=g^{-1}Xg$ for $X=(X_1, X_2, \cdots, X_m)$ and $g=(g_1, \cdots, g_m)$. Then we have $X_2=g_1^{-1}X_1g_1$, $X_3=g_2^{-1}X_2g_2, \cdots, X_m=g_{m-1}^{-1}X_{m-1}g_{m-1}$, and $X\sim(X_1, \cdots, X_1)$. This completes the proof.

In the rest of this section, we assume M is a cyclic extension of K.

Proposition 1.7. *For a unipotent element x in $G(K)$, there exists $X\in G(K)$ such that $X^l=x$, and x is contained in $N(G(M))$.*

Proof. We may assume $x\neq 1$. Then the Zariski closure of the group generated by x is isomorphic to G_a (cf. Remark in §7.4 of [1]). Our assertion follows from this.

For $x\in G(M)$, we denote by x_s, x_u the semi-simple, unipotent parts of x in the Jordan decomposition.

Proposition 1.8. *Let $x=x_sx_u$ be the Jordan decomposition of x. Then the following assertions hold.*

(1) *If $x\in N(G(M))$, then $x_s\in N(G(M))$.*

(2) *Assume $x_s=NY$ for $Y\in G(M)$. Then $x\in N(G(M))$ if and only if $x_u\in N_Y(G_{x_s}(M))$.*

Proof. Let $x=NX$ for $X\in G(M)$. Then ${}^{\sigma X}x=x$, and ${}^{\sigma X}x_s=x_s$, ${}^{\sigma X}x_u=x_u$ by the uniqueness of the Jordan decomposition. Hence x_s and x_u are contained in the set $G_x(K)$ of K-valued points of G_x. By Prop. 1.7, there exists $Y\in G_x(K)$ such that $N_X(Y)=x_u^{-1}$. By (1) of Prop. 1.3, we see $N(YX^{-1})=N_X(YXX^{-1})NX=x_u^{-1}x=x_s$. This proves (1).

(2) If $x_u=N_Y(Z)$ for $Z\in G_{x_s}(M)$, then $N(ZY)=N_Y(Z)NY=x_ux_s=x$. Conversely, if $x=NX$, then ${}^{\sigma X}(NY)=NY$, and $YX^{-1}\in G_{x_s}(M)$, therefore $XY^{-1}\in G_{x_s}(M)$ by (2) of Prop. 1.3. We see $N_Y(XY^{-1})=NX(NY)^{-1}=xx_s^{-1}=x_u$. This completes the proof.

§2. Hasse principle for σ-conjugacy

Let K be a finite algebraic number field, and M a cyclic extension of K of degree l. Let A be a semi-simple algebra over K and G the algebraic group over K such that $G(K)=A^{\times}$. We fix a generator σ of $\mathrm{Gal}(M/K)$, and consider the σ-conjugacy and the norm N in $G(M)$.

For a place v of K. let K_v be the completion of K at v and $M_v=M\otimes_K K_v$. Then we can extend σ to M_v by $\sigma\otimes id$. The field K_v and the K_v-algebra M_v satisfy the condition for K and M in §1. Hence we can define the σ-conjugacy and the norm in

$G(M_v)$. We denote them by $\underset{\sigma,v}{\sim}$ and the same letter N as in the global case. The usual conjugacy in $G(M_v)$ will be denoted by $\underset{v}{\sim}$. A class in $G(M)/\sim$ determines a class in $G(M_v)/\underset{v}{\sim}$ by the inclusion, and we have an injection $G(M)/\sim\rightarrow\prod_v G(M_v)/\underset{v}{\sim}$. In these notations, our main result is

Theorem 2.1. *The norm induces a bijection*

$$N\colon G(M)/\underset{\sigma}{\sim} \longrightarrow (G(M)/\sim)^{\sigma}\cap(\prod_v N(G(M_v))/\underset{v}{\sim}),$$

where the product is extended over all places of K.

By (2) of Cor. 1.2, we obtain

Corollary 2.2. *For $x\in G(M)$, if ${}^{\sigma}x\sim x$ and $x\in N(G(M_v))$ for all places v of K, there exists $X\in G(M)$ such that $NX=x$.*

As in the case of the usual conjugacy, a class in $G(M)/\underset{\sigma}{\sim}$ determines a class in $G(M_v)/\underset{\sigma,v}{\sim}$ for each v, and the diagram

$$\begin{array}{ccc} G(M)/\underset{\sigma}{\sim} & \xrightarrow{\quad N \quad} & G(M)/\sim \\ \downarrow & & \downarrow \\ \prod_v (G(M_v)/\underset{\sigma,v}{\sim}) & \xrightarrow{\quad N \quad} & \prod_v (G(M_v)/\underset{v}{\sim}) \end{array}$$

is commutative. Since both of the maps $N\colon G(M)/\underset{\sigma}{\sim}\rightarrow G(M)/\sim$ and $G(M)/\sim\rightarrow\prod_v(G(M_v)/\underset{v}{\sim})$ are injective, we obtain another type of Hasse principle for σ-conjugacy.

Proposition 2.3. *The natural map*

$$G(M)/\underset{\sigma}{\sim} \longrightarrow \prod_v(G(M_v)/\underset{\sigma,v}{\sim})$$

is injective. Namely, for $X, Y\in G(M)$, $X\underset{\sigma}{\sim}Y$ if and only if $X\underset{\sigma,v}{\sim}Y$ for all places v of K.

The proof of the injectivity in Th. 2.1 is easy. Let $A_M=A\otimes_K M$. Then $G(M)=A_M^{\times}$. For $x\in G(M)$, we put

$$A_x^M=\{z\in A_M \mid xz=zx\}.$$

Then A_x^M is an M-algebra and $G_x(M)=(A_x^M)^{\times}$. If $x=NX$ for $X\in G(M)$, $A_x=(A_x^M)^{\sigma_X}$ is a K-algebra and $A_x^M=A_x\otimes_K M$. It is well known that $H^1(\mathfrak{g}, (A_x\otimes_K M)^{\times})=1$, where $\mathfrak{g}$ is the group generated by σ_X. This proves the injectivity by Cor. 1.5.

We reduce the proof of the surjectivity to the special cases where x is semi-simple or unipotent. Let $x=x_s x_u$ be the Jordan decomposition and assume the conjugacy class of x is contained in the image of the norm for all places of K. Then by (2) of Cor. 1.2, $x\in N(G(M_v))$ for all places v of K. Let $M_v=M_1^m$ for a field M_1 and an integer m. We denote by τ the element of $\mathrm{Gal}(M_1/K_v)$ induced by σ^m. Let η, N_1 and N_2 be those defined for $n=m$. Then the following diagram is commutative.

$$\begin{array}{ccccc} G(M_v)/\underset{\sigma}{\sim} & \xrightarrow[\sim]{N_1} & G(M_v)/_\eta & \xrightarrow{N_2} & G(M_v)/\sim \\ & & \Delta\uparrow\wr & & \Delta\uparrow\wr \\ & & G(M_1)/\underset{\tau}{\sim} & \longrightarrow & G(M_1)/\sim \end{array}$$

By Prop. 1.6 and (1) of Prop. 1.8, $x_s \in N(G(M_v))$ for all v. By Prop. 3.1 (proved in §3), there exists $Y \in G(M)$ such that $x_s = NY$. By the proof of Prop. 1.8, we see $x_u \in N_Y(G_{x_s}(M_v))$ for all v. Since x_s is semi-simple, $G_{x_s}(K) = A_{x_s}^{\times}$ for a semi-simple algebra A_{x_s} over K. By Prop. 4.1 (proved in §4), we see there exists $Z \in G_{x_s}(M)$ such that $N_Y(Z) = x_u$ and again by (2) of Prop. 1.8, there exists $X \in G(M)$ such that $NX = x$. Thus the proof will be completed.

§ 3. Semi-simple case

Let the notation be as in §2. Throughout this section, we assume x is semi-simple, and we will prove the following special case of Th. 2.1.

Proposition 3.1. *For $x = x_s \in G(M)$, assume ${}^\sigma x \sim x$ and $x \in N(G(M_v))$ for all places v of K. Then one has $x \in N(G(M))$.*

We reduce the proof to the case where x is a regular element.

Lemma 3.2. *Assume ${}^\sigma x \sim x$. Then there exists $Y \in G(M)$ such that $Y\,{}^\sigma x Y^{-1} = x$, and NY and $x(NY)^{-1}$ are regular semi-simple elements.*

Proof. Let h be an element of $G(M)$ satisfying $h\,{}^\sigma x h^{-1} = x$ and let

$$Z_\sigma = \{ g \in G \mid g\,{}^\sigma x g^{-1} = x \}.$$

Then we have $Z_\sigma = G_x h = h\,{}^\sigma G_x$. Since G_x and $R_{M/K}(G_x)$ are connected, $R_{M/K}(G_x)(K)$ is Zariski dense in $R_{M/K}(G_x)$. Hence $R_{M/K}(Z_\sigma)(K)$ is Zariski dense in $R_{M/K}(Z_\sigma)$. We define a morphism $\tilde{N}$ of $R_{M/K}(Z_\sigma)$ to G as the composite of $R_{M/K}(Z_\sigma) \cong Z_\sigma \times {}^\sigma Z_\sigma \times \cdots \times {}^{\sigma^{l-1}} Z_\sigma \hookrightarrow G \times G \times \cdots \times G$ (the product of l copies of G)$\to G$. The last morphism is given by the multiplication in G. Then $\tilde{N}$ is defined over M, and its image is contained in G_x, because the condition $y_1\,{}^\sigma x y_1^{-1} = x$, $y_2\,{}^{\sigma^2} x y_2^{-1} = {}^\sigma x$, $\cdots$, $y_l\,{}^{\sigma^l} x y_l = {}^{\sigma^{l-1}} x$ implies $y_1 y_2 \cdots y_l x (y_1 y_2 \cdots y_l)^{-1} = x$. Hence $\tilde{N}$ gives rise to a morphism of $R_{M/R}(Z_\sigma)$ to G_x. For $z \in G_x$, put $\tilde{z} = (z(Nh)^{-1}h, {}^\sigma h, \cdots, {}^{\sigma^{l-1}} h)$. Then $\tilde{z} \in Z_\sigma \times {}^\sigma Z_\sigma \times \cdots \times {}^{\sigma^{l-1}} Z_\sigma$. Let $\tilde{z}'$ be the point of $R_{M/K}(Z_\sigma)$ corresponding to $\tilde{z}$ by the above isomorphism. Then $\tilde{N}(\tilde{z}') = z$. This shows $\tilde{N}$ is surjective. We note that if $z' \in R_{M/K}(Z_\sigma)(K)$ corresponds to $z \in Z_\sigma(M)$ under the morphism $R_{M/K}(Z_\sigma) \to Z_\sigma$, then $\tilde{N}(z') = N(z)$, and $\tilde{N}(R_{M/K}(Z_\sigma)(K)) = N(Z_\sigma(M))$.

Let S be the set of regular semi-simple elements in G. Then $G_x \cap S \cap Sx$ is a Zariski open subset of G_x defined over M(cf. [5], [7]). Let T be a maximal torus of G containing x. Then $T \subset G_x$ and for $y \in T$, $y \in G_x \cap S \cap Sx$ if and only if $\alpha(y) \neq 1$, $\alpha(yx^{-1}) \neq 1$ for every root α relative to T (cf. Prop. 3 in 3.5 of [7]). The set of such y is not empty. Hence $G_x \cap S \cap Sx$ is a non-empty open subset of G_x, and $\tilde{N}^{-1}(G_x \cap S \cap Sx)$ is also a non-empty open subset of $R_{M/K}(Z_\sigma)$. Since $R_{M/K}(Z_\sigma)(K)$ is dense in $R_{M/K}(Z_\sigma)$, there exists $Y\in Z_{\sigma}(M)$ such that $NY\in S$ and $NY\in Sx$. This Y satisfies the condition in our Lemma.

We note this lemma holds if G_x is connected, for example, if G_{der} is simply connected (cf. [2], [6]).

Let Y be as in Lemma 3.2, and let $y=NY$. By the remark after Prop. 1.3, G_y has a K-structure such that $y\in G_y(K)$. We see $x\in G_y(K)$, because ${}^{\sigma_Y}x=Y^{\sigma}xY^{-1}=x$.

Now we prove

Lemma 3.3. *Let x, Y, and y be as above. Then the following assertions hold.*

(1) *There exists $X\in G(M)$ such that $NX=x$ if and only if there exists $Z\in G_y(M)$ such that $N_Y(Z)=xy^{-1}$.*

(2) *For a place v of K, there exists $X_u\in G(M_v)$ such that $NX_v=x$ if and only if there exists $Z_v\in G_y(M_v)$ such that $N_Y(Z_v)=xy^{-1}$.*

Proof. We give a proof for (1). The assertion (2) can be proved in the same way. Let $NX=x$ for $X\in G(M)$. Then by Prop. 1.3, we have $N_Y(Z)=xy^{-1}$ for $Z=XY^{-1}$. We show $Z\in G_y(M)$. Since $N_Y(Z)=xy^{-1}$ and ${}^{\sigma_Y}(xy^{-1})=xy^{-1}$, $xy^{-1}=N_Y(Z)=Z^{\sigma_Y}(N_Y(Z))Z^{-1}=Z^{\sigma_Y}(xy^{-1})Z^{-1}=Zxy^{-1}Z^{-1}$. We see also that $ZxZ^{-1}=x$, because $Y^{\sigma}xY^{-1}=x$ and $X^{\sigma}xX^{-1}=x$. Hence we have $Zy^{-1}Z^{-1}=y^{-1}$ and $Z\in G_y(M)$. This completes the proof.

We note this lemma holds without any assumptions on G.

Let y be as above and put

$$A_y=\{a\in A\otimes_K M \mid ay=ya,\ {}^{\sigma_Y}a=a\}.$$

Then A_y is a commutative semi-simple algebra over K and $G_y(K)=A_y^{\times}$. By the above lemma, to prove Prop. 3.1, it is enough to show it in the case where A is a commutative semi-simple algebra over K. Such A is a direct product of finite extensions of K. Hence the proof of Prop. 3.1 is reduced to the following lemma.

Lemma 3.4. *Let S be a finite extension of K, and for $x\in(S\otimes_K M)^{\times}$ (resp. $(S\otimes_K M_v)^{\times}$), put $Nx=\prod_{i=1}^{l}{}^{\sigma^{i-1}}x$, where ${}^{\sigma}x=s\otimes{}^{\sigma}m$ for $s\otimes m\in S\otimes_K M$ (resp. $S\otimes_K M_v$). Let $x\in S^{\times}$. If $x\in(S\otimes_K M_v)^{\times}$ for all places v of K, then $x\in N(S\otimes_K M)^{\times}$.*

Proof. There exists a cyclic extension T of S of degree l/m for a divisor m of l such that $S\otimes_K M$ is isomorphic to the m-fold product of T, and the action of σ is given by ${}^{\sigma}(x_1, x_2, \cdots, x_m)=(x_2, x_3, \cdots, x_m, {}^{\tau}x_1)$ for a generator τ of Gal(T/S). Under this isomorphism, the subset S of $S\otimes_K M$ can be identified with the set $\{(x, x, \cdots, x) \mid x\in S\}$. For a place v of K, $S\otimes_K M_v$ is isomorphic to the m-fold product of $T_v=T\otimes_K K_v$ and the action of σ is given by the same formula as above. The assumption implies that there exists $X_{1,v}, X_{2,v}, \cdots, X_{m,v}\in T_v$ such that $N((X_{1,v}, X_{2,v}, \cdots, X_{m,v}))=(N_{T/S}(X_{1,v}\cdots X_{m,v}), \cdots, N_{T/S}(X_{1,v}\cdots X_{m,v}))$, where $N_{T/S}$ denotes the norm for T/S and its extension to T_v/S_v. The Hasse principle for the cyclic extension T/S asserts that there exists $X_1\in T^{\times}$ such that $N_{T/S}(X_1)=x$. Define an element X of $S\otimes_K M$ by $X=(X_1, 1, \cdots, 1)$. Then we see $NX=x$. This completes the proof of Lemma 3.4 and that of Prop. 3.1.

§ 4. Unipotent case

Let the notation be as in § 2. In this section, we will prove the following special case of Th. 2.1 and completes the proof of Th. 2.1.

Proposition 4.1. *Let* $x=x_u$, *and assume* $x\sim^v x$ *and* $x\in N(G(M_v))$ *for all places* v *of* K. *Then one has* $x\in N(G(M))$.

Proof. Let h be an element of $G(M)$ such that $h^\sigma x h^{-1}=x$. Let $y=Nh$. Then $H=G_y$ has a K-structure, and $x, y\in G_y(K)$. Hence H_x is also defined over K, and $x, y\in H_x(K)$. Let $N(X_v)=x$ for $X_v\in G(M_v)$. Then $N_h(X_v h^{-1})=xy^{-1}$ by Prop. 1.3. In the same way as in the proof of Lemma 3.3, we see $X_v h^{-1}\in G_y(M_v)\cap G_x(M_v)=H_x(M_v)$ and $xy^{-1}\in N_h(H_x(M_v))$. Let $y=y_s y_u$ be the Jordan decompositon of y. Then xy_u^{-1} is unipotent and $xy^{-1}=y_s^{-1}(xy_u^{-1})$ gives the Jordan decomposition of xy^{-1}. Let $H_x=L\cdot R_u(H_x)$ be a Levi decomposition of H_x with a reductive group L. Then $x_u y_u^{-1}\in R_u(H_x)(K)$ and we may take L so that $y_s^{-1}\in L(K)$. Since $y_s^{-1}(x_u y_u^{-1})\in N(H_x(M_v)), y_s^{-1}\in N(L(M_v))$. Now $L(K)=C^\times$ for a semi-simle algebra C over K. By Prop. 3.1, there exists $Y_s\in L(M)$ such that $N_h(X_s)=y_s^{-1}$. On the other hand, the element $y_u^{-1}x_u$ is contained in the center of H_x and is unipotent. Hence there exists Y_u in the center of H_x such that ${}^{\sigma h}(Y_u)=Y_u$ and $Y_u^l=y_u^{-1}x_u$. Let $Y=Y_s Y_u$. Then $N_h(Y)=N_h(Y_s)N_h(Y_u)=xy^{-1}$ and $N(Yh)=N_h(Y)Nh=xy^{-1}y=x$. This completes the proof.

DEPARTMENT OF MATHEMATICS
COLLEGE OF GENERAL EDUCATION
KYOTO UNIVERSITY

References

[1] A. Borel, Linear algebraic groups, W. A. Benjamin Inc., New York, 1969.
[2] R. Kottwitz, Rational conjugacy classes in reductive algebraic groups, Duke Math. J., **49** (1982), 785-806.
[3] R. P. Langlands, Base change for GL(2), Ann. of Math. Studies, No. 96, Princeton Univ. Press, 1980.
[4] H. Saito, Automorphic forms and algebraic extensions of number fields, Lectures in Math., Kyoto Univ., 1975.
[5] R. Steinberg, Regular elements of semisimple algebraic groups, Publ. Math. I. H. E. S., **25** (1965), 49-80.
[6] R. Steinberg, Endomorphisms of linear algebraic groups, Memoirs of Amer. Math. Soc., Providence, R. I., **80** (1968).
[7] R. Steinberg, Conjugacy classes in algebraic groups, Lecture Notes in Math., vol. 366, Springer-Verlag, Berlin, New York, Heidelberg, 1974.