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On the strongly hyperbolic systems II

—A reduction of hyperbolic matrices—

By

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§1. Introduction

This article is the continuation of the previous paper [12]. We shall study the strongly hyperbolic systems ($m\times m$-matrix) in more general cases.

Let $\Omega=(-T, T)\times R_x^l$ and we shall consider the Cauchy problem:

$$(1.1)\qquad \begin{cases} L[u]=\partial_t u-\sum_{k=1}^{l} A_k(t, x)\partial_{x_k}u-B(t, x)u=0 & \text{on } \Omega\,, \\ u(t_0, x)=u_0(x)\,, \qquad -T<t_0<T\,, \end{cases}$$

where $u(t, x)$ and $u_0(x)$ are m-vectors.

We consider (1.1) in the C^∞-category. Let $L_0=\partial_t-\sum_{k=1}^{l} A_k(t, x)\partial_{x_k}$, then we say that L_0 is a strongly hyperbolic system when the Cauchy problem (1.1) is uniformly C^∞-wellposed for any lower order term $B(t, x)$. For details see [12].

When the coefficients $A_x(t, x)$ are constant or the multiplicites of the characteristic roots of $A(t, x;\xi)=\sum_{k=1}^{l} A_k(t, x)\xi_k$ are constant for any $(t, x;\xi)\in\Omega\times R_\xi^l\setminus\{0\}$, we know the necessary and sufficient conditions for L_0 to be a strongly hyperbolic system ([3], [5]). On the other hand if we do not impose the assumptions on the characteristic roots in the case of variable coefficients, the situation will be much more complicated.

In [12] the author gave a necessary condition without any assumptions of the characteristic roots. But, in it, we assumed that the rank of $(\lambda I-A(t, x;\xi))=m-1$, where $\det(\lambda I-A(t, x;\xi))=0$. And the necessary condition for L_0 to be a strongly hyperbolic system was that the multiplicities of the characteristic roots are at most double at every point $(t, x;\xi)$.

It seems that the difficulties specific for systems will be appear when we drop the above assumption of rank. And instead of the above condition, if L_0 is a strongly hyperbolic system then it will hold that the orders (sizes) of the Jordan's blocks for any characteristic roots must be at most two at any point $(t, x;\xi)$. We will prove the above result in some restricted cases. Moreover when the orders of the Jordan's blocks are equal to two at a certain point we can give the following example.

Example. $L_0=\partial_t-A(t)\partial_x \qquad (l=1)$,

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$$A(t)=\begin{bmatrix} 0 & 1 & 0 & 0 \\ t^2 & 0 & t^p & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2t^2 & 0 \end{bmatrix} \qquad (m=4),$$

where p is a positive integer.

For this L_0 we can prove that L_0 is a strongly hyperbolic system if and only if $p\geqq 2$.

Now turoughont this paper we assume that the coefficients $A_k(t, x)$ and $B(t, x)$ depend only on time variable t. And instead of (1, 1) we study the Cauchy problem for the ordinary differential system depending on a parameter $\xi\in R^l_\xi$:

$$\text{(1.2)} \qquad \begin{cases} \partial_t\hat{u}(t;\xi)=(iA(t;\xi)+B(t))\hat{u}(t;\xi) \\ \hat{u}(t_0;\xi)=\hat{u}_0(\xi), \qquad -T<t_0<T. \end{cases}$$

where we denote the Fourier transform of $f(t, x)$ with respect to space variables x by $\hat{f}(t;\xi)$.

As is well-known, the following theorem is due to I. G. Petrowsky.

Theorem 1.1. ([11]) *In order that the Cauchy problem* (1.1) *is uniformly C^∞-well-posed it is necessary and sufficient that the following ineqaulity holds: There exist constants C and M independent of the initial plane $t=t_0$, which satisfy*

$$\text{(1.3)} \qquad |\hat{u}(t;\xi)|\leqq C(1+|\xi|)^M|\hat{u}_0(\xi)|,$$

for the solution $\hat{u}(t;\xi)$ of (1.2).

Owing to the above theorem our problem is now to find the condition on $A(t;\xi)=\sum_{k=1}^{l}A_k(t)\xi_k$ in order that the inequality (1.3) holds for any lower order term $B(t)$ in (1.2).

Since our purpose in this article is to give its necessary conditions pointwisely, we remark that the following assumption is not any restrictions of our situation. (See [4], [7], [12].)

(A) There exist $t_0\in(-T, T)$ and $\xi^0\in R^l_\xi\setminus\{0\}$ $(|\xi^0|=1)$ such that the eigenvalues of $A(t_0;\xi_0)$ are equal to zero.

Moreover without loss of generality we can take $t_0=0$. Thus throughout this paper we assume the assumption (A) with $t_0=0$ without mention of it.

Let $J=J(1)\oplus J(2)\oplus\cdots\oplus J(\nu)$ be a Jordan's form similar to $A(0;\xi^0)$ where $J(i)$ is the Jordan's block of order r_i with null eigenvalue $(r_1\geqq r_2\geqq\cdots\geqq r_\nu)$.

$$J(i)=\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix} \qquad (r_i\times r_i\text{-matrix})$$

We denote the largest order of Jordan's blocks by r, and the number of the Jordan's blocks of order r is denoted by s. Let $A(t;\xi^0)=A(0;\xi^0)+tA_1+t^2A_2(t)$ (Taylor expansion) and denote the (i, j)-entry of A_1 by $a_{i,j}$. We consider the $s\times s$-matrix K whose (i, j)-entry is $a_{ir,(j-1)r+1}$ $(1\leqq i,\ j\leqq s)$.

Now in order to propose our theorems we must impose some assumptions.

(A.1) The Jordan's form J does not contain the Jordan's blocks of order $r-1$.
(A.2) ν is at most 3.

Theorem 1.2. *Assume* (A.1) *or* (A.2). *If* $r\geqq 3$ *then* L_0 *is not a strongly hyperbolic system.*

Theorem 1.3. *Assume* (A.1) *or* (A.2). *Moreover assume that* K *is not null matrix. If* $r\geqq 2$ *then* L_0 *is not a strongly hyperbolic system.*

For the proofs of these theorems we employ a reduction of hyperbolic matrices. Here we say that a matrix $A(y)$ depending on $y\in\mathcal{O}$, $\mathcal{O}$ being an open set, is a hyperbolic matrix in $\mathcal{O}$ when the eigenvalues of $A(y)$ are real for any $y\in\mathcal{O}$. In § 2 we shall explain a reduction in general cases. Owing to this reduction we shall prove Theorem 1.2 and Theorem 1.3 under the assumption (A.1) in § 3. Theorem 1.2 and Theorem 1.3 under the assumption (A.2) will be proved in § 4.

§ 2. A reduction of hyperbolic matrices in general cases

We start with the system (1.2). Let $\xi=n\xi^0$, where n is a large parameter. Then our system can be written as follows.

$$\partial_t\hat{u}=(inA(t)+B)\hat{u}\,, \tag{2.1}$$

where $A(t)=A(t;\xi^0)=A_0+tA_1+t^2A_2(t)$. As is well-known the following proposition is due to Lax-Mizohata's Theorem ([6], [8]).

Proposition 2.1. *In order that the Cauchy problem* (1.1) *is uniformly* C^∞*-wellposed* $A(t)$ *must be a hyperbolic matrix, i.e., the characteristic roots of* $A(t)$ *are real for any* t.

Let us take the asymptotic transformation $t=n^{-\sigma}s$, where σ is a constant which will be determined afterward. And denote the non-singular matrix which transforms A_0 into the Jordan's form J by N_0, then (2.1) becomes

$$n^\sigma\partial_s v=(in(J+n^{-\sigma}\tilde{A}_1(s)+n^{-2\sigma}\tilde{A}_2(s\,;\,n))+\tilde{B})v\,, \tag{2.2}$$

where $v=N_0\hat{u}$, $\tilde{A}_1(s)=sN_0A_1N_0^{-1}$, $\tilde{A}_2(s\,;\,n)=s^2N_0A_2(n^{-\sigma}s)N_0^{-1}$ and $\tilde{B}=N_0BN_0^{-1}$.

For simplicity we only denote $\tilde{A}_1(s)$, $\tilde{A}_2(s\,;\,n)$ and $\tilde{B}$ by $A_1(s)$, $A_2(s\,;\,n)$ and B respectively.

$$n^\sigma\partial_s v=(in(J+n^{-\sigma}A_1(s)+n^{-2\sigma}A_2(s\,;\,n))+B)v\,. \tag{2.2$'$}$$

Our starting point is actually (2.2)'. We consider (2.2)' for $s\geqq 1$ and our purpose is to choose the lower order term B and the initial data at $s=s_0$ $(t=n^{-\sigma}s_0)$ which lead that the Cauchy problem is not C^∞-wellposed.

Definition 2.1. We say a $m\times m$-matrix A is (S)-type with respect to Jordan's form J when all the components of A except for $r_1+r_2+\cdots+r_i$-th row vector are zero $(1\leqq i\leqq\nu)$, where we used the notations in § 1.

Lemma 2.1. *Let $C(n)=J+C_1(n)$ be a $m\times m$-matrix which depends on a parameter n, where $C_1(n)$ is (S)-type with respect to J and can be expressed as follows:*

$$C_1(n)=n^{-\alpha_1}C_1+n^{-\alpha_2}C_2+\cdots \pmod{n^{-\infty}} \qquad as\ n\to\infty\,.$$

where $0<\alpha_1<\alpha_2<\cdots$. Then there exist a positive constant ε_0, a diagonal matrix $W(n)$, a constant matrix $\tilde{C}_1$ and a $\tilde{C}_2(n)$ which satisfy:

(i) $W(n)C(n)W(n)^{-1}=n^{-\varepsilon_0}(J+\tilde{C}_1+\tilde{C}_2(n))$,

(ii) $C_2(n)=o(1)$ *as* $n\to\infty$.

Apply this lemma to (2.2)′. Let $\tilde{v}=Wv$ then we can see

$$n^\sigma\partial_s\tilde{v}=(in^{1-\varepsilon_0}(J+\tilde{A}_1(s)+\tilde{A}_2(s\,;n))+\tilde{B})\tilde{v}\,. \tag{2.3}$$

Secondly, keeping in mind the change of variable $t=n^{-\sigma}s$ we propose the following

Lemma 2.2. *Assume $C(t)$ is a hyperbolic matrix in a neighborhood of $t=0$ and $C(0)$ is nilpotent. And assume that there exist an interval I_s, a positive constant ε and a non-singular matrix $N(s\,;n)$ such that*

$$N(s\,;n)C(n^{-\sigma}s)N(s\,;n)^{-1}=n^{-\varepsilon}(C_0(s)+C_R(s\,;n)) \qquad for\ s\in I_s\,,$$

where $C_R(s\,;n)=o(1)$ as $n\to\infty$.

If $\varepsilon<\sigma$, then $C_0(s)$ must be nilpotent for all $s\in I_s$.

Thirdly we employ the following

Lemma 2.3. *Let $C(s\,;n)$ be a $m\times m$-matrix given on an open interval I_s and n is a large parameter. Assume that $C(s\,;n)$ has the following expansion:*

$$C(s\,;n)=C_0(s)+n^{-\alpha_1}C_1(s)+n^{-\alpha_2}C_2(s)+\cdots \pmod{n^{-\infty}}\,,$$

for all $s\in I_s$ $(0<\alpha_1<\alpha_2<\cdots)$. Moreover assume that $C_0(s)$ is nilpotent for all $s\in I_s$, then there exist an open interval $\tilde{I}_s$ and a $m\times m$-matrix $N(s\,;n)$ which satisfies that

(i) $\det N(s\,;n)\neq 0$ *for all $s\in\tilde{I}_s$ and for large n.*

(ii) $N(s\,;n)C(s\,;n)N(s\,;n)^{-1}=J+\tilde{C}_1(s\,;n)$,

where J is a Jordan's form and $\tilde{C}_1(s\,;n)=o(1)$. Moreover $\tilde{C}_1(s\,;n)$ is (S)-type with respect to J.

Owing to these lemmas we can deform (2.3). At first, if $\varepsilon_0<\sigma$ in (2.3) then

$J+\tilde{A}_1(s)$ is nilpotent. So we can apply Lemma 2.3. Set $\tilde{\tilde{v}}=N(s;n)\tilde{v}$ then

$$n^{\sigma}\partial_s\tilde{\tilde{v}}=(in^{1-\varepsilon_0}(J_1+\tilde{\tilde{A}}_1(s;n))+\tilde{\tilde{B}}+n^{\sigma}(N(s;n))'_s N(s;n)^{-1})\tilde{\tilde{v}}\,, \tag{2.4}$$

where $\tilde{\tilde{A}}_1(s;n)=o(1)$ and $\tilde{\tilde{B}}=N\tilde{B}N^{-1}=NWBW^{-1}N^{-1}$.

Comparing (2.2)′ and (2.4) we can repeat a reduction similar to above steps. Observing this reduction we shall choose a lower order term such that the Cauchy problem is not C^∞-wellposed. In that process, the fact that we can choose a constant σ is a crucial point for our arguments.

Our purpose is to obtain the following proposition.

Proposition 2.2. *Assume that there exist a positive constant* ε_1 *and a matrix* $\tilde{N}(s;n)$ *which is products of some matrices of type* $W(n)$ *in Lemma* 2.1 *and* $N(s;n)$ *in Lemma* 2.3, *such that when we set* $w=\tilde{N}(s;n)v$, (2.2)′ *turns into*

$$\partial_s w(s;n)=n^{\varepsilon_1}(\tilde{A}_0(s)+\tilde{A}_R(s;n))w(s;n)\,, \tag{2.5}$$

where $\tilde{A}_R(s;n)=o(1)$ *as* $n\to\infty$.

Moreover there exists s_0 *such that* $\tilde{A}_0(s_0)$ *has at least an eigenvalue whose real part is positive, then the Cauchy problem* (1.1) *is not uniformly* C^∞*-wellposed.*

Proof. This proposition is a slight modification of Lax-Mizohata's Theorem (See [10].). So we explain this briefly. At first we can take a constant matrix N_0 such that $D_0=N_0\tilde{A}_0(s_0)N_0^{-1}$ is a triangular matrix and when $i<j$, the absolute value of the (i,j)-component can be taken small as we wish. Set $\tilde{w}=N_0 w$ then we obtain

$$\partial_s\tilde{w}=n^{\varepsilon_1}(D_0+N_0(\tilde{A}_0(s)-\tilde{A}_0(s_0))N_0^{-1}+N_0\tilde{A}_R N_0^{-1})\tilde{w}\,. \tag{2.6}$$

We denote the diagonal components of D_0 as follows:

$$\begin{cases}\text{Real part of } \lambda_1, \lambda_2, \cdots, \lambda_k \geqq \delta_1\ (>0)\,,\\ \text{Real part of } \lambda_{k+1}, \cdots, \lambda_m \leqq 0\,.\end{cases}$$

And we denote a energy form by

$$\Theta_n(s)=\{\exp(-\delta_2 n^{\varepsilon_1}(s-s_0))\}\Big(\sum_{i=1}^{k}|\tilde{w}_i(s;n)|^2-\sum_{j=k+1}^{m}|\tilde{w}_j(s;n)|^2\Big)\,,$$

where we denote the components of $\tilde{w}(s;n)$ by ${}^t(\tilde{w}_1,\cdots,\tilde{w}_m)$. Then we obtain the following inequality:

$$\Theta_n(s)\geqq\{\exp(\delta_3 n^{\varepsilon_1}(s-s_0))\}\Theta_n(s_0) \qquad \text{for } s\geqq s_0\,, \tag{2.7}$$

where δ_3 is a positive constant.

On the other hand, in the Cauchy problem (1.2) we shall take the initial data $\hat{u}_0(\xi)$ at $t=n^{-\sigma}s_0$ which assures $\Theta_n(s_0)=1$. Assume that the Cauchy problem (1.1) is C^∞-wellposed, then owing to (1.3) and the property of $N(s;n)$ we obtain

$$\Theta_n(s)\leqq C_1 n^{M_1} \qquad \text{for large } n\,, \tag{2.8}$$

where C_1 and M_1 are constants which do not depend on n.

The inequalities (2.7) and (2.8) are not compatible when $s>s_0$ and n is large. Our proof is thus complete. (Q. E. D.)

At the end of this section we shall give the proofs of the lemmas which are used in this section.

Proof of Lemma 2.1. At first we shall write concretely the entries of $C_1(n)$. Let $C^{i,j}(n)$ be the (i, j)-block of $C_1(n)$ corresponding to J, and let the last row vector of $C^{i,j}(n)$ be $(c_1^{i,j}, c_2^{i,j}, \cdots, c_{r_j}^{i,j})$. Remark that other row vectors are null. Denote the order of $c_k^{i,j}(n)$ by $-p_k^{i,j}$. (When $c_k^{i,j}(n)=0$ then $p_k^{i,j}=-\infty$.) And for a positive number ε, denote that

$$\begin{cases} q^{i,j}(\varepsilon)=\max_k(-p_k^{i,j}+(r_i-k+1)\varepsilon) & \text{for } i\neq j \\ q^{i,i}(\varepsilon)=0 \end{cases}$$

Define a positive number ε_0 in the following way: $\varepsilon_0=\max\{\varepsilon_1, \varepsilon_2\}$, where

$$\begin{cases} \varepsilon_1=\max\{\varepsilon\,;\ \max_\pi \sum_{i=1}^{\nu} q^{i,\pi(i)}(\varepsilon)\leqq 0\}, \\ \varepsilon_2=\max\{\varepsilon\,;\ \max_{i,k}(-p_k^{i,i}+(r_i-k+1)\varepsilon)\leqq 0\}, \end{cases}$$

where π runs through all the permutation of $\{1, 2, \cdots, \nu\}$. Then $q^{i,j}=q^{i,j}(\varepsilon_0)$ $(i\neq j)$, $(q^{i,i}=0)$ satisfies the conditions of Volevich's Lemma. (See [9], [13].) Thus there exist numbers β_i $(1\leqq\beta_i\leqq\nu)$ such that $q^{i,j}\leqq\beta_j-\beta_i$.

When we take the diagonal matrix of weight in the following way, we can see easily that (i), (ii) in the lemma hold.

$$W(n)=n^{\beta_1}W_{\varepsilon_0,r_1}(n)\oplus n^{\beta_2}W_{\varepsilon_0,r_2}(n)\oplus\cdots\oplus n^{\beta_\nu}W_{\varepsilon_0,r_\nu}(n)\,, \tag{2.9}$$

where $W_{\varepsilon,r}=\begin{bmatrix} 1 & & & \\ & n^\varepsilon & & \\ & & \ddots & \\ & & & n^{(r-1)\varepsilon} \end{bmatrix}$. (Q. E. D.)

Proof of Lemma 2.2. Let

$$\det(\lambda I-C(n^{-\sigma}s))=\lambda^m+c_1(s\,;\,n)\lambda^{m-1}+\cdots+c_m(s\,;\,n)$$

be the characteristic polynomial of $C(t)=C(n^{-\sigma}s)$. Owing to Proposition 2.1 (See also [2].), we know that $c_k(s\,;\,n)=O(n^{-k\sigma})$ $(1\leqq k\leqq m)$. Assume that $C_0(s)$ is not nilpotent, then there exists k_0 $(1\leqq k_0\leqq m)$ such that $p_{k_0}=k_0\varepsilon$ where we denote the order of $c_k(s\,;\,n)$ by $-p_k$ $(1\leqq k\leqq m)$. Thus we are led to the inequality: $k_0\sigma\leqq p_{k_0}=k_0\varepsilon$. This is a contrudiction to our hypothesis. (Q. E. D.)

For the proof of Lemma 2.3, see the previous paper ([12]). (See also [1], [4].)

§ 3. Proo of Theorem 1.2 and 1.3 under the assumption (A.1)

We start with (2.2)′ in § 2. Actually $A_1(s)$ in (2.2)′ is equal to sA_1, where A_1 is a constant matrix of (S)-type with respect to J. (See Definition 2.1.) For $i=1$ to ν, denote the $(r_1+r_2+\cdots r_i)$-th row vector by $(a_{i,1}, a_{i,2}, \cdots, a_{i,m})$, and define the $s\times s$ matrix as follows:

$$K=\begin{pmatrix} a_{1,1}a_{1,r+1}\cdots\cdots a_{1,(s-1)r+1} \\ a_{2,1}a_{2,r+1}\cdots\cdots a_{2,(s-1)r+1} \\ \\ a_{s,1}a_{s,r+1}\cdots\cdots a_{s,(s-1)r+1} \end{pmatrix},$$

where s is the number of the largest Jordan's block of order r.

Note that on account of Lemma 2.1 and Lemma 2.2, K is nilpotent when $r\geqq 2$. Here we shall consider (2.2)′ in two cases, i.e., the case when K is null matrix and the others.

(1) The case when $K\neq 0$.

Set $\varepsilon_1=\sigma/r$ and multiply the matrix of weight W_1 of the form (2.9) to (2.2)′ in §2 from left. Actually W_1 is given as follows:

$$W_1=W_{\varepsilon_1,r}(n)\oplus\cdots\oplus W_{\varepsilon_1,r}(n)$$
$$\oplus n^{(1/2)(r-r_{s+1})\varepsilon_1}W_{\varepsilon_1,r_{s+1}}(n)\oplus\cdots\oplus n^{(1/2)(r-r_\nu)\varepsilon_1}W_{\varepsilon_1,r_\nu}(n).$$

When we set $\tilde{v}=W_1v$ the system (2.2)′ becomes

$$n^\sigma\partial_s\tilde{v}=(in^{1-\varepsilon_1}(J+sA_1^{(0)}+n^{-\varepsilon_1}sA_1^{(1)}+n^{-(3/2)\varepsilon_1}sA_1^{(2)}(n)) \tag{3.1}$$
$$+in^{1-2\sigma}W_1A_2(s;n)W_1^{-1}+W_1BW_1^{-1})\tilde{v}.$$

Remark that from our hypothesis that $K\neq 0$, $A_1^{(0)}$ is not null matrix. Now we deform $J+sA_1^{(0)}$ to a new Jordan's form.

Lemma 3.1. *Let* $s\geqq 1$. *There exists a non-singular matrix* $N(s)$ *which satisfies the following* (i)~(iii):

(i) $N(s)\,(J+sA_1^{(0)})N(s)^{-1}=J_1$ *(a new Jordan's form)*

(ii) $\left(\frac{d}{ds}N(s)\right)N(s)^{-1}$ *is a diagonal matrix,*

(iii) *the* (i,j)*-entry of* $N(s)A_1^{(1)}N(s)^{-1}$ *vanishes if the same entry of* $A_1^{(1)}$ *vanishes.*

Proof. We can construct $N(s)$ in the form $I(s)N_0$, where $I(s)$ is a diagonal matrix whose entries are powers of s, and N_0 is a constant matrix of the form: $N_0=\tilde{N}_0\oplus I_{m-sr}$, where $\tilde{N}_0$ is a non-singular $sr\times sr$-matrix which is a blockwisely scalar matrix.

It is easy to see that N_0 is commutative with J and that N_0 changes K to a Jordan's form $\tilde{K}$ derived from $N_0A_1^{(0)}N_0^{-1}$ in the same way. Thus the lemma is proved.

(Q. E. D)

Set $\tilde{\tilde{v}}=N(s)\tilde{v}$, then we see that

$$n^\sigma\partial_s\tilde{\tilde{v}}=(in^{1-\varepsilon_1}(J_1+n^{-\varepsilon_1}\tilde{A}_1^{(1)}(s)+n^{-(3/2)\varepsilon_1}\tilde{A}_1^{(2)}(s;n)) \tag{3.2}$$
$$+in^{1-2\sigma}N(s)W_1A_2(s;n)W_1^{-1}N(s)^{-1}+NW_1BW_1^{-1}N^{-1}+n^\sigma(N)'_sN^{-1})\tilde{\tilde{v}}.$$

Here we remark that the largest order of the Jordan's blocks of the new Jordan's form J_1 can be denoted by k_1r ($k_1\geqq 2$), and also remark that for $1\leqq i,\ j\leqq s_1$ all the $(ik_1r,\ 1+(j-1)k_1r)$-entry of $\tilde{A}_1^{(1)}(s)$ are zero, where s_1 is the number of Jordan's blocks of order k_1r. Keeping in mind the new Jordan's form J_1 we divide the matrix $\tilde{A}_1^{(1)}(s)$ as follows: $\tilde{A}_1^{(1)}=\tilde{A}_1^{(10)}+\tilde{A}_1^{(11)}$, where $\tilde{A}_1^{(10)}$ is composed with the $(ik_1r,\ 2+(j-1)k_1r)$-entry of $\tilde{A}_1^{(1)}(s)$ $(1\leqq i,\ j\leqq s_1)$.

For this $\tilde{A}_1^{(10)}(s)$, similarly to Lemma 3.1, we obtain

Lemma 3.2. *Let* $s\geqq 1$. *There exists a non-singular matrix* $N_1(s)$ *which satisfies the following* (i)~(iii):

(i) $N_1(s)J_1=J_1N_1(s)$.

(ii) $\left(\frac{d}{ds}\right)N_1(s)\Big)N_1(s)^{-1}$ *is a diagonal matrix.*

(iii) *The* $s_1\times s_1$-*matrix whose* (i,j)-*entry is eqaul to the* $(ik_1r,\ 2+(j-1)k_1r)$-*entry of* $N_1\tilde{A}_1^{(1)}N_1^{-1}$ $(1\leqq i,\ j\leqq s_1)$ *is a Jordan's matrix with null eigenvalue.*

In (3.2) set $\tilde{\tilde{v}}=N_1(s)\tilde{v}$ and $\tilde{\tilde{A}}_1^{(10)}=N_1\tilde{A}_1^{(10)}N_1^{-1}$ then it holds that

$$n^{\sigma}\partial_s\tilde{\tilde{v}}=(in^{1-\varepsilon_1}(J_1+n^{-\varepsilon_1}\tilde{\tilde{A}}_1^{(10)})+N_1NW_1BW_1^{-1}N^{-1}N_1^{-1}$$
$$N_1(in^{1-2\varepsilon_1}(\tilde{A}_1^{(11)}(s)+n^{-(1/2)\varepsilon_1}\tilde{A}_1^{(2)}(s\,;\,n))+in^{1-2\sigma}NW_1A_2(s\,;\,n)W_1^{-1}N^{-1}$$
$$+n^{\sigma}(N)'_sN^{-1})N_1^{-1}+n^{\sigma}(N_1)'_sN_1^{-1})\tilde{\tilde{v}} \tag{3.3}$$

Now we choose the lower order term B. On account of the constructions of $N(s)$ and $N_1(s)$ it is easy to see

Proposition 3.1. *We can take a constant matrix B which is represented as follows:*

$$N_1(s)N(s)W_1BW_1^{-1}N(s)^{-1}N_1(s)^{-1}=n^{(r-1)\varepsilon_1}\tilde{B}(s)\,,$$

where all the entries except for the $(k_1r,\ 1)$-*entry are zero, and the* $(k_1r,\ 1)$-*entry is equal to* bs^{α} (*b is an arbitrary number we can take and* α *is an integer.*)

Taking account of Proposition 3.1, we can introduce another matrix of weight corresponding with J_1. Let $\varepsilon_2=(1-\sigma)/k_1r$ and denote that

$$W_2=W_{\varepsilon_2,k_1r}(n)\oplus n^{\bar{\varepsilon}}W_{\varepsilon_2,k_1r}(n)\oplus\cdots\oplus n^{(s_1-1)\bar{\varepsilon}}W_{\varepsilon_2,k_1r}(n)$$
$$\oplus n^{\bar{\varepsilon}_{s_1+1}}W_{\varepsilon_2,\tilde{r}_{s_1+1}}(n)\oplus\cdots\oplus n^{\bar{\varepsilon}_{s_1}}W_{\varepsilon_2,\tilde{r}_{\nu_1}}(n)\,, \tag{3.4}$$

where we denot that $J_1=J_1(1)\oplus J_1(2)\oplus\cdots\oplus J_1(\nu_1)$ and denote the order of Jordan's block $J_1(i)$ by $\tilde{r}_i$ $(s_1+1\leqq i\leqq\nu_1)$. Indeed for $1\leqq i\leqq s_1$ the order is k_1r.

We will take $\bar{\varepsilon}$, $\bar{\varepsilon}_i$ $(s_1+1\leqq i\leqq\nu_1)$ in order that our argument works well.

Set $w=W_2\tilde{\tilde{v}}$, then (3.3) changes into

$$n^{\sigma}\partial_sw=(in^{1-\varepsilon_1-\varepsilon_1}(J_1+i^{-1}\tilde{B}(s))+A_R(s\,;\,n))w\,, \tag{3.5}$$

where A_R is the remainder term, in our sense, such that

(3.6) $$A_R=in^{1-2\varepsilon_1}W_2\tilde{\tilde{A}}_1^{(10)}W_2^{-1}+in^{1-2\varepsilon_1}W_2N_1(\tilde{A}_1^{(11)}(s)+n^{-(1/2)\varepsilon_1}\tilde{A}^{(2)}(s\,;\,n))N_1^{-1}W_2^{-1}$$
$$+in^{1-2\sigma}W_2N_1NW_1A_2(s\,;\,n)W_1^{-1}N^{-1}N_1^{-1}W_2^{-1}$$
$$+n^\sigma W_1(N_1(N)'_sN^{-1}N_1^{-1}+(N_1)'_sN_1^{-1})W_1^{-1}\,.$$

The $A_R(s\,;\,n)$ is evaluated as follows.

Proposition 3.2. *There exist σ, $\bar{\varepsilon}$ and $\bar{\varepsilon}_k$ $(s_1+1\leqq i\leqq \nu_1)$ in* (3.4), *a positive number ε_0 and a constant C which do not depend on n such that*

(3.7) $$|A_R(s\,;\,n)|\leqq Cn^{1-\varepsilon_1-\varepsilon_2-\varepsilon_0}\qquad \textit{for lorge } n.$$

Precisely, for (3.6) to hold σ must satisfy the following inequality.

(3.8) $$\mathrm{Max}\left(\frac{1}{2},\ \frac{s_1(k_1r-1)-r}{s_1(k_1r+k_1-1)-r}\right)<\sigma<\frac{k_1r-1}{k_1r+k_1-1}\,.$$

It is easy to see that when $r\geqq 2$ we can take σ which satisfies (3.7). Owing to Proposition 2.2 we obtain Theorem 1.3 in this case.

(2) The case when $K=0$.

At first, recall that r is the largest order of Jordan's blocks of J. In this case set $\varepsilon_1=\sigma/(r-1)$ and we use the same W_1 as one in the previous case. Let $\tilde{v}=W_1v$ then (2.2)′ in § 2 becomes

(3.9) $$n^\sigma\partial_s\tilde{v}=(in^{1-\varepsilon_1}(J+sA_1^{(0)}+n^{-(1/2)\varepsilon_1}A_1^{(1)}(s\,;\,n))+in^{1-2\sigma}W_1A_2(s\,;\,n)W_1^{-1}+W_1BW_1^{-1})\tilde{v}\,.$$

Keeping in mind the assumption that $r\geqq 3$, we can see that the matrix $J+sA_1^{(0)}$ is nilpotent (Lemma 2.1 and Lemma 2.2). The following lemma plays an important role for our proof.

Lemma 3.3. *For $s\geqq 1$ there exists $N_0(s)$ such that*

(i) $\det N_0(s)\neq 0$.

(ii) $N_0(s)(J+sA_1^{(0)})=J_1N_0(s)$ *where J_1 is a new Jordan's form.*

(iii) $\dfrac{d}{ds}(N_0(s))N_0^{-1}$ *is a diagonal matrix.*

The proof of this lemma will be given in the last part of this section. set $\tilde{\tilde{v}}=N_0(s)\tilde{v}$ then we obtain

(3.10) $$n^\sigma\partial_s\tilde{\tilde{v}}=(in^{1-\varepsilon_1}(J_1+n^{-(1/2)\varepsilon_1}\tilde{A}_1^{(1)}(s\,;\,n))$$
$$+in^{1-2\sigma}N_0W_1A_2W_1^{-1}N_0^{-1}+N_0W_1BW_1^{-1}N_0^{-1}+n^\sigma(N_0)'_sN_0^{-1})\tilde{\tilde{v}}\,.$$

Here on account of the construction of $N_0(s)$ we can give the lower order term B (constant matrix), which satisfies

(3.11) $$N_0W_1BW_1^{-1}N_0^{-1}=n^{(r-1)\varepsilon_1}\tilde{B}(s)\,,$$

where the $(r, 1)$-entry or the $(2r-2, r-1)$-entry of $\tilde{B}(s)$ does not vanish and the non-zero component can be taken arbitrarily.

Observing the above fact set $\varepsilon_2=1/r(1-r\sigma/(r-1))$ and define that $W_2=W_{\varepsilon_2,R_1}(n)\oplus\cdots\oplus W_{\varepsilon_2,R_{\nu_2}}(n)$, where we denoted the order of the Jordan's block $J_1(i)$ of the Jordan's form $J_1=J_1(1)\oplus J_1(2)\oplus\cdots\oplus J_1(\nu_2)$ by R_i $(R_1\geqq R_2\geqq\cdots\geqq R_{\nu_2})$.

Let $w=W_2\tilde{v}$ then (3.10) becomes to

$$n^{\sigma}\partial_s w=in^{1-\varepsilon_1-\varepsilon_2}(J_1+i^{-1}\tilde{B}(s))w+A_R(s;n)w\,, \tag{3.12}$$

$$A_R=W_2(in^{1-(3/2)\varepsilon_1}\tilde{A}_1^{(1)}(s;n)+n^{1-2\sigma}N_0W_1A_2(s;n)W_1^{-1}N_0^{-1}+n^{\sigma}(N_0)'_sN_0^{-1})W_2^{-1}\,.$$

For the remainder term $A_R(s;n)$ it holds that

Proposition 3.3. *We can take σ such that there exist a positive constant ε_0 and a constant C which satisfy*

$$|A_R(s;n)|\leqq Cn^{1-\varepsilon_1-\varepsilon_2-\varepsilon_0}\,, \tag{3.13}$$

for large n and for $s\in I_s$ (I_s is a certain open interval.).

Actually we must take σ in such a way that

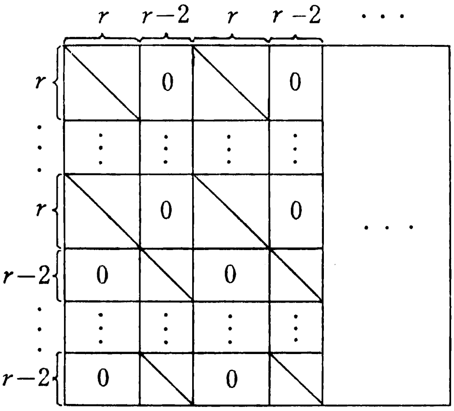
$$\operatorname{Max}\left\{\frac{2R_1(r-1)}{r(2R_1+1)},\ \frac{R_1(r-1)}{r(R_1+r-2)}\right\}<\sigma<\frac{r-1}{r}\,. \tag{3.14}$$

Taking account of the assumption $r\geqq 3$ we can take σ which satisfies the above inequality. Moreover noting that there exists $s=s_0$ such that $(iJ_1+\tilde{B}(s_0))$ has at most one eigenvalue whose real part is positive. Thus Proposition 2.2 derives Theorem 1.2 in this case. (Q. E. D.)

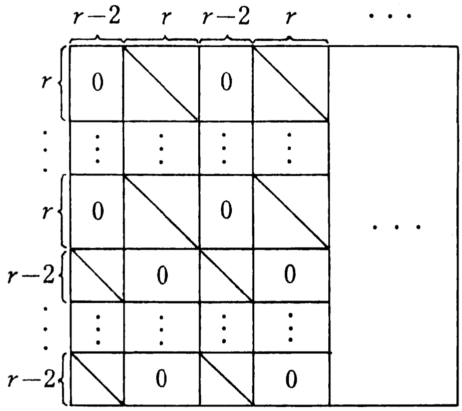
Proof of Lemma 3.3. For our argument to simlify we assume that J does not contain the Jordan's blocks of order less than $r-2$ and denote the number of the Jordan's blocks of order r, $r-2$ by s, s_1 recepectively. Also we assume that $A_1^{(0)}\neq 0$. In fact when $A_1^{(0)}=0$, $N(s)$ is nothing but the identity matrix.

Remark that the components of $A_1^{(0)}$ are zero except for $(ir,\ 2+(j-1)r)$-entry $(1\leqq i,\ j\leqq s)$, $(ir,\ sr+1+(j-1)(r-2))$-entry $(1\leqq i\leqq s,\ 1\leqq j\leqq s_1)$, and $(sr+i(r-2),\ 1+(j-1)r)$-entry $(1\leqq i\leqq s_1,\ 1\leqq j\leqq s)$.

Let $\{\vec{n}_1,\cdots,\vec{n}_{pr+q(r-2)}\}$ is one of the Jordan's chains of $(J+sA_1^{(0)})$ such that $p,q\geqq 1$, then the matrix $[\vec{n}_1,\cdots,\vec{n}_{pr+q(r-2)}]$ is given by one of the following figures.



(Figure 1)



(Figure 2)

In the above figures the oblique lines are the same numbers.

Observing this we can see that there exist integers α_i ($1 \leqq i \leqq pr+q(r-2)$) such that $\vec{n}_i = s^{\alpha_i}\vec{n}_i^{(0)}$, where $\vec{n}_i^{(0)}$ is a constant vector. This fact derives that there exist a non-singular constant matrix $N^{(0)}$ and a diagonal matrix $I(s)$ whose entries are powers of s such that $N_0 = I(s)N^{(0)}$. Thus Lemma 3.3 is proved. (Q. E. D.)

§ 4. Proof of Theorem 1.2 and 1.3 under the assumption (A.2)

Here we shall only argue in the case when $\nu=3$, that is, the number of Jordan's blocks is equal to three. In case of $\nu=1$ or $\nu=2$, the proof is much easier, so we omit it.

Though our argument is already given in § 2 in general cases, when we apply it we must need more delicate discussion according to cases. We shall give the proof as follows: The case when $r_1=r_2=r$, $r_3=r-1$ is in § 4.1, the case when $r_1=r$, $r_2=r_3=r-1$ is in § 4.2, and the case when $r_1=r$, $r_2=r-1$ and $r_3 \leqq r-2$ is in § 4.3, where we denote that $J=J(1)\oplus J(2)\oplus J(3)$ and the order of $J(i)$ are denoted by r_i ($i=1, 2, 3$).

4.1. $r_1=r_2=r$, $r_3=r-1$.

Let us recall our starting point (2.2)′. We know that $A_1(s)$ in that equation is actually sA_1 (A_1 is a constant matrix.), and without loss of generality we can regard that A_1 is (S)-type with respect to J. Keeping it in mind we denote the r-th row vector of A_1 by $(a_1, a_2, \cdots, a_{3r-1})$. Also the $2r$-th row vector and $(3r-1)$-th row vector is denoted by $(b_1, b_2, \cdots, b_{3r-1})$ and $(c_1, c_2, \cdots, c_{3-1})$ recepectively.

With those notations the 2×2-matrix K which was introduced in § 1 is now expressed by

$$K=\begin{bmatrix} a_1 & a_{r+1} \\ b_1 & b_{r+1} \end{bmatrix}.$$

Remark that by virtue of Lemma 2.1 and Lemma 2.2 K must be nilpotent. To carry out the reduction which was explained in § 2 we must divide our argument in two cases. One is the case when $K\neq 0$ and the other is when $K=0$.

(1) The case when $K\neq 0$.

At first we assume that $r\geqq 2$ in this case. Let $\varepsilon_1=\sigma/r$ and multiply the matrix of weight W_1 of the form (2.9) to (2.2)′ in § 2 from left, where

$$W_1=W_{\varepsilon_1, r}(n)\oplus W_{\varepsilon_1, r}(n)\oplus n^{(1/2)\varepsilon_1}W_{\varepsilon_1, r-1}(n).$$

Then (2.2)′ becomes the same system as (3.1).

We can also construct $N(s)$ which satisfies (i)~(iii) in Lemma 3.1, where $J_1=J_1(1)\oplus J_1(2)$, the order of $J_1(1)$ is $2r$ and the order of $J_1(2)$ is $r-1$. Thus we obtain the similar system to (3.2). Denoting that $\tilde{v}=N(s)W_1v$, we rewrite this.

$$(4.1)\qquad n^{\sigma}\partial_s\tilde{v}=(in^{1-\varepsilon_1}(J_1+n^{-(1/2)\varepsilon_1}A_1^{(1)}(s)+n^{-\varepsilon_1}A_1^{(2)}(s)+n^{-(3/2)}\varepsilon_1\tilde{A}_1^{(3)}(s\,;\,n))$$
$$+in^{1-2\sigma}NW_1A_2(s\,;\,n)W_1^{-1}N^{-1}+NW_1BW_1^{-1}N^{-1}+n^{\sigma}(N)'_sN^{-1})\tilde{v}\,.$$

After those multiplications $\tilde{A}_1^{(1)}$ and $\tilde{A}_1^{(2)}$ are also (S)-type with respect to J. We denote these entries as follows: The r-th row vector of $\tilde{A}_1^{(1)}$, $\tilde{A}_1^{(2)}$ is $(\tilde{a}_1^{(1)}, \cdots, \tilde{a}_{3r-1}^{(1)})$, $(\tilde{a}_1^{(2)}, \cdots, \tilde{a}_{3r-1}^{(2)})$ recepectively, the $2r$-th row vector is $(\tilde{b}_1^{(1)}, \cdots, \tilde{b}_{3r-1}^{(1)})$, $(\tilde{b}_1^{(2)}, \cdots, \tilde{b}_{3r-1}^{(2)})$ and the last $(3r-1)$-row is $(\tilde{c}_1^{(1)}, \cdots, \tilde{c}_{3r-1}^{(1)})$, $(\tilde{c}_1^{(2)}, \cdots, \tilde{c}_{3r-1}^{(2)})$. In those row vectors simple caluculation shows that the entries of $\tilde{A}_1^{(1)}$ are zero except for $\tilde{a}_{2r+1}^{(1)}$, $\tilde{b}_{2r+1}^{(1)}$, $\tilde{c}_1^{(1)}$ and $\tilde{c}_{r+1}^{(1)}$, and the entries of $\tilde{A}_1^{(2)}$ are also zero except for $\tilde{a}_2^{(2)}$, $\tilde{a}_{r+2}^{(2)}$, $\tilde{b}_2^{(2)}$, $\tilde{b}_{r+2}^{(2)}$ and $\tilde{c}_{2r+1}^{(2)}$.

By virtue of Lemma 2.1 and Lemma 2.2, when $s\geqq 1$ it holds that

$$(4.2)\qquad \begin{cases} \tilde{b}_{2r+1}^{(1)}\tilde{c}_1^{(1)}=0 \\ \tilde{a}_{2r+1}^{(1)}\tilde{c}_1^{(1)}+\tilde{b}_{2r+1}^{(1)}\tilde{c}_{r+1}^{(1)}=0\,. \end{cases}$$

Moreover from the same lemmas we also obtain

$$(4.3)\qquad \tilde{b}_2^{(2)}=0\,.$$

Now we can choose the lower order term B which possesses the similar property as explained in Proposition 3.1.

$$(4.4)\qquad N(s)W_1BW_1^{-1}N(s)^{-1}=n^{(r-1)\varepsilon_1}\tilde{B}(s)\,,$$

where the entries of $\tilde{B}(s)$ are zero except for only $(2r, 1)$-entry and it is expressed by bs^{α}. (b is an arbitrary number we can take and α is an integer.)

Taking account of (4.4) we introduce another matrix of weight corresponding with J_1. Let $\varepsilon_2=(1-\sigma)/2r$ and denote that

$$(4.5)\qquad W_2=W_{\varepsilon_2,2r}(n)\oplus n^{\bar{\varepsilon}}W_{\varepsilon_2,r-1}(n)\,.$$

Operate W_2 to (4.1) from ieft then we obtain

$$(4.6)\qquad n^{\sigma}\partial_s w=(in^{1-\varepsilon_1-\varepsilon_2}(J_1+i^{-1}\tilde{B}(s))+A_R(s\,;\,n))w\,,$$

where $w=W_2\tilde{v}$ and

$$\begin{aligned} A_R=\,&in^{1-\varepsilon_1}W_2(n^{-(1/2)\varepsilon_1}\tilde{A}_1^{(1)}+n^{-\varepsilon_1}\tilde{A}_1^{(2)}+n^{-(3/2)\varepsilon_1}\tilde{A}_1^{(3)})W_2^{-1} \\ &+in^{1-2\sigma}W_2NW_1A_2W_1^{-1}N^{-1}W_2^{-1}+n^{\sigma}W_2(N)'_sN^{-1}W_2^{-1}\,. \end{aligned}$$

For this remainder term it holds that

Proposition 4.1. *There exist σ, $\bar{\varepsilon}$ in* (4.5), *a positive number ε_0 and a constant C which do not depend on the parameter n such that*

$$(4.7)\qquad |A_R(s\,;\,n)|\leqq Cn^{1-\varepsilon_1-\varepsilon_2-\varepsilon_0}\qquad \textit{for large } n.$$

Actually we take $\bar{\varepsilon}$ according to three cases as follows.

$$-\frac{1}{2}\varepsilon_1+2r\varepsilon_2<\bar{\varepsilon}<\frac{3}{2}\varepsilon_1-(r-1)\varepsilon_2 \qquad \text{when } \tilde{b}_{2r+1}^{(1)}\neq 0\,.$$

$$r\varepsilon_2<\varepsilon_1 \ \text{ and } \ -\frac{1}{2}\varepsilon_1+r\varepsilon_2<\bar{\varepsilon}<\frac{1}{2}\varepsilon_1+\varepsilon_2 \qquad \text{when } \tilde{b}_{2r+1}^{(1)}=0,\ \tilde{a}_{2r+1}^{(1)}\neq 0.$$

$$-\frac{3}{2}\varepsilon_1+2r\varepsilon_2<\bar{\varepsilon}<\frac{1}{2}\varepsilon_1-(r-1)\varepsilon_2 \qquad \text{when } \tilde{b}_{2r+1}^{(1)}=0,\ \tilde{a}_{2r+1}^{(1)}=0\,.$$

In order that the above three inequalities hold, we must choose σ such that

$$(4.8)\qquad \frac{3r-1}{3r+3}<\sigma,\quad \frac{r}{r+2}<\sigma,\quad \frac{r-1}{r+1}<\sigma\,.$$

On the other hand we must impose the condition $\sigma<1-\varepsilon_1-\varepsilon_2$, that is σ must be taken such that

$$(4.9)\qquad \sigma<\frac{2r-1}{2r+1}\,.$$

Note that (4.8) and (4.9) are compatible when $r\geqq 2$. Thus owing to Proposition 2.2 we obtain Theorem 1.3 in this case. (Q. E. D.)

(2) The case when $K=0$.

At first we assume $r\geqq 3$ in this case. Remark that from Lemma 2.1 and Lemma 2.2 it holds that

$$(4.10)\qquad a_{2r+1}c_1+b_{2r+1}c_{r+1}=0\,.$$

Observing (4.10) we prove our theorem dividing four cases:

$(2)_1$ $|a_{2r+1}|+|b_{2r+1}|\neq 0\,,\qquad |c_1|+|c_{r+1}|\neq 0\,.$

$(2)_2$ $|a_{2r+1}|+|b_{2r+1}|\neq 0\,,\qquad |c_1|+|c_{r+1}|=0\,.$

$(2)_3$ $|a_{2r+1}|+|b_{2r+1}|=0\,,\qquad |c_1|+|c_{r+1}|\neq 0\,.$

$(2)_4$ $|a_{2r+1}|+|b_{2r+1}|=0\,,\qquad |c_1|+|c_{r+1}|=0\,.$

In the cases $(2)_1\sim(2)_3$ the arguments are parallel to the previous case when $K\neq 0$ and in the case $(2)_4$ we need not $N(s)$ and W_2. So we roughly explain our proof.

In the case $(2)_1$, set $\varepsilon_1=(2/(2r-1))\sigma$ (different from one in the previous case), and use the same W_1. And also use a similar $N(s)$ having the same property, but in this case J_1 in (4.1) is formed only one Jordan's block of order $3r-1$. Observing $N(s)$ we can choose the lower order term B such that $NW_1BW_1^{-1}N^{-1}=n^{(r-1)\varepsilon_1}\tilde{B}$, where the entries of $\tilde{B}$ except for only $(3r-1,1)$-entry are zero and the non-zero component can be arbitrarily given.

Set $\varepsilon_2=(1-r\varepsilon_1)/(3r-1)$ and operate $W_2=W_{\varepsilon_2,3r-1}(n)$, then we obtain the similar system as (4.6). And the corresponding inequality to (4.8) and (4.9) is as follows:

$$(4.11)\qquad \mathrm{Max}\left\{\frac{1}{2},\frac{r-1}{r+1}\right\}<\sigma<\frac{3r-2}{3r+1}\,.$$

This is compatible when $r\geqq 2$. Thus by the same argument as the previous one we obtain Theorem 1.3. Indeed Theorem 1.2 is true in this case.

In the case $(2)_2$ or $(2)_3$ the second Jordan's form is formed with two blocks $J_1=J_1(1)\oplus J_1(2)$, the order of $J_1(1)$ is $2r-1$ and the other one is r. Different from the case $(2)_1$, we must choose the lower order term B such that $N(s)W_1BW_1^{-1}N(s)^{-1}=n^{(r-(3/2))\varepsilon_1}\tilde{B}$, where the entries of $\tilde{B}$ except for $(2r-1,1)$-entry are zezo. So we set $\varepsilon_2=(1-\sigma)/(2r-1)$ and W_2 is given by $W_2=W_{\varepsilon_2,2r-1}(n)\oplus n^{(r-1)\varepsilon_1}W_{\varepsilon_2,r}(n)$, then we are led to the similar system as (4.6). In this case we must take σ in such a way that

(4.12) $$\operatorname{Max}\left\{\frac{1}{2}, \frac{r-2}{r-1}\right\}<\sigma<\frac{r-1}{r}.$$

Noting that (4.12) is compatible when $r\geqq 3$ we obtain Theorem 1.2 in this case.

In the case $(2)_4$, denote that

$$\tilde{K}=\begin{bmatrix} a_2 & a_{r+2} \\ b_2 & b_{r+2} \end{bmatrix}.$$

Then from Lemma 2.1 and Lemma 2.2 it holds that $\tilde{K}$ is nilpotent and $c_{2r+1}=0$. It is easy to see that there exists non-singular constant matrix N_0 such that $N_0J=JN_0$ and the corresponding matrix to $\tilde{K}$ is a Jordan's form.

Keeping it in mind, set $\varepsilon_1=1/r$ and denote that

$$\begin{cases} W_1=W_{\varepsilon_1,r}(n)\oplus n^{\xi}W_{\varepsilon_1,r}(n)\oplus n^{\xi}W_{\varepsilon_1,r-1}(n) & \text{when } \tilde{K}\neq 0, \\ W_1=W_{\varepsilon_1,r}(n)\oplus W_{\varepsilon_1,r}(n)\oplus W_{\varepsilon_1,r-1}(n) & \text{when } \tilde{K}=0. \end{cases}$$

Then (2.2)′ in §2 becomes

(4.13) $$n^{\sigma}\partial_s w=in^{1-(1/r)}(J+i^{-1}B)w+A_R(s\,;n)w,$$

where we denoted that $w=W_1N_0v$, and B was chosen in such a way that the entries except for only $(r, 1)$-entry are zero.

It is easy to see that we can take ξ and σ such that $A_R(s\,;n)$ can be a remainder term, i.e., $|A_R(s\,;n)|=o(n^{1-(1/r)})$ as $n\to\infty$. Actually we take σ which satisfies

(4.14) $$\operatorname{Max}\left\{\frac{1}{2}, \frac{1}{r}\left(r-\frac{5}{4}\right)\right\}<\sigma<\frac{r-1}{r}.$$

Note that (4.14) is compatible when $r\geqq 3$. Thus Theorem 1.2 holds in this case. (Q.E.D.)

4.2. $r_1=r$, $r_2=r_3=r-1$.

Hereafter we assume that $r\geqq 3$. At first we use the same notation of the components of A_1 which is (S)-type with recepect to J. $(a_1, \cdots, a_{3r-2})$ is the r-th row vector, $(b_1, \cdots, b_{3r-2})$ is the $(2r-1)$-th row vector and $(c_1, \cdots, c_{3r-2})$ is the $(3r-2)$-th row vector.

From Lemma 2.1 and Lemma 2.2 we can see

(4.15) $$a_1=0, \qquad a_{r+1}b_1+a_{2r}c_1=0.$$

Observing (4.15) we must divide our argument in the following cases:

(I) $|a_{r+1}|+|a_{2r}|\neq 0$. $|b_1|+|c_1|\neq 0$.

(II) $|a_{r+1}|+|a_{2r}|\neq 0$, $|b_1|+|c_1|=0$.

(III) $|a_{r+1}|+|a_{2r}|=0$, $|b_1|+|c_1|\neq 0$.

(IV) $|a_{r+1}|+|a_{2r}|=0$, $|b_1|+|c_1|=0$,

In the case (I), set $\varepsilon_1=2\sigma/(2r-1)$ and operate W_1 to (2.2)′ in §2 from left, where

(4.16) $$W_1=W_{\varepsilon_1,r}(n)\oplus n^{(1/2)\varepsilon_1}W_{\varepsilon_1,r-1}(n)\oplus n^{(1/2)\varepsilon_1}W_{\varepsilon_1,r-1}(n).$$

Then (2.2)′ becomes

(4.17) $$n^\sigma\partial_s\tilde{v}=(in^{1-\varepsilon_1}(J+sA_1^{(0)}+n^{-(1/2)\varepsilon_1}A_1^{(1)}(s;n))+in^{1-2\sigma}W_1A_2(s;n)W_1^{-1}+W_1BW^{-1})\tilde{v},$$

where $\tilde{v}=Wv$, and $A_1^{(1)}(s;n)$ is actually of the form: $A_1^{(1)}(s)+n^{-(1/2)\varepsilon_1}A_1^{(2)}(s)+n^{-\varepsilon_1}A_1^{(3)}(s)+n^{-(3/2)\varepsilon_1}A_1^{(4)}(s;n)$. Assume that $s\geqq 1$ and apply Lemma 2.3 to $J+sA_1^{(0)}+n^{-(1/2)\varepsilon_1}A_1^{(1)}(s;n)$ whose principal part $J+sA_1^{(0)}$ is nilpotent, then we have

Lemma 4.1. *There exists*

$$N(s;n)=N^{(0)}(s)+n^{-(1/2)\varepsilon_1}N^{(1)}(s)+n^{-\varepsilon_1}N^{(2)}(s)+n^{-(3/2)\varepsilon_1}N^{(3)}(s)$$

which satisfies

(i) $\det N^{(0)}(s)\neq 0$.

(ii) $N(s;n)(J+sA_1^{(0)}+n^{-(1/2)\varepsilon_1}A_1^{(1)}(s;n))$

$$=(J_1+n^{-(1/2)\varepsilon_1}\tilde{A}_1^{(1)}(s)+n^{-\varepsilon_1}\tilde{A}_1^{(2)}(s)+n^{-(3/2)\varepsilon_1}\tilde{A}_1^{(3)}(s)+n^{-2\varepsilon_1}\tilde{A}_1^{(4)}(s;n))N(s;n),$$

where J_1 is a Jordan's form with one block of order $3r-2$, and $\tilde{A}_1^{(1)}(s)\sim\tilde{A}_1^{(3)}(s)$ are (S)-type with recepect to J_1.

(iii) *When we expand $C(s;n)=(N(s;n))'_sN(s;n)^{-1}$ in such a way that*

$$C(s;n)=C^{(0)}(s)+n^{-(1/2)\varepsilon_1}C^{(1)}(s)+n^{-\varepsilon_1}C^{(2)}(s)+n^{-(3/2)\varepsilon_1}C^{(3)}(s;n),$$

where $C^{(3)}(s;n)=O(1)$ as $n\to\infty$, it holds that $C^{(0)}(s)$ is a diagonal matrix and $c_{i,j}^{(1)}=0$ when $i-j>r-1$ and $c_{i,j}^{(2)}=0$ when $i-j>2r-1$. Here we denoted the entries of $C^{(1)}(s)$ and $C^{(2)}(s)$ by $c_{i,j}^{(1)}$ and $c_{i,j}^{(2)}$ respectively ($1\leqq i,\ j\leqq 3r-2$).

Set $\tilde{\tilde{v}}=N(s;n)\tilde{v}$ then (4.17) becomes

(4.18) $$n^\sigma\partial_s\tilde{\tilde{v}}=(in^{1-\varepsilon_1}(J_1+\tilde{A}_1^{(1)}(s;n)+n^{-2\varepsilon_1}\tilde{A}_1^{(4)}(s;n))$$
$$+in^{1-2\sigma}NW_1A_2W_1^{-1}N^{-1}+NW_1BW_1^{-1}N^{-1}+n^\sigma(N)'_sN^{-1})\tilde{\tilde{v}},$$

where $\tilde{A}_1^{(1)}(s;n)=n^{-(1/2)\varepsilon_1}\tilde{A}_1^{(1)}(s)+n^{-\varepsilon_1}\tilde{A}_1^{(2)}(s)+n^{-(3/2)\varepsilon_1}\tilde{A}_1^{(3)}(s)$.

Keeping it in mind that $\tilde{A}_1^{(1)}(s;n)$ are (S)-type we denote the $(3r-2)$-th row vector of $\tilde{A}_1^{(1)}(s;n)$ by $\tilde{a}_k(s;n)$ ($1\leqq k\leqq 3r-2$). Then we obtain

Proposition 4.2. *There exists a positive constant C which do not depend on n such that $|\tilde{a}_k(s;n)|\leqq Cn^{-\alpha_k}$ for large n, where*

(4.19) $$\alpha_k=\mathrm{Min}\left\{\frac{(2r-3)(3r-k-1)}{2r-1}\sigma,\ 2\varepsilon_1\right\}\qquad \textit{for } 1\leqq k\leqq 3r-2.$$

Now concerning to the lower order term, we can take it as follows.

Proposition 4.3. *There exists a matrix B which satisfies*

(4.20) $$N(s;n)W_1BW_1^{-1}N(s;n)^{-1}=n^{(r-2)\varepsilon_1}(\tilde{B}(s)+\tilde{B}_R(s;n)),$$

where the entries of $\tilde{B}(s)$ *except only* $(3r-2, 1)$*-entry are zero and the non-zero component can be taken arbitrarily, and* $\tilde{B}_R(s; n)=o(1)$ *as* $n\to\infty$.

Set $\varepsilon_2=\dfrac{1-(r-1)\varepsilon_1}{3r-2}$ and define that $W_2=W_{\varepsilon_2, 3r-2}(n)$ then (4.18) becomes

$$n^\sigma \partial_s w=(in^{1-\varepsilon_1-\varepsilon_2}(J_1+i^{-1}\tilde{B}(s)+i^{-1}C_0(s))+A_R(s; n))w, \tag{4.21}$$

where $w=W_2\tilde{v}$ and the remainder term A_R is expressed as follows.

$$A_R(s; n)=in^{1-\varepsilon_1}W_2(\tilde{A}_1^{(1)}(s; n)+n^{-2\varepsilon_1}\tilde{A}_1^{(4)}(s; n))W_2^{-1}$$
$$+in^{1-2\sigma}W_2NW_1A_2W_1^{-1}N^{-1}W_2^{-1}+n^{(r-2)\varepsilon_1}W_2\tilde{B}_RW_2^{-1}+n^\sigma W_2((N)'_sN^{-1}-C_0(s))W_2^{-1}.$$

In the above expression $C_0(s)$ is composed with only $(3r-2, 1)$-entry of $\mathring{C}^{(3)}(s)$, where we denote in (iii) of Lemma 4.1 that $C^{(3)}(s; n)=\mathring{C}^{(3)}(1+o(1))$ as $n\to\infty$.

For the remainder term $A_R(s; n)$ we have

Proposition 4.4. *We can take* σ *such that there exist a positive constant* ε_0 *and a constant* C *which satisfy*

$$|A_R(s; n)|\leqq Cn^{1-\varepsilon_1-\varepsilon_2-\varepsilon_0} \quad \textit{for large } n. \tag{4.22}$$

Actually we can choose σ such that

$$\text{Max}\left\{\frac{1}{3r-4}, \frac{2r-1}{2(r+1)}\right\}<\sigma<\frac{r-1}{r}. \tag{4.23}$$

This can be compatible when $r\geqq 3$. Thus owing to Proposition 2.2, Theorem 1.2 in this case is proved. (Q.E.D.)

Secondly we shall step into the case of (II) and (III). Since in these cases the same arguments are developed, we treat only case (II).

We use the same matrix of weight W_1 as one in the previous case (I). So we are led to the system (4.17). Indeed the matrices $A_1^{(0)}(s)$, $A_1^{(1)}(s; n)$ and $A_2(s; n)$ are different from the previous ones. Recall that Lemma 4.1 played an important role. Now we propose a similar one.

Lemma 4.2. *There exists* $N(s; n)=N^{(0)}(s)+n^{-(1/2)\varepsilon_1}N^{(1)}(s)+n^{-\varepsilon_1}N^{(2)}(s)$ *which satisfies*

(i) $\det N^{(0)}(s)\neq 0$.

(ii) $N(s; n)(J+sA_1^{(0)}+n^{-(1/2)\varepsilon_1}A_1^{(1)}(s; n))$

$$=(J_1+n^{-(1/2)\varepsilon_1}\tilde{A}_1^{(1)}(s)+n^{-\varepsilon_1}\tilde{A}_1^{(2)})s)+n^{-(3/2)\varepsilon_1}\tilde{A}_1^{(3)}(s; n))N(s; n)$$

where J_1 *is a Jordan's form with two Jordan's blocks:* $J_1=J_1(1)\oplus J_1(2)$, *the order of* $J(1)$ *is* $2r-1$ *and the order of* $J(2)$ *is* $r-1$, *and* $\tilde{A}_1^{(1)}$ *and* $\tilde{A}_1^{(2)}$ *are (S)-type with respect to* J_1.

(iii) *When we expand* $C(s; n)=(N(s; n))'_sN(s; n)$ *in such a way that*

$$C(s; n)=C^{(0)}(s)+n^{-(1/2)\varepsilon_1}C^{(1)}(s)+n^{-\varepsilon_1}C^{(2)}(s)+n^{-(3/2)\varepsilon_1}C^{(3)}(s; n),$$

where $C^{(3)}(s; n)=O(1)$ *as* $n\to\infty$, *it holds that* $C^{(0)}(s)$ *is a diagonal matrix, moreover*

$c_{i,j}^{(1)}=0$ *when* $i-j>r-1$ *and* $c_{i,j}^{(2)}=0$ *when* $i-j\geqq 0$. *Here we denote the components of* $C^{(1)}(s)$ *and* $C^{(2)}(s)$ *by* $c_{i,j}^{(1)}$ *and* $c_{i,j}^{(2)}$ *respectively* $(1\leqq i,\ j\leqq 3r-2)$.

Thus we obtain the similar system as (4.18). But in this case we must furthermore divide in two cases when $\tilde{b}_{2r}^{(1)}=0$ or not, where we denote that, for $k=1, 2$, the $(2r-1)$-th row vector of $\tilde{A}_1^{(k)}(s)$ by $(\tilde{b}_1^{(k)}, \cdots, \tilde{b}_{3r-2}^{(k)})$ and the last $(3r-2)$-th row vector of $\tilde{A}_1^{(k)}(s)$ by $(\tilde{c}_1^{(k)}, \cdots, \tilde{c}_{3r-2}^{(k)})$. Here we remark that the entries of $\tilde{A}_1^{(1)}(s)$ are zero except for $\tilde{b}_{r+1}^{(1)}$, $\tilde{b}_{2r}^{(1)}$, $\tilde{c}_{r+1}^{(1)}$ and $\tilde{c}_{2r}^{(1)}$, and the entries of $\tilde{A}_1^{(2)}(s)$ are also zero except for $\tilde{b}_2^{(2)}$ and $\tilde{c}_2^{(2)}$.

When $\tilde{b}_{2r}^{(1)}\neq 0$ we must proceed our reduction of hyperbolic matrices.

In the case $(\mathrm{II})_1: \tilde{b}_1^{(1)}\neq 0$, set $\varepsilon_2=\dfrac{\sigma}{(r-1)(2r-1)}$ and operate $W_2=W_{\varepsilon_2, 2r-1}(n)\oplus n^{r\varepsilon_2}W_{\varepsilon_2, r-1}(n)$ to the similar system as (4.18) then we have

$$n^{\sigma}\partial_s\tilde{\tilde{v}}=(in^{1-\varepsilon_1-\varepsilon_2}(J_1+\tilde{\tilde{A}}_1^{(0)}(s)+n^{-(r-2)\varepsilon_2}\tilde{\tilde{A}}_1^{(1)}(s;n)) \tag{4.24}$$
$$+W_2(in^{1-2\sigma}NW_1A_2W_1^{-1}N^{-1}+NW_1BW_1^{-1}N^{-1}+n^{\sigma}(N)'_sN^{-1})W_2)\tilde{\tilde{v}}\,,$$

where $\tilde{\tilde{v}}=W_2\tilde{v}$. Let $N_1(s)$ be the non-singular matrix which transform $J_1+\tilde{\tilde{A}}_1^{(0)}$ to a new Joran's form J_2 of order $3r-2$. Denote $\tilde{\tilde{\tilde{v}}}=N_1(s)\tilde{\tilde{v}}$ then (4.24) becomes

$$n^{\sigma}\partial_s\tilde{\tilde{\tilde{v}}}=in^{1-\varepsilon_1-\varepsilon_2}(J_2+n^{-(r-2)\varepsilon_2}\tilde{\tilde{\tilde{A}}}_1^{(1)}(s;n)) \tag{4.25}$$
$$+N_1W_2(in^{1-2\sigma}NW_1A_2W_1^{-1}N_1^{-1}+NW_1BW_1^{-1}N^{-1}$$
$$+n^{\sigma}C(s;n))W_2^{-1}N_1^{-1}+n^{\sigma}(N_1)'_sN_1^{-1})\tilde{\tilde{\tilde{v}}}\,.$$

At last we choose the lower order term B which satisfies

$$N_1W_2NW_1BW_1^{-1}N^{-1}W_2^{-1}N_1^{-1}=n^{(r-(3/2))\varepsilon_1+2(r-1)\varepsilon_2}(\tilde{B}(s)+\tilde{B}_R(s;n))\,, \tag{4.26}$$

where the entries of $\tilde{B}(s)$ except for only $(3r-2, 1)$-entry are zero and the non-zero component can be taken arbitrarily, and $\tilde{B}_R(s;n)=o(1)$ as $n\to\infty$.

Set $\varepsilon_3=\dfrac{1}{3r-1}\left(1-\dfrac{r\sigma}{r-1}\right)$ and define $W_3=W_{\varepsilon_3, 3r-2}(n)$ then (4.25) becomes

$$n^{\sigma}\partial_s w=(in^{1-\varepsilon_1-\varepsilon_2-\varepsilon_3}(J_2+i^{-1}\tilde{B}(s)+i^{-1}C_0(s))+A_R(s;n))w\,, \tag{4.27}$$

where $w=W_3\tilde{\tilde{\tilde{v}}}$ and the remainder term A_R is expressed as follows.

$$A_R=in^{1-\varepsilon_1-(r-1)\varepsilon_2}W_3\tilde{\tilde{\tilde{A}}}_1^{(1)}(s;n)W_3^{-1}+n^{(r-(3/2))\varepsilon_1+2(r-1)\varepsilon_2}W_3\tilde{B}_R(s;n)W_3^{-1}$$
$$+W_3N_1W_2(in^{1-2\sigma}NW_1A_2W_1^{-1}N^{-1}+n^{\sigma}\tilde{C}(s;n))W_2^{-1}N_1^{-1}W_3^{-1}+n^{\sigma}W_3(N_1)'_sN_1^{-1}W_3^{-1}\,.$$

In the above expression the term $C_0(s)$ is the matrix whose non-zero component is only $(3r-2, 1)$-one, and it is equal to the same component of $N_1C^{(1)}(s)N_1^{-1}$, while $\tilde{C}(s;n)$ is the remainder of $C(s;n)$.

For $A_R(s;n)$ we also obtain the same inequality as (4.22) and instead of (4.23) we must impose the inequality

$$\mathrm{Max}\left\{\frac{2r-1}{2r+2}, \frac{2r-1}{4r-3}\right\}<\sigma<\frac{r-1}{r}\,. \tag{4.28}$$

Indeed (4.28) is compatible when $r\geqq 3$. Thus Theorem 1.2 is proved.

In the case $(\mathrm{II})_2: \tilde{b}_{2r}^{(1)}=0$, we need not the second reduction which was used in the previous case $(\mathrm{II})_1$. From Lemma 2.1 and Lemma 2.2 it is easy to see that

$$\tilde{b}_2^{(2)}=\tilde{b}_{r+1}^{(1)}=\tilde{c}_{2r}^{(1)}=0\,. \tag{4.29}$$

Observing (4.29) we can choose the lower order term which, similar to (4.26), satisfies

$$NW_1BW_1^{-1}N^{-1}=n^{(r-(3/2))\varepsilon_1}(\tilde{B}(s)+\tilde{B}_R(s\,;\,n))\,, \tag{4.30}$$

where the entries of $\tilde{B}(s)$ except for only $(2r-1, 1)$-entry are zero and non-zero component can be taken arbitrarily, and $\tilde{B}_R(s\,;\,n)=o(1)$ as $n\to\infty$.

Set $\varepsilon_2=(1-\sigma)/(2r-1)$ and define $W_2=W_{\varepsilon_2,2r-1}(n)\oplus W_{\varepsilon_2,r-1}(n)$ then, instead of (4.27), we obtain

$$n^{\sigma}\partial_s w=(in^{1-\varepsilon_1-\varepsilon_2}(J_1+i^{-1}\tilde{B}(s))+A_R(s\,;\,n))w\,, \tag{4.31}$$

where $w=W_2\tilde{v}$ and

$$A_R=in^{1-\varepsilon_1}W_2(\tilde{A}_1^{(1)}(s\,;\,n)+n^{-2\varepsilon_1}\tilde{A}_1^{(1)}(s\,;\,n))W_2^{-1}+in^{1-2\sigma}W_2NW_1A_2W_1^{-1}N^{-1}W_2^{-1}$$
$$+n^{(r-(3/2))\varepsilon_1}W_2\tilde{B}_RW_2^{-1}+n^{\sigma}W_2C(s\,;\,n)W_2^{-1}\,.$$

For this A_R we can regard it as a remainder term when we take σ which satisfies the same inequality (4.28). Thus Theorem 1.2 is proved in this case. (Q. E. D.)

Now we come to the case (IV). Let $\tilde{K}$ be the 2×2-matrix such that

$$\tilde{K}=\begin{bmatrix} b_{r+1} & b_{2r} \\ c_{r+1} & c_{2r}\end{bmatrix},$$

then from Lemma 2.1 and Lemma 2.2 it follows that

$$a_2=0,\quad \text{and}\quad \tilde{K}\ \text{is nilpotent.} \tag{4.32}$$

We remark that without loss of generality $\tilde{K}$ can be regarded as a Jordan's form of order two with the eigenvalues are zero, that is, there exists a non-singular constant matrix N_0 such that the corresponding 2×2-matrix derived from $N_0A_1N_3^{-1}$ is a Jordan's form.

Keeping this in mind, we take the lower order term B such that the entries except for $(r, 1)$-entry are zero.

Set $\varepsilon=1/r$ and define the matrix of weight W as follows.

$$W=W_{\varepsilon,r}(n)\oplus n^{\varepsilon}W_{\varepsilon,r-1}(n)\oplus n^{1}W_{\varepsilon,r-1}(n)\,. \tag{4.33}$$

Operate W to (2.2)′ in § 2, then when we denote $w=Wv$ it holds that

$$n^{\sigma}\partial_s w=(in^{1-\varepsilon}(J+i^{-1}B)+A_R(s\,;\,n))w\,, \tag{4.34}$$
$$A_R(s\,;\,n)=inW(sn^{-\sigma}A_1+n^{-2\sigma}A_2(s\,;\,n))W^{-1}\,.$$

For A_R we can obtain

Proposition 4.5. *There exist* σ, $\tilde{\varepsilon}$ *and* $\tilde{\tilde{\varepsilon}}$ *in* (4.33), *a positive constant* ε_0 *and a constant* C *such that*

$$|A_R(s\,;\,n)|\leqq Cn^{1-\varepsilon-\varepsilon_0} \qquad \textit{for large } n.$$

In order that the above inequality holds we must actually take σ which satisfies that

$$\text{(4.35)} \qquad \operatorname{Max}\left\{\frac{1}{2},\ \frac{2r-3}{2r}\right\}<\sigma<\frac{r-1}{r}\,.$$

The above inequality is compatible when $r\geqq 3$. Thus Theorem 1.2 is proved in this case. (Q. E. D.)

4.3. $r_1=r,\ r_2=r-1,\ r_3\leqq r-2.$

Even though we must divide our argument in two cases: one is $r_3=r-2$, the other is $r_3<r-2$, we only treat the case when $r_3=r-2$. In the case when $r_3<r-2$ the argument is much easier.

We use the same notation of the components of A_1 as one in 4.2. Keeping it in mind that A_1 is (S)-type with respect to J, denote that $(a_1, \cdots, a_{3r-3})$ is the r-th row vector, $(b_1, \cdots, b_{3r-3})$ is the $(2r-1)$-th row vector and $(c_1, \cdots, c_{3r-3})$ is the last $(3r-3)$-th row vector.

From Lemma 2.1 and Lemma 2.2 it is easy to see that

$$\text{(4.36)} \qquad a_1=0\,, \qquad a_{r+1}b_1=0\,.$$

Observing (4.36) we must divide our argument in the following way.

(I) $a_{r+1}\neq 0$

(II) $b_1\neq 0$

(III) $a_{r+1}=b_1=0$

In the cases of (I) and (II) we can develope the same argument. Set $\varepsilon_1=2\sigma/(2r-1)$ and define W_1 in such a way that

$$W_1=W_{\varepsilon_1,r}(n)\oplus n^{(1/2)\varepsilon_1}W_{\varepsilon_1,r-1}(n)\oplus n^{\varepsilon_1}W_{\varepsilon_1,r-2}(n)\,.$$

Denote $\tilde{v}=W_1v$ then (2.2)′ becomes

$$\text{(4.37)} \qquad n^{\sigma}\partial_s\tilde{v}=(in^{1-\varepsilon_1}(J+sA_1^{(0)}+n^{-(1/2)\varepsilon_1}sA_1^{(1)}+n^{-\varepsilon_1}sA_1^{(2)}+n^{-(3/2)\varepsilon_1}sA_1^{(3)}(n))$$
$$+in^{1-2\sigma}W_1A_2(s\,;\,n)W_1^{-1}+W_1BW_1^{-1})\tilde{v}\,.$$

Secondly apply Lemma 2.3 to $J+sA_1^{(0)}+n^{-(1/2)\varepsilon_1}sA_1^{(1)}+n^{-\varepsilon_1}sA_1^{(2)}+n^{-(3/2)\varepsilon_1}sA_1^{(3)}(n)$, then we obtain

Lemma 4.3. *Let* $s\geqq 1$. *There exists* $N(s\,;\,n)=N^{(0)}(s)+n^{-(1/2)\varepsilon_1}N^{(1)}(s)+n^{-\varepsilon_1}N^{(2)}(s)$ *which satisfies following* (i)$\sim$(iii).

(i) $\det N^{(0)}(s)\neq 0$.

(ii) $N(s;n)(J+sA_1^{(0)}+n^{-(1/2)\varepsilon_1}sA_1^{(1)}+n^{-\varepsilon_1}sA_1^{(2)}+n^{-(3/2)\varepsilon_1}sA_1^{(3)}(n))$

$$=(J_1+n^{-(1/2)\varepsilon_1}\tilde{A}_1^{(1)}(s)+n^{-\varepsilon_1}\tilde{A}_1^{(2)}(s)+n^{-(3/2)\varepsilon_1}\tilde{A}_1^{(3)}(s;n))N(s;n)$$

Where J_1 is a Jordan's form with two Jordan's blocks: $J_1=J_1(1)\oplus J_1(2)$, where the order of $J_1(1)$ is $2r-1$ and the order of $J_1(2)$ is $r-2$, moreover $\tilde{A}_1^{(1)}(s)$ and $\tilde{A}_1^{(2)}(s)$ are (S)-type with respect to J_1.

(iii) *When we expand $C(s;n)=(N(s;n))'_sN(s;n)^{-1}$ in such a way that*

$$C(s;n)=C^{(0)}(s)+n^{-(1/2)\varepsilon_1}C^{(1)}(s)+n^{-\varepsilon_1}C^{(2)}(s)+n^{-(3/2)\varepsilon_1}C^{(3)}(s;n),$$

where $C^{(3)}(s;n)=O(1)$ as $n\to\infty$, it holds that $c_{i,j}^{(1)}=0$ when $i-j>r-1$. Here we denoted the entries of $C^{(1)}(s)$ by $c_{i,j}^{(1)}$ $(1\leq i,\ j\leqq 3r-3)$.

Set $\tilde{\tilde{v}}=N(s;n)\tilde{v}$ then (4.37) becomes

$$n^{\sigma}\partial_s\tilde{\tilde{v}}=(in^{1-\varepsilon_1}(J_1+n^{-(1/2)\varepsilon_1}\tilde{A}_1^{(1)}(s)+n^{-\varepsilon_1}\tilde{A}_1^{(2)}(s)+n^{-(3/2)\varepsilon_1}\tilde{A}^{(3)}(s;n)) \tag{4.38}$$
$$+in^{1-2\sigma}NW_1A_2W_1^{-1}N^{-1}+NW_1BW_1^{-1}N^{-1}+n^{\sigma}C(s;n))\tilde{\tilde{v}}.$$

Here we also apply Lemma 2.1 and Lemma 2.2. Then we obtain

$$\tilde{b}_{r+1}^{(1)}=\tilde{b}_2^{(2)}=0, \tag{4.39}$$

where, for $k=1,2$, we denoted the $(2r-1)$-th row and the $(3r-3)$-th row vector of $\tilde{A}_1^{(k)}(s)$ by $(\tilde{b}_1^{(k)},\cdots,\tilde{b}_{3r-3}^{(k)})$ and $(\tilde{c}_1^{(k)},\cdots,\tilde{c}_{3r-3}^{(k)})$ recepectively. We remark that the entries of $\tilde{A}_1^{(1)}(s)$ are zero except for $\tilde{b}_{r+1}^{(1)}$, $\tilde{b}_{2r}^{(1)}$ and $\tilde{c}_{r+1}^{(1)}$ and the entries of $\tilde{A}_1^{(2)}(s)$ are also zero except for $\tilde{b}_2^{(2)}$.

Now we choose our lower order term.

Proposition 4.6. *There exists a matrix B such that*

$$NW_1BW_1^{-1}N^{-1}=n^{(r-(3/2))\varepsilon_1}(\tilde{B}(s)+\tilde{B}_R(s;n)), \tag{4.40}$$

where the entries of $\tilde{B}(s)$, except for $(2r-1,1)$-th entry, are zero and $\tilde{B}_R(s;n)=o(1)$ as $n\to\infty$.

Set $\varepsilon_2=(1-\sigma)/(2r-1)$ and define that $W_2=W_{\varepsilon_2,2r-1}(n)\oplus n^{\xi}W_{\varepsilon_2,r-2}(n)$ then (4.38) becomes

$$n^{\sigma}\partial_s w=in^{1-\varepsilon_1-\varepsilon_2}(J_1+i^{-1}\tilde{B}(s)+C_0(s))w+A_R(s;n)w, \tag{4.41}$$

where $w=W_2\tilde{\tilde{v}}$ and the remainder term A_R is expressed as follows.

$$A_R(s;n)=in^{1-(3/2)\varepsilon_1}W_2(\tilde{A}_1^{(1)}+n^{-(1/2)\varepsilon_1}\tilde{A}_1^{(2)}+n^{-\varepsilon_1}\tilde{A}_1^{(3)})W_2^{-1}$$
$$+in^{1-2\sigma}W_2NW_1A_2W_1^{-1}NW_2^{-1}+n^{(r-(3/2))\varepsilon_1}W_2\tilde{B}_R(s;n)W_2^{-1}+n^{\sigma}W_2\tilde{C}(s;n)W_2^{-1}.$$

In the above expression the term $C_0(s)$ is the matrix whose non-zero component is only $(2r-1,1)$-one, it is equal to the same component of $C^{(2)}(s)$, while $\tilde{C}(s;n)$ is the remainder of $C(s;n)$.

Owing to Lemma 4.3 and (4.38) we obtain

Proposition 4.7. *There exist σ, ξ in the definition of W_2, a positive constant ε_0 and a constant C such that*

(4.42) $$|A_R(s;n)|\leqq Cn^{1-\varepsilon_1-\varepsilon_2-\varepsilon_0} \quad \textit{for large } n.$$

In order to obtain the estimate (4.42) we must actually take σ such that

(4.43) $$\operatorname{Max}\left\{\frac{1}{2}, \frac{3r-3}{3r+1}\right\}<\sigma<\frac{r-1}{r}.$$

The inequality (4.43) is compatible when $r\geqq 3$. Thus Theorem 1.2 is proved.

(Q. E. D.)

Secondly, in the case (III), from Lemma 2.1 and Lemma 2.2 we can see further conditions of A_1.

(4.44) $$a_2=b_{r+1}=0, \qquad a_{2r-1}c_1\doteqdot 0.$$

Set $\varepsilon=1/r$ and define that

$$W=W_{\varepsilon,r}(n)\oplus n^{(1/2)\varepsilon}W_{\varepsilon,r-1}(n)\oplus n^{\varepsilon}W_{\varepsilon,r-2}(n).$$

Operate W to (2.2)' from left, then we obtain

(4.45) $$n^{\sigma}\partial_s w=in^{1-\varepsilon}(J+i^{-1}B)w+A_R(s;n)w,$$

where $w=Wv$ and A_R is expressed as follows.

$$A_R(s;n)=inW(sn^{-\sigma}A_1+n^{-2\sigma}A_2(s;n))W^{-1}.$$

For this A_R we can regard it as a remainder term when we take σ in such a way that

(4.46) $$\operatorname{Max}\left\{\frac{1}{2}, \frac{2r-3}{2r}\right\}<\sigma<\frac{r-1}{r}.$$

It is easy to see that $r\geqq 3$ makes (4.46) compatible. Thus Theorem 1.2 is proved in this case. (Q. E. D.)

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