J. Math. Kyoto Univ. (JMKYAZ)
30-1 (1990) 149-192

Fourier integral operators of infinite order on Gevrey spaces applications to the Cauchy problem for certain hyperbolic operators*

By

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Fourier integral operators on Gevrey spaces have been recently used by various authors[1)] in the study of problems for partial differential operators. In particular they seem to be an appropriate tool for studying well-posedness and propagation of singularities for the Cauchy problem for weakly hyperbolic operators. We consider here Fourier integral operators with amplitude of infinite order, i.e. of a suitable exponential growth in the dual space variables, since parametrices of a simple form can be represented through them for some operators.

The analogous pseudo-differential case has been studied in [24] and applied in [19]. Analytic pseudo-differential operators of infinite order have been considered by L. Boutet de Monvel [2] and in a series of papers by T. Aoki [1]. Finite order analytic pseudo-differential operators are described in [22] and the analogous Gevrey case is studied by Boutet de Monvel-P. Krée [3], S. Hashimoto-T. Matsuzawa-Y. Morimoto [7], V. Iftimie [10] and L.R. Volevič [23].

In section 1 we introduce spaces of locally Gevrey symbols of infinite and of finite order and prove some related properties. The exponential growth allowed to symbols of infinite order requires an appropriate definition of the related oscillatory integrals by means of suitable cut-off functions. Formal series of symbols are also considered. Fourier integral operators with amplitude of infinite order are studied in section 2, mainly following section 1 of [8]. A result on the formal series equivalent to the amplitude of the operator obtained by composition of a pseudo-differential and a Fourier integral operator of infinite orders is proved. Section 2 ends with the study of the action of a Fourier integral operator of infinite order and of its transpose on the Gevrey wave front set of an ultradistribution. Section 3 contains the construction, as a Fourier integral operator of infinite order, of a parametrix to the Cauchy problem for the hyperbolic operator

*) Work supported by CEE research contract *n.* ST2J-0096-2-1 (EDB).

1) See for example [5], [6], [15], [18], [20], [21].

Communicated by Prof. S. Mizohata July 20, 1987. Revised May 6 1988.

(1) $$P=(D_t-\lambda(t,x,D_x))^m+\sum_{j=1}^{m}a_j(t,x,D_x)(D_t-\lambda(t,x,D_x))^{m-j},$$

where $\lambda(t,x,\xi)$ is a real-valued symbol of order one and $a_j, j=1,\dots,m$, are symbols of order pj, $p\in[0,1[$. As functions of x they are analytic in a convex open set Ω of R^n, while as functions of t, the a_j are only continuous and λ is C^{m-1}. A formal series of symbols equivalent to the amplitude of the required parametrix is obtained by solving transport equations derived by the application of the composition theorem of section 2. The necessary estimates for the solutions of the transport equations are proved following [4], where the case $\lambda=0$ is treated. With the aid of the parametrix obtained in this way a semiglobal existence theorem for the solution of the Cauchy problem for the operator (1) is proved. Similar results are shown for an initial value problem for the transpose of the operator (1). Using these results it is possible to prove an uniqueness theorem for the solution of the Cauchy problem for (1) and a representation formula for $u\in C([0,T];G^{(\sigma)}(\Omega))$. By means of this formula the propagation of Gevrey singularities of u when $Pu\in C([0,T];G^{(\sigma)}(\Omega))$ is proved.

In the case when $\Omega=R^n$ and the symbols λ and a_j are in $G_b^{(\sigma)}(R^{n+1})$[2] as functions of (t,x), the well-posedness of the Cauchy problem for P and the propagation of the singularities of the initial values has been proved by K. Taniguchi [21], with the aid of a fundamental solution to the Cauchy problem, and by S. Mizohata [17] by using the energy method.

Part of the results of the present paper have been described in [5] without proof.

Finally we wish to thank Professor Kazuo Taniguchi for his criticism to a crucial point of a previous version of *n*.3 of this paper.

0. Main notation

For $x=(x_1,\dots,x_n)\in R^n$ we set $D_x=(D_{x_1},\dots,D_{x_n})$, $D_j=-i\partial/\partial_{x_j}$, $j=1,\dots,n$, and for $\alpha\in Z_+^n$, Z_+ the set of non negative integers, we let $D_x^\alpha=D_{x_1}^{\alpha_1}\cdots D_{x_n}^{\alpha_n}$, $|\alpha|=\alpha_1+\cdots+\alpha_n$. If $x\in R^n$ we also write $\langle x,\xi\rangle=\sum_{j=1}^{n}x_j\xi_j$.

For a given open set $X\subset R^n$ and given $\sigma>1$, $A>0$ we denote by $G_b^{(\sigma),A}(X)$ the Banach space of all complex-valued functions $\varphi\in C^\infty(X)$ such that

$$\|\varphi\|_{X,A}=\sup_{\substack{x\in X\\ \alpha\in Z_+^n}}A^{-|\alpha|}\alpha!^{-\sigma}|D_x^\alpha\varphi(x)|<+\infty$$

and set

$$G_b^{(\sigma)}(X)=\varinjlim_{A\to+\infty}G^{(\sigma),A}(X),\quad G^{(\sigma)}(X)=\varprojlim_{X'\subset\subset X}G_b^{(\sigma)}(X')$$

2) See section 0 for notation.

and

$$G_0^{(\sigma)}(X) = \varinjlim_{X' \subset\subset X} \varinjlim_{A \to +\infty} G_b^{(\sigma),A}(X') \cap C_0^\infty(X'),$$

where X' are relatively compact open subsets of X.

The dual spaces of $G^{(\sigma)}(X)$ and $G_0^{(\sigma)}(X)$, called spaces of ultradistributions of Gevrey type σ, will be denoted by $G^{(\sigma)\prime}(X)$ and $G_0^{(\sigma)\prime}(X)$ respectively. As is well known the former can be identified with the subspace of the ultradistributions of $G_0^{(\sigma)\prime}(X)$ with compact support.

For functions or ultradistributions u with compact support we consider the Fourier transform $\tilde{u}$ defined by

$$\tilde{u}(\xi) = \int_{R^n} e^{-i\langle x,\xi\rangle} u(x)dx, \qquad \xi \in R^n,$$

when $u \in L^1(R^n)$ and by $\tilde{u}(\xi) = u(e^{-i\langle \cdot,\xi\rangle})$, when $u \in G^{(\sigma)\prime}(X)$.

We shall also denote by σ-singsupp u, the smallest closed subset of X such that the ultradistribution u is in $G^{(\sigma)}$ in the complement and by $WF_{(\sigma)}(u)$ the complement in $X \times R^n \setminus \{0\}$ of the set of (x_0, ξ_0) such that there exist a neighborhood U of x_0, a conic neighborhood Γ of ξ_0 in $R^n \setminus \{0\}$ and a function $\chi \in G_0^{(\sigma)}(X)$ equal to one in U such that for some positive constants c and h

$$|(\widetilde{\chi u})(\xi)| \leqq c \exp(-h|\xi|^{1/\sigma}), \qquad \xi \in \Gamma. \tag{0.1}$$

If V is a topological vector space, $\mathscr{A}$ a subset of R^k and $m \in Z_+$, we shall denote by $\mathscr{B}^m(\mathscr{A}; V)$ the set of all V-valued functions on $\mathscr{A}$ which are bounded on $\mathscr{A}$ together with all their derivatives up to the order m. We shall also write $\mathscr{B}(\mathscr{A}; V)$ instead $\mathscr{B}^0(\mathscr{A}; V)$ and set $\mathscr{B}^m_{loc}(\mathscr{A}; V) = \bigcap_{A' \subset\subset A} \mathscr{B}^m(\mathscr{A}'; V)$.

1. Symbols of infinite order of Gevrey type and related oscillatory integrals

1.1. Symbols of infinite order

Definition 1.1.1. Let X be an open set of R^ν and let $R^N_{B_0} = \{\xi \in R^N; |\xi| > B_0\}$, $B_0 \geqq 0$. For $\sigma > 1$, $\mu \in [1, \sigma]$, $A > 0$, $B \geqq 0$ we denote by $S_b^{\infty,\sigma,\mu}(X \times R^N_{B_0,B}; A)$ the space of all complex-valued functions a defined on $X \times R^N_{B_0}$ such that for every $\varepsilon > 0$

$$\|a\|_{X,\varepsilon}^{A,B_0,B} = \sup_{\substack{\alpha \in Z_+^N \\ \beta \in Z_+^\nu}} \sup_{\substack{x \in X \\ |\xi| > B|\alpha|^\sigma + B_0}} A^{-|\alpha|-|\beta|} \alpha!^{-\mu} \beta!^{-\sigma} (1+|\xi|)^{|\alpha|} \exp(-\varepsilon|\xi|^{1/\sigma}) \times |D_\xi^\alpha D_x^\beta a(x,\xi)| < +\infty.$$

Endowed with the topology defined by the family of seminorms $\|a\|_{X,\varepsilon}^{A,B_0,B}$, $\varepsilon > 0$, $S_b^{\infty,\sigma,\mu}(X \times R^N_{B_0,B}; A)$ is a Fréchet space. Moreover $\|a\|_{X,\varepsilon}^{A',B_0',B'} \leqq \|a\|_{X,\varepsilon}^{A,B_0,B}$ if $A \leqq A'$, $B_0 \leqq B_0'$, $B \leqq B'$ and $a \in S_b^{\infty,\sigma,\mu}(X \times R^N_{B_0,B}; A)$.

Proposition 1.1.2. *Let X' be a relatively compact open subset of X ($X' \subset\subset X$) and let $A < A'$, $B_0 \leqq B'_0$, $B \leqq B'_0$, Then every subset $\mathscr{A}$ of $C^\infty(X \times R^N)$ which is bounded in $C^\infty(X' \times R^N)$ and in $S_b^{\infty,\sigma,\mu}(X' \times R^N_{B_0,B}; A)$ is relatively compact in $S_b^{\infty,\sigma,\mu}(X' \times R^N_{B'_0,B'}; A')$.*

Proof. For given positive ε, r let $W_{\varepsilon,r} = \{a \in S_b^{\infty,\sigma,\mu}(X' \times R^{N}_{B'_0,B'}; A'); \|a\|^{A',B'_0,B'}_{X',\varepsilon} < r\}$. Since $\mathscr{A}$ is bounded in $S_b^{\infty,\sigma,\mu}(X' \times R^N_{B_0,B}; A)$, for every $\eta > 0$ there exists $c_\eta > 0$ such that $\|a\|^{A,B_0,B}_{X',\eta} \leqq c_\eta$ for every $a \in \mathscr{A}$.

If $m \in Z_+$ is such that $(A/A')^m < r/2c_\varepsilon$, then for every $\alpha \in Z^N_+$, $B \in Z^\nu_+$ such that $|\alpha| + |\beta| \geqq m$ and for any $a \in \mathscr{A}$

$$\begin{aligned}(1.1.1)\quad &\sup_{x\in X'} \sup_{|\xi| > B'|\alpha|^\sigma + B'_0} A'^{-|\alpha|-|\beta|}\alpha!^{-\mu}\beta!^{-\sigma}(1 + |\xi|)^{|\alpha|}\exp(-\varepsilon|\xi|^{1/\sigma}) \\ &\qquad \times |D^\alpha_\xi D^\beta_x a(x, \xi)| \leqq (A/A')^{|\alpha|+|\beta|}\|a\|^{A,B_0,B}_{X',\varepsilon} < r/2.\end{aligned}$$

Choose now $\rho > 0$ such that $\exp(-\varepsilon\rho^{1/\sigma}/2) < r/2c_{\varepsilon/2}$ and $B'|\alpha|^\sigma + B'_0 < \rho$ when $|\alpha| \leqq m$. Then for $|\alpha| + |\beta| \leqq m$ and any $a \in \mathscr{A}$

$$(1.1.2)\quad \sup_{x\in X'} \sup_{|\xi| > \rho} A'^{-|\alpha|-|\beta|}\alpha!^{-\mu}\beta!^{-\sigma}(1 + |\xi|)^{|\alpha|}\exp(-\varepsilon|\xi|^{1/\sigma})|D^\alpha_\xi D^\beta_x a(x, \xi)| < r/2.$$

On the other hand, since $\mathscr{A}$ is bounded in $C^\infty(X' \times R^N)$, there exist $a_1, \dots, a_k \in \mathscr{A}$ such that

$$(1.1.3)\quad \mathscr{A} \subset \bigcup_{i=1}^{k} \{a_i + V\},$$

where $V = \{a \in C^\infty(X' \times R^N); \sup_{|\alpha|+|\beta|\leqq m} \sup_{\substack{x\in X' \\ |\xi|\leqq\rho}} |D^\alpha_\xi D^\beta_x a(x, \xi)| \leqq r\inf(A'^m, 1)(1 + \rho)^{-m}\}$.

Since form (1.1.1), (1.1.2), (1.1.3) it follows that $\mathscr{A} \subset \bigcup_{i=1}^{k} \{a_i + W_{\varepsilon,r}\}$, the proposition is proved in view of well known results about Fréchet spaces.

Definition 1.1.3. Let X be an open set in R^ν and let $\sigma > 1$, $\mu \in [1, \sigma]$. We let

$$S^{\infty,\sigma,\mu}_{b,N}(X) = \varinjlim_{A,B_0,B\to+\infty} S_b^{\infty,\sigma,\mu}(X \times R^N_{B_0,B}; A)$$

and

$$S^{\infty,\sigma,\mu}_{N}(X) = \varprojlim_{X'\to X} S^{\infty,\sigma,\mu}_{b,N}(X'),$$

where X' denotes a relatively compact open subset of X.

We shall also write

$$\tilde{S}^{\infty,\sigma,\mu}_{N}(X) = \varprojlim_{X'\to X} \tilde{S}^{\infty,\sigma,\mu}_{b,N}(X') = \varprojlim_{X'\to X} \varinjlim_{A,B_0\to+\infty} S_b^{\infty,\sigma,\mu}(X' \times R^N_{B_0,0}; A).$$

From Proposition 1.1.2 it follows

Lemma 1.1.4. *Let $\{a_k\}_{k\in Z^+}$ be a sequence in $C^\infty(X \times R^N)$ which is bounded in $S_N^{\infty,\sigma,\mu}(X)$ and assume that $a_k \to a$ in $C^\infty(X \times R^N)$ as $k \to +\infty$. Then $a \in S_N^{\infty,\sigma,\mu}(X)$ and $a_k \to a$ in $S_N^{\infty,\sigma,\mu}(X)$ as $k \to +\infty$.*

Corollary 1.1.5. *Let $\mathscr{A}$ be a subset of R^k and let $a \in C(\mathscr{A}, C^\infty(X \times R^N)) \cap \mathscr{B}(\mathscr{A}; S_N^{\infty,\sigma,\mu}(X))$. Then $a \in C(\mathscr{A}; S_N^{\infty,\sigma,\mu}(X))$.*

For $a \in S_N^{\infty,\sigma,\mu}(X)$ we shall also need to estimate

$$\|a\|_{X',\varepsilon,m}^{A,B_0,B} = \sup_{\substack{\alpha\in Z_+^N \\ \beta\in Z_+^\nu}} \sup_{\substack{x\in X' \\ |\xi|>B(|\alpha|+m)^\sigma+B_0}} A^{-|\alpha|-|\beta|}\alpha!^{-\mu}\beta!^{-\sigma}(1+|\xi|)^{|\alpha|+m}\exp(-\varepsilon|\xi|^{1/\sigma}) \times |D_\xi^\alpha D_x^\beta a(x,\xi)|,$$

where $X' \subset\subset X$, $A > 0$, $B_0 \geqq 0$, $B \geqq 0$, $\varepsilon > 0$, $m \geqq 0$.

The following propositions are easy to prove.

Proposition 1.1.6. *Let $a \in S_b^{\infty,\sigma,\mu}(X \times R_{B_0,B}^N; A)$ and let $\alpha \in Z_+^N$, $\beta \in Z_+^\nu$. Then for every $\varepsilon > 0$, $m \geqq 0$*

$$\|D_\xi^\alpha D_x^\beta a\|_{X,\varepsilon,|\alpha|+m}^{A',B_0,B} \leqq A'^{|\alpha|+|\beta|}\alpha!^\mu\beta!^\sigma\|a\|_{X,\varepsilon,m}^{A,B_0,B},$$

where $A' \geqq 2^\sigma A$.

Proposition 1.1.7. *If $a_i \in S_b^{\infty,\sigma,\mu}(X \times R_{B_{0i},B_i}^N; A_i)$, $i = 1,2$, then for every $\varepsilon_i > 0$, $m_i \geqq 0$, $i = 1,2$*

$$\|a_1 a_2\|_{X,\varepsilon,m_1+m_2}^{A,B_0,B} \leqq \|a_1\|_{X,\varepsilon_1,m_1}^{A_1,B_{01},B_1}\|a_2\|_{X,\varepsilon_2,m_2}^{A_2,B_{02},B_2},$$

where $A \geqq A_1 + A_2$, $B_0 \geqq \sup(B_{01}, B_{02})$, $B \geqq \sup(B_1, B_2)$, $\varepsilon \geqq \varepsilon_1 + \varepsilon_2$.

Example 1.1.8. Let $\chi \in C^\infty(R^N)$ be such that $\chi(0) = 1$ and there exist positive constants c_0, c_1, h such that for every $\xi \in R^N$ and $\alpha \in Z_+^N$

$$|D^\alpha\chi(\xi)| \leqq c_0 c_1^{|\alpha|}\alpha!(1 + |\xi|)^{-|\alpha|}\exp(-h|\xi|^{1/\sigma}).$$ [3)]

If $a \in S_N^{\infty,\sigma,\mu}(X) \cap C^\infty(X \times R^N)$ and $\rho \in [0, 1]$, then by Proposition 1.1.7 with $m_1 = m_2 = 0$ the set $\mathscr{A} = \{a_\rho(x, \xi) = \chi(\rho\xi)a(x, \xi), \rho \in [0, 1]\}$ is bounded in $S_N^{\infty,\sigma,\mu}(X)$. Since $a_\rho \to a$ in $C^\infty(X \times R^N)$ as $\rho \to 0$, then by Lemma 1.1.4 $a_\rho \to a$ in $S_N^{\infty,\sigma,\mu}(X)$ as $\rho \to 0$.

Proposition 1.1.9. *Let $a \in S_b^{\infty,\sigma,\mu}(X \times R_{B_0,0}^N)$ and let*

$$P(D_x) = \sum_{|\beta|=0}^{\infty} c_\beta D_x^\beta$$

be an ultradifferential operator on $G^{(\sigma)}(X)$ with constant coefficients, i.e. [4)] *let $c_\beta \in \mathbf{C}$ and assume that for every $h > 0$ there exists $H_h > 0$ such that*

[3)] For example let $\chi(\zeta) = \exp(-(1 + \zeta^2)^{1/2\sigma})$, $\zeta \in C^N$, $|\mathrm{Im}\zeta| < c(1 + |\mathrm{Re}\zeta|)$, $c \in]0, 1/2]$.

[4)] See [12], p. 47.

$$|c_\beta| \leqq H_h h^{|\beta|} \beta!^{-\sigma}, \quad \beta \in Z^\nu_+.$$

Then for $h < (2^\delta \nu A)^{-1}$, $A' \geqq 2^\delta A$, $\varepsilon > 0$

$$\| P(D_x) a \|^{A', B_0, 0}_{X, \varepsilon} \leqq H_h (1 - 2^\sigma \nu A h)^{-1} \| a \|^{A, B_0, 0}_{X, \varepsilon}.$$

We now define spaces of symbols of finite order similar to those defined in [7] and [10]. They can also be considered as subspaces of the spaces defined above.

Definition 1.1.10. For $\sigma > 1$, $\mu \in [1, \sigma]$, $A > 0$, $B_0 \geqq 0$, $B \geqq 0$, $m \in R$, X an open subset of R^ν we denote by $S^{m,\sigma,\mu}_b(X \times R^N_{B_0,B}; A)$ the Banach space of all complex-valued functions a defined on $X \times R^N_{B_0}$ with the norm

$$|a|^{A,B_0,B}_{X,m} = \sup_{\substack{\alpha \in Z^N_+ \\ \beta \in Z^\nu_+}} \sup_{\substack{x \in X \\ |\xi| > B|\alpha|^\sigma + B_0}} A^{-|\alpha| - |\beta|} \alpha!^{-\mu} \beta!^{-\sigma} (1 + |\xi|)^{-m + |\alpha|} |D^\alpha_\xi D^\beta_x a(x, \xi)| < +\infty.$$

We then define

$$S^{m,\sigma,\mu}_{b,N}(X) = \varinjlim_{A, B_0, B \to +\infty} S^{m,\sigma,\mu}_b(X \times R^N_{B_0,B}; A)$$

and

$$S^{m,\sigma,\mu}_N(X) = \varprojlim_{X' \to X} S^{m,\sigma,\mu}_{b,N}(X'),$$

where X' are relatively compact open subsets of X, and denote by $\tilde{S}^{m,\sigma,\mu}_{b,N}(X)$, $\tilde{S}^{m,\sigma,\mu}_N(X)$ the subspaces obtained by letting $B = 0$ in the definitions above.

The following simple propositions will be used.

Proposition 1.1.11. *Let* $a \in S^{p,\sigma,\mu}_b(X)$, $p \in [0, 1/\sigma[$. *Then* $e^a \in S^{\infty,\sigma,\mu}_{b,N}(X)$ *and the map* $a \to e^a$ *is continuous from* $S^{p,\sigma,\mu}_{b,N}(X)$ *to* $S^{\infty,\sigma,\mu}_{b,N}(X)$.

Proposition 1.1.12. *Let* $a \in S^{1,\sigma,\mu}_b(X \times R^N_{B_0,B}; A)$. *Then for every* $\beta \in Z^\nu_+$, $e^{-a} D^\beta_x e^a \in S^{|\beta|,\sigma,\mu}_b(X \times R^N_{B_0,B}; |\beta| A)$ *and for every* $\varepsilon > 0$

$$\| e^{-a} D^\beta_x e^a \|^{|\beta| A, B_0, B}_{X, \varepsilon} \leqq (dA)^{|\beta|} \beta!^\sigma \varepsilon^{-\sigma|\beta|} (|a|^{A,B_0,B}_{X,1} + 1)^{|\beta|},$$

where d *is a constant which depends only on* ν, N, σ.

The proof of this proposition is obtained by using Faà di Bruno formula for the derivatives of the composite functions and the estimate

$$(1.1.4) \qquad \sum_{\substack{0 \neq \gamma \leqq \beta \\ \eta_\gamma \geqq 0 \\ \Sigma \gamma \eta_\gamma = \beta}} \frac{|\Sigma \eta_\gamma|!}{\prod_\gamma \eta_\gamma!} \leqq d'^{|\beta|},$$

where $d' > 0$ depends only on ν.

In the same way it can also be proved

Proposition 1.1.11′. *Let* $\phi \in S^{q,\sigma,\mu}_b(X \times R^N_{B_0,B}; A)$ *be real-valued and let*

$q \in [0, \mu/\sigma[$. *Then* $e^{i\phi} \in S^{\infty,\sigma,\mu}(X \times R^N_{B0,B}; A')$, $A' \geqq c(\mu, v, N)A$ *and the map* $\phi \to e^{i\phi}$ *is continuous from* $S^{q,\sigma,\mu}_{b,N}(X)$ *to* $S^{\infty,\sigma,\mu}_{b,N}(X)$.

With the same arguments used for proving Lemma 5.3 of [11] we can obtain

Proposition 1.1.13. *Let* $Q = \sum_{h=1}^{v} a_h(x, \xi)\partial_{x_h} + \sum_{i=1}^{N} b_i(x, \xi)\partial_{\xi_i} + a_0(x, \xi) + b_0(x, \xi)$, a_h, $b_i \in C^\infty(X \times R^N_{B0})$, $h = 0,\ldots, v$; $i = 0,\ldots, N$, *and assume that there exist positive constants* c_0, C_0 *such that for every* $\alpha \in Z^N_+$, $\beta \in Z^v_+$, $(x, \xi) \in X \times R^N_{B0}$

$$|D^\alpha_\xi D^\beta_x a_h(x, \xi)| \leqq c_0 C_0^{|\alpha|+|\beta|}(1 + |\xi|)^{-|\alpha|}|\alpha|!^\mu |\beta|!^\sigma, \ h = 1,\ldots, v,$$
$$|D^\alpha_\xi D^\beta_x b_i(x, \xi)| \leqq c_0 C_0^{|\alpha|+|\beta|}(1 + |\xi|)^{-|\alpha|+1}|\alpha|!^\mu |\beta|!^\sigma, \ i = 1,\ldots, N,$$
$$|D^\alpha_\xi D^\beta_x a_0(x, \xi)| \leqq c_0 C_0^{|\alpha|+|\beta|+1}(1 + |\xi|)^{-|\alpha|}|\alpha|!^\mu (|\beta| + 1)!^\sigma,$$
$$|D^\alpha_\xi D^\beta_x b_0(x, \xi)| \leqq c_0 C_0^{|\alpha|+|\beta|+1}(1 + |\xi|)^{-|\alpha|}(|\alpha| + 1)!^\mu |\beta|!^\sigma.$$

Let $a \in C^\infty(X \times R^N_{B0})$ *and assume that there exist constants* $A \geqq kC_0$, $k > 1$, $B \geqq 0$, $m \geqq 0$, $p \geqq 0$, $q \geqq 0$, $\mu \in [1, \sigma]$ *such that for every* $\varepsilon \geqq 0$

$$\sup_{\substack{\alpha\in Z^N_+\\ \beta\in Z^v_+}} \sup_{\substack{x\in X\\ |\xi|>B(|\alpha|+m)^\sigma+B_0}} A^{-|\alpha|-|\beta|-m}(1 + |\xi|)^{|\alpha|+m}(|\alpha| + p)!^{-\mu}(|\beta| + q)!^{-\sigma}\exp(-\varepsilon|\xi|^{1/\sigma})$$
$$\times |D^\alpha_\xi D^\beta_x a(x, \xi)| = c_a(\varepsilon, A, B, m, p, q) < +\infty.$$

Then

$$\sup_{\substack{\alpha\in Z^N_+\\ \beta\in Z^v_+}} \sup_{\substack{x\in X\\ |\xi|>B(|\alpha|+m+j)^\sigma+B_0}} (2^\sigma A)^{-|\alpha|-|\beta|-j}A^{-m}(1 + |\xi|)^{|\alpha|+m}(|\alpha| + p)!^{-\mu}$$
$$\times (|\beta| + q)!^{-\sigma}\exp(-\varepsilon|\xi|^{1/\sigma})|D^\alpha_\xi D^\beta_x Q^j a(x, \xi)|$$
$$\leqq c_a(\varepsilon, A, B, m, p, q)(c_0 ck(k-1)^{-1})^j j!^\sigma 2^{\sigma(p+q)},$$

where $c = v + N + 2$ *and* $j!^\sigma$ *is replaced by* $j!^\mu$ *when* $a_h(x, \xi) \equiv 0$, $h = 1,\ldots, v$.

Remark 1.1.13′. Proposition 1.1.13 with c_0 replaced by $c_0(1 + |\xi|)^{-1}$ also holds and will be used in the sequel.

Lemma 1.1.14. *Let* Y *be an open subset of* R^{v_1} *and let* $\phi = (\phi', \phi'')$ *be a* C^∞ *map from* $Y \times R^{N_1}_{D0}$ *to* $X \times R^N_{B0}$, $D_0 > 0$, $B_0 > 0$. *Assume that there exist positive constants* c_0, C_0, $\mu \in [1, \sigma]$, $R > B_0 D_0^{-1}$ *such that*

i) $|\phi''(y, \eta)| \geqq R|\eta|$, $(y, \eta) \in Y \times R^{N_1}_{D0}$;

ii) *for every* $\gamma \in Z^{N_1}_+$, $\delta \in Z^{v_1}_+$, $(y, \eta) \in Y \times R^{N_1}_{D0}$

$$|D^\gamma_\eta D^\delta_y \phi'_h(y, \eta)| \leqq c_0(C_0/(1 + |\eta|))^{|\gamma|}C_0^{|\delta|}\gamma!^\mu \delta!^\sigma, \qquad h = 1,\ldots, v,$$
$$|D^\gamma_\eta D^\delta_y \phi''_i(y, \eta)| \leqq c_0|\eta|(C_0/(1 + |\eta|))^{|\gamma|}C_0^{|\delta|}\gamma!^\mu \delta!^\sigma, \qquad i = 1,\ldots, N.$$

Suppose that $a \in C^\infty(X \times R^N_{B_0})$ *and that there exist* $A > 0, B \geqq 0, p, q, m \geqq 0$ *such that for every* $\varepsilon \geqq 0$, $c_a(\varepsilon, A, B, m, p, q) < +\infty$ [5]. *Then for every* $\gamma \in Z^{\nu_1}_+$, $\delta \in Z^{N_1}_+$, $(y, \eta) \in Y \times R^{N_1}_{D_0}$, $|\eta| > R^{-1}B(|\gamma| + |\delta| + m)^\delta + D_0$, $\varepsilon > 0$,

$$|D^\gamma_\eta D^\delta_y(a \circ \phi)(y, \eta)| \leqq c_a(\varepsilon, A, B, m, p, q) 2^{\sigma(p+q)} \exp(\varepsilon(c_0|\eta|)^{1/\sigma}) A'^{|\delta|} \times (A'/(1 + |\eta|))^{|\gamma|+m} p!^\mu q!^\sigma \gamma!^\sigma \delta!^\sigma,$$

where $A' \geqq 2^\sigma \tilde{c}_0 \tilde{A} \tilde{R} k(k-1)^{-1}(\nu + 1 + N)(N_1 \nu_1)^\sigma$, $\tilde{c}_0 = \sup(c_0 C_0, 1)$, $\tilde{A} \geqq \sup(A, k2^\sigma C_0)$, $\tilde{R} = \sup(1, R^{-1})$, $k > 1$, *and* $\gamma!^\sigma$ *is replaced by* $\gamma!^\mu$ *when* ϕ' *does not depend on* η.

The proof of this lemma can be obtained as an application of Proposition 1.1.13 and Remark 1.1.13′, by letting Q be equal to

$$\chi_j = \sum_{h=1}^{\nu} \partial_{y_j} \phi'_h \partial_{x_h} + \sum_{i=1}^{N} \partial_{y_j} \phi''_i \partial_{\xi_i} + \partial_{y_j}, \ j = 1, \dots, \nu_1$$

or

$$H_\ell = \sum_{h=1}^{\nu} \partial_{\eta_\ell} \phi'_h \partial_{x_h} + \sum_{i=1}^{N} \partial_{\eta_\ell} \phi''_i \partial_{\xi_i} + \partial_{\eta_\ell}, \ \ell = 1, \dots, N_1$$

and noting that

$$\partial^\gamma_\eta \partial^\delta_y (a \circ \phi)(y, \eta) = (H_1^{\gamma_1} \cdots H_{N_1}^{\gamma_{N_1}} X_1^{\delta_1} \cdots X_{\nu_1}^{\delta_{\nu_1}} a(x, \xi))|_{\substack{x = \phi'(y,\eta) \\ \xi = \phi''(y,\eta)}} \ ^{6)}.$$

Corollary 1.1.15. *Let the hypotheses* i) *and* ii) *of Lemma* 1.1.14 *be satisfied. Furthermore assume that*

iii) $a \in S^{\infty,\sigma,\mu}_b(X \times R^N_{B_0,0}; A)$;

or

iii′) $a \in S^{\infty,\sigma,\mu}_b(X \times R^N_{B_0,B}; A)$, $B > 0$ *and* ϕ'' *does not depend on* y,

then $a \circ \phi \in S^{\infty,\sigma,\sigma}_b(Y \times R^{N_1}_{D_0,R^{-1}B}; A')$ [7] *and* $\|a \circ \phi\|^{A',D_0,R^{-1}B}_{Y,c_0^{1/\sigma}\varepsilon} \leqq \|a\|^{A,B_0,B}_{X,\varepsilon}$,

where A' *is as in Lemma* 1.1.14.

The following definitions of formal series of symbols and of equivalence of formal series of symbols, given in [24] when $\mu = 1$, are needed.

Definition 1.1.16. A series $\sum_{j \geqq 0} a_j(x, \xi)$, $a_j \in S^{\infty,\sigma,\mu}_b(X \times R^N_{B_0,B}; A)$, is called a formal series of symbols in $S^{\infty,\sigma,\mu}_b(X \times R^N_{B_0,B}; A)$ if for every $\varepsilon > 0$

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5) Notation as in Proposition 1.1.13.

6) See the proof of Lemma 5.4 of [11].

7) $a \circ \phi \in S^{\infty,\sigma,\mu}_b(Y \times R^{N_1}_{D_0,R^{-1}B}; A')$ if ϕ' does not depend on η.

$$(1.1.5) \quad \sup_{j\in Z_+}\sup_{\substack{\alpha\in Z_+^N\\ \beta\in Z_+^v}}\sup_{\substack{x\in X'\\ |\xi|>B(|\alpha|+j)^\sigma+B_0}} A^{-|\alpha|-|\beta|-j}\alpha!^{-\mu}(j!\beta!)^{-\sigma}(1+|\xi|)^{|\alpha|+j} \times \exp(-\varepsilon|\xi|^{1/\sigma})|D_\xi^\alpha D_x^\beta a_j(x,\xi)| < +\infty.$$

The Fréchet space of all formal series of symbols in $S_b^{\infty,\sigma,\mu}(X \times R_{B_0,B}^N; A)$ with the right hand sides of (1.1.5) as seminorms will be denoted by $FS_b^{\infty,\sigma,\mu}(X \times R_{B_0,B}^N; A)$. We also set

$$FS_N^{\infty,\sigma,\mu}(X) = \varinjlim_{X'\subset\subset X}\ \varinjlim_{A,B_0,B\to+\infty} FS_b^{\infty,\sigma,\mu}(X \times R_{B_0,B}^N; A).$$

We can identify $S_N^{\infty,\sigma,\mu}(X)$ with the subspace of all $\sum_{j\geqq 0} a_j \in FS_N^{\infty,\sigma,\mu}(X)$ such that $a_j = 0$ for every $j > 0$.

Definition 1.1.17. We shall say that two series $\sum_{j\geqq 0} a_j$, $\sum_{j\geqq 0} b_j$ in $FS_N^{\infty,\sigma,\mu}(X)$ are equivalent and write $\sum_{j\geqq 0} a_j \sim \sum_{j\geqq 0} b_j$ if for every open set $X' \subset\subset X$ there exist constants $A > 0$, $B_0 \geqq 0$, $B \geqq 0$ such that for every $\varepsilon > 0$

$$\sup_{s\in Z_+}\sup_{\substack{\alpha\in Z_+^N\\ \beta\in Z_+^v}}\sup_{\substack{x\in X'\\ |\xi|>B(|\alpha|+s)^\sigma+B_0}} A^{-|\alpha|-|\beta|-s}\alpha!^{-\mu}(\beta!s!)^{-\sigma}(1+|\xi|)^{|\alpha|+s}\exp(-\varepsilon|\xi|^{1/\sigma}) \times |D_\xi^\alpha D_x^\beta \sum_{j<s}[a_j(x,\xi) - b_j(x,\xi)]| < +\infty.$$

Let now $\{g_j\}_{j\in Z_+}$ be a sequence in $C_0^\infty(R^N)$ such that

$$(1.1.6) \quad \begin{cases} 0 \leqq g_j(\xi) \leqq 1 \text{ for } \xi\in R^N,\ g_j(\xi) = 1 \text{ for } |\xi| \leqq 2,\ g_j(\xi) = 0 \text{ for } |\xi| \geqq 3 \\ |D^\alpha g_j(\xi)| \leqq (cj)^{|\alpha|},\ |\alpha| \leqq j,\ \xi\in R^N, \end{cases}$$

where c is a positive constant [8] and for any given $R > 0$ consider the sequence $\{\phi_j\}_{j\in Z_+}$ in $C^\infty(R^N)$ defined by

$$\phi_0(\xi) = 1 - g_0(\xi/R),\ \phi_j(\xi) = 1 - g_j(\xi/Rj^\sigma),\ j \geqq 1.$$

With the aid of this sequence we can prove as in [24]

Lemma 1.1.18. *Let* $\sum_{j\geqq 0} a_j \in FS_N^{\infty,\sigma,\mu}(X)$. *Then for every* $X' \subset\subset X$ *there exists* $R>0$ *such that* $\sum_{j\geqq 0} \phi_j a_j|_{X'} = a_{X'} \in S_N^{\infty,\sigma,\mu}(X)$ *and* $a_{X'} \sim \sum_{j\geqq 0} a_j|_{X'}$ *in* $FS_N^{\infty,\sigma,\mu}(X')$.

From this lemma by means of a partition of unity in $G_0^{(\sigma)}(X)$ related to a locally finite covering of X by relatively compact open subsets we obtain

Corollary 1.1.19. *For every* $\sum_{j\geqq 0} a_j \in FS_N^{\infty,\sigma,\mu}(X)$ *there exists* $a \in S_N^{\infty,\sigma,\mu}(X)$ *such*

[8] For the existence of a sequence with these properties see [16] [22].

that $a \sim \sum_{j \geqq 0} a_j$ *in* $FS_N^{\infty,\sigma,\mu}(X)$.

We also note

Proposition 1.1.20. [24]. *Let* $a \sim 0$ *in* $FS_N^{\infty,\sigma,\mu}(X)$. *Then for every open set* $X' \subset \subset X$ *there exist constants* $A > 0$, $B_0 \geqq 0$, $h > 0$ *such that*

$$\sup_{\beta \in Z_+^\nu} \sup_{\substack{x \in X' \\ |\xi| > B_0}} A^{-|\beta|}\beta!^{-\sigma}\exp(h|\xi|^{1/\sigma})|D_x^\beta a(x, \xi)| < +\infty.$$

From Corollary 1.1.15 it also follows

Theorem 1.1.21. *Let* $\sum_{j \geqq 0} a_j \in FS_N^{\infty,\sigma,\mu}(X)$ *and let* $\phi = (\phi', \phi'')$ *be as in Lemma* 1.1.14. *Assume that one of the following conditions is satisfied*

i) $\sum_{j \geqq 0} a_j \in F\tilde{S}_N^{\infty,\sigma,\mu}(X) = \varprojlim_{X' \subset \subset X} \varinjlim_{A, B_0 \to +\infty} FS_b^{\infty,\sigma,\mu}(X \times R_{B_0,0}^N; A)$;

ii) ϕ'' *does not depend on* y.

Then $\sum_{j \geqq 0} (a_j \circ \phi) \in FS_{N_1}^{\infty,\sigma,\sigma}(Y)$ [9].

1.2. Oscillatory integrals. Let X be an open set of R^ν and let $\phi \in C^\infty(X \times R^N) \cap \tilde{S}_N^{1,\sigma,\mu}(X)$ be real-valued and such that for every $X' \subset \subset X$ there exist constants $B_0 \geqq 0$, $M_\phi > 0$ such that

$$\theta(x, \xi) = (|\nabla_x \phi|^2 + |\xi|^2 |\nabla_\xi \phi|^2)^{-1} \leqq M_\phi |\xi|^{-2}, \ (x, \xi) \in X' \times R_{B_0}^N. \tag{1.2.1}$$

If $a \in L^1(X' \times R^N)$ for every $X' \subset \subset X$, the integral

$$I_\phi(au) = \iint e^{i\phi(x,\xi)} a(x, \xi) u(x) dx\, đ\xi, \ đ\xi = (2\pi)^{-N} d\xi \tag{1.2.2}$$

is well defined for every $u \in G_0^{(\sigma)}(X)$. In order to define $I_\phi(au)$ as a continuous extension of this integral for every $a \in C^\infty(X \times R^N) \cap S_N^{\infty,\sigma,\mu}(X)$, we consider for any given $R > 0$ the sequence $\{\psi_j\}_{j \in Z_+}$ in $C_0^\infty(R^N)$ defined by

$$\psi_0(\xi) = g_1(\xi/R), \ \psi_j(\xi) = g_{j+1}(\xi/(R(j+1)^\sigma)) - g_j(\xi/(Rj^\sigma)), \ j \geqq 1, \tag{1.2.3}$$ [10]

where $\{g_j\}_{j \in Z_+}$ is a sequence in $C_0^\infty(R^N)$ with the properties (1.1.6).

Note that $\{\psi_j\}$ is a partition of unity in $C_0^\infty(R^N)$ and that

$$\operatorname{supp}\psi_0 \subset \{|\xi| \leqq 3R\}, \ \operatorname{supp}\psi_j \subset \{2Rj^\sigma \leqq |\xi| \leqq 3R(j+1)^\sigma\}, \ j \leqq 1,$$

$$|D^\alpha \psi_j(\xi)| \leqq 2(c/(Rj^{\sigma-1}))^{|\alpha|}, \ |\alpha| \leqq j, \ \xi \in R^N. \tag{1.2.4}$$

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9) $\sum_{j \geqq 0} (a_j \circ \phi) \in FS_{N_1}^{\infty,\sigma,\mu}(Y)$ if ϕ' does not depend on η.

10) See [24].

Note also that if we let

$$a_h = i\theta\partial_{x_h}\phi,\ h = 1,\dots,\nu,\qquad b_j = i|\xi|^2\theta\partial_{\xi_j}\phi,\ j = 1,\dots,N,$$

(1.2.5) $$a_0 = \sum_{h=1}^{\nu} \partial_{x_h} a_h,\quad b_0 = \sum_{j=1}^{N} \partial_{\xi_j} b_j,$$

then the transpose ${}^tL$ of the operator

(1.2.6) $$L = \sum_{h=1}^{\nu} a_h\partial_{x_h} + \sum_{j=1}^{N} b_j\partial_{\xi_j} + a_0 + b_0$$

leaves $e^{i\phi}$ unchanged. Thus if $u \in G_0^{(\sigma)}(X)$ and $a \in C^\infty(X \times R^N) \cap S_N^{\infty,\sigma,\mu}(X)$ we have

$$I_\phi(\psi_j au) = \iint e^{i\phi(x,\xi)} L^j(\psi_j(\xi)a(x,\xi)u(x))dx\bar{d}\xi,\ j = 1,\dots$$

and

(1.2.7) $$I_\phi(au) = \sum_{j\geqq 0} I_\phi(\psi_j au),$$

when $a \in L^1(X' \times R^N)$, for every $X' \subset\subset X$.

With the aid of Lemma 1.1.14 it can be easily proved

Proposition 1.2.1. *For every $X' \subset\subset X$ there is a constant $A_\phi > 0$ such that for every $\alpha \in Z_+^N$, $\beta \in Z_+^\nu$, $(x,\xi) \in X' \times R_{B_0}^N$*

$$|D_\xi^\alpha D_x^\beta a_h(x,\xi)| \leqq c_\phi C_\phi^{|\alpha|+|\beta|}(1+|\xi|)^{-|\alpha|-1}\alpha!^\mu\beta!^\sigma,\qquad h = 1,\dots,\nu,$$

$$|D_\xi^\alpha D_x^\beta b_j(x,\xi)| \leqq c_\phi C_\phi^{|\alpha|+|\beta|}(1+|\xi|)^{-|\alpha|}\alpha!^\mu\beta!^\sigma,\qquad j = 1,\dots,N,$$

where a_h and b_j are defined in (1.2.5), $c_\phi = d_1 A_\phi|\phi|_{X',1}^{A_\phi,B_0,0}$, $C_\phi = d_2 A_\phi(1 + A_\phi|\phi|_{X',1}^{A_\phi,B_0,0} M_\phi)$, *$B_0$ and M_ϕ as in* (1.2.1), *d_1, d_2 positive constants independent of ϕ.*

Proposition 1.2.2. *Let L be the operator in* (1.2.6) *and let ψ_j, $j \geqq 1$, be defined as in* (1.2.3). *Assume that $a \in S_{b,N}^{\infty,\sigma,\mu}(X' \times R_{B_0,B}^N; A_a)$, $u \in G_0^{(\sigma),A_u}(X')$, $X' \subset\subset X$. Then for $(x,\xi) \in X' \times \operatorname{supp}\psi_j$, $R \geqq \sup(1, B_0 + B)$, $\varepsilon \in]0, 1/(6R^{1/\sigma})[$*

$$|L^j(\psi_j au)(x,\xi)| \leqq \|a\|_{X',\varepsilon}^{A_a,R,R}\|u\|_{X',A_u}(d_3 c_\phi R^{-1}A')^j$$

where $d_3 > 0$ depends only on ν, N, σ, μ, $A' \geqq \sup(A_a + A_u + c, C_\phi)$ and c, c_ϕ, C_ϕ are as in (1.2.4) *and in Propostion* 1.2.1 *respectively.*

Proof. It is easily seen that for $(x,\xi) \in X' \times \operatorname{supp}\psi_j$, $|\alpha| \leqq j$ and $R \geqq \sup(1, (B_0 + B)/2)$, $\varepsilon > 0$

$$|D_\xi^\alpha D_x^\beta(\psi_j(\xi)a(x,\xi)u(x))| \leqq 2e^{\mu j}\|a\|_{X',\varepsilon}^{A_a,R,R}\|u\|_{X',A_u}(A_a + A_u + 2^{\sigma+2}c)^{|\alpha|+|\beta|}$$
$$\times |\alpha|!^\mu|\beta|!^\sigma(1+|\xi|)^{-|\alpha|}\exp(\varepsilon|\xi|^{1/\sigma}).$$

The desired estimate follows from this estimate and Proposition 1.2.1 by applying Remark 1.1.13′ for $m = p = q = 0$ and $k = 2$.

From Proposition 1.2.2 it follows easily that if $a \in C^\infty(X \times R^N) \cap S_N^{\infty,\sigma,\mu}(X)$, then for every $X' \subset\subset X$ there exist $R > 0$ such that the series at the right hand side of (1.2.7) is convergent. More precisely we have

Lemma 1.2.3. Let $a \in C^\infty(X \times R^N) \cap S_b^{\infty,\sigma,\mu}(X' \times R^N_{B_0,B}; A_a)$, $X' \subset\subset X$ and let $u \in G_0^{(\sigma),A_u}(X')$. Choose $h > 1$ such that $R = 2k e^{\sigma N} d_3 c_0 A' \geqq \sup(1, B_0 + B)$, where d_3, c_ϕ, A' are as in Proposition 1.2.2. Then for any $\varepsilon \in]0, 1/(6R^{1/\sigma})[$

$$(1.2.8) \quad \sum_{j \geqq 0} |I_\phi(\psi_j au)| \leqq d_4 |X'| (kc_\phi A')^N \|u\|_{X',A_u} (\sup_{X' \times \{|\xi| \leqq 3R\}} |a(x, \xi)| + \|a\|_{X',\varepsilon}^{A_a,R,R}),$$

where $d_4 > 0$ depends only on v, N, σ, μ and $|X'|$ is the Lebesgue measure of X'.

For $a \in C^\infty(X \times R^N) \cap S_N^{\infty,\sigma,\mu}(X)$ consider now $a_\rho(x, \xi) = \chi(\rho\xi) a(x, \xi)$, $\rho \in [0, 1]$, where χ is as in the Example 1.1.8. Since the set $\{a_\rho; \rho \in [0, 1]\}$ is bounded in $S_N^{\infty,\sigma,\mu}(X)$, for every $X' \subset\subset X$ there exist $A_a > 0$, $B_0 \geqq 0$, $B \geqq 0$ such that $\{a_\rho; \rho \in [0, 1]\}$ is bounded in $S_b^{\infty,\sigma,\mu}(X' \times R^N_{B_0,B} A_a)$ and (1.2.1) holds. Since (1.2.7) holds for a_ρ, from Lemma 1.2.3 it follows that for every $X' \subset\subset X$ there exist $d_5 > 0$ and $R > 0$ independent of ρ and $u \in G_0^{(\sigma),A_u}(X')$ such that for every $\varepsilon \in]0, 1/(6R^{1/\sigma})[$

$$|I_\phi(a_\rho u) - \sum_{j \geqq 0} I_\phi(\psi_j au)| \leqq d_5 \|u\|_{X',A_u} (\sup_{X' \times \{|\xi| \leqq 3R\}} |(a - a_\rho)(x, \xi)| + \|a - a_\rho\|_{X',\varepsilon}^{A_a,R,R}).$$

Since, as we have remarked in Example 1.1.8, $a_\rho \to a$ in $C^\infty(X \times R^N)$ and in $S_N^{\infty,\sigma,\mu}(X)$ when $\rho \to 0+$, we conclude that $\lim_{\rho \to 0+} I_\phi(a_\rho u) = \sum_{j \geqq 0} I_\phi(\psi_j au)$ when R in (1.2.3) is chosen as indicated above. This gives sense to

Definition 1.2.4. Let $\phi \in C^\infty(X \times R^N) \cap \tilde{S}_N^{1,\sigma,\mu}(X)$ be real-valued and let $a \in C^\infty(X \times R^N) \cap S_N^{\infty,\sigma,\mu}(X)$ and $u \in G_0^{(\sigma)}(X)$. Assume that for every $X' \subset\subset X$ (1.2.1) is satisfied. Then define

$$(1.2.9) \quad I_\phi(au) := Os - \iint e^{i\phi(x,\xi)} a(x, \xi) u(x) dx đ\xi = \lim_{\rho \to 0+} I_\phi(a_\rho u) = \sum_{j \geqq 0} I_\phi(\psi_j au),$$

where a_ρ is as in Example 1.1.8 and the number R in (1.2.3) is chosen as indicated in Lemma 1.2.3.

From the estimate of Lemma 1.2.3 it also follows

Theorem 1.2.5. *Let ϕ be as in Definition 1.2.4. Then the bilinear map*

$$(C^\infty(X \times R^N) \cap S_N^{\infty,\sigma,\mu}(X)) \times G_0^{(\sigma)}(X) \ni (a, u) \longrightarrow I_\phi(au)$$

defined by (1.2.9) is separately continuous for the topology of $(C(X \times R^N) \cap S_N^{\infty,\sigma,\mu}(X)) \times G_0^{(\sigma)}(X)$, uniformly with respect to ϕ on the bounded subset of $\tilde{S}_N^{1,\sigma,\mu}(X)$ where (1.2.1) holds uniformly. The same map is also continuous with respect to ϕ for the topology of $C^\infty(X \times R^N)$ on the bounded subset of $\tilde{S}_N^{1,\sigma,\mu}(X)$ where (1.2.1) holds uniformly.

Assume now that in place of (1.2.1) the function ϕ satisfies

$$(1.2.1') \qquad |\nabla_x \phi(x, \xi)|^{-2} \leqq M_\phi |\xi|^{-2}, \ (x, \xi) \in X' \times R^N_{B_0},$$

where M_ϕ and B_0 depend on X'. In this case by using the operator

$$L'(x, \xi, \partial_x) = \sum_{h=1}^{\nu} a'_h \partial_{x_h} + a'_0,$$

where $a'_h = i\,|\nabla_x \phi|^{-2} \partial_{x_h} \phi$, $a'_0 = \sum_{h=1}^{\nu} \partial_{x_h} a'_h$, in place of the operator L in (1.2.6), we can express $I_\phi(au)$ as a repeated integral. First we prove

Proposition 1.2.6. *Let $\phi \in C^\infty(X \times R^{N_1}) \cap \tilde{S}^{1,\sigma,\mu}_{N_1}(X)$ be real-valued and let (1.2.1′) be satisfied. Assume that $a(\cdot, \xi) \in G_0^{(\sigma)}(X')$, $X' \subset\subset X$, $\xi \in R^N_{B_0}$, and that there exists $A_a > 0$ such that for every $\varepsilon > 0$*

$$\sup_{\beta \in Z^\nu_+} \sup_{X' \times R^N_{B_0}} A_a^{-|\beta|} \beta!^{-\sigma} |D^\beta_x a(x, \xi)| \exp(-\varepsilon |\xi|^{1/\sigma}) = c'_a(A_a, \varepsilon) < +\infty.$$

Then there exists a positive constant d_6 independent of ϕ, a, ε such that for every $\varepsilon > 0$ and $|\xi| > B_0$, $|\eta| > B_\phi$

$$\left| \int e^{i\phi(x,\eta)} a(x, \xi) dx \right| \leqq |X'| c'_a(A', \varepsilon) \exp(\varepsilon |\xi|^{1/\sigma} - d_6 (c'_\phi A')^{-1/\sigma} |\eta|^{1/\sigma}).$$

Here $A' \geqq \sup(A_a, C'_\phi)$ and c'_ϕ, C'_ϕ depend on ϕ in the same way as c_ϕ and C_ϕ in Proposition 1.2.1.

Proof. Note that

$$\int e^{i\phi(x,\eta)} a(x, \xi) dx = \int e^{i\phi(x,\eta)} (L'(x, \eta, \partial_x))^j a(x, \xi) dx, \ j \geqq 1,$$

and that the estimates in Proposition 1.2.1 for a_h also hold for the coefficients a'_h, $h = 1, \dots, \nu$, of L' with constants c'_ϕ, C'_ϕ with the same properties as c_ϕ and C_ϕ. Applying Remark 1.1.13′ for $N = 0$, $m = p = q = 0$, $k = 2$

$$\left| \int e^{i\phi(x,\eta)} a(x, \xi) dx \right| \leqq |X'| c'_a(A', \varepsilon) (c'_\phi 2(\nu + 2) 2^\sigma A' (1 + |\eta|)^{-1} j^\sigma)^j \exp(\varepsilon |\xi|^{1/\sigma}),$$

$j = 1, \dots,$ where $A' \geqq \sup(A_a, C'_\phi)$. Thus the proposition is proved by taking the l.u.b. with respect to j of the right hand side of the last estimate.

From Proposition 1.2.6 it follows

Lemma 1.2.7. *Let $\phi \in C^\infty(X \times R^N) \cap \tilde{S}^{1,\sigma,\mu}_N(X)$ be real-valued and let $a \in C^\infty(X \times R^N) \cap S^{\infty,\sigma,\mu}_b(X' \times R^N_{B_0,B}; A_a)$ and $u \in G_0^{(\sigma),A_u}(X')$, $X' \subset\subset X$. Assume that* (1.2.1′) *holds. Then there exists $d_7 > 0$, independent of ϕ, a such that for every $\varepsilon \in]0, \varepsilon_0[$, $\varepsilon_0 = 2^{-1} d_6 (c'_\phi A)^{-1/\sigma}$*

$$(1.2.10)\qquad \left|\int d\xi \int e^{i\phi(x,\xi)}a(x,\xi)u(x)dx\right| \leqq d_7|X'|\ \|u\|_{X',A_u}(\|a\|_{X',\varepsilon}^{A_a,B_0,B}(c'_\phi A')^N + B_0^N \sup_{X'\times\{|\xi|\leqq B_0\}} |a(x,\xi|),$$

where d_6, c'_ϕ *are as indicated in Proposition* 1.2.6 *and* $A' \geqq \sup(A_a + A_u, C'_\phi)$.

If $a\in C^\infty(X\times R^N)\cap S_N^{\infty,\sigma,\mu}(X)$ and for every $X'\subset\subset X$ (1.2.1′) is satisfied, we can apply Lemma 1.2.7 to $a_\rho - a$, where a_ρ is as in Example 1.1.8 and conclude that for every $u\in G_0^{(\sigma)}(X)$

$$(1.2.11)\qquad I_\phi(au) = \lim_{\rho\to 0+} I_\phi(a_\rho u) = \int d\xi \int e^{i\phi(x,\xi)}a(x,\xi)u(x)dx.$$

Finally for a real-valued $\phi\in C^\infty(X\times R^N)\cap \tilde{S}_N^{1,\sigma,\mu}(X)$ let

$$X_\phi = \{x\in X\,;\,\exists\, M_\phi(x)>0,\ B_0(x)\geqq 0;\ |\nabla_\xi\phi|^{-2}\leqq M_\phi,\ |\xi|>B_0\},$$

and

$$L'' = \sum_{j=1}^N b''_j(x,\xi)\partial_{\xi_j} + b''_0(x,\xi),\ x\in X_\phi,\ \xi\in R^N_{B_0(x)},$$

where $\quad b''_j = i|\nabla_\xi\phi|^{-2}\partial_{\xi_j}\phi,\ b''_0 = \sum_{j=1}^N \partial_{\xi_j} b''_j.$

If we define

$$I_\phi(a(x,\cdot)) = \int e^{i\phi(x,\xi)}a(x,\xi)d\xi,\ x\in X_\phi$$

for $a(x,\cdot)\in L^1(R^N)$, we can prove an estimate analogous to (1.2.8.) for $\sum_{j\geqq 0}|I_\phi(\psi_j a(x,\cdot))|$, when $a\in C^\infty(X\times R^N)\cap S_b^{\infty,\sigma,\mu}(X'\times R^N_{B_0,B})$, $x\in X'\subset\subset X_\phi$, by using similar arguments as in the proof of Lemma 1.2.3 and replacing L by L''. As a consequence we can define

$$(1.2.12)\qquad I_\phi(a(x,\cdot)) := Os\text{-}\int e^{i\phi(x,\xi)}a(x,\xi)d\xi = \lim_{\rho\to 0+} I_\phi(a_\rho(x,\cdot)) = \sum_{j\geqq 0} I_\phi(\psi_j a(x,\cdot)),\ x\in X_\phi,$$

when $a\in C^\infty(X\times R^N)\cap S_N^{\infty,\sigma,\mu}(X)$.

Moreover for $x\in X'\subset\subset X$

$$|I_\phi(a(x,\cdot)| \leqq d'_4(kc_\phi A')^N(\sup_{X'\times\{|\xi|\leqq 3R\}}|a(x,\xi)| + \|a\|_{X',\varepsilon}^{A_a,R,R}),$$

where d'_4 is independent of ϕ, a, X' and A', R are as in Lemma 1.2.3 for $A_u = 0$. Thus a result similar to Theorem 2.5 holds for $I_\phi(a(x,\cdot))$, $x\in X_\phi$, and from Definition 1.2.4 it follows that

$$I_\phi(au) = \int I_\phi(a(x, \cdot))u(x)dx, \quad u \in G_0^{(\sigma)}(X_\phi). \tag{1.2.13}$$

2. Fourier integral operators of infinite order on Gevrey spaces

Definition 2.1. Let Ω be an open set of R^n and let $\sigma > 1$ and $\mu \in [1, \sigma]$. A function a defined on $\Omega \times \Omega \times R^n$ will be called an amplitude of infinite order of type (σ, μ) on Ω if

(i) $$a \in \mathscr{B}_{loc}(R^n; G^{(\sigma)}(\Omega \times \Omega)),$$

(ii) $$a \in C^\infty(\Omega \times \Omega \times R^n) \cap S_n^{\infty,\sigma,\mu}(\Omega \times \Omega).$$

The set of all amplitudes of infinite order of type (σ, μ) on Ω will be denoted by $\boldsymbol{a}^{\infty,\sigma,\mu}(\Omega \times \Omega)$. The subset of $\boldsymbol{a}^{\infty,\sigma,\mu}(\Omega \times \Omega)$ defined by i), ii) when $S_n^{\infty,\sigma,\infty}(\Omega \times \Omega)$ is replaced by $\tilde{S}_n^{\infty,\sigma,\mu}(\Omega \times \Omega)$ or by $S_{b,n}^{\infty,\sigma,\mu}(\Omega \times \Omega)$ will be denoted by $\tilde{\boldsymbol{a}}^{\infty,\sigma,\mu}(\Omega \times \Omega)$ and $\boldsymbol{a}_b^{\infty,\sigma,\mu}(\Omega \times \Omega)$ respectively [11].

Similarly we shall denote by $\boldsymbol{a}^{m,\sigma,\mu}(\Omega \times \Omega)$, $\tilde{\boldsymbol{a}}^{m,\sigma,\mu}(\Omega \times \Omega)$, $\boldsymbol{a}_b^{m,\sigma,\mu}(\Omega \times \Omega)$, $m \in R$, the sets of all amplitudes of order m and type (σ, μ) on Ω defined by i), ii) when $S_n^{\infty,\sigma,\mu}(\Omega \times \Omega)$ is replaced by $S_n^{m,\sigma,\mu}(\Omega \times \Omega)$, $\tilde{S}_n^{m,\sigma,\mu}(\Omega \times \Omega)$, $S_b^{m,\sigma,\mu}(\Omega \times \Omega)$ respectively [12].

Definition 2.2. A real-valued function $\varphi \in \tilde{\boldsymbol{a}}^{1,\sigma,\mu}(\Omega \times \Omega)$ such that for every $\Omega' \subset\subset \Omega$ there exist $B \geqq 0$ and $M_\varphi > 0$ such that for $(x, y, \xi) \in \Omega' \times \Omega' \times R^n_{B_\varphi}$

$$(|\nabla_x \varphi|^2 + |\xi|^2 |\nabla_\xi \varphi|^2)^{-1} \leqq M_\varphi |\xi|^{-2}, \quad (|\nabla_y \varphi|^2 + |\xi|^2 |\nabla_\xi \varphi|^2)^{-1} \leqq M_\varphi |\xi|^{-2}$$

will be called a phase function on Ω.

If φ is a phase function on Ω and $a \in \boldsymbol{a}^{\infty,\sigma,\mu}(\Omega \times \Omega)$, we define the Fourier integral operator A on $G_0^{(\sigma)}(\Omega)$ as an oscillatory integral in the sense of Definition 1.2.4.:

$$(Au)(x) = Os\text{-}\iint e^{i\varphi(x,y,\xi)} a(x, y, \xi) u(y) dy đ\xi, \quad x \in \Omega, \quad u \in G_0^{(\sigma)}(\Omega). \tag{2.1}$$

Proposition 2.3. *Let φ, a, u as in* (2.1) *and let $\beta \in Z_+^n$. Denote*

$$a_\beta(x, y, \xi) = e^{-i\varphi(x,y,\xi)} D_x^\beta (e^{i\varphi(x,y,\xi)} a(x, y, \xi)).$$

Then

$$D_x^\beta(Au)(x) = Os\text{-}\iint e^{i\varphi(x,y,\xi)} a_\beta(x, y, \xi) u(y) dy đ\xi, \quad x \in \Omega, \tag{2.2}$$

and for every $\Omega' \subset\subset \Omega$ there exists $C > 0$ such that

$$\sup_{x \in \Omega'} |D_x^\beta (Au)(x)| \leqq C^{|\beta|+1} \beta!^\sigma.$$

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[11] See Definition 1.1.3.

[12] See Definition 1.1.10.

Proof. First note that by Proposition 1.1.12, $a_\beta \in \boldsymbol{a}^{|\beta|,\sigma,\mu}(\Omega \times \Omega)$ and hence $a_\beta \in \boldsymbol{a}^{\infty,\sigma,\mu}(\Omega \times \Omega)$. Suppose that $u \in G_0^{(\sigma)}(\Omega'')$, $\Omega'' \subset\subset \Omega$, and that $\varphi \in S_b^{1,\sigma,\mu}(\Omega' \times \Omega'' \times R^n_{B_\varphi,0}; A_\varphi)$, $a \in S_b^{\infty,\sigma,\mu}(\Omega' \times \Omega'' \times R^n_{B_0,B}; A_a)$. Then by Propositions 1.1.6, 1.1.7 and 1.1.12

$$\sup_{x\in\Omega'} \| a_\beta(x,\cdot,\cdot)\|^{A',B_0',B}_{\Omega'',\varepsilon} \leqq \| a_\beta \|^{A',B_0',B}_{\Omega'\times\Omega'',\varepsilon} \leqq (dC)^{|\beta|}\beta!^{\sigma} \| a \|^{A_a,B_0,B}_{\Omega'\times\Omega'',\varepsilon'},$$

where $d \geqq 1$ depends only on n and σ, $\varepsilon' = \min(\varepsilon, 1)/2$, $c = \sup(A_\varphi \varepsilon'^{-\sigma}(|\varphi|^{A_\varphi,B_\varphi 0}_{\Omega\times\Omega'',1} + 1), A_a, 1)$, $A' \geqq |\beta| A_\varphi + A_a$, $B_0' \geqq \sup(B_\phi, B_0)$. Thus by Lemma 1.2.3 the series $\sum_{j\geqq 0} D_x^\beta I_{\varphi(x,\cdot,\cdot)}(\psi_j a(x,\cdot,\cdot)u)$ is uniformly convergent on Ω' to the right hand side of (2.2) and the proposition is proved in view of Lemma 1.2.3 and of the assumption i) in Definition 2.1.

The same result of Proposition 2.3 holds for the transpose ${}^tA$ of A defined by

$$({}^tAv)(x) = Os\text{-}\iint e^{i\varphi(y,x,\xi)} a(y, x, \xi) v(y) dy d\xi, \quad x \in \Omega, \quad v \in G_0^{(\sigma)}(\Omega).$$

This leads to

Theorem 2.4. *Let φ be a phase function on Ω and let $a \in \boldsymbol{a}^{\infty,\sigma,\mu}(\Omega \times \Omega)$. Then (2.1) defines a continuous linear map from $G_0^{(\sigma)}(\Omega)$ to $G^{(\sigma)}(\Omega)$, which extends to a continuous linear map from $G^{(\sigma)\prime}(\Omega)$ to $G_0^{(\sigma)\prime}(\Omega)$ with kernel $K_A \in G_0^{(\sigma)\prime}(\Omega \times \Omega)$ defined by*

$$K_A(w) = Os\text{-}\iint e^{i\varphi(x,y,\xi)} a(x, y, \xi) w(x, y) dx dy d\xi, \quad w \in G_0^{(\sigma)}(\Omega \times \Omega).$$

With the same arguments used for proving Proposition 2.3 it can be proved that

$$I_{\varphi(x,y,\cdot)}(a(x, y, \cdot)) = Os\text{-}\int e^{i\varphi(x,y,\xi)} a(x, y, \xi) d\xi \in G^{(\sigma)}(R_\varphi \cup \complement P_{\Omega\times\Omega}(\text{supp } a)),$$

where

$$R_\varphi = \{(x, y) \in \Omega \times \Omega; \exists B_\varphi \geqq 0, M_\varphi > 0; |\nabla_\xi \varphi|^{-2} \leqq M_\varphi, |\xi| > B_\varphi\},$$

$P_{\Omega\times\Omega}(\text{supp } a))$ denotes the projection of supp a on $\Omega \times \Omega$, and the oscillatory integral is defined according to (1.2.12).

Thus from (1.2.13) it follows that $K_A \in G^{(\sigma)}(R_\phi \cup \complement P_{\Omega\times\Omega}(\text{supp } a))$ and we have

Theorem 2.5. *Let φ be a phase function on Ω and let $a \in \boldsymbol{a}^{\infty,\sigma,\mu}(\Omega \times \Omega)$. Then $K_A \in G^{(\sigma)}(R_\varphi \cup \complement P_{\Omega\times\Omega}(\text{supp } a))$ and for $u \in G^{(\sigma)\prime}(\Omega)$*

$$\begin{aligned} \sigma\text{-sing supp } Au &\subset (\complement R_\varphi \cap P_{\Omega\times\Omega}(\text{supp } a)) \circ (\sigma\text{-sing supp } u) \\ &= \{x \in \Omega; \exists y \in \sigma\text{-sing suppu}, (x, y) \in \complement R_\varphi \cap P_{\Omega\times\Omega}(\text{supp } a)\}. \end{aligned}$$

Definition 2.6. A continuous linear map from $G_0^{(\sigma)}(\Omega)$ to $G^{(\sigma)}(\Omega)$ is said a σ-regularizing operator in Ω if it extends to a continuous linear map from $G^{(\sigma)\prime}(\Omega)$ to $G^{(\sigma)}(\Omega)$, i.e. by [13] if and only if $K_A \in G^{(\sigma)}(\Omega \times \Omega)$.

Theorem 2.7. *Let $\varphi \in L^\infty_{loc}(R^n; G^{(\sigma)}(\Omega \times \Omega))$ be a real-valued function such that for every pair of relatively compact open subset Ω', Ω'' of Ω there exist $A_\varphi > 0$, $B_\varphi > 0$ such that*

$$(2.2) \qquad |D_x^\beta D_y^\gamma \varphi(x, y, \xi)| \leqq A_\varphi^{|\alpha+\beta|+1}(\beta!\gamma!)^\sigma(1 + |\xi|), \ (x, y, \xi) \in \Omega' \times \Omega'' \times R^n_{B_\varphi}.$$

Then a continuous linear map A from $G_0^{(\sigma)}(\Omega)$ to $G^{(\sigma)}(\Omega)$ is σ-regularizing in Ω if and only if it can be represented in the form (2.1) *where* i) $a \in L^1_{loc}(R^n; G^{(\sigma)}(\Omega \times \Omega))$; ii) *for every Ω', $\Omega'' \subset\subset \Omega$ there exist $A_a > 0$, $B_a \geqq 0$, $h > 0$ such that for every $\beta, \gamma \in Z^n_+$*

$$(2.3) \quad |D_x^\beta D_y^\gamma a(x, y, \xi)| \leqq A_a^{|\gamma+\beta|+1}(\beta!\gamma!)^\sigma \exp(-h|\xi|^{1/\sigma}), \ (x, y, \xi) \in \Omega' \times \Omega'' \times R^n_{B_a}.$$

Proof. If A is a σ-regularizing operator on Ω, then A can be represented in the form (2.1) with

$$a(x, y, \xi) = \exp(-i\varphi(x, y, \xi))K_A(x, y)r_0\exp(-(1 + |\xi|^2)^{1/2}),$$

where K_A is the kernel of A and $r_0^{-1} = \int \exp(-(1 + |\xi|^2)^{1/2})\bar{d}\xi$.

From (2.2) by using Faà di Bruno formula and (1.1.4) it follows that

$$(2.4) \qquad |D_x^\beta D_y^\gamma \exp(-i\varphi(x, y, \xi))| \leqq \tilde{A}^{|\beta+\gamma|+1}(\beta!\gamma!)^\sigma \exp(|\xi|/2),$$

$(x, y, \xi) \in \Omega' \times \Omega'' \times R^n_{B_\varphi}$, where $\tilde{A}$ is a constant which depends only on A_φ, σ, n. Since $K_A \in G^{(\sigma)}(\Omega \times \Omega)$, (2.4) gives a bound of the form (2.3) for a.

Conversely, suppose that A can be represented in the form (2.1) and that a satisfies the conditions i) and ii) above.

From (2.2) and (2.3) it follows:

$$(2.5) \quad |D_x^\beta D_y^\gamma(e^{i\varphi(x,y,\xi)}a(x, y, \xi))| \leqq A_a A_\varphi(\beta!\gamma!)^\sigma(A_a + cA_\varphi)^{|\beta+\gamma|}\exp\left(-\frac{h}{2}|\xi|^{1/\sigma}\right),$$

$(x, y, \xi) \in \Omega' \times \Omega'' \times R^n_B$, where $B = \max(B_a, B_\varphi, 1)$ and c is a constant that depends only on σ, h, n. Hence $e^{i\varphi}a \in L^1(R^n, G^{(\sigma)}(\Omega \times \Omega))$ and

$$K_A(x, y) = \int e^{i\varphi(x,y,\xi)}a(x, y, \xi)\bar{d}\xi \in G^{(\sigma)}(\Omega' \times \Omega'')$$

i.e. A is a σ-regularizing operator.

From this theorem and Proposition 1.1.20 it follows in particular

Corollary 2.8. *Let φ satisfy the conditions of Theorem* 2.7 *and let $a \sim 0$ in $FS_n^{\infty,\sigma,\mu}(\Omega \times \Omega)$. Then the operator* (2.1) *is σ-regularizing in Ω.*

We restrict now ourselves to consider operators (2.1) with phase functions $\varphi(x, y, \xi) = \phi(x, \xi) - \langle y, \xi\rangle$ and amplitudes a independent of y. The sets of these amplitudes will be denoted by $\boldsymbol{a}^{\infty,\sigma,\mu}(\Omega)$, etc. instead of $\boldsymbol{a}^{\infty,\sigma,\mu}(\Omega \times \Omega)$, $\boldsymbol{a}^{m,\sigma,\mu}(\Omega \times \Omega)$, respectively. We shall also assume that $\phi \in \mathscr{P}_{loc}^{(\sigma,\mu)}(\Omega)$, where, following [14] we define $\mathscr{P}_{loc}^{(\sigma,\mu)}(\Omega)$ by

Definition 2.9. $\mathscr{P}_{loc}^{(\sigma,\mu)}(\Omega)$, will denote the set of all real-valued functions ϕ defined on $\Omega \times R^n$ such that $\phi \in \tilde{\boldsymbol{a}}^{1,\sigma,\mu}(\Omega)$ and for every $\Omega' \subset\subset \Omega$, $\phi \in \mathscr{P}(\tau(\Omega'))$ i.e. there exist $\tau(\Omega') \in [0, 1[$ and $B_0 > 0$ such

$$\sum_{|\alpha+\beta| \leqq 2} \sup_{\substack{x\in\Omega' \\ |\xi| \geqq B_0}} |D_\xi^\alpha D_x^\beta[\phi(x, \xi) - \langle x, \xi\rangle]|(1 + |\xi|)^{|\alpha|-1} \leqq \tau(\Omega'). \tag{2.6}$$

As it is shown in [14] all functions $\varphi = \phi(x, \xi) - \langle y, \xi\rangle$, $\phi \in \mathscr{P}_{loc}^{(\sigma,\mu)}(\Omega)$, are phase functions on Ω according to Definition 2.2. Moreover if $\boldsymbol{a} \in a^{\infty,\sigma,\mu}(\Omega)$, then by (1.2.11) the operator A defined by (2.1) can be written as:

$$(Au)(x) = \int e^{i\phi(x,\xi)} a(x, \xi)\tilde{u}(\xi)đ\xi, \quad u \in G_0^{(\sigma)}(\Omega). \tag{2.7}$$

Note that (2.7) is defined for every $u \in \mathscr{S}'$ such that for a positive constant h

$$\sup_{\xi\in R^n} |\tilde{u}(\xi)\exp(h|\xi|^{1/\sigma})| < +\infty,$$

and that $Au \in G^{(\sigma)}(\Omega)$ also in this case.

The following result on composition of operators of type (2.7) will be used later.

Theorem 2.10. *Let P_1 and P_2 be defined on $G_0^{(\sigma)}(\Omega)$ by*

$$(P_1 u)(x) = \int e^{i<x,\xi>} p_1(x, \xi)\tilde{u}(\xi)đ\xi,$$

$$(P_2 u)(x) = \int e^{i\phi(x,\xi)} p_2(x, \xi)\tilde{u}(\xi)đ\xi, \quad x \in \Omega,$$

where $\phi \in \mathscr{P}_{loc}^{(\sigma,\mu)}(\Omega)$, $p_1 \in \tilde{\boldsymbol{a}}^{\infty,\sigma,1}(\Omega)$, $p_2 \in \boldsymbol{a}^{\infty,\sigma,\mu}(\Omega)$, and let $\Omega' \subset\subset \Omega$ and $h \in G_0^{(\sigma)}(\Omega'')$, Ω'' a convex subset of Ω, $h \equiv 1$ on open neighborhood of Ω'.

Then there exists P defined on $G_0^{(\sigma)}(\Omega)$ by

$$(P\,u)(x) = \int e^{i\phi(x,\xi)} p(x, \xi)\tilde{u}(\xi)đ\xi$$

and a σ-regularizing operator on Ω', $R_{\Omega'}$, such that

$$(P_1 h P_2 u)(x) = (P\,u)(x) + (R_{\Omega'}\,u)(x), \quad x \in \Omega', \quad u \in G_0^{(\sigma)}(\Omega'),$$

where $p \in \boldsymbol{a}^{\infty,\sigma,\mu}(\Omega)$ and

$$p(x, \xi) \sim \sum_{j \geqq 0} q_j(x, \xi) \ \textit{in} \ FS_n^{\infty,\sigma,\mu}(\Omega),$$

$$q_j(x, \xi) = \sum_{|\alpha|=j} \alpha!^{-1} D_y^\alpha((\partial_\xi^\alpha p_1)(x, \tilde{\nabla}_x \phi(x, y, \xi)) p_2(y, \xi))|_{y=x},$$

$$\tilde{\nabla}_x \phi(x, y, \xi) = \int_0^1 \nabla_x \phi(y + \theta(x - y), \xi) d\theta.$$

Proof. In view of (2.6) we can estimate the derivatives of $\partial_\xi^\alpha p_1(x, \tilde{\nabla}_x \phi(x, y, \xi))$ using Lemma 1.1.14. Hence, with the aid of Propositions 1.1.6 and 1.1.7 we prove easily that $\sum_{j \geqq 0} q_j \in FS_n^{\infty,\sigma,\mu}(\Omega')$.

If $u \in G_0^{(\sigma)}(\Omega)$, in view of Proposition 1.2.6, with a change in the order of integrations we have

$$(P_1 h P_2 u)(x) = \int e^{i\phi(x,\xi)} q(x, \xi) \tilde{u}(\xi) d\!\!{}^-\xi, \quad x \in \Omega',$$

where

$$q(x, \xi) = \int d\!\!{}^-\xi' \int e^{i\varphi(x,x',\xi,\xi')} p_1(x, \xi') h(x') p_2(x', \xi) dx',$$

$$\varphi(x, x', \xi, \xi') = \langle x - x', \xi' \rangle - (\phi(x, \xi) - \phi(x', \xi)).$$

Let χ_N, $N \in Z_+$, be a C^∞ function on R^n such that $\chi_N(\eta) = 1$ for $|\eta| \leqq 1/2$, $\chi_N(\eta) = 0$ for $|\eta| \geqq 2/3$, $|\partial^{\alpha+\beta} \chi_N| \leqq a_0^{|\alpha+\beta|+1} \beta!^\sigma N^{|\alpha|}$ for $|\alpha| \leqq N$, $\beta \in Z_+^n$, where the constant a_0 depends only on n. Define

$$\chi(\xi'; \xi) = \chi_{[(1+|\xi|)^{1/\sigma}]}(\xi'/((1 - \tau)(1 + |\xi|))), \quad \xi, \xi' \in R^n,$$

where for $r \in R^+$, $[r]$ denotes the integral part of r and $\tau = \tau_{\Omega''}$ is the constant in Definition 2.9 corresponding to Ω''.

Then we have:

$$|\partial_{\xi'}^{\alpha+\beta} \chi(\xi'; \xi)| \leqq a_0(a_0/(1 - \tau))^{|\alpha+\beta|} \beta!^\sigma (1 + |\xi|)^{-(1-1/\sigma)|\alpha| - |\beta|} \quad \text{for } \xi, \xi' \in R^n, \ |\alpha| \leqq [(1 + |\xi|)^{1/\sigma}], \ \beta \in Z_+^n. \tag{2.8}$$

Set now

$$q(x, \xi) = r_0(x, \xi) + \tilde{q}(x, \xi), \tag{2.9}$$

$$r_0(x, \xi) = \int e^{i\varphi(x,x',\xi,\xi')} p_1(x, \xi') h(x') p_2(x', \xi) \chi(\xi'; \xi) dx' d\!\!{}^-\xi', \tag{2.10}$$

$$\tilde{q}(x, \xi) = \int d\!\!{}^-\xi' \int e^{i\varphi(x,x',\xi,\xi')} p_1(x, \xi') h(x') p_2(x', \xi)(1 - \chi(\xi'; \xi)) dx'.$$

Since $\varphi(x, x', \xi, \xi') = \langle x - x', \xi' - \tilde{\nabla}_x \phi(x, x', \xi) \rangle$, the change of variables $z = x' - x$, $\zeta = \xi' - \tilde{\nabla}_x \phi(x, x', \xi)$ gives

(2.11) $$\tilde{q}(x, \xi) = r_1(x, \xi) + \tilde{q}_1(x, \xi),$$

where

(2.12) $$r_1(x, \xi) = \int đ\zeta \int e^{-i<z,\zeta>}(1 - \chi(\zeta; \xi))(1 - \chi(\zeta + \tilde{\nabla}_x\phi(x, x + z, \xi); \xi)) \times p_1(x, \zeta + \tilde{\nabla}_x\phi(x, x + z, \xi))h(x + z)p_2(x + z, \xi)dz,$$

$$\tilde{q}_1(x, \xi) = \int đ\zeta \int e^{-i<z,\zeta>}\chi(\zeta; \xi)(1 - \chi(\zeta + \tilde{\nabla}_x\phi(x, x + z, \xi); \xi)) \times p_1(x, \zeta + \tilde{\nabla}_x\phi(x, x + z, \xi))h(x + z)p_2(x + z, \xi)dz.$$

If we let:

$$a(x, y, \xi, \eta) = \chi(\eta - \xi; \xi)(1 - \chi(\eta - \xi + \tilde{\nabla}_x\phi(x, y, \xi); \xi)) \times p_1(x, \eta - \xi + \tilde{\nabla}_x\phi(x, y, \xi))h(y)p_2(y, \xi),$$

then we have

$$\tilde{q}_1(x, \xi) = \int đ\zeta \int e^{-i<z,\zeta>} a(x, x + z, \xi; \xi + \zeta)\, dz.$$

Now the Taylor expansion of $a(x, x + z, \xi, \xi + \zeta)$ in the last argument about the point ξ and integration by parts give

(2.13) $$\tilde{q}_1(x, \xi) = \sum_{|\alpha| < j} \alpha!^{-1} D_y^\alpha \partial_\eta^\alpha a(x, y, \xi, \eta)|_{\substack{y=x\\ \eta=\xi}} + j \sum_{|\alpha| = j} \alpha!^{-1} \int_0^1 (1 - \theta)^{j-1} r_{\alpha,\theta}(x, \xi)d\theta, j \in Z_+,$$

where

(2.14) $$r_{\alpha,\theta}(x, \xi) = \iint e^{-i<z,\zeta>}(D_y^\alpha \partial_\eta^\alpha a)(x, x + z, \xi, \xi + \theta\zeta)\, dz\, đ\zeta.$$

From (2.6) it follows that

$$|\tilde{\nabla}_x\phi(x, y, \xi)| \geqq 2(1 - \tau)(1 + |\xi|)/3,$$

for $(x, y, \xi) \in \Omega' \times \Omega'' \times R^N, |\xi| > B = \max\left(B_0, \dfrac{2 + \tau}{1 - \tau}\right)$, where B_0 and τ are the constants in Definition 2.9 corresponding to Ω''.

Hence

$$\chi(\eta - \xi + \tilde{\nabla}_x\phi(x, y, \xi); \xi) = 0 \qquad \text{for } |\eta - \xi| \text{ small enough and } |\xi| > B$$

and

$$\chi(\eta - \xi; \xi) = 1 \qquad \text{for } |\eta - \xi| < (1 - \tau)/2.$$

Thus

$$\sum_{|\alpha|=h} \alpha!^{-1} D_y^\alpha \partial_\eta^\alpha a(x, y, \xi, \eta)|_{\substack{y=x\\ \eta=\xi}} = q_h(x, \xi),\ (x, \xi) \in \Omega' \times R_B^n, \tag{2.15}$$

and (2.13), (2.15) give

$$\begin{aligned} \tilde{q}_1(x, \xi) - \sum_{h \geq 0} \phi_h(\xi) q_h(x, \xi) \\ = \sum_{j \geq 0} \psi_j(\xi)(j+1) \sum_{|\alpha|=j+1} \alpha!^{-1} \int_0^1 (1-\theta)^j r_{\alpha,\theta}(x, \xi) d\theta \\ = r_2(x, \xi),\ (x, \xi) \in \Omega' \times R_B^n, \end{aligned} \tag{2.16}$$

where the functions ϕ_j are as in Lemma 1.1.18 and the ψ_j are defined by (1.2.3).

In view of (2.9), (2.11), (2.16) we conclude that

$$q(x, \xi) = \sum_{h \geq 0} \phi_h(\xi) q_h(x, \xi) + \sum_{j=0}^{2} r_j(x, \xi),\ (x, \xi) \in \Omega' \times R_B^n,$$

where by Lemma 1.1.18 $p_{\Omega'}(x, \xi) = \sum_{h \geq 0} \phi_h(\xi) q_h(x, \xi) \in S_n^{\infty,\sigma,\mu}(\Omega')$. It remains to prove that the operators R_j, $j = 0,1,2$, defined by:

$$(R_j u)(x) = \int e^{i<x,\xi>} r_j(x, \xi) \hat{u}(\xi) đ\xi,\ u \in G_0^{(\sigma)}(\Omega'),$$

are σ-regularizing on Ω'.

Let

$$\begin{aligned} \tilde{r}_0(x, \xi) &= e^{i(\phi(x,\xi) - <x,\xi>)} r_0(x, \xi) \\ &= \iint e^{i\tilde{\varphi}(x,x',\xi,\xi')} p_1(x, \xi') h(x') p_2(x', \xi) \chi(\xi'; \xi) dx' đ\xi', \end{aligned}$$

where $\quad \tilde{\varphi}(x, x', \xi, \xi') = \langle x - x', \xi' \rangle - \langle x, \xi \rangle + \phi(x', \xi)$.

If $\phi \in S_b^{1,\sigma,\mu}(\Omega'' \times R_{B_0,0}^n; A_\phi'')$, then:

$$|\tilde{\varphi}(x, \cdot, \cdot, \xi')|_{\Omega'',1}^{A_\phi'',B_0,0} \leq 2\max(2 \sup_{x' \in \Omega''} |x'|,\ |\phi|_{\Omega'',1}^{A_\phi'',B_0,0}),$$

uniformly for $(x, \xi') \in \Omega' \times \{\xi' \in R^n; |\xi'| \leq 2(1-\tau)(1+|\xi|/3\}$. Moreover, in view of (2.6), $|\nabla_{x'} \tilde{\varphi}(x, x', \xi, \xi')| > (1-\tau)|\xi|/6$ for $(x, x', \xi) \in \Omega' \times \Omega'' \times R_B^n$, $|\xi'| \leq 2(1-\tau)(1+|\xi|)/3$. Hence, from Proposition 1.2.6, we have for every $\varepsilon > 0$:

$$\begin{aligned} \left| \int e^{i\tilde{\varphi}(x,x',\xi,\xi')} h(x') p_2(x', \xi) dx' \right| \leq |\Omega''|\ \|p_2\|_{\Omega'',\varepsilon}^{A_{p_1}'',B_0',p_2,B_{p_2}''} \\ \times |h|_{\Omega'',A_h} \exp(\varepsilon - d_6(\tilde{c}_{\tilde{\varphi}} A)^{-1/\sigma} |\xi|^{1/\sigma}), \end{aligned} \tag{2.17}$$

$x\in\Omega', |\xi| > \tilde{B} = \max(B, B''_{0,p_2})$, $|\xi'| \leqq 2(1+\tau)(1+|\xi|)/3$, where

$$A \geqq \sup(A''_{p_2} + A_h, \tilde{C}_{\tilde{\varphi}}),\ \tilde{c}_{\tilde{\varphi}} = d_1 A''_\phi\ 2\max(2\sup_{\Omega''}|x'|, |\phi|^{A''_\phi, B_0, 0}_{\Omega'', 1}),$$

$$\tilde{C}_{\tilde{\varphi}} = d_2 A''_\phi (1 + A''_\phi 2\max(2\sup_{\Omega''}|x'|, |\phi|^{A''_\phi, B_0, 0}_{\Omega'', 1}) 3\sigma/(1-\tau)^2),$$

where d_1, d_2, d_6 are constants independent of ϕ (see Propositions 1.2.6 and 1.2.1).

Since $\operatorname{supp}\chi(\cdot\,; \xi) \subset \{\xi' \in R^n; |\xi'| \leqq 2(1-\tau)(1+|\xi|)/3\}$, from (2.17) it follows for every $\varepsilon > 0$:

$$(2.18)\qquad |\partial_x^\beta \tilde{r}_0(x, \xi)| \leqq A_\varepsilon^{|\beta|+1} \beta!^\sigma \exp((2\varepsilon - d)|\xi|^{1/\sigma}),\ (x, \xi) \in \Omega' \times R^n_{B_1},$$

where $B_1 = \max(3/2(1-\tau), B'_{0,p_1}, B'_{0,p_2}, B)$,

$$A_\varepsilon = \max\Big(2ea_0\omega_n \|p_2\|^{A''_{p_2}, B''_{0,p_2}, B''_{p_2}}_{\Omega'', \varepsilon} |h|_{\Omega'', A_h} d_{\sigma,n} (\tilde{c}_{\tilde{\varphi}} A)^n$$

$$\times \max(\|p_1\|^{A'_{p_1}, B'_{0,p_1}, 0}_{\Omega', \varepsilon}, |p_1|^{A'_0, p_1}_{\Omega' \times CR^n_{B'_{0,p_1}}}), (12/d_6)^\sigma \tilde{c}_{\tilde{\varphi}} A + \max(A''_{0,p_1}, A'_{p_1})\Big),$$

$d = d_6/2(\tilde{c}_{\tilde{\varphi}} A)^{1/\sigma}$, where

$$|p_1|^{A'_0, p_1}_{\Omega' \times CR^n_{B'_{0,p_2}}} = \sup_{\beta \in Z^n_+} \sup_{\substack{x \in \Omega' \\ |\xi'| \leqq B'_{0,p_2}}} \frac{|\partial_x^\beta p_1(x, \xi)|}{A'^{|\beta|}_{0,p_2} \beta!^\sigma},$$

ω_n denotes the measure of the n-dimensional unitary sphere and $d_{\sigma,n}$ is a constant that depends only on σ and n.

By a suitable choice of ε in (2.18) we obtain a bound for $\partial_x^\beta \tilde{r}_0(x, \xi)$ of the form (2.3). Since (2.2) is satisfied by $\varphi(x, y, \xi) = \phi(x, \xi) - \langle y, \xi \rangle$, Theorem 2.7 implies that R_0 is a σ-regularizing operator.

Consider now $r_1(x, \xi)$. We can write

$$\partial_x^\beta r_1(x, \xi) = Os\text{-}\int (1 - \chi(\zeta; \xi)) d\zeta \int e^{-i\langle z, \zeta\rangle} b_{(\beta)}(x, z, \xi, \zeta) dz,$$

where

$$b_{(\beta)}(x, z, \xi, \zeta) = \partial_x^\beta((1 - \chi(\zeta + \tilde{\nabla}_x \phi(x, x+z, \xi); \xi)) p_1(x, \zeta + \tilde{\nabla}_x \phi(x, x+z, \xi))$$
$$\times h(x+z) p_2(x+z, \xi)).$$

Using Proposition 1.1.14 we get

$$|\partial_z^\gamma \partial_x^\beta (1 - \chi(\zeta + \tilde{\nabla}_x \phi(x, x+z, \xi); \xi))| \leqq a_0 A'^{|\beta+\gamma|} (\beta+\gamma)!^\sigma,$$

for $(x, z, \xi, \zeta) \in \Omega' \times (\Omega'' - \Omega') \times R^n_B \times R^n$, where

$$(2.19)\quad A' = 2^{\sigma+1}(2n+1) n^{2\sigma} \sup(a_0, 2^{\sigma+2} A_\phi) \sup(\|\phi\|^{A_\phi, B_0, 0}_{\Omega'', 1} (1-\tau)^{-1} 4A_\phi^2, 1).$$

Thus, in view of inequality

$$|\zeta + \tilde{\nabla}_x\phi(x, x+z, \xi)| \geqq (1-\tau)(1+|\xi|)/2 \text{ on supp } b_{(\beta)},$$

$$|\partial_z^\gamma \partial_1^\beta p_1(x, \zeta + \tilde{\nabla}_x \phi(x, x+z, \xi))| \leqq \|p_1\|_{\Omega',\varepsilon}^{A'_{p_1}, B'_{0,p_1}, 0} A''^{|\beta+\gamma|}(\beta+\gamma)!^\sigma$$

$$\times \exp(\varepsilon(c_0(|\xi| + |\zeta|))^{1/\sigma}),$$

for $\varepsilon > 0$, $(x, z, \xi, \zeta) \in \Omega' \times (\Omega'' - \Omega') \times R^n_{B_0} \times R^n_1$, where

$$B_0 = \max((1-\tau)B'_{0,p_1}/2, B), \ c_0 = 1 + \|\phi\|_{\Omega'',1}^{A_\phi, B_0, 0} 2A_\phi,$$

$$A'' = (1 + 2^\sigma A_\phi + 2^{\sigma+1} A_\phi^2 \|\phi\|_{\Omega'',1}^{A_\phi, B_0, 0})(A'_{p_1} + 2^{2\sigma+1} A_\phi)$$

$$\times (1-\tau)^{-1} 2^{3\sigma+2} n^{2\sigma}(2n+1).$$

Hence, for every $\varepsilon > 0$, $(x, z, \xi, \zeta) \in \Omega' \times (\Omega'' - \Omega') \times R^n_{B_2} \times R^n_1$,

$$B_2 = \max(B, (1-\tau)B'_{0,p_1}/2, B''_{0,p_2}),$$

$$|\partial_z^\gamma b_{(\beta)}(x, z, \xi, \zeta)| \leqq a_0 \|p_1\|_{\Omega',\varepsilon}^{A'_{p_1}, B'_{0,p_1}, 0} |h|_{\Omega'', A_h} \|p_2\|_{\Omega'',\varepsilon}^{A'_{p_2}, B'_{0,p_2}, B''_{p_2}}$$

$$\times \exp(\varepsilon(c_0(|\xi| + |\zeta|))^{1/\sigma}) \exp(\varepsilon|\xi|^{1/\sigma})(A' + A'' + A_h + A''_{p_2})^{|\beta+\gamma|}(\beta+\gamma)!^\sigma.$$

This inequality allows us to use Proposition 1.2.6, thus obtaining

$$\text{(2.20)} \qquad \int |e^{-i<z,\zeta>} b_{(\beta)}(x, z, \xi, \zeta) dz| \leqq |\Omega'' - \Omega'| \tilde{c}(\varepsilon) \exp(2\varepsilon(c_0|\xi|)^{1/\sigma})$$

$$\times \exp((\varepsilon c_0^{1/\sigma} - d')|\zeta|^{1/\sigma}),$$

for $(x, \xi, \zeta) \in \Omega' \times R^n_{B_2} \times R^n_1$, $\varepsilon > 0$, where

$$\tilde{c}(\varepsilon) = a_0 |h|_{\Omega'', A_h} \|p_1\|_{\Omega',\varepsilon}^{A'_{p_1}, B'_{0,p_1}, 0} \|p_2\|_{\Omega'',\varepsilon}^{A'_{p_2}, B'_{0,p_2}, B''_2} A^{*|\beta|} \beta!^\sigma,$$

$$A^* = 2^n(A' + A'' + A_h + A''_{p_2}), \ d' = d_6(\bar{c}\tilde{A})^{-1/\sigma}, \ \tilde{A} \geqq \sup(A^*, \bar{C}),$$

$\bar{c}$, $\bar{C}$ positive constants independent of p_1, p_2, ϕ, h, ε. Since $|\zeta|^{1/\sigma} > |\zeta|^{1/\sigma}/2 + ((1-\tau)|\xi|/2)^{1/\sigma}/2$ on supp $(1 - \chi(\cdot; \xi))$, for $(x, \xi) \in \Omega' \times R^n_{B_0}$ and $\zeta \in \text{supp}(1 - \chi(\cdot; \varepsilon))$ we get from (2.20) and $\varepsilon = \dfrac{d'}{8}\left(\dfrac{1-\tau}{2c_0}\right)^{1/\sigma}$

$$\left|\int e^{-i<z,\zeta>} b_{(\beta)}(x, z, \xi, \zeta) dz\right| \leqq |\Omega'' - \Omega'| a_0 |h|_{\Omega'', A_h} \|p_1\|_{\Omega',\varepsilon}^{A'_{p_1}, B'_{0,p_1}, 0} \|p_2\|_{\Omega'',\varepsilon}^{A'_{p_2}, B'_{0,p_2}, B''_{p_2}}$$

$$\times A^{*|\beta|} \beta!^\sigma \exp\left(-\frac{d'}{4}\left(\frac{1-\tau}{2}|\xi|\right)^{1/\sigma} - \frac{d'}{4}|\xi|^{1/\sigma}\right).$$

Hence

$$|\partial_x^\beta r_1(x, \xi)| \leqq C^* A^{*|\beta|} \beta!^\sigma \exp\left(-\frac{d'}{4}\left(\frac{1-\tau}{2}|\xi|\right)^{1/\sigma}\right), \ (x, \xi) \in \Omega' \times R^n_{B_2},$$

where

$$C^* = |\Omega'' - \Omega'| \ |h|_{\Omega', A_h} \|p_1\|_{\Omega', \varepsilon}^{A_h, B'_0, p_1, 0} \ \|p_2\|_{\Omega'', \varepsilon}^{A''_{p_2}, B'_{0}, p_2, B''_{p_2}}$$

$$\times a_0(1 + a_0) \int \exp(-d'|\zeta|^{1/\sigma}/4) d\zeta$$

and as in the case of r_0, this proves that R_1 is a σ-regularizing operator on Ω'.

It remains to prove that this is also the case for R_2. To this end let us estimate $\partial_x^\beta \partial_y^\gamma \partial_\eta^\alpha a(x, y, \xi, \eta)$ for $|\alpha| = j$, $(x, y, \xi, \eta) \in \Omega' \times R^n \times \operatorname{supp}\psi_j \times R^n$. Choose $R > \max(B_2/2, 2^{\sigma-1})$ in the definition of ψ_j and note that on supp a

$$(2.21) \quad |\eta - \xi| \leqq \frac{2}{3}(1-\tau)(1+|\xi|); \quad |\eta - \xi + \tilde{V}_x \phi(x, y, \xi)| \geqq \frac{1}{2}(1-\tau)(1+|\xi|)$$

and that

$$(2.22) \qquad \xi \in \operatorname{supp}\psi_j, \ |\alpha| = j + 1 \Rightarrow |\alpha| < |\xi|^{1/\sigma}$$

By using Proposition 1.1.14, from (2.8), (2.21), (2.22) it follows that for $|\alpha| = j + 1$, $(x, y, \xi, \eta) \in (\Omega' \times R^n \times \operatorname{supp}\psi_j \times R^n) \cap \operatorname{supp} a$

$$|\partial_x^\beta \partial_y^\gamma \partial_\eta^\alpha (1 - \chi(\eta - \xi + \tilde{V}_x \phi(x, y, \xi); \xi)| \leqq a_0(a_0/(1-\tau))^{|\alpha|}(1 + |\xi|)^{-(1-1/\sigma)|\alpha|} A'^{|\beta+\gamma|} \times (\beta + \gamma)!^\sigma,$$

where A' is defined by (2.21), and

$$|\partial_x^\beta \partial_y^\gamma \partial_\eta^\alpha p_1(x, \eta - \xi + \tilde{V}_x \phi(x, y, \xi))| \leqq \|p_1\|_{\Omega', \varepsilon}^{A'_{p_1}, B'_0, p_1} \ A'''^{|\beta+\gamma+\alpha|} \alpha! \beta!^\sigma \gamma!^\sigma$$

$$\times (1 + |\xi|)^{-|\alpha|} \exp(\varepsilon(4c_0|\xi|)^{1/\sigma}),$$

where $A''' = 2^{\sigma+2}(2n+1)(4n^2)^\sigma(1-\tau)^{-1}\tilde{c}_0 \tilde{A}$, $\tilde{c}_0 = \operatorname{supp}(c_0 C_0, 1)$,

$$\tilde{A} = \sup(A'_{p_1}, 2^{\sigma+1} C_0), \ c_0 = 2(1 + \|\phi\|_{\Omega'', 1}^{A_\phi, B_0, 0} A_\phi), \ C_0 = 2^\sigma A_\phi.$$

Hence from Proposition 1.1.7

$$|\partial_x^\beta \partial_y^\gamma \partial_\eta^\alpha a(x, y, \xi, \eta)| \leqq \tilde{\tilde{c}}_\varepsilon A^{**|\alpha+\beta+\gamma|}(\beta+\gamma)!^\sigma (1 + |\xi|)^{-(1-1/\sigma)|\alpha|} \exp(\varepsilon d''|\xi|^{1/\sigma}),$$
$$(x, y, \xi, \eta) \in (\Omega' \times R^n \times \operatorname{supp}\psi_j \times R^n) \cap \operatorname{supp} a, \ |\alpha| = j + 1,$$

where:

$$\tilde{\tilde{c}}_\varepsilon = a_0^2 \|p_1\|_{\Omega',\varepsilon}^{A'_{p_1},B'_{0,p_1},0} |h|_{\Omega'',A_h} \|p_2\|_{\Omega',\varepsilon}^{A''_{p_2},B''_{0,p_2},B''_{p_2}},$$

$$A^{**} = \max\left(A' + A''' + A_h + A''_{p_2}, 3\left(\frac{2a_0}{1-\tau} + A'''\right)\right), d^* = (4c_0)^{1/\sigma} + 1.$$

Thus

$$|\partial_x^\beta \partial_z^\gamma (D_y^\alpha \partial_\eta^\alpha a)(x, x+z, \xi, \xi+\theta\zeta))| \leqq \tilde{c}_\varepsilon (2n)^{|\beta|} A^{**|2\alpha+\beta+\gamma|} (\beta+\gamma+\alpha)!^\sigma$$
$$\times (1+|\xi|)^{-(1-1/\sigma)|\alpha|} \exp(\varepsilon d''|\xi|^{1/\sigma}),$$
$$(x, z, \xi, \zeta) \in (\Omega' \times R^n \times \operatorname{supp}\psi_j \times R^n) \cap \operatorname{supp} a, \ |\alpha| = j+1.$$

From this estimate, by Proposition 1.2.6 it follows

$$\left|\int e^{-i<z,\zeta>} \partial_x^\beta (D_y^\alpha \partial_\eta^\alpha a)(x, x+z, \xi, \xi+\theta\zeta) dz\right| \leqq |\Omega'' - \Omega'| \tilde{\tilde{c}}_\varepsilon 2^n (2nA^{**})^{2|\alpha|+|\beta|} \tag{2.23}$$
$$\times (\beta+\alpha)!^\sigma (1+|\xi|)^{-(1-1/\sigma)|\alpha|} \exp(\varepsilon d''|\xi|^{1/\sigma} - (\bar{c}\tilde{\tilde{A}}^{-1/\sigma}|\zeta|^{1/\sigma})),$$

for $(x, \xi, \zeta) \in \Omega' \times \operatorname{supp}\psi_j \times R^n$, $|\alpha| = j+1$, where $\tilde{\tilde{A}} \geqq \sup(A^{**}, \tilde{\tilde{C}})$ and $\bar{c}$, $\tilde{\tilde{C}}$ are positive constants independent of p_1, p_2, ϕ, h, ε.

In view of (2.14), since $\xi \in \operatorname{supp}\psi_j$, $|\alpha| = j+1$ imply $\alpha!^{\sigma-1}(1+|\xi|)^{-(1-1/\sigma)|\alpha|} < (2e)^{\sigma-1}(2R)^{-(1-1/\sigma)(j+1)}$, (2.23) gives

$$|\partial_x^\beta r_{\alpha,\theta}(x, \xi)| \leqq |\Omega'' - \Omega'| \tilde{\tilde{\tilde{c}}}_\varepsilon (2nA^{**})^{2|\alpha|+|\beta|} \beta!^\sigma (2R)^{-(1-1/\sigma)(j+1)} \alpha!$$
$$\times \exp(2\varepsilon d''(3R)^{1/\sigma}(j+1) - \varepsilon d''|\xi|^{1/\sigma}),$$
$$(x, \xi) \in \Omega' \times \operatorname{supp}\psi_j, |\alpha| = j+1, \ \tilde{\tilde{\tilde{c}}}_\varepsilon = \tilde{\tilde{c}}_\varepsilon (2e)^{\sigma-1} 2^n \int \exp(-(\bar{c}\tilde{\tilde{A}})^{-1/\sigma}|\zeta|^{1/\sigma}) d\zeta.$$

With the aid of this estimate with $\varepsilon = 1/R$ and R such that $\exp(6d'') 4n^3 (A^{**})^2 (2R)^{-(1-1/\sigma)} < 1$, we obtain that

$$|D_x^\beta r_2(x, \xi)| = \sum_{j \geqq 0} (j+1) \left|\psi_j(\xi) \sum_{|\alpha|=j+1} \int_0^1 \frac{(1-\theta)^{j-1}}{\alpha!} \partial_x^\beta r_{\alpha,\theta}(x, \xi) d\theta\right|$$
$$\leqq M_0 M^{|\beta|} \beta!^\sigma \exp\left(-\frac{d''}{R}|\xi|^{1/\sigma}\right),$$

for $(x, \xi) \in \Omega' \times R^n_{B_2}$, where $M_0 = \tilde{\tilde{\tilde{c}}}_{1/R} |\Omega'' - \Omega'| \sum_{j \geqq 1} \left(\frac{4n^3 (A^{**})^2 e^{6d''}}{(2R)^{1-1/\sigma}}\right)^j$ and $M = 2nA^{**}$. This proves that R_2 is a σ-regularizing operator on Ω'.

Remark 2.10′. Note that, as it is clear from the proof given above, the norm in $G^{(\sigma)}(\Omega' \times \Omega')$ of the kernel of the regularizing operator $R_\mathscr{A}$ in Theorem 2.10 remains bounded when p_1, p_2, h vary in bounded subsets of $\tilde{a}^{\infty,\sigma,1}(\Omega'')$, $\tilde{a}^{\infty,\sigma,\mu}(\Omega'')$, $G_0^{(\sigma)}(\Omega'')$ respectively and ϕ varies in a bounded subset of $\tilde{a}^{1,\sigma,\mu}(\Omega'')$ where $\tau(\Omega'')$ is bounded by a constant smaller than 1.

Lemma 2.11. *Let $\phi\in\mathscr{P}^{(\sigma,\mu)}_{loc}(\Omega)$ be homogeneous of degree one with respect to ξ and let A be defined by* (2.7) *and $u\in G^{(\sigma)'}(\Omega)$. Assume that for $x^\circ\in\Omega$ there exists $\bar r>0$ such that $\tau(B(x^\circ,\bar r))<1/2$.* [13] *Then*

i) *there exists $D>0$ and $C>0$ such that for every $\xi\in R^n_D$ and $x\in B(x^\circ,\bar r)=V$ there exists a unique $\eta\in R^n_C$ such that $\xi=\nabla_x\phi(x,\eta)$;*

ii) *let $\xi^\circ\in R^n_D$ and $\xi^\circ=\nabla_x\phi(x^\circ,\eta^\circ)$.* If $(\nabla_\eta\phi(x^\circ,\eta^\circ),\eta^\circ)\notin WF_{(\sigma)}(u)$, *then $(x^\circ,\xi^\circ)\notin WF_{(\sigma)}(Au)$.*

Proof. i) Let $C>B_0$ be such that $\tau(V)<C/2(1+C)$, $V=B(X^\circ,\bar r)\subset\Omega$ and let $D=C(C-\tau(V)(1+C))/(C-2\tau(V)(1+C))$.

Then i) follows by noting that for every $x\in V$, $|\xi|\geqq D$, $F(\eta)=\xi+\eta-\nabla_x\phi(x,\eta)$ is a contraction mapping on $\{\eta\in R^n;\ |\xi|C/D\leqq|\eta|\leqq|\xi|C/(C-\tau(V)(1+C))\}$. Moreover the mapping $(x,\xi)\to\eta=\nabla_x\phi^{-1}(x,\xi)$ is continuous on $V\times R^n_D$.

ii) Let now $\xi^\circ\in R^n_D$ and $\xi^\circ=\nabla_x\phi(x^\circ,\eta^\circ)$, $y^\circ=\nabla_\eta\phi(x^\circ,\eta^\circ)$. Since $(y^\circ,\eta^\circ)\notin WF_{(\sigma)}(u)$, there exist $r_1>0$, $\chi_1\in G^{(\sigma)}_0(B(y^\circ,r_1))$, $\chi_1=1$ on $B(y^\circ,r_1/2)$, a conic neighborhood Γ_1 of η° and two positive constants C_0, c_0 such that

$$|(\chi_1\tilde u)(\eta)|\leqq C_0\exp(-c_0|\eta|^{1/\sigma}),\ \eta\in\Gamma_1. \tag{2.24}$$

Moreover for any neighborhood $\Gamma_1''\subset\subset\Gamma_1$ of η° we can choose $r<\bar r$ and a conic neighborhood Γ_2 of ξ° such that

$$\nabla_x\phi(x,\eta)\notin\Gamma_2\cap R^n_D,\ x\in B(x^\circ,r)=V,\ \eta\notin\Gamma_1''\cap R^n_C.$$

Next choose $\psi\in G^{(\sigma)}(R^n)$ such that $\operatorname{supp}\psi\subset\Gamma_1$, $\psi\equiv1$ on a conic neighborhood Γ_1' of η° such that $\Gamma_1''\subset\subset\Gamma_1'\subset\subset\Gamma_1$ and

$$|\psi^{(\alpha)}(\eta)|\leqq L^{|\alpha|+1}\alpha!^\sigma(1+|\eta|)^{-|\alpha|},\ \alpha\in Z^n_+,\ \eta\in R^n_{B_0},$$

for a positive L and let $\chi_2\in G^{(\sigma)}_0(V')$, $\chi_2(x^\circ)\neq0$. We can write

$$\begin{aligned}(\widetilde{\chi_2Au})(\xi)&=u({}^t(A\psi(D))\chi_2e^{-i<\cdot,\xi>})+u({}^t(A(I-\psi(D)))\chi_2e^{-i<\cdot,\xi>})\\&=v_1(\xi)+v_2(\xi).\end{aligned}$$

Since for a positive constant c_1

$$|\nabla_x\phi(x,\eta)-\xi|\geqq c_1(|\eta|+|\xi|),\ x\in V',\ \eta\notin\Gamma_1',\ |\eta|\geqq C,\ \xi\in\Gamma_2\cap R^n_D,$$

by Proposition 1.2.6 there exist positive constants C', c' such that for such ξ, η

$$\left|\int e^{i\phi(x,\eta)-i<x,\xi>}a(x,\eta)\chi_2(x)dx\right|\leqq C'\exp(-c'(|\eta|+|\xi|)^{1/\sigma}).$$

Thus there exists $H>0$ such that for every $\xi\in\Gamma_2\cap R^n_D$

$$\begin{aligned}{}^t(A(I-\psi(D)))\chi_2e^{-i<\cdot,\xi>}&=\iint e^{i\phi(x,\eta)-i<y,\eta>}a(x,\eta)(1-\psi(\eta))\chi_2(x)e^{-i<x,\xi>}dx\,đ\eta\\&\in G^{(\sigma),H}_b(R^n),\end{aligned}$$

[13] See Definition 2.9.

with norm estimated by $0(\exp(-c'|\xi|^{1/\sigma}))$. This proves that

$$|v_2(\xi)| \leqq C_2 \exp(-c'|\xi|^{1/\sigma}), \quad \xi \in \Gamma_2 \cap R^n_D, \tag{2.25}$$

for a positive constant C_2. Next write

$$\begin{aligned} v_1(\xi) &= \widehat{\chi_2 A\psi(D)(\chi_1 u)}(\xi) + (1-\chi_1)u({}^t(A\psi(D))\chi_2 e^{-i<\cdot,\xi>}) \\ &= v_{11}(\xi) + v_{12}(\xi). \end{aligned}$$

By (2.24) and the remark following (2.7), $A\psi(D)(\chi_1 u) \in G^{(\sigma)}(\Omega)$ and this implies that v_{11} satisfies an estimate of type (2.25) for every $\xi \in R^n$. Finally since for a suitable choice of r and Γ_1

$$\nabla_\eta \phi(x, \eta) \in B(y^\circ, r_1/4), \quad x \in V', \quad \eta \in \Gamma_1 \cap R^n_C$$

by Theorem 2.5.

$$K_{A\psi(D)} = \int e^{i\phi(x,\eta) - i<y,\eta>} a(x, \eta)\psi(\eta) d\eta \in G^{(\sigma)}(V' \times \complement B(y^\circ, r_1/2)).$$

Thus for every $\xi \in R^n$, ${}^t(A\psi(D))\chi_2 e^{-i<\cdot,\xi>} \in G^{(\sigma)}(\complement B(y^\circ, r_1/2))$ with norms estimated by $O(\exp(-c''|\xi|^{1/\sigma}))$, c'' a positive constant. This proves that r_2 too satisfies an estimate of type (2.25) for every $\xi \in R^n$ and we can conclude that $(x^\circ, \xi^\circ) \notin WF_{(\sigma)}(Au)$.

Lemma 2.12. *Let $\phi \in \mathscr{P}^{(\sigma,\mu)}_{loc}$ be homogeneous of degree one with respect to ξ for $|\xi|$ large and let A be defined by* (2.7) *and $u \in G^{(\sigma)'}(\Omega)$. Assume that Ω is convex and that for $x^\circ \in \Omega$ there exists $r_0 \in]0, \operatorname{dist}(x_0, \complement\Omega)[$ such that $\tau_0 = \tau(B(x^\circ, r_0)) < r_0$. Then:*

i) *there exists $r \in]0, r_0[$ such that for every $(x, \eta) \in B(x_0, r) \times R^n_{B_0}$, there exists a unique $y \in \Omega$ such that $\nabla_\eta \phi(y, \eta) = x$; if we let $y = \nabla_\eta \phi^{-1}(x, \eta)$, then $\nabla_\eta \phi^{-1}(x, \eta)$ is uniformly continuous (in the conic sense) in $B(x_0, r) \times R^n_{B_0}$ with values in $B(x_0, r_0)$;*

ii) *let $x^\circ = \nabla_\eta \phi(y^\circ, -\xi^\circ)$. If $(y^\circ, -\nabla_x \phi(y^\circ, -\xi^\circ)) \notin WF_{(\sigma)}(u)$, then $(x^\circ, \xi^\circ) \notin WF_{(\sigma)}({}^t Au)$.*

Proof. Let $r > 0$ be such that $r + \tau_0 < r_0$. For every $(x, \eta) \in B(x^\circ, r) \times R^n_{B_0}$, $F(y) = x + y - \nabla_\eta \phi(y, \eta)$ is a contraction mapping on $B(x^\circ, r_0)$. Hence there exists a unique $y \in B(x^\circ, r_0)$ such that $x = \nabla_\eta \phi(y, \eta)$. If $x = \nabla_\eta \phi(y', \eta)$, $y' \in \Omega$ then $F(y') = y'$ and

$$|y - y'| = |F(y) - F(y')| \leqq \tau(\Omega')|y - y'|, \quad \Omega' \subset\subset \Omega, \quad y, y' \in \Omega'.$$

Hence $y = y'$.

Let $y_1 = \nabla_\eta \phi^{-1}(x_1, \eta_1)$, $y_2 = \nabla_\eta \phi^{-1}(x_2, \eta_2)$,

$(x_1, \eta_1), (x_1, \eta_2) \in B(x_0, r) \times R^n_{B_0}$, $|\eta_1| = |\eta_2| > B_0$.

Then,

$y_1 - y_2 = x_2 - x_1 + \nabla_\eta J(y_1, \eta_1) - \nabla_\eta J(y_2, \eta_2)$, $J(y, \eta) = \phi(y, \eta) = \phi(y, \eta) - \langle y, \eta \rangle$.

Hence

$$|y_1 - y_2| \leqq |x_2 - x_1| + \tau_0 |y_1 - y_2| + \tau_0 |\eta_1 - \eta_2|$$

and

$$|y_1 - y_2| \leqq (1 - \tau_0)^{-1} |x_2 - x_1| + \tau_0 (1 - \tau_0)^{-1} |\eta_1 - \eta_2|.$$

This completes the proof of i).

ii) Let $\eta^\circ = -\nabla_x \phi(y^\circ, -\xi^\circ)$. Since $(y^\circ, \eta^\circ) \notin WF_{(\sigma)}(u)$, there exist $\tilde{r} \in]0, r_0[$ and a conic neighborhood Γ_1 of η_0 such that for every $\chi_1 \in G_0^{(\sigma)}(B(y_0, \tilde{r}))$, $\chi_1(y_0) \neq 0$:

$$|\widetilde{(\chi_1 u)}(\eta)| \leqq C \exp(-c|\eta|^{1/\sigma}), \qquad \eta \in \Gamma_1,\ C, c > 0. \tag{2.26}$$

Let $\chi_1' \in G_0^{(\sigma)}(B(y_0, r_1))$, $r_1 < \tilde{r}$, $\chi_1' \equiv 1$ in a neighborhood of $B(y_0, r_1/2)$ and $\Gamma_1', \Gamma_1'', \Gamma_2''$ open cones such that

$$\eta_0 \in \Gamma_1'' \subset\subset \Gamma_1' \subset\subset \Gamma_1; \qquad \xi_0 \in \Gamma_2''.$$

Using the continuity of $\nabla_y \phi(y, \eta)$ and $\nabla_\eta^{-1} \phi(x, \eta)$ we can choose $r_1, r_2 \in]0, r[$ and Γ_2'' such that:

$$-\nabla_y \phi(y, \eta) \in \Gamma_1'' \cap R^n_{B_0},\ \forall (y, \eta) \in B(y_0, r_1) \times ((-\Gamma_2'') \cap R^n_{B_0}) \tag{2.27}$$

and

$$\nabla_\eta \phi^{-1}(x, \eta) \in B(y_0, r/8),\ \forall (x, \eta) \in B(x_0, r_2) \times ((-\Gamma_2'') \cap R^n_{B_0}). \tag{2.28}$$

Now choosing $\chi_1 \in G_0^{(\sigma)}(B(y_0, r_1/2))$, $\chi_1 \equiv 1$ in a neighborhood of $B(y_0, r_1/4)$, $\chi_2 \in G_0^{(\sigma)}(B(x_0, r_2))$, $\chi_2(x_0) \neq 0$ and $\psi \in G^{(\sigma)}(R^n)$ with $\operatorname{supp}\psi \subset \Gamma_1$, $\psi \equiv 1$ on Γ_1',

$$|\psi^{(\alpha)}(\eta)| \leqq L^{|\alpha|+1} \alpha!^\sigma (1 + |\eta|)^{-|\alpha|},\ \alpha \in Z^n_+,\ \eta \in R^n_{B_0},\ L > 0,$$

we can write

$$\begin{aligned}(\chi_2\, {}^t\! A u)(\xi) &= (\psi(D)\chi_1 u)(\chi_1' A \chi_2 e^{-i\langle \cdot, \xi \rangle}) \\ &\quad + ((I - \psi(D))\chi_1 u)(\chi_1' A \chi_2 e^{-i\langle \cdot, \xi \rangle}) + ((1 - \chi_1)u)(A \chi_2 e^{-i\langle \cdot, \xi \rangle}) \\ &= v_0(\xi) + v_1(\xi) + v_2(\xi).\end{aligned}$$

In view of (2.26), $\psi(D)\chi_1 u \in G^{(\sigma)}(R^n)$. Thus ${}^t\! A \chi_1' \psi(D)\chi_1 u \in G^{(\sigma)}(\Omega)$ and since $v_0(\xi) = \int e^{-i\langle x, \xi \rangle} \chi_2(x)({}^t\! A \chi_1' \psi(D)\chi_1 u)(x)\, dx$, we conclude that

$$|v_0(\xi)| \leqq C' \exp(-c'|\xi|^{1/\sigma}),\ \xi \in R^n, \tag{2.29}$$

for certain positive constants C', c'.

If we let Γ_2, Γ_2' be open cones such that, $\xi_0 \in \Gamma_2 \subset\subset \Gamma_2' \subset\subset \Gamma_2''$ and choose $\phi \in G^{(\sigma)}(R^n)$ with $\operatorname{supp}\phi \subset R^n \backslash (-\Gamma_2')$, $\phi \equiv 1$ on $R^n \backslash (-\Gamma_2'')$, $|\phi^{(\alpha)}(\eta)| \leqq M^{|\alpha|+1} \alpha!^\sigma (1$

$+|\eta|)^{-|\alpha|}$, $\alpha \in Z_n^+$, $\eta \in R^n_{B_0}$, $M > 0$, we can write

$$v_1(\xi) = ((I - \psi(D))\chi_1 u)(\chi_1' A\phi(D)\chi_2 e^{-i<\cdot,\xi>})$$
$$+ ((I - \psi(D))\chi_1 u)(\chi_1' A(I - \phi(D))\chi_2 e^{-i<\cdot,\xi>}) = v_1'(\xi) + v_1''(\xi).$$

Since, for a positive constant $\bar{c}$,

$$|\xi + \eta| \geqq \bar{c}(|\xi| + |\eta|),\ \forall(\xi, \eta) \in \Gamma_2 \times \operatorname{supp}\phi,$$

there exist positive constants C_1, c_1 such that

$$\phi(\eta)|\tilde{\chi}_2(\xi + \eta)| \leqq C_1 \exp(-c_1(|\xi|^{1/\sigma} + |\eta|^{1/\sigma})),\ \xi \in \Gamma_2.$$

Moreover, for every η, the function $y \to e^{i\phi(y,\eta)}\chi_1'(y)a(y, \eta)$ is in $G_0^{(\sigma,H)}$ with H independent of η and a norm estimated by $O(\exp(\varepsilon|\eta|^{1/\sigma}))$, for every $\varepsilon > 0$. Hence, for $\xi \in \Gamma_2$ $\chi_1' A\phi(D)\chi_2 e^{-i<\cdot,\xi>} \in G_0^{(\sigma)}(B(y_0, r_1))$ with a norm estimated by $O(\exp(-c_1|\xi|^{1/\sigma}))$. Thus v_1' satisfies an estimate of type (2.29) for every $\xi \in \Gamma_2$.

Since in view of (2.27)

$$|\nabla_y\phi(y, \eta) + \zeta| \geqq c''(|\eta| + |\zeta|),\ (y, \eta) \in B(y_0, r_1) \times ((-\Gamma_2'') \cap R^n_{B_0}),\ \zeta \in \operatorname{supp}(1 - \psi),$$

c'' a positive constant, by Proposition 1.2.6 there exist positive constants C_3, c_3 such that

$$\left|\int e^{i\phi(y,\eta) + i<y,\zeta>}(1 - \phi(\eta))a(y, \eta)\chi_1'(y)dy\right|$$
$$\leqq C_3 \exp(-c_3(|\eta|^{1/\sigma} + |\zeta|^{1/\sigma})),\ \zeta \in \operatorname{supp}(1 - \psi).$$

Hence

$$g(\eta) = \int (1 - \psi(\zeta))\widetilde{(\chi_1 u)}(\zeta)\left(\int e^{i\phi(y,\eta) + i<y,\zeta>}(1 - \phi(\eta))\, a(y, \eta)\chi_1'(y)dy\right)d\!\!\bar{}\,\zeta$$

can be estimated by $O(\exp(-c_3|\eta|^{1/\sigma}))$.

This proves that on estimate of type (2.29) is valid for

$$v_1''(\xi) = \int e^{-i<x,\xi>}\chi_2(x)\left(\int e^{-i<x,\eta>} g(\eta)d\!\!\bar{}\,\eta\right)dx.$$

Letting now

$$v_2(\xi) = ((1 - \chi_1)u)(A\phi(D)\chi_2 e^{-i<\cdot,\xi>}) + ((1 - \chi_1)u)(A(I - \phi(D))\chi_2 e^{-i<\cdot,\xi>})$$
$$= v_2'(\xi) + v_2''(\xi),$$

the arguments used for v_1' prove that an estimate of type (2.29) holds also for v_2'. Since, as a consequence of (2.28),

$$|\nabla_\eta\phi(y, \eta) - x| > r(1 - \tau_0)/8,\ \forall(x, \eta) \in B(x_0, r_2) \times ((1 - \Gamma_2'') \cap R^n_{B_0}),$$

$$y \in \complement\,\overline{B(y_0, r/4)},$$

from Theorem 2.5 it follows that the kernel of $A(I-\varphi(D))$ is in $G^{(\sigma)}(B(x_0, r_2) \times \complement\overline{B(y_0, r/4)})$. This implies that $A(I-\varphi(D))\chi_2 e^{-i<\cdot,\xi>} \in G^{(\sigma)}(\complement\overline{B(y_0, r/4)})$ with norms estimated by $O(\exp(-c_4|\xi|^{1/\sigma})$. Since $\mathrm{supp}(1-\chi_1)u \subset \complement\overline{B(y_0, r/4)}$, this proves that $v_2''(\xi)$ satisfies an estimate of type (2.29) for every $\xi \in R^n$.

Summing up what we have proved above, we can conclude that an estimate of type (2.29) holds for $(\widehat{\chi_2{}^t Au})(\xi)$ for every $\xi \in \Gamma_2$. This completes the proof of the lemma.

In view of Proposition 1.1.11′ we immediately have

Corollary 2.13. *Let A be given by* (2.7) *and assume that there exists $\psi \in \mathscr{P}_{loc}^{(\sigma,\mu)}(\Omega)$ homogeneous of degree one with respect to ξ, for $|\xi|$ large, such that $\phi - \psi \in \tilde{a}^{q,\sigma,\mu}(\Omega)$, $q \in [0, \mu/\sigma[$. Then Lemmas* 2.11 *and* 2.12 *hold for A, replacing ϕ by ψ in* i), ii).

Remark 2.14. By means of a $G_0^{(\sigma)}$ partition of unity $\{\chi_h\}_{h\in Z_+}$ in Ω related to a locally finite covering $\{\Omega_h\}_{h\in Z_+}$ of Ω with relatively compact open subsets Ω_h of Ω we define

$$(2.30) \qquad a^\natural(x, y, \xi) = \sum_{h,k}{}' \chi_h(x) a(x, \xi) \chi_k(y), \quad a \in a^{\infty,\sigma,\mu}(\Omega),$$

and for $\phi \in \mathscr{P}_{loc}^{(\sigma,\mu)}(\Omega)$

$$(2.31) \qquad (A^\natural u)(x) = Os\text{-}\iint e^{i\phi(x,\xi) - i<y,\xi>} a^\natural(x, y, \xi) u(y) dy đ\xi, \quad u \in G_0^{(\sigma)}(\Omega),$$

where $\sum_{h,k}{}'$ runs over the $(h, k) \in Z_+^2$ such that $\Omega_h \cap \Omega_k \neq \emptyset$. For every $\Omega' \subset\subset \Omega$, $A^\natural u$, $u \in G_0^{(\sigma)}(\Omega')$, can be written in the form (2.7) with an amplitude given as the product of $a(x, \xi)$ and a function $\chi_{\Omega'} \in G_0^{(\sigma)}(\Omega)$ equal to one on Ω'. Moreover if $\Omega_0' \subset\subset \Omega'$ we can choose the covering $\{\Omega_h\}_{h\in Z_+}$ such that

$$(2.32) \qquad \mathrm{supp} A^\natural u \subset \Omega' \text{ for every } u \in G_0^{(\sigma)}(\Omega_0').$$

From this remark it follows that if we assume that Ω is convex, Theorem 2.10 holds for the operator $P_1 P_2^\natural$, where $P_2^\natural$, is defined according to (2.31).

Remark 2.15. As in the C^∞ and in finite order cases it could be easily seen that if in Theorem 2.10 $p_i \sim \sum_{h\geq 0} p_{i,h}$ in $FS_n^{\infty,\sigma,\mu}(\Omega)$, $i = 1,2$, then

$$q_j(x, \xi) = \sum_{\ell=0}^{j} \sum_{|\alpha|+h=\ell} \alpha!^{-1} D_y^\alpha (\partial_\xi^\alpha p_{1,h}(x, \tilde{\nabla}_x \phi(x, y, \xi)) p_{2,j-\ell}(y, \xi))|_{y=x}.$$

Remark 2.16. From Remark 2.14 it follows that the results of Lemmas 2.11 and 2.12 and of Corollary 2.13 also hold for operators defined as in (2.31).

3. Applications of Fourier integral operators of infinite order to the Cauchy problem for certain hyperbolic operators with a characteristic of constant multiplicity

In this section we consider operators of the form

$$(3.1)\quad P(t, x, D_t, D_x) = (D_t - \lambda(t, x, D_x))^m + \sum_{j=1}^{m} a_j(t, x, D_x)(D_t - \lambda(t, x, D_x))^{m-j},$$

assuming that for every $t \in [0, T]$, $\lambda(t, x, D_x)$, $a_j(t, x, D_x)$ are pseudo-differential operators in an open convex set $\Omega \subset R^n$ satisfying:

i) $\lambda(t, x, \xi)$ is a real-valued function on $[0, T] \times \Omega \times R^n$ such that $\lambda \in C^{m-1}([0, T]; C^\infty(\Omega \times R^n)) \cap \mathscr{B}^{m-1}([0, T]; \tilde{S}_n^{1,1,1}(\Omega))$;

ii) $a_j \in C([0, T]; C^\infty(\Omega \times R^n)) \cap \mathscr{B}([0, T]; \tilde{S}_n^{pj,1,1}(\Omega))$, $j = 1, \ldots, m$, $p \in [0, 1[$

From i) and Proposition 1 of [25] it follows that for every $\Omega' \subset\subset \Omega$ there exists $T' \in]0, T]$ such that for every $s \in [0, T']$, $y \in \Omega'$, $\eta \in R^n$ there exists a unique solution $x(t, s; y, \eta) \in \Omega$, $\xi(t, s; y, \eta) \in R^n$ of the system

$$(3.2)\qquad \begin{cases} dx/dt = -\nabla_\xi \lambda(t, x, \xi) \\ x(s, s; y, \eta) = y \end{cases}, \quad \begin{cases} d\xi/dt = \nabla_x \lambda(t, x, \xi) \\ \xi(s, s; y, \eta) = \eta \end{cases}$$

and

$$(3.3_1)\qquad x \in C^m([0, T']^2; C^\infty(\Omega' \times R^n)) \cap \mathscr{B}^m([0, T']^2; \tilde{S}_n^{0,1,1}(\Omega')),$$

$$(3.3_2)\qquad \xi \in C^m([0, T']^2; C^\infty(\Omega' \times R^n)) \cap \mathscr{B}^m([0, T']^2; \tilde{S}_n^{1,1,1}(\Omega')),$$

$$(3.4)\qquad |x(t, s; y, \eta) - y| \le c_0|t - s|, \ |\xi(t, s; y, \eta) - \eta| \le c_0|t - s|(1 + |\eta|)$$

for a suitable constant $c_0 > 0$ and every $(t, s; y, \eta) \in [0, T']^2 \times \Omega' \times R^n$.

From (3.4) it follows that if Ω'' is an open convex set such that $\Omega' \subset\subset \Omega'' \subset\subset \Omega$, then for T' sufficiently small $x(t, s; y, \eta) \in \Omega''$ for $(t, s; y, \eta) \in [0, T']^2 \times \Omega' \times R^n$.

For T'' sufficiently small we can consider the solution $\phi(t, s; x, \eta)$ of the eikonal equation

$$(3.5)\qquad \begin{cases} \partial_t \phi(t, s; x, \eta) = \lambda(t, x, \nabla_x \phi(t, s; x, \eta)) \\ \phi(s, s; x, \eta) = \langle x, \eta \rangle \end{cases}$$

in $[0, T'']^2 \times \Omega'' \times R^n$. It follows that $\phi(t, s) \in \mathscr{P}(\tau_{\Omega''}|t - s|)$ for $(t, s; x, \eta) \in [0, T'']^2 \times \Omega'' \times R^n$ and

$$(3.6)\qquad \phi \in C^m([0, T'']^2; C^\infty(\Omega'' \times R^n)) \cap \mathscr{B}^m([0, T'']^2; \tilde{S}_n^{1,\sigma,\sigma}(\Omega'')),$$

for every $\sigma \ge 1$. Moreover if $x = x(t, s; y, \eta)$, $\xi = \xi(t, s; y, \eta)$ are defined in $[0, T'']^2 \times \Omega' \times R^n$, $T'' < \mathrm{dist}(\Omega', \complement\Omega'')/\tau_{\Omega''}$ by $y = \nabla_\eta \phi(t, s; x, \eta)$, $\xi = \nabla_x \phi(t, s; x, \eta)$, then $(x(t, s; y, \eta), \xi(t, s; y, \eta)) \in \Omega'' \times R^n$ is the solution of (3.2) for $(t, s; y, \eta) \in [0, \inf(T', T'')]^2 \times \Omega' \times R^n$.

We shall slso consider the solution of (3.2) when $(y, \eta)\in\hat{\Omega}' \times R^n$, where $\Omega' \subset \subset \hat{\Omega}' \subset\subset \Omega$. Let $d = \mathrm{dist}(\Omega', \complement\hat{\Omega}')$ and write $T_{\Omega'}$ instead of T' in this case. For $(t, s; x, \eta)\in[0, T_{\Omega'}]^2 \times \Omega' \times R^n$ define on $\bar{\hat{\Omega}}'$ the function $F(y) = x - x(t, s; y, \eta) + y$. If $T_{\Omega'} < \inf(d/c_0, 1/2c_0)$, then F is a contraction mapping on $\bar{\hat{\Omega}}'$. Thus on Ω' there exists the inverse function $x \to y(t, s; x, \eta)$ of the function $y \to x(t, s; y, \eta)$ and $y(t, s; x, \eta)$ has the property (3.3_1).

Theorem 3.1. *Assume that the operator P given by* (3.1) *satisfies* i), ii), iii). *Then there exist $e^{(\ell)}(t, s; x, y, \xi)$, $\ell = 0,\dots,m-1$ such that for every $\Omega' \subset\subset \Omega$ and a suitable $T_{\Omega'}\in]0, T]$,*

$$D_t^j e^{(\ell)}\in C([0, T_{\Omega'}]^2; C^\infty(\Omega' \times \Omega' \times R^n))\cap\mathscr{B}([0, T_{\Omega'}]^2; \boldsymbol{a}^{\infty,\sigma,\sigma}(\Omega' \times \Omega')),$$

$j = 0,\dots,m$ for every $\sigma\in]1,1/p[$, and the Fourier integral operators $E^{(\ell)}(t, s)$, $\ell = 0,\dots,m-1$, $t, s\in[0, T_{\Omega'}]^2$, defined by

$$(E^{(\ell)}(t, s)u)(x) = Os\text{-}\iint e^{i\phi(t,s;x,\eta)-i<y,\eta>} e^{(\ell)}(t, s; y, \eta)u(y)dyd\eta, \tag{3.7}$$

$$x\in\Omega',\ u\in G_0^{(\sigma)}(\Omega'),$$

where ϕ is the solution of (3.5), *satisfy the equations*

$$\begin{cases} P(t, x, D_t, D_x) E^{(\ell)}(t, s) = R_{\Omega'}^{(\ell)} & \text{on } G_0^{(\sigma)}(\Omega') \\ D_t^j E^{(\ell)}(s, s) = \delta_\ell^j I, & 0 \le j \le \ell, \end{cases} \tag{3.8}$$

where $R_{\Omega'}^{(\ell)}(t, s)$ have their kernel in $C([0, T_{\Omega'}]; G^{(\sigma)}(\Omega' \times \Omega'))$ and I is the identity operator. Moreover $T_{\Omega'}$ can be chosen so that $D_t^j E^{(\ell)}(t, s)u\in C([0, T_{\Omega'}]^2; G_0^{(\sigma)}(\Omega))$ for every $u\in G_0^{(\sigma)}(\Omega')$, $j = 0,\dots,m$ and if $\Omega_0' \subset\subset \Omega'$, $E^{(\ell)}(t, s; y, \eta)$ can be defined so as

$$\operatorname{supp} E^{(\ell)}(t, s)u \subset \Omega' \text{ for every } u\in G_0^{(\sigma)}(\Omega_0'),\ (t, s)\in[0, T_{\Omega'}]^2. \tag{3.8'}$$

Proof. Let $E(t, s)$ be Fourier integral operator of type (2.1) with phase functions as in (3.7) and recall that for every $\Omega' \subset\subset \Omega$ the solution $\phi(t, s; x, \eta)$ of (3.5) has the property (3.6) where T'' and Ω'' are replaced by $T_{\Omega'}$ and Ω' respectively, and is in $\mathscr{P}(\tau_{\Omega'}|t-s|)$, $\tau_{\Omega'}\in[0, 1[$.

By using Remark 2.14 the amplitudes $e(t, s; x, y, \eta)$ of $E(t, s)$ can be defined by (2.30) where $a(x, \xi)$ is replaced by $e(t, s; x, \eta)$. Note also that in view of Corollary 2.8, for every $(t, s)\in[0, T_{\Omega'}]^2$ $P(t, x; D_t, D_x) E(t, s)$ will be equal to *a* σ-regularizing operator on Ω' for every $\Omega' \subset\subset \Omega$ if we determine $e(t, s; x, \eta)$ such that $P(t, x, D_t, D_x) E(t, s) \sim 0$ for every $\Omega' \subset\subset \Omega$ and every $(t, s)\in[0, T_{\Omega'}]^2$. To this end, assuming that the amplitude $e(t, s; x, \eta)$ is given by a formal series $\sum_{h\ge0} e_h(t, s; x, \eta)$, we note that by Remarks 2.14 and 2.15 there exists $T_{\Omega'}\in[0, T[$ such that for every $(t, s)\in[0, T_{\Omega'}]^2$ the operator $(D_t - \lambda(t, x, D_x)) E(t, s)$ on $G_0^{(\sigma)}(\Omega')$ is equal up to an operator with kernel in $C^{m-1}([0, T_{\Omega'}]^2, G^{(\sigma)}(\Omega' \times \Omega'))$ to a Fourier integral operator with the same phase function of $E(t, s)$ and amplitude

$$b(t, s; x, \eta) = \hat{b}(t, s; y(t, s; x, \eta), \eta) \sim \sum_{h \geq 0} \hat{b}_h(t, s; y(t, s; x, \eta), \eta),$$

$$(t, s; x, \eta) \in [0, T_{\Omega'}]^2 \times \Omega' \times R^n,$$

where

$$\hat{b}_0(t, s; y, \eta) = (D_t - \hat{q}(t, s; y, \eta))\hat{e}_0(t, s; y, \eta)$$

$$\hat{b}_h(t, s; y, \eta) = (D_t - \hat{q}(t, s; y, \eta))\hat{e}_h(t, s; y, \eta) - \sum_{r=0}^{h-1} \sum_{|\beta| \leq h-r+1} \hat{p}_{\beta,h-r}(t, s; y, \eta) D_y^\beta \hat{e}_r(t, s; y, \eta),$$

$$h \geq 1, \ (t, s; y, \eta) \in [0, T_{\Omega'}]^2 \times \hat{\Omega}' \times R^n.$$

Here $y(t, s; x, \eta)$ is the inverse function of the solution $x(t, s; y, \eta)$ of (3.2) when $(t, s; y, \eta) \in [0, T_{\Omega'}]^2 \times \hat{\Omega}' \times R^n$, $\Omega' \subset\subset \Omega$, as indicated above,

$$\hat{e}(t, s; y, \eta) = e(t, s; x(t, s; y, \eta), \eta),$$

(3.9) $$\hat{e}_h(t, s; y, \eta) = e_h(t, s; x(t, s; y, \eta), \eta), \qquad h \geq 0$$

$$\hat{q}(t, s; y, \eta) = q(t, s; x(t, s; y, \eta), \eta),$$

(3.10) $$q(t, s; x, \eta) = -2^{-1} i \sum_{j,k=1}^{n} \partial^2_{\xi_j \xi_k} \lambda(t, x, \nabla_x \phi(t, s; x, \eta)) \partial^2_{x_j x_k} \phi(t, s; x, \eta),$$

and

$$\sum_{|\beta| \leq h-r+1} \hat{p}_{\beta,h-r}(t, s; y, \eta) D_y^\beta \hat{e}_r(t, s; y, \eta)$$

$$= \sum_{|\alpha| = h-r+1} \alpha!^{-1} D_z^\alpha (\partial_\xi^\alpha \lambda(t, x, \tilde{\nabla}_x \phi(t, s; x, z, \eta))) \hat{e}_r(t, s; y(t, s; z, \eta), \eta)_{|z = x(t,s;y,\eta)},$$

$$\tilde{\nabla}_x \phi(t, s; x, z, \eta) = \int_0^1 \nabla_x \phi(t, s; z + \theta(x - z), \eta) d\theta.$$

Thus by repeatedly using Remarks 2.14 and 2.15 we see that there exists $T_{\Omega'} \in [0, T[$ such that up to an operator with kernel in $C([0, T_{\Omega'}]^2; G^{(\sigma)}(\Omega' \times \Omega'))$, $P(t, x, D_t, D_x) E(t, s)$ is equal on $G_0^{(\sigma)}(\Omega')$ to an operator with the same phase function as $E(t, s)$ and amplitude $d(t, s; x, \eta) \sim \sum_{h \geq 0} d_h(t, s; x, \eta)$ such that

$$\hat{d}_0(t, s; y, \eta) = [(D_t - \hat{q})^m + \sum_{j=1}^{m} \hat{a}_j (D_t - \hat{q})^{m-j}] \hat{e}_0,$$

$$\hat{d}_h(t, s; y, \eta) = [(D_t - \hat{q})^m + \sum_{j=1}^{m} \hat{a}_j (D_t - \hat{q})^{m-j}] \hat{e}_h - \sum_{r=0}^{h-1} \sum_{\ell=1}^{m} \sum_{i+j=\ell} \sum_{|\beta| \leq h-r+i} \hat{d}^{(m-j)}_{\beta,h-r,i} D_y^\beta D_t^{m-\ell} \hat{e}_r, \ h = 1,2,\ldots.$$

Here $\hat{a}_j(t, s; y, \eta) = a_j(t, x(t, s; y, \eta), \xi(t, s; y, \eta))$ and by Corollary 1.1.15

$$(3.11)\qquad \begin{aligned}&\hat{a}_j \in C([0, T_{\Omega'}]^2; C^\infty(\hat{\Omega}' \times R^n)) \cap \mathscr{B}([0, T_{\Omega'}]^2; \tilde{S}_n^{pj,1,1}(\hat{\Omega}')),\\ &D_t^{j-1}\hat{q} \in C([0, T_{\Omega'}]^2; C^\infty(\hat{\Omega}' \times R^n)) \cap \mathscr{B}([0, T_{\Omega'}]^2; \tilde{S}_n^{0,1,1}(\hat{\Omega}')),\\ &\quad j = 1, \dots, m.\end{aligned}$$

We note that from i) and ii)

$$(D_t - \hat{q})^m + \sum_{j=1}^m \hat{a}_j (D_t - \hat{q})^{m-j} = D_t^m + \sum_{j=1}^m \hat{b}_j D_t^{m-j},$$

for suitable $\hat{b}_j$, satisfy (3.11).

Moreover there exist positive constants R, A, C such that

$$(3.12)\qquad \begin{aligned}|D_y^\gamma D_\eta^\delta \hat{d}_{\beta,h-r,i}^{(m-j)}(t, s; y, \eta)| &\le R^{m-j} C^{(m-j)(h-r+i)} A^{(m-j+1)(h-r+i)-|\beta|+|\gamma|+|\delta|}\\ &\times (1 + |\eta|)^{pj-h+r-|\delta|}(h - r + i - |\beta| + |\gamma| + |\delta|)!.\end{aligned}$$

Thus the equation $P(t, x, D_t, D_x)E(t, s) \sim 0$ will be satisfied if for $(t, s; y, \eta) \in [0, T_{\Omega'}]^2 \times \hat{\Omega}' \times R^n$ the functions $\hat{e}_h(t, s; y, \eta)$, $h \ge 0$ are solutions of the equations

$$(3.13_0)\qquad [D_t^m + \sum_{j=1}^m \hat{b}_j(t, s; y, \eta) D_t^{m-j}]\hat{e}_0 = 0,$$

$$(3.13_h)\qquad \begin{aligned}&[D_t^m + \sum_{j=1}^m \hat{b}_j(t, s; y, \eta) D_t^{m-j}]\hat{e}_h\\ &= \sum_{r=0}^{h-1} \sum_{k=1}^m \sum_{i+j=k} \sum_{|\beta| \le h-r+i} \hat{d}_{\beta,h-r,i}^{(m-j)} D_y^\beta D_t^{m-k} \hat{e}_r, \quad h \ge 1.\end{aligned}$$

For $\ell = 0, \dots, m-1$, consider now the solution $e_0^{(\ell)}$ of (3.13_0) satisfying the initial conditions

$$(3.14_0)\qquad D_t^j e_0^{(\ell)}(s, s; x, \eta) = \delta_j^\ell, \qquad j = 0, \dots, m-1.$$

It is easy to see that there exist positive constants c, c', D_0, which depend on the coefficients $\hat{b}_j$ such that for $(y, \eta) \in \hat{\Omega}' \times R_{D_0}^n$, $(t, s) \in [0, T_{\Omega'}]^2$

$$(3.15)\qquad \begin{aligned}&|D_t^j \hat{e}_0^{(\ell)}(t, s; y, \eta)| \le |t - s|^{\ell-j}/(\ell - j)! \exp(mc'(1 + |\eta|)^p |t - s|), \; j = 0, \dots, \ell\\ &|D_t^j \hat{e}_0^{(\ell)}(t, s; y, \eta)| \le mc(1 + |\eta|)^{p(m-\ell)} |t - s|^{m-j}/(m - j)! \exp(mc'(1 + |\eta|)^p |t - s|),\\ &\quad j = \ell + 1, \dots, m.\end{aligned}$$

Moreover in the same way as in Proposition 2.1 of [4] it can be proved that there exists a constant $A > 0$ such that for $|\alpha + \beta| > 0$

$$(3.16)\qquad \begin{aligned}|D_\eta^\alpha D_y^\beta D_t^j \hat{e}_0^{(\ell)}(t, s; y, \eta)| &\le (2^{m+1}c)^{|\alpha+\beta|} A^{|\alpha+\beta|} \alpha! \beta! (1 + |\eta|)^{-|\alpha|}\\ &\times \exp(mc'(1 + |\eta|)^p |t - s|) \Bigg[(1 - \delta_0^{m-\ell+1}) mc \sum_{i=1}^{m|\alpha+\beta|-\ell-1}\\ &\times (1 + |\eta|)^{p(i+m-\ell)} |t - s|^{m-j+i}/(m - j + i)!\end{aligned}$$

$$+ \sum_{i=m-\ell}^{m|\alpha+\beta|} (1 + |\eta|)^{pi}|t - s|^{i+\ell-j}/(i + \ell - j)! \Bigg],$$

$$j = 0,\dots,m - 1,\ \ell = 0,\dots,m - 1.$$

If we let

$$\hat{g}_h^{(\ell)}(t, s; y, \eta) = \sum_{r=0}^{h-1} \sum_{k=1}^{m} \sum_{i+j=k} \sum_{|\beta| \le h-r+k} \hat{d}^{(m-j)}_{\beta,h-r,i}(t, s; y, \eta)$$
$$\times D_y^\beta D_t^{m-k} \hat{e}_r^{(\ell)}(t, s; y, \eta),$$

then the solution $\hat{e}_h^{(\ell)}$, $h \ge 1$, $\ell = 0,\dots,m - 1$ of (3.13_h) satisfying the initial conditions

(3.14_h) $$D_t^j \hat{e}_h^{(\ell)}(s, s; y, \eta) = 0 \qquad j = 0,\dots,m - 1.$$

is given by

(3.17) $$\hat{e}_h^{(\ell)}(t, s; y, \eta) = \int_s^t \hat{e}_0^{(m-1)}(t, s, \tau; y, \eta)\hat{g}_h^{(\ell)}(\tau, s; y, \eta)d\tau,$$

where $\hat{e}_0^{(m-1)}(t, s, \tau; y, \eta)$ is the solution of (3.13_0) satisfying the initial conditions

$$D_t^j \hat{e}_0^{(m-1)}(\tau, s, \tau; y, \eta) = \delta_j^{m-1}, \qquad j = 0,\dots,m - 1.$$

For this function the estimates (3.15), (3.16) with $\ell = m - 1$ and $t - \tau$ in place of $t - s$ also hold for every $(t, s, \tau) \in [0, T_{\Omega'}]^3$.

With the aid of the representation formula (3.17) and using the estimates (3.12), (3.15), (3.16) and Lemma 5.1 of [11] we can prove by induction on h that there exist positive constants C_e, A_e, B_e such that for every $\sigma \in]1, 1/p[$, $p' \in [p, 1/\sigma[$, $|\eta| > B_h(h + |\delta|)^\sigma$, $(t, s, y) \in [0, T_{\Omega'}]^2 \times \hat{\Omega}'$, $\gamma, \delta \in Z_+^n$

$$|D_t^j D_\eta^\delta D_y^\gamma \hat{e}_h^{(\ell)}(t, s; y, \eta)| \le C_e^{|\gamma+\delta|+(m+2)h} A_e^{|\gamma+\delta|+(m+1)h}(1 + |\eta|)^{-h-|\delta|}$$
$$\times (|\gamma + \delta| + (m + 1)h)!(mh)!^{-\sigma p'}\exp(mc'(1 + |\eta|)^p|t - s|)\Bigg[(1 - \delta_0^{m-1-\ell})mc$$
$$\times \sum_{i=1}^{(|\gamma+\delta|+(m+2)h)m-\ell-1} (1 + |\eta|)^{p'(i+m-\ell)}|t - s|^{m-j+i}/(m - j + i)!$$
$$+ \sum_{i=m-\ell}^{(|\gamma+\delta|+(m+2)h)m} (1 + |\eta|)^{p'i}|t - s|^{i+\ell-j}/(i + \ell - j)! \Bigg],$$

$$j, \ell = 0,\dots,m - 1,\ h \ge 1.$$

Since for every h

$$(|\gamma + \delta| + (m + 1)h)!(mh)!^{-\sigma p'} \le 2^{|\gamma+\delta|}c^{h(1-\sigma p')}h!^{m(1-\sigma p')+1-\delta}h!^\sigma|\gamma + \delta|!,$$

c a positive constant, if we choose p' such that $1 - \sigma p' \in]0, (\sigma - 1)/m[$ we easily sec that $\sum_{h\ge 0} D_t^j \hat{e}_h^{(\ell)} \in \mathscr{B}([0, T_{\Omega'}]^2; FS_n^{\infty,\sigma,\sigma}(\hat{\Omega}'))$, $j = 0,\dots,m - 1$. So by (3.9) and Theorem 1.1.21 $\sum_{h\ge 0} D_t^j e_h^{(\ell)} \in \mathscr{B}([0, T_{\Omega'}]^2; FS_n^{\infty,\sigma,\sigma}(\hat{\Omega}'))$ and by Lemma 1.1.18 we conclude that there exists $e^{(\ell)}(t, s; x, \eta)$, $\ell = 0,\dots,m - 1$ such that

$D_t^j e_h^{(\ell)} \in C([0, T_{\Omega'}]^2; C^\infty(\Omega' \times R^n)) \cap \mathscr{B}([0, T_{\Omega'}]^2; a^{\infty,\sigma,\sigma}(\Omega'))$ and

$$(3.18) \qquad D_t^j e^{(\ell)}(s, s; x, \eta) = \delta_j^\ell, \ \ell = 0,\dots,m-1, \ j = 0,\dots,m-1.$$

Moreover for every $(t, s) \in [0, T_{\Omega'}]^2$, $P(t, x, D_t, D_x) E^{(\ell)}(t, s)$ is a σ-regularizing operator on Ω', if $E^{(\ell)}$ is the Fourier integral operator in (3.7) with amplitude

$$(3.19) \qquad e^{(\ell)}(t, s; x, y, \eta) = \sum_{h,k}{}' \chi_h(x) e^{(\ell)}(t, s; x, \eta) \chi_k(y) \text{ as in (2.30).}$$

It is also easy to verify that $E^{(\ell)}$ satisfies the initial conditions in (3.8), as a consequence of (3.18).

The following corollary is a simple consequence of Theorem 3.1.

Corollary 3.1. *If the hypotheses of Theorem* 3.1 *are satisfied, the operators* $W_k(t, s)$, $k = 0, 1,\dots, m-1, (t, s) \in [0, T_{\Omega'}]^2$ *defined on* $G_0^{(\sigma)}(\Omega')$ *by*:

$$W_{m-1}(t, s) = E^{(m-1)}(t, s),$$

$$W_k(t, s) = E^{(k)}(t, s) - \sum_{h=k-1}^{m-1} W_h(t, s)(D_t^h E^{(k)})|_{t=s}, \qquad k = 0,\dots,m-2$$

are solutions of:

$$\begin{cases} P(t, x, D_t, D_x) W_k(t, s) = R_{k,\Omega'}(t, s) \\ D_t^j W_k(s, s) \qquad\qquad = \delta_k^j I, \qquad j = 0,\dots,m-1 \end{cases}$$

and satisfy:

$$(3.20) \qquad \operatorname{supp} W_k(t, s) u \subset \Omega' \textit{ for every } u \in G_0^{(\sigma)}(\Omega'_0), \ (t, s) \in [0, T_{\Omega'}]^2,$$

$\Omega'_0 \subset \subset \Omega$, *if the partition of unity* $\{\chi_h\}$ *in* (3.19) *in suitably chosen.*

With the aid of the Fourier integral operators $W_k(t, s)$ obtained in Corollary 3.1 and by solving a Volterra type integral equation we can prove as in [4] the following

Theorem 3.2. *Let the notation be as in Theorem* 3.1. *Assume that P given by* (3.1) *satisfies conditions* i), ii) *and let* $\Omega'_0 \subset \subset \Omega' \subset \subset \Omega$. *Then for every* $s \in [0, T_{\Omega'}]$ *there exists a continuous linear map* $L_{\Omega'}$ *from* $C([0, T_{\Omega'}]; G^{(\sigma)}(\Omega) \times (G_0^{(\sigma)}(\Omega'))^m)$ *to* $C([0, T_{\Omega'}]; G_0^{(\sigma)}(\Omega'))$ *and from* $C([0, T]; G^{(\sigma)\prime}(\Omega)) \times (G^{(\sigma)\prime}(\Omega'))^m$ *to* $C([0, T_{\Omega'}]; G^{(\sigma)\prime}(\Omega'))$ *such that*:

i) $$L_{\Omega'}(f, g_0,\dots, g_{m-1})(t, x) = i\chi(x)(f(t, x) - \sum_{k=0}^{m-1} (R_{k,\Omega'}(t, s) g_k)(x)) + h(t, x),$$

where $\chi \in G_0^{(\sigma)}(\Omega')$, $\chi \equiv 1$ *on* Ω'_0, $h \in C([0, T_{\Omega'}]; G_0^{(\sigma)}(\Omega'))$;

ii) $$u(t, x) = \int_s^t (W_{m-1}(t, s') L_{\Omega'}(f, g_0,\dots, g_{m-1})(s', \cdot))(x) ds' + \sum_{k=0}^{m-1} (W_k(t, s) g_k)(x)$$

is a solution in $C^m([0, T_{\Omega'}]; G^{(\sigma)}(\Omega'))$ *(Resp. in* $C^m([0, T_{\Omega'}]; G_0^{(\sigma)\prime}(\Omega'))$ *of the Cauchy* *problem*:

$$(3.21)\qquad \begin{cases} P(t, x, D_x, D_t)u = f(t, x) & (t, x) \in [0, T_{\Omega'}] \times \Omega'_0 \\ D_t^j u(s, x) = g_j(x) & j = 0, \ldots, m-1, \ x \in \Omega'_0. \end{cases}$$

Moreover from Lemma 2.11, Corollary 2.13 and Remark 2.16 it follows easily, by using property (3.20) of the operators W_k,

Theorem 3.3. *Let* P *in* (3.1) *satisfy conditions* i), ii) *and* iii) $\lambda - \lambda^{(1)} \in C^{m-1}([0, T]; C^\infty(\Omega \times R^n)) \cap \mathscr{B}([0, T]; \tilde{S}_n^{q,\sigma,\sigma}(\Omega))$, $q \in [0, 1[$, *where*

$$\lambda^{(1)}(t, x, \xi) = \overline{\lim_{\rho \to +\infty}} \rho^{-1} \lambda(t, x, \rho\xi).$$

Assume that in Theorem 3.2 $f \in C([0, T]; G^{(\sigma)}(\Omega))$ *and* $g_j \in G^{(\sigma)\prime}(\Omega)$, $j = 0, \ldots, m-1$. *Take* $\Omega' \subset \subset \Omega$, Ω' *open and convex, and* $W_k(t, s)$, $k = 0, \ldots, m-1$, *such that* $\operatorname{supp} g_j \subset \Omega'$, $j = 0, \ldots, m-1$ *and* $\operatorname{supp} \sum_{k=0}^{m-1} W_k(t, s) g_k \subset \Omega'$. *Then there exist* $T' \in]0, T[$ *and* $D > 0$ *such that for the function* u *given by* ii) *of Theorem* 3.2 *and for every* (t, s)$\in [0, T']^2$

$$\bigcup_{j=1}^{m-1} WF_{(\sigma)}(D_t^j u(t, \cdot)) \subset \{(x^{(1)}(t, s; y, \eta), \xi^{(1)}(t, s; y, \eta)) \in \Omega' \times R_D^n;$$

$$(y, \eta) \in \bigcup_{j=1}^{m-1} WF_{(\sigma)} g_j\},$$

where $(x^{(1)}(t, s; y, \eta), \xi^{(1)}(t, s; y, \eta))$ *is the solution of* (3.2) *when* λ *is replaced by* $\lambda^{(1)}$.

In order to prove an uniqueness theorem for the solution of the Cauchy problem (3.21) and a representation formula for any $u \in C^m([0, T]; G^{(\sigma)\prime}(\Omega))$, we consider the transpose of the operator (3.1)

(3.1′)

$${}^tP = {}^ta_m + (-D_t - {}^t\lambda)({}^ta_{m-1} + (-D_t - {}^t\lambda)(\cdots {}^ta_2 + (-D_t - {}^t\lambda)({}^ta_1 - D_t - {}^t\lambda),$$

where ${}^t\lambda$, ${}^ta_j$, $j = 1, \ldots, m$, denote the transpose of the operators λ, a_j in (3.1) respectively. By a result in [7] for every $\Omega' \subset \subset \Omega$ there exist pseudo-differential operators λ', a'_j, in Ω', with the same properties as λ and a_j respectively, such that up to σ-regularizing operators with kernel in $C([0, T]; G^{(\sigma)}(\Omega' \times \Omega'))$, ${}^t\lambda = -\lambda'$, ${}^ta_j = a'_j$.

Let $(x'(t, s; y, \eta), \xi'(t, s; y, \eta))$, $\phi'(t, s; x, \eta)$, $(t, s) \in [0, T_{\Omega'}]^2$, be the solutions of (3.2) and (3.5) when λ is replaced by λ'. We prove

Lemma 3.4. *Let* λ', a'_j, $j = 1, \ldots, m$, *satisfy conditions* i), ii). *Then for every* $\Omega' \subset \subset \Omega$ *there exists* $T_{\Omega'} \in]0, T]$ *and for every* $(t, s) \in [0, T_{\Omega'}]^2$ *a* $m \times m$ *matrix of Fourier integral operators of infinite order on* Ω', $E'^{j,(k)}(t, s)$, $j, k = 0, \ldots, m-1$

defined as in (3.7) *where* ϕ *is replaced by* ϕ', *such that*

i) *their amplitudes* $e'^{j,(k)}(t, s; x, y, \eta)$ *defined as in* (2.30) *are in* $C([0, T_{\Omega'}]^2; C^\infty(\Omega' \times \Omega' \times R^n)) \cap \mathscr{B}([0, T_{\Omega'}]^2; a^{\infty,\sigma,1}(\Omega' \times \Omega'))$ *together with their first derivative with respect to* t;

ii) $E'^{j,(k)}(s, s) = -i\delta_j^{(k)}$;

iii) *up to* σ*-regularizing operators with kernel in* $C([0, T_{\Omega'}]^2; G^{(\sigma)}(\Omega' \times \Omega'))$

$$a'_j E'^{0,(k)} + (-D_t + \lambda')E'^{j-1,(k)} = \begin{cases} E'^{j,(k)} & \textit{for } j = 1,\dots,m-1 \\ 0 & \textit{for } j = m; \end{cases}$$

iv) *for the operators* $E'^{j,(k)}$ *the property given by* (3.8′) *holds.*

Proof. Let Ω' and $\hat{\Omega}'$ as at the beginnig of this section and set

$$e'^{j,(k)}(t, s; x, \eta) \sim \sum_{h \geq 0} e_h'^{j,(k)}(t, s; x, \eta), \qquad j, k = 0,\dots,m-1,$$

$$\hat{e}_h'^{j,(k)}(t, s; y, \eta) = e_h'^{j,(k)}(t, s; x'(t, s; y, \eta), \eta), \qquad h \geq 0, \tag{3.22}$$

$$\hat{f}_h'^{j,(k)}(t, s; y, \eta) = \exp\Big(-i \int_s^t \hat{q}'(\tau, s; y, \eta)d\tau\Big)\hat{e}_h'^{j,(k)}(t, s; y, \eta), \tag{3.23}$$

$$\hat{q}'(t, s; y, \eta) = q'(t, s; x(t, s; y, \eta), \eta), \ (t, s; y, \eta) \in [0, T_{\Omega'}]^2 \times \hat{\Omega}' \times R^n,$$

where q' is given by (3.10) with λ', ϕ' in place of λ, ϕ respectively. In view of Remark 2.14, application of Corollary 2.15 and Remark 2.16 shows that by Corollary 2.8 iii) is satisfied if for $k = 0,\dots,m-1$

$$\begin{cases} \hat{a}'_j \hat{f}_0'^{0,(k)} - D_t \hat{f}_0'^{j-1,(k)} = \begin{cases} \hat{f}_0'^{j,(k)} & \text{for } j = 1,\dots,m-1 \\ 0 & \text{for } j = m \end{cases} \\ \hat{f}_0'^{j-1,(k)}(s, s) = -i\delta_{j-1}^{(k)}, \end{cases} \tag{3.24$_0$}$$

$$\begin{cases} \hat{a}'_j \hat{f}_h'^{0,(k)} - D_t \hat{f}_h'^{j-1,(k)} = \displaystyle\sum_{r=0}^{h-1}\Big[\sum_{|\beta| \leq h-r} \hat{a}'_{\beta,h-r,j}\hat{f}_r'^{0,(k)} + \sum_{|\beta| \leq h-r+1} \hat{p}'_{\beta,h-r} \\ \qquad \times D_y^\beta \hat{f}_r'^{j-1,(k)}\Big] + \begin{cases} \hat{f}_h'^{j,(k)} & \text{for } j = 1,\dots,m-1 \\ 0 & \text{for } j = m \end{cases} \\ \hat{f}_h'^{j-1,(k)}(s, s) = 0, \qquad h \geq 1, \end{cases} \tag{3.24$_h$}$$

where $\hat{a}'_j(t, s; y, \eta) = a'_j(t, x(t, s; y, \eta), \eta)$, $j = 1,\dots,m$ and

$$\hat{a}'_{\beta,h-r,j} \in C([0, T_{\Omega'}]^2; C^\infty(\hat{\Omega}' \times R^n)), \ \hat{p}'_{\beta,h-r} \in C^{m-1}([0, T_{\Omega'}]^2; C^\infty(\hat{\Omega}' \times R^n))$$

depend on a'_j and λ' and for every $\gamma, \delta \in Z_+^n$, $(t, s; y, \eta) \in [0, T_{\Omega'}]^2 \times \hat{\Omega}' \times R^n$ satisfy the estimates

$$|D_y^\gamma D_\eta^\delta \hat{a}'_{\beta,h-r,j}(t, s; y, \eta)| \leq C_c C_n^{h-r} A_c^{2(h-r)-|\beta|+|\gamma|+|\delta|}$$
$$\times (h - r - |\beta| + |\gamma| + |\delta|)!(1 + |\eta|)^{pj-h-r-|\delta|},$$

$$|D_y^\gamma D_\eta^\delta \hat{p}'_{\beta,h-r}(t, s; y, \eta)| \leq C_c C_n^{h-r+1} A_c^{2(h-r+1)-|\beta|+|\gamma|+|\delta|}$$
$$\times (h - r + 1 - |\beta| + |\gamma| + |\delta|)!(1 + |\eta|)^{-h+r-|\delta|},$$

where C_c, A_c, are positive constants that depend on the coefficients a_j', λ' and C_n is a positive constant that depends only on n.

(3.24_h), $h \geq 0$ imply that for $j = 1,\dots,m$

$$(3.25_0) \qquad \hat{f}_0'^{j-1,(k)}(t, s) + \int_s^t \sum_{\ell=0}^{m-j} (-i)^{\ell+1} \frac{(t-\tau)^\ell}{\ell!} (\hat{a}'_{\ell+j} \hat{f}_0'^{0,(k)})(\tau, s) d\tau$$
$$= -iH(k-j+1)(-i)^{k-j+1}(t-s)^{k-j+1}/(k-j+1)!,$$

where $H(z) = 0$ for $z < 0$ and $H(z) = 1$ for $z \geq 0$ and

$$\hat{f}_h'^{j-1,(k)}(t, s) + \int_s^t \sum_{\ell=0}^{m-j} (-i)^{\ell+1} \frac{(t-\tau)^\ell}{\ell!} (\hat{a}'_{\ell+j} \hat{f}_h'^{0,(k)})(\tau, s) d\tau$$
$$= -i \sum_{r=0}^{h-1} \Bigg[\sum_{|\beta| \leq h-r} \int_s^t \sum_{\ell=0}^{m-j} (-i)^\ell \frac{(t-\tau)^\ell}{\ell!} (\hat{a}'_{\beta,h-r,\ell+j} D_y^\beta \hat{f}_r'^{0,(k)})(\tau, s) d\tau$$
$$+ \sum_{|\beta| \leq h-r+1} \int \sum_{\ell=0}^{m-1} (-i)^\ell \frac{(t-\tau)^\ell}{\ell!} (\hat{p}'_{\beta,h-r} D_y^\beta \hat{f}_r'^{\ell+j-1,(k)})(\tau, s) d\tau \Bigg], \quad h \geq 1.$$

Hence by induction on h it can be proved that for $h \geq 1$, $j = 1,\dots,m$

$$(3.25_h) \qquad \hat{f}_h'^{j-1,(k)}(t, s; y, \eta) + \int_s^t \sum_{\ell=0}^{m-j} (-i)^{\ell+1} \frac{(t-\tau)^\ell}{\ell!} (\hat{a}'_{\ell+j} \hat{f}_h'^{0,(k)})(\tau, s) d\tau$$
$$= \sum_{r=0}^{h-1} \sum_{i=0}^{h-r} \sum_{|\beta| \leq h-r+i} \int_s^t q_{\beta,h,r,i}^{j-1}(t, \tau, s; y, \eta) D_y^\beta \hat{f}_r'^{0,(k)}(\tau, s; y, \eta) d\tau$$
$$+ H(k-j+1) g_h^{j-1,(k)}(t, s; y, \eta),$$

where $q_{\beta,h,r,i}^{j-1} \in C([0, T_{\Omega}']^3; C^\infty(\hat{\Omega}' \times R^n))$, $g_h^{j-1,(k)} \in C^m([0, T_{\Omega}']^2; C^\infty(\hat{\Omega}' \times R^n))$ are such that there exist positive constnats C, A such that for every γ, $\delta \in Z_+^n$

$$(3.26) \qquad |D_y^\gamma D_\eta^\delta q_{\beta,h,r,i}^{j-1}(t, \tau, s; y, \eta)| \leq$$
$$\leq C^{h-r+1} A^{2(h-r+i)-|\beta|+|\gamma|+|\delta|} (1 + |\eta|)^{pj-h+r-|\delta|}$$
$$\times \sum_{\ell=0}^{m-j} 2^\ell \frac{(1+|\eta|)^p |t-\tau|)^\ell}{(\ell+1)!} |t-\tau|^i (h-r+i-|\beta|+|\gamma|+|\delta|)!,$$

$$(3.27) \qquad |D_y^\gamma D_\eta^\delta g_h^{j-1,(k)}(t,s; y, \eta)| \leq$$
$$\leq 2^{k-j+1} \sum_{i=1}^{h} C^{h+i} A^{2(h+i)+|\gamma|+|\delta|} (1 + |\eta|)^{-h-|\delta|} (h+i+|\gamma|+|\delta|)!$$
$$\times |t-s|^{k-j+i+1}/(k-j+i+1)!.$$

For $j = 1$, (3.25_h) $h \geq 0$ yield a system of Volterra type integral equations which has the first equation (i.e. (3.25_0)) and all the left hand sides as the system (3.4_h) in [4]. Note also that for $j = 1$, $h \geq 1$, $h \geq 1$, $\sigma \in]1, 1/p[$, $p' \in [p, 1/\sigma[$ and $|\eta| > Bh^\sigma$, $B^p > CA^2$, the estimates (3.26) and (3.27) can be written

(3.26′) $$|D_y^\gamma D_\eta^\delta q^0_{\beta,h,r,i}(t,\tau,s;y,\eta)| \le C^{h-r+1}A^{2(h-r)-|\beta|+|\gamma|+|\delta|}(1+|\eta|)^{p-h+r-|\delta|} \times \sum_{\ell=0}^{m-1} 2^\ell \frac{((1+|\eta|)^{p'}|t-\tau|)^{\ell+i}}{(\ell+i)!} h^{-\sigma p' i}(h-r+i-|\beta|+|\gamma|+|\delta|)!,$$

(3.27′) $$|D_y^\gamma D_\eta^\delta g_h^{0,(k)}(t,s;y,\eta)| \le 2^k(2C)^h A^{2h+|\gamma|+|\delta|}(1+|\eta|)^{-h-|\delta|}\exp((1+|\eta|)^{p'}|t-s|) \times (2h+|\gamma|+|\delta|)!h!^{-\sigma p'}.$$

On the other hand as in [4] the following estimates hold for the resolvent kernel $K(t,\tau,s;y,\eta)$ of each of the equations (3.25_h), $h\ge 0$, $j=1$ and for $\hat{f}_0^{\prime 0,(k)}$

$$|D_y^\gamma D_\eta^\delta K(t,\tau,s;y,\eta)| \le C_c(3A_c/2)^{|\gamma|+|\delta|}(|\gamma|+|\delta|)!(1+|\eta|)^{p-|\delta|}\exp(c_a'(1+|\eta|)^p|t-s|),$$

$$|D_y^\gamma D_\eta^\delta \hat{f}_0^{\prime 0,(k)}(t,s;y,\eta)| \le C_c(2A_c)^{|\gamma|+|\delta|}(|\gamma|+|\delta|)!(1+|\eta|)^{-\delta}\exp(c'(1+|\eta|)^p|t-s|)|t-s|^k/k!,$$

$k=0,\dots,m-1$, $\gamma,\delta\in Z^n_+$, C_c, A_c positive constants that depend on a_j'.

Thus, using (3.26′), (3.27′) it can be proved by induction on h that there exist positive constants B, C', c', A' such that for every $h\ge 0$, $|\eta|>Bh^\sigma$, γ, $\delta\in Z^n_+$, $(t,s,y)\in[0,T_{\Omega'}]^2\times\hat{\Omega}'$

$$|D_y^\gamma D_\eta^\delta \hat{f}_h^{\prime 0,(k)}(t,s;y,\eta)| \le C'^{h+1}A'^{2h+|\gamma|+|\delta|}(1+|\eta|)^{-h-|\delta|}\exp(c'(1+|\eta|)^{p'}|t-s|) \times (2h+|\gamma|+|\delta|)!h!^{-\sigma p'}\sum_{\ell=1}^{(m+2)h} 2^\ell \frac{(1+|\eta|)^{p'\ell}|t-s|^{k+\ell}}{(k+\ell)!}.$$

This proves, as in the case of Theorem 3.1, that $\sum_{h\ge 0}\hat{f}_h^{\prime 0,(k)}\in\mathscr{B}([0,T_{\Omega'}]^2;FS_n^{\infty,\sigma,1}(\hat{\Omega}'))$ $k=0,\dots,m-1$. It follows by (3.25_h) that is also the case for $\sum_{h\ge 0}\hat{f}_h^{\prime j-1,(k)}$, $j=2,\dots,m$, and by (3.24_h) for $\sum_{h\ge 0} D_t\hat{f}^{\prime j-1,(k)}$. Thus from (3.23), (3.22) and in view of Proposition 1.1.11 and Theorem 1.2.21, we have that $\sum_{h\ge 0} e_h^{j,(k)}$, $\sum_{h\ge 0} D_t e_h^{j,(k)}$, $j,k=0,\dots,m-1$ are in $\mathscr{B}([0,T_{\Omega'}];FS_n^{\infty,\sigma,1}(\Omega'))$. This proves the lemma in view of Corollary 1.1.19.

Let now $\Omega'\subset\subset\Omega$ and let $E'^{j,(k)}(t,s)$, $k=0,\dots,m-1$, be the operators in Lemma 3.4 for Ω' replaced by U. Set

$$v(t,s)=\int_s^t\sum_{k=0}^{m-1}(E'^{0,(k)}(t,s')\varphi_k(s',\cdot))(x)ds',\quad (t,s)\in[0,T_U]\times U,$$

$\varphi_k\in C([0,T];G_0^{(\sigma)}(\Omega'))$. By property vi) in Lemma 3.4 the operators $E'^{j-1,(k)}(t,s)$ can be determined so as for every $\varphi\in C([0,T];G_0^{(\sigma)}(\Omega'))$ and $(t,s)\in[0,T_U]^2$

$$\text{supp}\, E'^{j-1,(k)}(t, s)\varphi(s, \cdot) \subset U \subset\subset \Omega, \qquad k = 0,\dots, m-1. \tag{3.28}$$

Let

$$\begin{aligned} R_j^{(k)} &= {}^t a_j E'^{0,(k)} + (-D_t - {}^t\lambda)E'^{j-1,(k)} \\ &= \begin{cases} E'^{j,(k)} & j = 1,\dots,m-1 \\ 0 & j = m \end{cases}. \end{aligned} \tag{3.29}$$

By solving the system of Volterra type integral equations

$$\varphi_j(t, x) + \int_s^t ds' \int_\Omega \sum_{k=0}^{m-1} \chi(x) R_{j+1}^{(k)}(t, s'; x, y)\varphi_k(s', y)dy = 0,$$

$$j = 0,\dots, m-2,$$

$$\varphi_{m-1}(t, x) + \int_s^t ds' \int_\Omega \sum_{k=0}^{m-1} \chi(x) R_m^{(k)}(t, s'; x, y)\varphi_k(s', y)dy$$

$$= \chi(x)f(t, x),\ (t, x) \in [0, T_U] \times \Omega',$$

where $\chi \in G_0^{(\sigma)}(\Omega')$, $\chi = 1$ on $\Omega'_0 \subset\subset \Omega'$ and $R_j^{(k)} \in C([0, T_U]^2; G^{(\sigma)}(U \times U))$ are the kernels of the σ-regularizing operators defined in (3.29), we prove the following

Theorem 3.5. *Let P satisfy conditions* i), ii) *and let ${}^tP$ be defined by* (3.1′). *Let $\Omega'_0 \subset\subset \Omega' \subset\subset U \subset\subset \Omega$ and let $E'^{0,(k)}(t, s)$ be the operators determined in Lemma* 3.4 *that satisfy* (3.28) *when Ω' is replaced by U. Then for every $s_0 \in [0, T_U]$ there exists a continuous linear map $L'_{\Omega'}$ from $C([0, T_U]; G^{(\sigma)}(\Omega))$ to $C([0, T_U]; G_0^{(\sigma)}(\Omega'; C^m))$ and from $C([0, T_U]; G_0^{(\sigma)\prime}(\Omega))$ to $C([0, T_U]; G^{(\sigma)\prime}(\Omega'; C^m))$ such that*

i) $(L'_{\Omega'}f)_k(t, x) = \delta_{k,m-1}\chi(x)f(t, x) + g_k(t, x),\ k = 0,\dots, m-1$
where $\chi \in G_0^{(\sigma)}(\Omega')$, $\chi = 1$ an Ω'_0 and $g_k \in C([0, T_U]; G_0^{(\sigma)}(\Omega'))$;

ii) $v(t, x) = \int_{s_0}^t \sum_{k=0}^{m-1} (E'^{0,(k)}(t, s')(L'_{\Omega'}f)_k(s', \cdot))(x)ds'$
is a solution in $C^m([0, T_U]; G_0^{(\sigma)}(U))$ [resp. in $C^m([0, T_U]; G^{(\sigma)\prime}(U))$] of the initial value problem

$$\begin{cases} {}^tP(t, x, D_t, D_x)v = f(t, x),\ (t, x) \in [0, T_{\Omega'}] \times \Omega'_0, \\ v(s_0, x) = 0, \\ {}^ta_jv + (-D_t - {}^t\lambda)(\cdots(-D_t - {}^t\lambda)({}^ta_1 - D_t - {}^t\lambda)\cdots)v|_{t=s_0} = 0 \\ j = 1,\dots, m-1. \end{cases} \tag{3.30}$$

We use this result for proving the following uniqueness theorem.

Theorem 3.6. *Let the operator P given by* (3.1) *satisfy conditions* i), ii) *and* iv) *for every $t \in [0, T]$, $D_t^j\lambda(t, x, D_x)$, $j = 0,\dots, m$, are properly supported operators on $G_0^{(\sigma)}(\Omega)$.*

Assume that $\Omega'_0 \subset\subset U \subset\subset \Omega$ and $u \in C^m([0, T]; G^{(\sigma)\prime}(\Omega'_0))$. Then there exists $T' \in [0, T[$ such that if for a fixed $s \in [0, T']$, $D_t^ju(s, x) = 0, j = 0,\dots, m-1$, on

Ω_0' and $Pu = 0$ in $[0, T'] \times U$, then $u = 0$ on $[0, T'] \times \Omega$.

Proof. By Theorem 3.5 there exists $T' \in [0, T[$ such that for every $f \in C([0, T]; G_0^{(\sigma)}(\Omega_0'))$ and every $s_0 \in [0, T']$ there exists a solution $v \in C^m([0, T']; G_0^{(\sigma)}(U))$ of the problem (3.30) on $[0, T'] \times \Omega_0'$. Then the theorem is proved since

$$\int_s^{s_0} \langle u(t, \cdot), f(t, \cdot) \rangle \, dt = \int_s^{s_0} \langle u(t, \cdot), {}^tPv(t, \cdot) \rangle \, dt$$

$$= \int_s^{s_0} \langle Pu(t, \cdot), v(t, \cdot) \rangle \, dt = 0$$

for every $f \in C([0, T']; G_0^{(\sigma)}(\Omega'))$ and every $s_0 \in [0, T']$.

By means of the operators $E'^{0,(k)}(t, s)$, $k = 0, \ldots, m-1$, we can also obtain a representation formula for any $u \in C^m([0, T]; G^{(\sigma)\prime}(\Omega))$ that we shall use for improving the result of Theorem 3.3. In fact we can easily prove

Lemma 3.7. *Let the conditions of Theorem 3.6 on the operator P given by (3.1) be satisfied and let $\Omega' \subset\subset \Omega$. Then there exist $T' \in [0, T[$ such that for every $u \in C^m([0, T']; G^{(\sigma)\prime}(\Omega'))$ satisfying $D_t^j u(s, x) = 0$, $j = 0, \ldots, m-1$, for $x \in \Omega$ and a fixed $s \in [0, T']$ the following identity holds on $\tilde{\Omega}'$*

$$((D_t - \lambda)^{m-k-1} u)(t, x) = -\int_s^t {}^t\tilde{E}'^{0,(k)}(s', t)(Pu)(s', \cdot) \, ds'$$

$$+ \sum_{j=1}^{m} \int_s^t \tilde{R}_j^{(k)}(s', t)(D_{s'} - \lambda)^{m-j-1} u(s', \cdot) \, ds',$$

$k = 0, \ldots, m-1$, where $\tilde{E}'^{0,(k)}$ are defined as in Lemma 3.4 when Ω' is replaced by $\tilde{\Omega}'$, $\tilde{R}_j^{(k)}$ are the σ-regularizing operators on $\tilde{\Omega}'$ defined as in (3.29) and $\tilde{\Omega}' \supset\supset \operatorname{supp}(D_t - \lambda)^j u \cup \Omega'$, $j = 0, \ldots, m-1$.

With the aid of this Lemma we finally prove.

Theorem 3.8. *Let the operator P given by (3.1) satisfy conditions* i), ii). *Furthermore assume that* iii) *of Theorem 3.3 and* iv) *of Theorem 3.6 holds.*

Let $u \in C^m([0, T]; G^{(\sigma)\prime}(\Omega'))$ and $Pu \in C([0, T]; G^{(\sigma)}(\Omega))$. Then there exists $T'' \in]0, T]$ and $D > 0$ such that for every $(s, t) \in [0, T'']$

$$(3.31) \qquad \bigcup_{j=0}^{m-1} WF_{(\sigma)}(D_t^j u(t, \cdot)) = \{(x^{(1)}(t, s; y, \eta), \xi^{(1)}(t, s; y, \eta)) \in \Omega_0' \times R_D^n;$$

$$(y, \eta) \in \bigcup_{j=0}^{m-1} WF_{(\sigma)}(D_t^j u(s, \cdot))\},$$

where $(x^{(1)}(t, s; y, \eta), \xi^{(1)}(t, s; y, \eta))$ is the solution of the system (3.2) when λ is replaced by $\lambda^{(1)}$ and $\Omega_0' = \bigcup_{t \in [0, T]} \operatorname{supp} u(t, \cdot) \subset\subset \Omega'$.

Proof. Let $W_k(t, s)$, $k = 0, \dots, m-1$ be the operators determined in Corollary 3.1′ satisfying (3.8′). Then

$$v(t, x) = \sum_{k=0}^{m-1} (W_k(t, s) D_t^k u(s, \cdot))(x) \in C^m([0, T_{\Omega'}]; G^{(\sigma)'}(\Omega')),$$

$D_t^j v(s, x) = D_t^j u(s, x)$, $x \in \Omega'$, $s \in [0, T_{\Omega'}]$ and $Pv \in C([0, T_{\Omega'}]; G^{(\sigma)}(\Omega'))$.

Moreover as in Theorem 3.3

$$\text{(3.32)} \qquad \bigcup_{j=0}^{m-1} WF_{(\sigma)}(D_t^j v(t, \cdot)) \subset \{(x^{(1)}(t, s; y, \eta), \xi^{(1)}(t, s; y, \eta)) \in \Omega' \times R_D^n;$$
$$(y, \eta) \in \bigcup_{j=0}^{m-1} WF_{(\sigma)}(D_t^j u(s, \cdot))\}, \ t \in [0, T_{\Omega'}].$$

Lemma 3.7 can now be applied to $u - v$, for $(s, t) \in [0, T'']$, $T'' = \inf(T_{\Omega'}, T')$. If $\tilde{E}'^{0,(k)}$ are defined so that $E'^{0,(k)}(s', t)\varphi \in C([0, T'']^2; G_0^{(\sigma)}(\Omega'))$, we see that ${}^t E'^{0,(k)}(s', t) P(u - v)(s'', \cdot) \in C([0, T'']^2; G^{(\sigma)}(\Omega_0'))$. Hence by Lemma 3.7

$$(D_t - \lambda)^j (u - v) \in C^{m-j}([0, T'']; G^{(\sigma)}(\Omega_0')), \ j = 0, \dots, m-1,$$

and by induction on j: $D_t^j u - D_t^j v \in C^{m-j}([0, T'']; G^{(\sigma)}(\Omega_0'))$, $j = 0, \dots, m-1$. Thus

$$\bigcup_{j=0}^{m-1} WF_{(\sigma)}(D_t^j u(t, \cdot)) = \bigcup_{j=0}^{m-1} WF_{(\sigma)}(D_t^j v(t, \cdot)) \cap (\Omega_0' \times R^n), \ t \in [0, T'']$$

which implies by (3.32) that in (3.31) the left hand side is included in the right hand side. Since s and t can be interchanged, (3.31) is proved.

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