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Simple transcendental extensions of valued fields III: The uniqueness property

By

Michel MATIGNON and Jack OHM

Let (K_0, v_0) be a valued field and x be an indeterminate over K_0. For any t in $K_0(x)\setminus K_0$, one can define an extension $v_0{}^t$ of v_0 to a valuation of $K_0(t)$ by:

$$\text{for all } a_0, \cdots, a_n \text{ in } K_0,\ v_0{}^t(a_0+a_1t+\cdots+a_nt^n)=\inf\{v_0(a_i)\,|\,i=0, \cdots, n\}.$$

We are concerned here with the

Uniqueness problem: Given a valuation v of $K_0(x)$ which extends $v_0{}^t$, does there exist a t' in $K_0(x)\setminus K_0$ such that v extends $v_0{}^{t'}$ uniquely?

We proved in [9] that the answer is "yes" if $\mathrm{rk}\, v_0$ is 1. We shall show here that the answer is also "yes" if v_0 is henselian, and that the answer is "no" in general.

The henselian result, which is proved in section 3, follows from the theorem that, for v_0 henselian, $v_0{}^t$ extends uniquely to a valuation of $K_0(x)$ whenever t has the form $t=f(x)^m/b$, where $f(x)$ is irreducible in $K_0[x]$, b is in K_0, and m is $\geqq 1$.

The negative result is proved in section 2. It follows from the observation that an affirmative answer to the uniqueness problem is equivalent to a fundamental equality, $E=IRD^h$, relating some numerical invariants of the extension v/v_0. By studying these invariants punctually at t, we show that if there exists a t such that $v/v_0{}^t$ is unique, then $v/v_0{}^{t'}$ is also unique for every t' of minimal deg such that v extends $v_0{}^{t'}$.

The D^h that appears in the above equality is called the henselian defect of v/v_0. The notion of defect, which is central to the ideas of section 2, is introduced in section 1.

We have come to this work from two directions. One is that of the equality $E=IRD^h$, which had been conjectured to hold under certain hypotheses in [12]; the present paper puts [12] in its proper setting and completes the proof of its conjectures. The other direction is that of the uniqueness problem for function fields: an affirmative answer to this problem is given in [7] for v_0 rk 1 complete and is a key step in the proof of the genus reduction inequality of that paper. See also the introduction to [7] for some historical remarks on this problem.

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0. Notation and terminology

Throughout the paper $(K_0, v_0) \subset (K_0(x), v)$ will denote a simple transcendental (abbreviated tr.) extension of valued fields, i.e. x is tr. over K_0, v is a valuation of $K_0(x)$, and $v_0 = v | K_0$. Sometimes it will be convenient to abbreviate this notation to $(K_0(x)/K_0, v)$ or merely $K_0(x)/K_0$ or v/v_0. The corresponding valuation rings, value groups, and residue fields will be denoted $V_0 \subset V$, $G_0 \subset G$, and $k_0 \subset k$ respectively).

Moreover, k_0' will denote the algebraic closure of k_0 in k.

We use $(\ )^*$ for image under the residue map of a given valuation; thus, if s is in $K_0(x)$ and $v(s) \geqq 0$, then s^* is the image of s under the v-residue map $V \to V/m_v = k$. We shall also refer to s^* as the v-residue of s.

Let t be in $K_0(x) \setminus K_0$.

We call t residually tr. for v/v_0 if t^* is tr. over k_0.

The extension $(K_0, v_0) \subset (K_0(x), v)$ is called residually tr. if there exists a residually tr. t in $K_0(x)$.

$$\deg t = [K_0(x) : K_0(t)], \qquad \text{or equivalently,}$$

$$= \max\{\deg f(x), \deg g(x)\},$$

when t is, written in the form $t = f(x)/g(x)$ with $f(x)$, $g(x)$ relatively prime elements of $K_0[x]$; cf. [18, p. 197].

v_0^t (the inf extension of v_0 w.r.t. t)=the valuation of $K_0(t)$ determined by: for all $a_0, \cdots, a_n$ in K_0, $v_0^t(a_0 + a_1 t + \cdots + a_n t^n) = \inf\{v_0(a_i) | i = 0, \cdots, n\}$. The residue field of v_0^t is $k_0(t^*)$, with t^* tr. over k_0; and the value group is G_0. Moreover, $v | K_0(t) = v_0^t$ iff t is residually tr. for v/v_0. (Cf. [2 pp. 160-161])

K_0^{alg} will be a fixed algebraic closure of K_0. If w_0 is any extension of v_0 to K_0^{alg}, then the residue field of w_0 is the algebraic closure k_0^{alg} of k_0 and the value group is the divisible hull G_0^{div} of G_0.

$(K_0, v_0)^\wedge$ denotes completion and $(K_0, v_0)^h$ henselization.

We use $\subset$ for inclusion and $<$ for proper inclusion.

1. Defect

Fix throughout section 1 a finite algebraic extension of valued fields $(L_0, w_0) \subset (L, w)$, with value groups $H_0 \subset H$ and residue fields $l_0 \subset l$.

We use L^h to denote henselization and $L^\wedge$ to denote completion. Recall that both L^h and $L^\wedge$ have the same residue field and value group as L and that any finite generating set for L/L_0 is also a generating set for L^h/L_0^h and for $L^\wedge/L_0^\wedge$; see [1, pp. 175-179] or [3, p. 131] for henselizations and [17, pp. 45-47, §§4 and 5] or [2, p. 121] for completions.

A set of extensions $w_1 = w, w_2, \cdots, w_n$ of w_0 to L is called a complete set of extensions of w_0 to L if every extension of w_0 to L is equivalent to one of these and no two of these are equivalent. Similarly, a set of extensions $w_{i_1} = w, w_{i_2}, \cdots, w_{i_m}$ of w_0

to L is called a complete set of independent extensions of w_0 to L if every extension of w_0 to L is dependent on one of these and no two of these are dependent. (Recall that two extensions are dependent if either their valuation rings have a common valuation overring $<L$ or both are trivial.)

1.1. The fundamental equalities.

$$\text{i)}\quad [L: L_0]=\textstyle\sum_{i=1}^{n}[(L, w_i)^h : (L_0, w_0)^h]$$

(Apply [3, p. 125, (17.3)] and the fact that, in the terminology of that reference, L_0^h/L_0 is a separable, "allowable" extension.)

$$\text{ii)}\quad [L: L_0]=\textstyle\sum_{j=1}^{m}[(L, w_{i_j})^\wedge : (L_0, w_0)^\wedge]Q^\wedge(w_{i_j}/w_0)$$

(cf. [17, p. 49, Lemma 5]), where $Q^\wedge(w_{i_j}/w_0)=[L: L_0]_{ins}/[(L, w_{i_j})^\wedge : (L_0, w_0)^\wedge]_{ins}$ ($[\ ]_{ins}$ denotes deg of inseparability).

The first sum is taken over a complete set of extensions $w_1=w, \cdots, w_n$ of w_0 to L, and the second sum over a complete set of independent extensions $w_{i_1}=w, \cdots, w_{i_m}$ of w_0 to L.

We shall call $[(L, w)^h : (L_0, w_0)^h]$ (resp. $[(L, w)^\wedge : (L_0, w_0)^\wedge]$) the henselian (resp. completion) degree of the extension $(L_0, w_0)\subset(L, w)$, and $Q^\wedge(w/w_0)$ the inseparability quotient. Note that $Q^\wedge(w/w_0)=1$ whenever $L_0^\wedge/L_0$ or L/L_0 is separable, and in general equals p^i for some $i\geqq 0$, where $p=\operatorname{char} L_0$; cf. [19, p. 119, Cor. 2, and p. 114, Lem.]. For a discrete rk 1 example with $[L: L_0]=Q^\wedge(w/w_0)=p>0$, see [20, p. 62] or [2, p. 187, Exercise 3].

1.2. Definition of the defects. Let $e=[H: H_0]$ and $f=[1: 1_0]$. We define three notions of defect for the extension $(L_0, w_0)\subset(L, w)$, the defect, the henselian defect, and the completion defect, resp., as follows:

$$\text{i)}\quad def(w/w_0)=[L: L_0]/ef$$

$$\text{ii)}\quad def^h(w/w_0)=[(L, w)^h : (L_0, w_0)^h]/ef$$

$$\text{iii)}\quad def^\wedge(w/w_0)=[(L, w)^\wedge : (L_0, w_0)^\wedge]/ef$$

We can now restate the fundamental equalities of 1.1 in terms of these defects:

$$\begin{aligned}[L: L_0]&=def(w/w_0)ef\\ &=\textstyle\sum_{i=1}^{n} def^h(w_i/w_0)e_i f_i\\ &=\textstyle\sum_{j=1}^{m} def^\wedge(w_{i_j}/w_0)e_{i_j}f_{i_j}Q^\wedge(w_{i_j}/w_0).\end{aligned}$$

Note that each defect is a rational number $\geqq 1$. Much of the usefulness of these notions is due to the following classical

Theorem. *Let* $p=\operatorname{char} l_0$ *if* $\operatorname{char} l_0>0$ *and* $p=1$ *if* $\operatorname{char} l_0=0$.

i) (Ostrowski) *If* $\operatorname{rk} w_0=1$, *then* $def^\wedge(w/w_0)=p^i$ *for some* $i\geqq 0$; *and if* w_0 *is discrete* rk 1, *then* $def^\wedge(w/w_0)=1$ (cf. [15, p. 355] and [2, p. 148, Cor. 2].)

ii) (E. Artin—Ostrowski) *If* rk w_0 *is arbitrary, then* $def^h(w/w_0)=p^i$ *for some* $i\geqq 0$ (cf. [1, p. 180, Prop. 15], or [2, p. 190, Exercise 9]).

It is easy to give examples to show $def^\wedge(w/w_0)$ is not related to the residue characteristic if rk $w_0>1$; in fact, in 2.5 below we give discrete rk 2 examples, in, say, residue characteristic 0 for which $def^\wedge(w/w_0)$ is an arbitrarily prescribed rational number $\geqq 1$. Note that by ii) of the theorem $def^h(w/w_0)=1$ for such examples.

1.3. Comparison of the defects. Since $L_0\subset L_0^h$ and $L_0\subset L_0^\wedge$, both $def^h(w/w_0)$ and $def^\wedge(w/w_0)$ are $\leqq def(w/w_0)$. As for the relationship between def^h and $def^\wedge$, we have the

Proposition. $def^h(w/w_0)\leqq def^\wedge(w/w_0)Q^\wedge(w/w^0)$, *and* $=$*holds if* $(L_0, w_0)^\wedge$ *is henselian* (*in particular*, $=$*holds if* rk $w_0=1$).

Proof. We shall form our henselizations and completions inside the fixed field $(L^{alg})^\wedge$, where L^{alg} is an algebraic closure of L, with w extended arbitrily.

We must show: $[L^h: L_0^h]\leqq[L^\wedge: L_0^\wedge]Q^\wedge(w/w_0)$, and $=$holds if $L_0^\wedge$ is henselian. By 1.1, since w^h is the unique extension of w_0^h to L^h,

$$[L^h: L_0^h]=[L^{h\wedge}: L_0^{h\wedge}]Q^\wedge(w^h/w_0^h). \tag{1.3.1}$$

To prove the inequality, it suffices to show

$$[L^\wedge: L_0^\wedge]Q^\wedge(w/w_0)\geqq[L^{h\wedge}: L_0^{h\wedge}]Q^\wedge(w^h/w_0^h), \quad \text{or}$$

$$[L^\wedge: L_0^\wedge]([L: L_0]_{ins})/[L^\wedge: L_0^\wedge]_{ins}\geqq[L^{h\wedge}: L_0^{h\wedge}]([L^h: L_0^h]_{ins}/[L^{h\wedge}: L_0^{h\wedge}]_{ins}), \quad \text{or}$$

$$[L^\wedge: L_0^\wedge]_{sep}[L: L_0]_{ins}\geqq[L^{h\wedge}: L_0^{h\wedge}]_{sep}[L^h: L_0^h]_{ins}.$$

But L_0^h/L_0 is separable, so $[L: L_0]_{ins}=[L^h: L_0^h]_{ins}$ (cf. [19, p. 119, Cor. 2]). Therefore we must show $[L^\wedge: L_0^\wedge]_{sep}\geqq[L^{h\wedge}: L_0^{h\wedge}]_{sep}$, which follows from $L_0^\wedge\subset L_0^{h\wedge}$ (cf. [19, p. 114, Lem. 1]).

Finally, if $L_0^\wedge$ is henselian, theu $L_0^{h\wedge}=L_0^\wedge$, and (1.3.1) becomes $[L^h: L_0^h]=[L^\wedge: L_0^\wedge]Q^\wedge(w^h/w_0^h)$. But $Q^\wedge(w^h/w_0^h)=Q^\wedge(w/w_0)$ since as noted above, $[L: L_0]_{ins}=[L^h: L_0^h]_{ins}$. Q.E.D.

As we mentioned in 1.2, there exist examples with $def^h=1$ and $def^\wedge$ an arbitrary rational number $\geqq 1$. On the other hand, the example mentioned in 1.1 is a discrete rk 1 example with $def^\wedge=1$ and $def^h=p>0$.

1.4. The uniqueness criteria. We shall say w/w_0 is unique (resp., dependence unique) if all extension of w_0 to L are equivalent to (resp., dependent on) w. By the fundamental equalities of 1.2, the following are equivalent:

i) w/w_0 is unique (resp., dependence unique)

ii) $def(w/w_0)=def^h(w/w_0)$ (resp., $=def^\wedge(w/w_0)Q^\wedge(w/w_0)$)

iii) $[L: L_0]=efdef^h(w/w_0)$ (resp., $=efdef^\wedge(w/w_0)Q^\wedge(w/w_0)$).

Remark. w/w_0 is dependence unique iff there exists a valuation w' of L whose valuation ring contains the valuation ring of w and is such that $w'/(w'|L_0)$ is unique (cf. [20, p. 65]).

It is always possible to embed the value group of an extension w of w_0 to L in a fixed divisible hull H_0^{div} of H_0; this has the effect of replacing the valuation w by an equivalent one. But it is easily seen that if two extensions w_1, w_2 of w_0 to L both have value groups contained in the same H_0^{div} and are equivalent, then they are equal. Thus, by requiring that all extensions under consideration should have values in H_0^{div}, we could replace equivalence by equality.

2. The fundamental invariants

Throughout section 2 $(K_0, v_0)\subset(K_0(x), v)$ will denote a residually *tr.* extension of valued fields with value groups $G_0\subset G$ and residue fields $k_0\subset k$. (Here x is a single indeterminate.) In addition, we let

$$\mathcal{T}=\{t \text{ in } K_0(x) \mid t \text{ is residually } tr.\}, \quad \text{and}$$

$$\min\mathcal{T}=\{t \text{ in } \mathcal{T} \mid \text{ for all } t' \text{ in } \mathcal{T},\ \deg t\leqq\deg t'\}.$$

Note that for all t in $\mathcal{T}$, $v|K_0(t)=v_0^t$, the inf extension of v_0 w.r.t. t (cf. § 0). It follows that index(v/v_0^t)=index$(v|v_0)$:=$[G: G_0]$; we shall denote this index by I. We define an additional set of rational numbers $\geqq 1$ as follows:

2.1. Definition. For t in $\mathcal{T}$,

$$R(t) \text{ (residue deg at } t)=[k: k_0(t^*)],$$

$$\begin{cases} E(t) \text{ (extension deg at } t)=[K_0(x): K_0(t)], \\ E^h(t) \text{ (henselian extension deg at } t)=[K_0(x)^h: K_0(t)^h], \\ E^\wedge(t) \text{ (completion extension deg at } t)=K_0(x)^\wedge: K_0(t)^\wedge], \end{cases}$$

$$\begin{cases} D(t) \text{ (defect at } t)=E(t)/IR(t), \\ D^h(t) \text{ (henselian defect at } t)=E^h(t)/IR(t), \\ D^\wedge(t) \text{ (completion defect at } t)=E^\wedge(t)/IR(t). \end{cases}$$

Moreover, if $A(t)$ denotes any one of the above symbols, we define

$$A=\inf\{A(t) \mid t \text{ in } \mathcal{T}\}\text{; e.g. } D^h=\inf\{D^h(t) \mid t \text{ in } \mathcal{T}\}.$$

In this way we obtain a set of "invariants" I, R, E, E^h, $E^\wedge$, D, D^h, $D^\wedge$ for the extension $(K_0, v_0)\subset(K_0(x), v)$, called respectively the index, residue deg, extension deg, henselian extension deg, etc.

Each of the expressions of 2.1 may be regarded as defining a function from $\mathcal{T}$ to the {rationals $\geqq 1$}.

2.2. Theorem. *$D^h(t)$ is constant on $\mathcal{T}$, i.e. for all t in $\mathcal{T}$, $D^h(t)=D^h$; and the remaining functions of* 2.1 *are constant on* $\min\mathcal{T}$ *and attain their infs there, i.e. for all t in* $\min\mathcal{T}$, $R=R(t)$, $E=E(t)$, $E^h=E^h(t)$, *etc.*

Proof, i) $D^h(t)$ is constant on $\mathcal{T}$.

This was proved for $D\hat{}(t)$, in the rk 1 case, in [9, Thm. 2.5]; and the same proof yields i), modulo the observation that Lemma 2.5.3 of [9] was not really needed there, only the weaker observation that if K_1 is a finite algebraic extension of K_0 and v is extended arbitrarily to $K_1(x)$, then for t, t' in $\mathcal{T}$, the extensions $K_0(t)\subset K_1(t)$ and $K_0(t')\subset K_1(t')$ are isomorphic as valued field extensions. (Actually it is also easy to see that Lemma 2.5.3 of [9] remains valid if $def\hat{}$ is replaced by def^h; cf. [14, §4.1].)

ii) $R(t)$ is constant on $\min\mathcal{T}$ and attains its inf there.

This is a consequence of the Ruled Residue Theorem [13]: For t in $\min\mathcal{T}$, $k=k_0'(t^*)$ and $R(t)=[k_0':k_0]$ (where k_0' is the algebraic closure of k_0 in k).

iii) $E(t)$ is constant on $\min\mathcal{T}$ and attains its inf there (i.e., $E=\deg\min\mathcal{T}$).

This is immediate from the definition of $\min\mathcal{T}$.

iv) $E^h(t)$ is constant on $\min\mathcal{T}$ and attains its inf there.

This is by i), ii), and the equality $E^h(t)=IR(t)D^h$.

v) $E\hat{}(t)$ is constant on $\min\mathcal{T}$ and $=E$ there.

By [9, Theorem 2.1], $E\hat{}(t)=E(t)$ on $\min\mathcal{T}$; and by iii), $E(t)=E$ on $\min\mathcal{T}$.

vi) $D(t)$ and $D\hat{}(t)$ are constant on $\min\mathcal{T}$ and $=E/IR$ there.

This is by ii), iii), and v).

It remains to prove $E\hat{}(t)$, $D(t)$, and $D\hat{}(t)$ attain their infs on $\min\mathcal{T}$, or in view of v) and vi), that $E=E\hat{}$ and $D\hat{}=D=E/IR$. These equalities are a corollary to

2.2.1. Proposition. *For all t in $\mathcal{T}$, $E\hat{}(t)/R(t)\geqq E/R$.*

Proof. Let t be in $\mathcal{T}$ and t_0 be in $\min\mathcal{T}$. By definition of $\min\mathcal{T}$, $E(t_0)=E$; and as noted in ii) above, $R(t_0)=R=[k_0':k_0]$ and $k=k_0'(t_0^*)$, where k_0'=algebraic closure of k_0 in k. Let $m=[k:k_0'(t^*)]$ (=deg of t_0^* over $k_0'(t^*)$), Then

$$R(t)=[k:k_0(t^*)]=[k:k_0'(t^*)][k_0'(t^*):k_0(t^*)]=mR\,.$$

Therefore it suffices to show $E\hat{}(t)\geqq mE$, which follows from the

Claim: The mE elements of

$$\mathcal{S}=\{x^i t_0^j \mid i=0,\cdots,E-1;\ j=0,\cdots,m-1\}$$

are linearly independent over $K_0(t)\hat{}$.

We need the following lemma, whose proof will be given later.

2.2.2. Lemma. *The extension* deg E' *of* $K_0\hat{}(x)/K_0\hat{}$ *=the extension* deg E *of* $K_0(x)/K_0$.

By the lemma, x is of deg E over $K_0\hat{}(t_0)$. Since also t_0 is *tr.* over $K_0\hat{}$ (because t_0 is residually *tr.*), it follows that $\mathcal{S}$ is linearly independent over $K_0\hat{}$. Therefore by [9, Cor. 1.6], we need only check that t satisfies the condition (inf/$K_0\langle\mathcal{S}\rangle$) of that reference, where $K_0\langle\mathcal{S}\rangle$ denotes the K_0-vector space spanned by $\mathcal{S}$, i.e. we must show:

if $b_0, \cdots, b_n$ are in $K_0\langle\mathcal{S}\rangle$, then $v(b_0+b_1t+\cdots+b_nt^n)=\inf\{v(b_i)\mid i=0, \cdots, n\}$.

Write $b_i=b_{i0}+b_{i1}t_0+\cdots+b_{i,m-1}t_0^{m-1}$ $(i=0, \cdots, n)$, where b_{ij} are in $K_0+K_0x+\cdots+K_0x^{E-1}$. By the lemma in the proof of [9, Theorem 2.1], $v(b_i)=\inf\{v(b_{ij})\mid j=0, \cdots, m-1\}$. Therefore if b is an element of least value from $\{b_{ij}\mid i=0, \cdots, n; j=0, \cdots, m-1\}$, then $\inf\{v(b_i)\mid i=0, \cdots, n\}=v(b)$. Thus, if $s=b_0+b_1t+\cdots+b_nt^n$, we must prove $(s/b)^*\neq 0$ (where * denotes image in k). We have

$$(s/b)^*=(b_0/b)+(b_1/b)^*t^*+\cdots+(b_n/b)^*t^{*n}$$
$$=[(b_{00}/b)^*+(b_{01}/b)^*t_0^*+\cdots+(b_{0,m-1}/b)^*t_0^{*m-1}]+\cdots+[\ \ ]t^*+\cdots+[\ \ ]t^{*n}.$$

But $\deg b_{ij}/b$ is $<E$, and by definition of E this implies each $(b_{ij}/b)^*$ is in k_0', the algebraic closure of k_0 in k. On the other hand, $\{t_0^{*i}t^{*j}\mid i=0, \cdots, m-1; j=0, \cdots, n\}$ is linearly independent over k_0', since t^* is *tr.* over k_0' and t_0^* is of $\deg m$ over $k_0'(t^*)$. Thus, $(s/b)^*\neq 0$. Q. E. D. for the Proposition.

2.2.3. Corollary. *$E=E^\wedge$, and $D=D^\wedge=E/IR\geqq D^h$.*

Proof. Proposition 2.2.1 is equivalent to: $D^\wedge\geqq E/IR$. On the other hand, since, for t in $\mathcal{T}$, $K_0(t)\subset K_0(t)^\wedge$ and $K_0(t)\subset K_0(t)^h$, we have $E(t)\geqq E^\wedge(t)$ and $E(t)\geqq E^h(t)$. Therefore, $E(t)/IR(t)\geqq E^\wedge(t)/IR(t)$ and $E(t)/IR(t)\geqq E^h(t)/IR(t)$. By taking infs over $\mathcal{T}$ of these inequalities, we conclude $D\geqq D^\wedge$ and $D\geqq D^h$.

But by vi) above, $D\leqq E/IR$. Putting these observations together, we obtain $D=D^\wedge=E/IR\geqq D^h$.

To see $E=E^\wedge$, note that 2.2.1 yields $E^\wedge(t)\geqq(E/R)R(t)$ on $\mathcal{T}$, and hence by taking infs over $\mathcal{T}$, we have $E^\wedge\geqq E$. Conversely, as noted above, $E(t)\geqq E^\wedge(t)$ on $\mathcal{T}$ so $E\geqq E^\wedge$.

Proof of Lemma 2.2.2. Since any residually *tr.* element t of $K_0(x)$ is also a residually *tr.* element of $K_0^\wedge(x)$, and since $[K_0^\wedge(x): K_0^\wedge(t)]\leqq[K_0(x): K_0(t)]$, we conclude $E'\leqq E$. Thus, it remains to show: for any residually *tr.* element t' of $K_0^\wedge(x)$, there exists a residually *tr.* t in $K_0(x)$ such that $[K_0^\wedge(x): K_0^\wedge(t')]\geqq E(t)$.

Write $t'=a'/b'$, where a', b' are relatively prime elements of $K_0^\wedge[x]$, and note that $[K_0^\wedge(x): K_0^\wedge(t')]=\max\{\deg a', \deg b'\}$ (cf. [18, p. 197]). Now write $a'=a_0'+a_1'x+\cdots+a_n'x^n$, where the a_i' are in $K_0^\wedge$ and $a_n'\neq 0$. Since K_0 is dense in $K_0^\wedge$, there exist a_i in K_0 such that $v(a_i'x^i-a_ix^i)>v(b')$ and $a_n\neq 0$. Let $a=a_0+\cdots+a_nx^n$. Then $v(a'/b'-a/b')>0$, so $(a'/b')^*=(a/b')^*$. Thus a/b' is residually *tr.* We can now repeat this procedure, using b'/a, to obtain an element $t=b/a$ in $K_0(x)$ which is residually *tr.* and such that $E(t)\leqq\max\{\deg a, \deg b\}\leqq\max\{\deg a', \deg b'\}$. Q. E. D.

This concludes the proof of Theorem 2.2.

Remark. In [9] we defined the defect D of v/v_0 to be E/IR and then proved that $D^\wedge(t)=E/IR$ for t in $\min\mathcal{T}$, while the crux of 2.2 is that, in fact, E/IR is the inf of $D^\wedge(t)$ on $\mathcal{T}$.

2.3. D^h and the property (U). We shall say

i) t in $\mathfrak{T}$ has the uniqueness property (U) (resp., the dependence uniqueness property (U_{dep})) if v/v_0^t is unique (resp., dependence unique), and

ii) the extension $(K_0, v_0)\subset(K_0(x), v)$ has the uniqueness property (U) (resp., the dependence uniqueness property (U_{dep})) if there exists t in $\mathfrak{T}$ with (U) (resp., (U_{dep})).

By 1.4 applied to the extension $(K_0(t), v_0^t)\subset(K_0(x), v)$, for t in $\mathfrak{T}$ the following are equivalent:

(2.3.1)
i) t has (U)
ii) $D(t)=D^h$
iii) $E(t)=IR(t)D^h \quad (=E^h(t))$.

2.3.2. Corollary. *The following are equivalent:*

i) *the extension* $(K_0, v_0)\subset(K_0(x), v)$ *has* (U)

ii) $D=D^h$

iii) $E=IRD^h \quad (=E^h)$

iv) *every element of* $\min \mathfrak{T}$ *has* (U).

Proof. i)→ii): Suppose t_0 in $\mathfrak{T}$ has (U). Then $D\leqq D(t_0)=D^h$, the equality by 2.3.1. On the other hand, $D\geqq D^h$ by 2.2.3. ii)↔iii): $E/IR=D$ by 2.2.3. ii)→iv): $D(t)=D$ on $\min \mathfrak{T}$ by 2.2, so 2.3.1 applies. iv)→i): By definition. Q. E. D.

2.3.4. Corollary. *There exists a residually tr. simple tr. extension which does not have* (U).

Proof. In [11, p. 218, §7.2] there is given a discrete rk 2 example with $E=2$ and $IR=1$; and the residue char is 0, so $D^h=1$. (Note that this example is optimal in the sense that every rk 1 extension has (U) by [9, Cor. 2.2].)

2.4. $D^\wedge$ and the property (U_{dep}). The statements 2.3.1 and 2.3.2 have analogues for (U_{dep}) and $D^\wedge$ as follows:

For t in $\mathfrak{T}$ the following are equivalent:

(2.4.1)
i) t has (U_{dep})
ii) $D(t)=D^\wedge(t)$
iii) $E(t)=IR(t)D^\wedge(t) \quad (=E^\wedge(t))$.

This follows from 1.4 applied to the extension $(K_0(t), v_0^t)\subset(K_0(x), v)$, provided one first proves that the inseparability quotient $Q^\wedge(v/v_0^t)$ which appears in 1.4 is 1, i.e.

2.4.2. Lemma. *For t in T,* $1=Q^\wedge(v/v_0^t)$ $(=[K_0(x): K_0(t)]_{ins}/[K_0(x)^\wedge: K_0(t)^\wedge]_{ins})$.

Before proving this lemma, we shall give the analogue of 2.3.2:

2.4.3. *The following are true:*

i) *the extension* $(K_0, v_0) \subset (K_0(x), v)$ *has* (U_{dep})

ii) $D = D\hat{}$

iii) $E = IRD\hat{} \quad (= E\hat{})$

iv) *every element of* $\min \mathcal{T}$ *has* (U_{dep}).

Proof. i) is by [9, Cor. 2.2], ii) and iii) by 2.2.3, and iv) by the equalities $D(t) = D = D\hat{} = D\hat{}(t)$ for t in $\min \mathcal{T}$, which come from 2.2 and 2.2.3.

Proof of Lemma 2.4.2. It suffices to prove $Q\hat{}(v/v_0^t)$ is independent of t in $\mathcal{T}$, since we know by [9, Thm. 2.1] that $E(t) = E\hat{}(t)$ for t in $\min \mathcal{T}$, which implies $Q\hat{}(v/v_0^t) = 1$ for t in $\min \mathcal{T}$.

Let $K_0(t)^{\mathrm{sep}}$ denote the separable closure of $K_0(t)$ in $K_0(x)$ and $p = \mathrm{char}\, K_0$. Since x is purely inseparable over $K_0(t)^{\mathrm{sep}}$, it follows that $K_0(t)^{\mathrm{sep}} = K_0(x^{p^i})$ for some $i \geqq 0$. (This is because the only fields L between $K_0(x^{p^i})$ and $K_0(x)$ are $K_0(x^{p^i}), K_0(x^{p^{i-1}}), \cdots, K_0(x^p), K_0(x)$. For, $[K_0(x) : L] = p^j$ for some j in $\{0, \cdots, i\}$, since $[K_0(x) : L] \mid p^i$. But the irreducible polynomial for x over $K_0(x^{p^i})$ is $X^{p^i} - x^{p^i} = (X - x)^{p^i}$, and the irreducible polynomial for x over L must be a factor of this polynomial, hence must be $(X - x)^{p^j}$; so x^{p^j} is in L.)

First consider the case that x itself is residually *tr*. Then x^{p^i} is also residually *tr.*, so the residue field extension corresponding to $K_0(x^{p^i}) \subset K_0(x)$ is $k_0(x^{*p^i}) \subset k_0(x^*)$. Therefore $[K_0(x) : K_0(t)]_{ins} = [K_0(x) : K_0(x^{p^i})] = [k_0(x^*) : k_0(x^{*p^i})] \leqq [K_0(x)\hat{} : K_0(x^{p^i})\hat{}] = [K_0(x)\hat{} : K_0(t)\hat{}]_{ins}$. Since $[K_0(x) : K_0(t)]_{ins} \geqq [K_0(x)\hat{} : K_0(t)\hat{}]_{ins}$ is immediate from $K_0(t) \subset K_0(t)\hat{}$, we have proved $Q\hat{}(v/v_0^t) = 1$ when x is residually tr.

Consider next the general case. There exists a finite algebraic extension K_1 of K_0 such that (after extending v arbitrarily to $K_1(x)$), $K_1(x)/K_1$ has a residually *tr.* generator (cf. [9, Proof of 2.5]).

$$\begin{array}{ccc} K_1(t) & \text{———} & K_1(x) \\ | & & | \\ K_0(t) & \text{———} & K_0(x)\,. \end{array}$$

Now use the multiplicative property of $Q\hat{}$: $Q\hat{}(K_1(x)/K_1(t))Q\hat{}(K_1(t)/K_0(t)) = Q\hat{}(K_1(x)/K_0(x))Q\hat{}(K_0(x)/K_0(t))$. (Note that we have changed notation here from e.g. $Q\hat{}(v/v_0^t)$ to $Q\hat{}(K_0(x)/K_0(t))$.) We have seen above that $Q\hat{}(K_1(x)/K_1(t)) = 1$, so it remains to observe that $Q\hat{}(K_1(t)/K_0(t))$ is independent of t in $\mathcal{T}$. But this follows from the fact that for any t, t' in $\mathcal{T}$, the extensions $K_0(t) \subset K_1(t)$ and $K_0(t') \subset K_1(t')$ are isomorphic as valued field extensions. Q. E. D.

In view of 1.3 and 2.2, Lemma 2.4.2 yields also the

2.4.4. Corollary. *If* $\mathrm{rk}\, v_0 = 1$, *then* $D\hat{}(t)$ *is constant on* $\mathcal{T}$ *and* $= D^h$.

A second proof of Lemma 2.4.2 may be obtained by replacing K_1 in the above proof by $K_0\hat{}$. As in the above proof, one must first verify that $Q\hat{}(K_0\hat{}(x)/K_0\hat{}(t)) = 1$, which by 1.1 follows from the

2.4.5. Proposition. *Let $K_0 \subset K_0(t)$ be a valued field extension with K_0 complete and t residually tr. Then $K_0(t)\hat{}$ is separable over $K_0(t)$.*

Proof. This was proved in the discrete rk 1 case in [5, p. 48, Lemma 3.1] and in the rk 1 case in [7, p. 183, Cor. 1] or [6, p. 18], for a detail proof); these proofs carry over to the present case of arbitrary rk valuations, provided one has the

Lemma. *Same hypothesis as* 2.4.5. *If K_0' is a finite algebraic extension of K_0, then $K_0'(t)$ and $K_0(t)\hat{}$ are linearly disjoint overt $K_0(t)$ (or, equivalently, $[K_0'(t): K_0(t)]=[K_0(t)\hat{} : K_0(t)\hat{}]$).*

Proof. Since $[K_0'(t): K_0(t)]=[K_0' : K_0]$, it suffices to show: if $s_0=1, s_1, \cdots, s_n$ is a vector space basis of K_0'/K_0, then $s_0, \cdots, s_n$ are linearly independent over $K_0(t)\hat{}$. By [9, Cor. 1.6] one need only check that t satisfies the condition $(\inf/K_0s_0+\cdots+K_0s_n)$ of that reference. But t is residually *tr.* and any element of value 0 which is a quotient of elements of $K_0s_0+\cdots+K_0s_n$ is residually algebraic, so the condition follows as in [9, Lemma to 2.1].

2.4.6. Remark. This second proof is of special interest, since by iteration of 2.4.5 it yields the Theorem (in the terminology of 2.7 below): Let $(K_0, v_0)\subset(K, v)$ be a residually *tr.* valued function field. Then $Q\hat{}(v/v_0{}^t)$ is independent of residually *tr.* basis t in $\mathcal{T}$. (See also 2.7.4.)

2.5. Example. Let (K_0, v_0) be any non-trivally valued field, and let v be the inf extension (to $K_0(x)$) of v_0 w.r.t. x. This is the most elementary kind of residually *tr.* extension. For any t in $\mathcal{T}$, the residue fields corresponding to $K_0\subset K_0(t)\subset K_0(x)$ are $k_0\subset k_0(t^*)\subset k_0(x^*)$, and the value groups are all equal. Thus, $1=E=I=R=D=D\hat{}=D^h$, and therefore $D\hat{}(t)=D(t)=1$ on min T. What do $D(t)$ and $D\hat{}(t)$ look like on the rest of $\mathcal{T}$?

First, $D(t)$ assumes all rational numbers $\geqq 1$. To see this, choose a in K_0 such that $v_0(a)>0$; and let $t_{m/n}=ax^m+x^n$. Then $t_{m/n}{}^*=x^{*n}$, so $R(t_{m/n})=n$. Therefore $D(t_{m/n})=E(t_{m/n})/IR(t_{m/n})=m/n$.

Secondly, $D\hat{}(t)=1$ on T if rk $v_0=1$, and assumes all rational numbers $\geqq 1$ if rk $v_0>1$. For, if rk $v_0=1$, we know by 1.3 that $D\hat{}(t)Q\hat{}(v/v_0{}^t)=D^h=1$; and this implies $D\hat{}(t)=1$. On the other hand, if rk $v_0>1$, then there exists a valuation ring $<K_0$ which properly contains the valuation ring of v_0; call the associated valuation u_0, and let u be the inf extension of u_0 w.r.t. x. Then u and v define the same topology on $K_0(x)$ and hence produce the same $K_0(x)\hat{}$. Choose a in K_0 such that $u_0(a)=0$ and $v_0(a)>0$, and let $t_{m/n}=ax^m+x^n$. As above, we see that the u-residue deg of $K_0(t)\subset K_0(x)$ is m, and this forces $E\hat{}(t_{m/n})$ $(=[K_0(x)\hat{} : K_0(t_{m/n})\hat{}])$ to be m. Similarly, the v-residue deg of $K_0(t)\subset K_0(x)$ is n, so $R(t_{m/n})=n$. Therefore $D\hat{}(t_{m/n})=E\hat{}(t_{m/n})/IR(t_{m/n})=m/n$.

2.6. The property (U) implies the more general uniqueness property of the following theorem; it is in this form that it plays a key role in the proof of the genus reduction inequality of [7] (see also 2.7.5).

Theorem. *Let* (K_0, v_0) *be a valued field and* $v_1, \cdots, v_n$ *be non-equivalent extensions of* v_0 *to* $K_0(x)$ *such that each* v_i/v_0 *is residually tr. and has* (*U*). *Then there exists t in* $K_0(x)\setminus K_0$ *such that* $v_1, \cdots, v_n$ *is a complete set of extensions of* $v_0{}^t$ *to* $K_0(x)$.

The proof is virtually the same as the rk 1 proof (for 1-dim function fields) given in [16, p. 19, Cor. 1] (and which has antecedents in the work of Lamprecht, Mathieu, and Matignon): for the convenience of the reader we shall sketch it here.

Proof. By hypothesis there exist $t_1, \cdots, t_n$ in $K_0(x)$ such that v_i extends $v_0{}^{t_i}$ uniquely. Consider first the case that $v_2(t_1)=0$. Since $v_1/v_0{}^{t_1}$ is unique, $v_2|K_0(t_1)\neq v_0{}^{t_1}$. This implies there exists t_1' in $K_0[t_1]$ having the form $t_1'=a_1t_1+\cdots+a_{n-1}t_1{}^{n-1}+t_1{}^n$ such that $v_2(t_1')>0$, where $n>1$ and $a_i=0$ or a_i is an element of K_0 of value 0. Let for the moment * denote v_1-residue; since $[k_0(t_1{}^*): k_0(t_1'{}^*)]=\deg$ of $t_1'{}^*$ in $k_0(t_1{}^*)=n=\deg t_1'$ in $K_0(t_1)=[K_0(t_1): K_0(t_1')]$, $v_0{}^{t_1}/v_0{}^{t_1'}$ is unique. Since $v_1/v_0{}^{t_1}$ is also unique, this implies $v_1/v_0{}^{t_1'}$ is unique. Note also that if $v_3(t_1)\neq 0$, then $v_3(t_1')\neq 0$. By replacing t_1 by t_1', we may therefore assume $v_2(t_1)\neq 0$; more generally, by repeating this procedure, we may assume $v_i(t_j)\neq 0$ for $i\neq j$.

Now let $t_1''=(1+t_2{}^2)/(1+t_1+t_1{}^2)$. As before, we see that $v_1/v_0{}^{t_1''}$ is unique; this uses the observation that the v_1-residue deg of the extension $K_0(t_1'')\subset K_0(t_1)$ =the deg of the extension =2. Also, for $i\neq 1$, $v_i(t_1)\neq 0$ implies $v_i(t_1'')=0$ and the v_i-residue of t_1'' is 1. Thus, by replacing t_1 by t_1'', we may further assume t_1 is v_i-residually equal to 1 for $i\geqq 2$; and in general, by repeating this procedure for the other t_j, we may assume t_j is v_i-residually equal to 1 for $j\neq i$.

Claim. $t=t_1\cdots t_n$ satisfies the theorem. Note first that all the v_i contract to $v_0{}^t$ on $K_0(t)$ since t is residually *tr.* for v_i; in fact, the v_i-residue of t equals the v_i-residue of t_i. Moreover, obviously $[K_0(x): K_0(t)]\leqq\sum_{i=1}^n[K_0(x): K_0(t_i)]$. Now fix i and consider v_i/v_0. Let $I_i(t)=\operatorname{index} v_i/v_0$ at t, etc., as in 2.1. Then by 1.4 $[K_0(x): K_0(t_i)]=I_i(t_i)R_i(t_i)D_i{}^h(t_i)$, since $v_i/v_0{}^t$ is unique. But this expression equals $I_i(t)R_i(t)D_i{}^h(t)$, for $I_i(\ )$ and $D_i{}^h(\ )$ are independent of choice of residually *tr.* element for v_i/v_0 (cf. 2.2) and $R_i(t_i)=R(t)$ because the v_i-residue of t equals the v_i-residue of t_i. Thus, $[K_0(x): K_0(t)]\leqq\sum_{i=1}^n I_i(t)R_i(t)D_i{}^h(t)$, which by 1.2 implies $v_1, \cdots, v_n$ are the only extensions of $v_0{}^t$ to $K_0(x)$. Q.E.D.

2.7. Remarks on residually tr. functian fields. Let $(K/K_0, v)$ be a valued function field of dim n with constant field K_0, i.e. K is a finitely generated extension of K_0 of deg of *tr.* n, v is a valuation of K, and K_0 is algebraically closed in K. Let $k_0\subset k$ and $G_0\subset G$ denote as before the residue fields and value groups for $K_0\subset K$.

The previous definitions carry over with self-evident minor changes to this situation. For example, a *tr.* basis $t=(t_1, \cdots, t_n)$ of K/K_0 is called *residually tr.*, or is said to be a *residually tr. basis*, if $v(t_i)\geqq 0$ $(i=1, \cdots, n)$ and $t^*=(t_1{}^*, \cdots, t_n{}^*)$ is a *tr.* basis of k/k_0; and $(K/K_0, v)$ is called a *residually tr. function field* if there exists such a residually $tr.$ basis. As before, we let $\mathcal{T}=\{t|t$ is a residually $tr.$ basis of $(K/K_0, v)\}$ and $\min \mathcal{T}=\{t$ in $\mathcal{T}|$ for all t' in $\mathcal{T}$, $[K:K_0(t)]\leq[K:K_0(t')]\}$. The functions defined in 2.1 can now be defined analogously for residually $tr.$ function fields, e.g. $I=[G:G_0]$, and for t in $\mathcal{T}$, $R(t)=[k:k_0(t^*)]$, $E^h(t)=[K^h:K_0(t)^h]$, and $D^h(t)=E^h(t)/IR(t)$.

2.7.1. Kuhlmann and the second author [14] have independently proved that $D^h(t)$ is constant on $\mathcal{T}$ (hence can be denoted D^h). (Kuhlmann announces this result in [4]).

2.7.2. The set $\min\mathcal{T}$ no longer seems to play for function fields the strong role that it does in the simple $tr.$ case. Although $E(t)$ is still constant on $\min \mathcal{T}$ by definition, the remaining functions of 2.1 need not be. It is not clear if these functions attain their infs on $\min\mathcal{T}$; in fact, in the case of $D(t)$ and $D\hat{}(t)$ it is not clear if they attain their infs anywhere on $\mathcal{T}$, or even if their infs D and $D\hat{}$ are rational numbers.

Moreover, we have seen that $D=D\hat{}$ in the simple $tr.$ case. The first author [8] has proved that for 1-dim function fields $(K/K_0, v)$ with infinite residue field k_0, the equality $D=D\hat{}Q\hat{}$ holds (where $Q\hat{}$ is given by 2.4.6). Conjecture: this equality holds for arbitrary function fields.

As for R, one can no longer expect to have $R=[k_0':k_0]$, where k_0' is the algebraic closure of k_0 in k; for, if this were the case, we would have $k=k_0'(t^*)$ for some t in T, i.e. k/k_0 would be ruled. Thus, $[k_0':k_0]$ must now be considered a separate invariant.

2.7.3. It is easy to give examples of residually $tr.$ function fields with (U) such that not every element of $\min\mathcal{T}$ has (U).

Example. Let k_0 be a field of char$\neq 2$ and z be an indeterminate; let $K_0=k_0(z)$ and v_0 be the z-adic valuation of $k_0(z)/k_0$ (i.e. the valuation whose ring is $k_0[z]_{(z)}$); let $K=K_0(x, y)$, where $y^2=1+z(x^2+1)$; let $v_0{}^x$ be the inf extension of v_0 w.r.t. x; and let v be an extension of $v_0{}^x$ to K.

Claim: $v_0{}^x$ has two extensions to K. For $(y-1)(y+1)=z(x^2+1)$ has value 1 and $(y+1)-(y-1)=2$ has value 0 under any extension; so one of $y-1$, $y+1$ has value 1 and the other value 0 under the extension. Since these values are reversed under the $K_0(x)$-conjugation map $y\to -y$, v and its conjugate are distinct. This also shows $k=k_0(x^*)$.

Say $v(y-1)=1$. Then x is in $\min\mathcal{T}$, and we have just seen that x does not have (U). On the other hand, $(y-1)/z$ is also in $\min\mathcal{T}$ and does have (U), since $((y-1)/z)^*=(x^{*2}+1)/2$ implies $R((y-1)/z)=2$.

If the same construction is used for the function field defined by $y^2=1+zx^2+zx^3$, we do not know if the resulting $(K/K_0, v)$ has (U). If $K_0=k_0(z)$ is replaced by $K_0=k_0((z))$, then by [7, p. 197, Thm. 3] the extension has (U); but we still do not know an explicit t in $\mathcal{T}$ with (U).

2.7.4. It was proved in [7, p. 191, Cor. 1] that $D\hat{}(t)$ is constant on $\mathcal{T}$ for a function field $(K/K_0, v)$ with v of rk 1; the proof involves relating $D\hat{}(t)$ to a certain collection of vector spaces. Another proof can now be effected by putting together the equality $D^h(t)=D^\wedge(t)Q^\wedge(v/v_0{}^t)$ of 1.3 (which holds in rk 1), the theorem mentioned in 2.7.1 above that $D^h(t)$ is constant on $\mathfrak{T}$, and the remark of 2.4.6 that $Q^\wedge(v/v_0{}^t)$ is constant on $\mathfrak{T}$.

2.7.5. The theorem of 2.6 holds more generally for 1-dim function fields. The only new fact needed is that D^h is constant on T (c.f. 2-7-1); also, the inequality $[K_0(x): K_0(t)]\leqq\sum_{i=1}^n[K_0(x): K_0(t_i)]$ appearing in the proof is no longer obvious but requires an argument as in [16, p. 20, Proof of Cor. 1].

2.7.6. Let $(K/K_0, v)$ be a 1-dim residually *tr.* function field with constant field K_0, and assume v is of rk 1. Polzin [16], by showing that the property (U) is equivalent to a certain "descent of Skolem" property, has proved: $(K/K_0, v)$ has (U) if either

i) $v|K_0$ is henselian,

ii) K/K_0 is of genus 0, or

iii) k_0 is algebraic over a finite and v is discrete.

Moreover, there exist examples (of arbitrarily prescribed genus>0) which do not have (U).

It should be emphasized that all of these assertions are extremely difficult to prove and rely heavily on previous results and on techniques from algebraic and rigid analytic geometry; in particular, iii) is an application of deep results of Rumely, Roquette, Moret-Bailly, and Szpiro. Our counterexample of 2.3.4 shows ii) no longer holds for valuations of rk>1, and an analogous example shows the same for iii). We conjecture that i) holds for arbitrary valuations; we shall prove this for the simple *tr.* case in the next section.

3. The henselian theorem

We fix throughout section 3 the following notation: (K_0, v_0) is a valued field with value group G_0 and residue field k_0, x is an indeterminate over K_0, and t is in $K_0(x)\setminus K_0$.

3.1. Generalized inf extensions (cf. [11, p. 209, Prop. 4.3]).

Let γ be an element of a totally ordered group containing G_0. The inf extension of v_0 w.r.t. (t, γ) is the valuation v of $K_0(t)$ determined by: for every integer $n\geqq 0$ and all $a_0, \cdots, a_n$ in K_0,

$$v(a_0+a_1t+\cdots+a_nt^n)=\inf\{v(a_i)+i\gamma\,|\,i=1, \cdots, n\}.$$

We denote the inf extension of v_0 w.r.t. (t, γ) by $v_0{}^{(t,\gamma)}$; if $\gamma=0$, we drop the reference to γ and merely write $v_0{}^t$.

The value group of $v_0{}^{(t,\gamma)}$ is generated by G_0 and γ. We shall only be concerned here with the case that γ is a torsion element mod G_0, or equivalently, the case that γ is in $G_0{}^{\text{div}}$. If γ is of order m mod G_0, then there exists b in K_0 such that $v_0(b)=m\gamma=v_0{}^{(t,\gamma)}(t^m)$, and it is easily seen that the $v_0{}^{(t,\gamma)}$-residue of t^m/b is *tr.* over k_0 and the residue field of $v_0{}^{(t,\gamma)}$ is $k_0((t^m/b)^*)$.

One usually checks that an extension of v_0 to $K_0(t)$ is some $v_0{}^{(t,\gamma)}$ by means of the

3.1.1. Proposition. *Let γ be an element of $G_0{}^{\text{div}}$ of order m mod G_0, let v be an extension of v_0 to $K_0(t)$, and let w be a further extension of v to $K_0{}^{\text{alg}}(t)$. Then the* *following are equivalent*:

i) $v=v_0^{(t,\gamma)}$,

ii) *there exists* b *in* K_0 *of value* $m\gamma$ *such that the* v*-residue of* t^m/b *is tr. over* k_0,

iii) *there exists* a *in* K_0^{alg} *of value* γ *such that the* w*-residue of* t/a *is tr. over* k_0.

3.2. Theorem. *Let* $t(x)$ *be in* $K_0[x]\setminus K_0$.

i) *For every root* r *in* K_0 *of* $t(x)$ *and every* β *in* G_0^{div}, *there exists a unique* $\gamma=\gamma(r, \beta)$ *in* G_0^{div} *such that* $v_0^{(x-r,\gamma)}(t)=\beta$. *Moreover, if* $\gamma_0=\gamma(r, 0)$, *then* $v_0^{(x-r,\gamma_0)}$ *extends* v_0^t.

ii) *If* $t(x)=a(x-r_1)\cdots(x-r_n)$, *where* a, r_i *are in* K_0, *and* $\gamma_i=\gamma(r_i, 0)$, *then* $v_i=: v_0^{(x-r_i,\gamma_i)}$ *extends* v_0^t, *and every extension of* v_0^t *to* $K_0(x)$ *is equivalent to some* $v_i (i=1, \cdots, n)$.

Proof. We can write $t=a_1(x-r)+\cdots+a_n(x-r)^n$, a_i in K_0. Define

$$(\#)\quad \gamma=\max\{(\beta-v_0(a_i))/i \mid i=1, \cdots, n\}.$$

The equality $v_0^{(x-r,\gamma)}(t)=\beta$ is equivalent to: $i\gamma+v_0(a_i)\geqq\beta$ for all i in $\{1, \cdots, n\}$ and $=$ holds for some i; and this is equivalent to (#).

For the second assertion of i), we must show the $v_0^{(x-r,\gamma_0)}$-residue of t is *tr*. over k_0. Extend v_0 to a valuation w_0 of K_0^{alg}. Then $w_0^{(x-r,\gamma_0)}$ extends $v_0^{(x-r,\gamma_0)}$. Choose b in K_0^{alg} such that $w_0(b)=\gamma_0$, and write

$$(\#\#)\quad t=ba_1((x-r)/b)+\cdots+b^n a_n((x-r)/b)^n.$$

We have $v_0^{(x-r,\gamma_0)}(t)=0$ iff $w_0^{(x-r,\gamma_0)}(t)=0$ iff $w_0(b^i a_i)\geqq 0$ for i in $\{1, \cdots, n\}$ and $=$holds for some i. Now apply the $w_0^{(x-r,\gamma_0)}$-residue map to (##). Since the residue of $(x-r)/b$ is *tr*. over k_0^{alg}, it follows that the residue of t is also *tr*. over k_0^{alg}, and hence *tr*. over k_0.

For the proof of ii) note first that v_i extends v_0^t by i). Let v be any extension of v_0^t to $K_0(x)$. By replacing v by an equivalent valuation, we may assume the value group of v is contained in G_0^{div}. Let w be a further extension of v to a valuation of $K_0^{\mathrm{alg}}(x)$ having value group G_0^{div}, and let $w_0=w\mid K_0^{\mathrm{alg}}$. Since the value group of K_0^{alg} is G_0^{div}, there exists b_i in K_0^{alg} such that $w(x-r_i)=w_0(b_i)$. Write

$$t=(ab_1\cdots b_n)[(x-r_1)/b_1]\cdots[(x-r_n)/b_n].$$

The v-residue of t is *tr*. over k_0 because v extends v_0^t; and therefore the w-residue of $(x-r_i)/b_i$ must be *tr*. over k_0 for some i. Then $w=w_0^{(x-r_i, w_0(b_i))}$ for that i. Hence $v=v_0^{(x-r_i, w_0(b_i))}$. Finally, since $v(t)=0$, the uniqueness property of i) yields $w_0(b_i)=\gamma_i$.

Q. E. D.

The question remains as to how many of the extensions v_i of v_0^t in 3.2-ii) are equivalent; we shall settle this later (in 3.6) since the result is not used below.

3.3. Theorem. *Let* (K_0, v_0) *be a henselian valued field; let* x *be an indeterminate; let* $t=f(x)^m/b$, *where* $f(x)$ *is irreducible of* $\deg>0$ *in* $K_0[x]$, $0\neq b$ *in* K_0, *and* $m\geqq 1$; *and let* v_0^t *be the inf extension of* v_0 *w.r.t.* t. *Then the valuation* v_0^t *of* $K_0(t)$ *extends uniquely* (*up to equivalence*) *to a valuation of* $K_0(x)$.

Proof. By replacing our valuations by equivalent ones when necessary, we may assume all our values lie in a fixed G_0^{div}.

Since v_0 is henselian, v_0 has a unique extension w_0 to K_0^{alg}, and then w_0^t is the unique extension of v_0^t to $K_0^{\mathrm{alg}}(t)$.

If $r_1, \cdots, r_n$ are the roots (in K_0^{alg}) of $f(x)$ and $w_i =: w_0^{(x-r_i,\gamma_i)}$ $(i=1, \cdots, n)$ are the extensions of w_0^t to $K_0^{\mathrm{alg}}(x)$ given by 3.2-ii), then every extension of w_0^t to $K_0^{\mathrm{alg}}(x)$ is one of these w_i. Thus, to prove v_0^t has a unique extension to $K_0(x)$, it suffices to prove that the w_i all restrict to the same valuation of $K_0(x)$.

$$\begin{array}{ccc} (K_0^{\mathrm{alg}}(t),\ w_0^t) & \text{——} & (K_0^{\mathrm{alg}}(x),\ w_i) \\ | & & | \\ (K_0(t),\ v_0^t) & \text{———} & K_0(x) \end{array}$$

We shall show w_1 and w_2 both restrict to the same valuation on $K_0(x)$. Since $f(x)$ is irreducible in $K_0[x]$, there exists a K_0-automorphism σ of K_0^{alg} which takes r_1 to r_2. Extend σ to an automorphism τ of $K_0^{\mathrm{alg}}(x)$ by defining $\tau(x)=x$. Since w_0/v_0 is unique, σ is an isometry of $(K_0^{\mathrm{alg}}, w_0)$ onto $(K_0^{\mathrm{alg}}, w_0)$ (i.e. $w_0=w_0\sigma$), and it follows that τ is an isometry of $(K_0^{\mathrm{alg}}(x), w_0^{(x-r_1,\gamma_1)})$ onto $(K_0^{\mathrm{alg}}(x), w_0^{(x-r_2,\gamma_1)})$. But $K_0(x)$ is left element-wise fixed under τ, so $w_1=w_0^{(x-r_1,\gamma_1)}$ and $w_0^{(x-r_2,\gamma_1)}$ must restrict to the same valuation on $K_0(x)$.

Thus, it remains to prove $w_2=w_0^{(x-r_2,\gamma_1)}$, or equivalently, $\gamma_1=\gamma_2$. This follows from the uniqueness assertion of 3.2-i) since $w_0^{(x-r_2,\gamma_1)}(t)=w_0^{(x-r_1,\gamma_1)}(t)=0$ and $w_2(t)=0$.

Q. E. D.

3.3.1. Corollary. *A residually tr. extension* $(K_0, v_0)\subset(K_0(x), v)$ *with* (K_0, v_0) *henselian has property* (U). *More precisely, there exists a residually tr. element t in* $K_0[x]$ *of the form* $t=f(x)^m/b$, *where* $f(x)$ *is irreducible of* $\deg>0$ *in* $K_0[x]$, $0\neq b$ *is in* K_0, *and* $m\geqq 1$; *and any such residually tr. t has* (U).

Proof. Let t_0 be any residually *tr.* element of $K_0(x)$, and write $t_0=f_1\cdots f_q/g_1\cdots g_p$, where f_i, g_j are irreducible in $K_0[x]$. Since $[G: G_0]$ is finite, there exists an integer $m\geqq 1$ such that all the f_i^m and g_j^m have values in G_0. Therefore there exist b_i, c_j in K_0 such that $v(f_i^m/b_i)=0$ and $v(g_j^m/c_j)=0$. Then the equality $t_0^m=(c_1\cdots c_p/b_1\cdots b_q)(f_1^m/b_1)\cdots(f_q^m/b_q)/(g_1^m/c_1)\cdots(g_p^m/c_p)$ implies some f_i^m/b_i or g_j^m/c_j is residually *tr.* Now apply 3.3.

3.3.2. Corollary. *Let* (K_0, v_0) *be henselian, and let* $v_1, \cdots, v_n$ *be a set of non-equivalent extensions of* v_0 *to* $K_0(x)$ *such that each* v_i/v_0 *is residually tr. Then there exists a t in* $K_0(x)\setminus K_0$ *such that* $v_1, \cdots, v_n$ *is a complete set of extensions of* v_0^t *to* $K_0(x)$ (*where* v_0^t *is the inf extension of* v_0 *w.r.t. t*).

Proof. Apply 3.3.1 and 2.6.

3.4. Remark. If $(K_0, v_0)\subset(K_0(x), v)$ is a residually *tr.* extension with (K_0, v_0) henselian, then $E=IRD^h$ by 2.3.2 and 3.3.1. It was conjectured in [12] that $E=IR$ if

v_0 is henselian and of residue char 0; since residue char 0 implies $D^h=1$ (cf. 1.2) this now follows.

The equality $E=IRD^h$ (or equivalently, the property (U)) also holds if $\mathrm{rk}\, v_0=1$; this was proved in [9, Cor. 2.4.1] by using completions. It is also possible to derive this rk 1 case from the henselian case by observing that the extensions $K_0\subset K_0(x)$ and $K_0{}^h\subset K_0{}^h(x)$ have the same E, I, R, D^h. This is true for I and R since passing to the henselization does not change the residue field or value group, and it is true for D^h by the definition of D^h and 2.2. As for E, in the rk 1 case K_0 is dense in $K_0{}^h$, and as in 2.2.2 this implies the extension deg of $K_0\subset K_0(x)$ equals the extension deg of $K_0{}^h\subset K_0{}^h(x)$.

3.5. Even if the extension $(K_0, v_0)\subset(K_0(x), v)$ has (U), it is not possible without the henselian hypothesis to conclude every residually *tr.* t of the form $t=f(x)^m/b$ (as in 3.3.1) has (U). Consider the following discrete rk 1 example: Let y be an indeterminate over a field k_0, let v_0 be the y-adic valuation of $K_0=k_0(y)$ over k_0, and let v be the inf extension (to $K_0(x)$) of v_0 w. r. t. x. Then $t=yx^2+x+1$ is residually *tr.*, since $t^*=x^*+1$. But $I=R(t)=1$ and $E(t)=2$, and $D^h=1$ since v_0 is discrete rk 1; so $E(t)\neq IR(t)D^h$, and therefore (by 2.3.1) t does not have (U), or, in other words, $v_0{}^t$ does not extend uniquely to $K_0(x)$.

One can still ask, however, if an extension with (U) necessarily contains at least one residually *tr.* t of the form $t=f(x)^m/b$ having (U). We shall show next that this is so.

3.5.1. Theorem. *If a residually tr. extension $(K_0, v_0)\subset(K_0(x), v)$ has (U), then $K_0(x)$ contains a residually tr. element t with (U) of the form $t=f(x)^m/b$, where $f(x)$ is irreducible in $K_0[x]\setminus K_0$, $0\neq b$ is in K_0, and $m\geqq 1$.*

Proof. We need the following lemma from [10] (it would take us too far afield to give its proof here):

Lemma (cf. [10]). Min $\mathfrak{T}$ *contains an element t of the form $t=f(x)^n/g(x)$ with the following properties: $f(x)$ is irreducible in $K_0[x]$; $g(x)$ is in $K_0[x]$ and* $\deg g(x)<\deg f(x)$; *$n\geqq 1$; and there exists an integer $q\geqq 1$ and an element $b\neq 0$ in K_0 such that f^{nq}/b is residually tr. and g^q/b is residually alg.*

We claim the residually *tr.* element $t_0=f^{nq}/b$ is the element of the theorem. Since t is in min $\mathfrak{T}$, t has (U) by 2.3.2; and since deg=residue deg for the extension $K_0(t^q)\subset K_0(t)$, t^q also has (U). Therefere by 2.3.1 $E(t^q)=IR(t^q)D^h$. But $E(t^q)=E(t_0)$, and we shall show below that $R(t^q)=R(t_0)$; so it follows that $E(t_0)=IR(t_0)D^h$, which by 2.3.1 implies t_0 has (U).

Thus, it only remains to prove $R(t^q)=R(t_0)$. Since $t^{*q}=(b/g^q)^*t_0{}^*$ and $(g^q/b)^*$ is in $k_0{}'$ (=algebraic closure of k_0 in k), we have $k_0{}'(t_*{}^q)=k_0{}'(t_0{}^*)$. Then $R(t^q)=[k:k_0(t^{*q})]=[k:k_0{}'(t^{*q})][k_0{}'(t^{*q}):k_0(t^{*b})]=[k:k_0{}'(t_0{}^*)][k_0{}'(t_0{}^*):k_0(t_0{}^*)]=[k:k_0(t_0{}^*)]=R(t_0)$. Q. E. D.

3.6. Let (K_0, v_0) be a valued field, let x be an indeterminate, and let t be an element of $K_0[x]\setminus K_0$ such that $t=a(x-r_1)\cdots(x-r_n)$, where a, r_i are in K_0.

We have seen in 3.2 that there exists a unique γ_i in G_0^{div} such that $v_0^{(x-r_i,\gamma_i)}(t)=0$, and that if $v_i=: v_0^{(x-r_i,\gamma_i)}$, then each v_i extends v_0^t and every extension of v_0^t to $K_0(x)$ is equivalent to one of these v_i's. The following theorem completes the descriptions of these extensions of v_0^t to $K_0(x)$.

3.6.1. Theorem (continuation of 3.2). *Let* $\mathcal{D}_i=\{d \text{ in } K_0 \mid v_0(d-r_i)\}\geqq\gamma_i$ *i.e.* $\mathcal{D}_i$ *is the disc of center* r_i *and radius* γ_i. *Then* $v_i=v_j$ *iff* $\mathcal{D}_i=\mathcal{D}_j$. *Moreover, the index of* v_i/v_0^t (*=order of* $\gamma_i \bmod G_0$) *times the residue deg of* v_i/v_0^t *equals the number of* r_q*'s, counted with mutiplicity, in* $\mathcal{D}_i$.

Proof. Since the theorem is trivial if $n=1$, let us assume $n\geqq 2$, and then, for ease of reading, that $i=1$ and $j=2$. The first assertion of the theorem follows from the

3.6.2. Lemma. *The following are equivalent*:

i) $v_1=v_2$.

ii) $v_0(r_1-r_2)\geqq\gamma_1=\gamma_2$.

iii) $\mathcal{D}_1=\mathcal{D}_2$.

iv) $\mathcal{D}_1\cap\mathcal{D}_2\neq\emptyset$.

Proof. i)⇒ii): By definition of v_2, the equality $x-r_1=(x-r_2)-(r_1-r_2)$ implies $v_2(x-r_1)=\inf\{\gamma_2, v_0(r_1-r_2)\}$. Since $v_1=v_2$, then $\gamma_1=v_1(x-r_1)=v_2(x-r_1)=\inf\{\gamma_2, v_0(r_1-r_2)\}$. Therefore, $\gamma_1\leqq\gamma_2$ and $\gamma_1\leqq v_0(r_1-r_2)$; and by symmetry, $\gamma_2\leqq\gamma_1$.

ii)⇒i): By extending v_0 to K_0^{alg}, we are reduced to considering the case that γ_1, γ_2 are in G_0. Let $b\in K_0^{\mathrm{alg}}$ be such that $v_0(b)=\gamma_1=\gamma_2$. By applying the v_1-residue map to the expression $(x-r_2)/b=(x-r_1)/b-(r_2-r_1)/b$, we conclude that $(x-r_2)/b$ is residually *tr.* for v_1. By 3.1.1 this implies $v_1=v_0^{(x-r_2,\gamma_1)}$. But $\gamma_1=\gamma_2$, so $v_0^{(x-r_2,\gamma_1)}=v_2$.

ii)⇒iii)⇒iv): Immediate.

iv)⇒ii): We may suppose d is in $\mathcal{D}_1\cap\mathcal{D}_2$ and $\gamma_2\leqq\gamma_1$. Then $v_0(r_2-r_1)\geqq\inf\{v_0(r_2-d), v_0(r_1-d)\}\geqq\inf\{\gamma_2, \gamma_1\}=\gamma_2$. It remains to show $\gamma_1=\gamma_2$.

By definition of v_1

(1) $$v_1(x-r_j)=\inf\{v_1(x-r_1), v_0(r_1-r_j)\}=\inf\{\gamma_1, v_0(r_1-r_j)\}.$$

Similarly,

(2) $$v_2(x-r_j)=\inf\{\gamma_2, v_0(r_2-r_j)\}.$$

But $v_0(r_2-r_j)\geqq\inf\{v_0(r_1-r_j), v_0(r_2-r_1)\}$, and = holds if $v_0(r_1-r_j)\neq v_0(r_2-r_1)$. Since $v_0(r_2-r_1)\geqq\gamma_2$, it follows from (2) that

(3) $$v_2(x-r_j)=\inf\{\gamma_2, v_0(r_1-r_j)\}.$$

Therefore by (1) and (3) $\gamma_2\leqq\gamma_1$ implies

(4) $$v_1(x-r_j)\geqq v_2(x-r_j).$$

By taking $j=1$ in (2), (4), we have

(5) $$\gamma_1 = v_1(x-r_1) \geqq v_2(x-r_1) = \gamma_2.$$

By (4) $v_1(t) \geqq v_2(t)$; and by (5), if $\gamma_1 > \gamma_2$, then $v_1(t) > v_2(t)$. Since by hypothesis $v_1(t) = v_2(t)$, we therefore have $\gamma_1 = \gamma_2$. Q. E. D.

We now turn to the proof of the second assertion of 3.6.1. First consider the case that γ_1 is in G_0. Then there exists b in K_0 such that $v_1(x-r_1) = v_1(b) = \gamma_1$. Let $t_1 = (x-r_1)/b$. The residue field extension for $v_1/v_0{}^t$ is $k_0(t_1{}^*)/k_0(t^*)$, so it remains to verify that $[k_0(t_1{}^*) : k_0(t^*)] = m$, the number of r_j in $\mathscr{D}_1$. This will follow from the

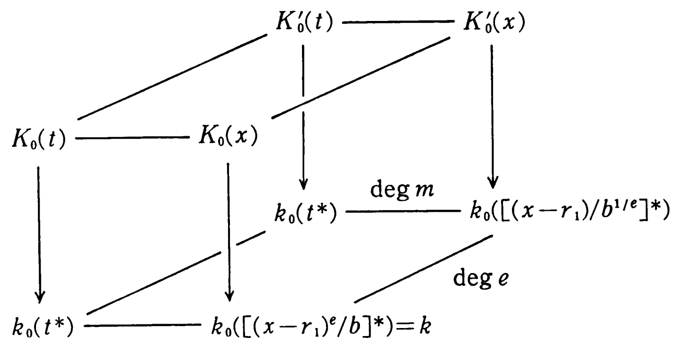
Claim: t^* is a polynomial of deg m in $k_0[t_1{}^*]$. We may suppose $r_1, \cdots, r_m$ are in $\mathscr{D}_1$ and $r_{m+1}, \cdots, r_n$ are not in $\mathscr{D}_1$. Write

(#) $$t = a[(x-r_1)][(x-r_1)-(r_2-r_1)]\cdots[(x-r_1)-(r_n-r_1)]$$
$$= [ab^m(r_{m+1}-r_1)\cdots(r_n-r_1)][t_1][t_1-(r_2-r_1)/b]\cdots$$
$$[t_1-(r_m-r_1)/b][((x-r_1)/(r_{m+1}-r_1))-1]\cdots[((x-r_1)/(r_n-r_1))-1].$$

Since $v_0(r_j-r_1) \geqq \gamma_1$ iff r_j is in $\mathscr{D}_1$, $v_0((r_j-r_1)/b) \geqq 0$ for $j=1, \cdots, m$, and $v_1((x-r_1)/(r_j-r_1)) > 0$ for $j=m+1, \cdots, n$. The claim now follows by applying the v_1-residue map to the expression (#).

Now consider the case that γ_1 is not in G_0, and let e be the order of $\gamma_1 \bmod G_0$ By 3.1 the residue field of v_1 is now $k_0([(x-r_1)^e/b]^*)$, where b is an element of K_0 such that $v_0(b) = e\gamma_1$.

Note that v_0 has a unique extension (of index e) to $K_0(b^{1/e}) =: K_0{}'$ and v_1 a unique extension (of residue deg e\ to $K_0{}'(x)$:



By applying the previous case to the extension $K_0{}' \subset K_0{}'(x)$, we conclude that t^* is of deg m in $k_0([(x-r_1)/b^{1/e}]^*)$. Therefore $m = e[k : k_0(t^*)]$. Q. E. D.

3.6.3. Corollary. *If* $t = a(x-r_1)\cdots(x-r_n)$, a, r_i *in* K_0, $n \geqq 1$, *then every extension* v *of* $v_0{}^t$ *to* $K_0(x)$ *has* $\mathrm{def}^h(v/v_0{}^t) = 1$.

Proof. Le $v_1, \cdots, v_m$ be a complete set of extensions of $v_0{}^t$ to $K_0(x)$. By 3.6.1, $n = \sum_{i=1}^m [\mathrm{index}(v_i/v_0{}^t)][\text{residue deg}(v_i/v_0{}^t)]$. But $n = [K : K_0(t)]$, so by 1.2 this implies $\mathrm{def}^h(v_i/v_0{}^t) = 1$ $(i=1, \cdots, m)$. Q. E. D.

3.6.4. Example. Let $K_0=k_0(z)$; let v_0 be the z-adic valuation of $k_0(z)/k_0$; and let $t=(1/z^5)(x-r_1)(x-r_2)(x-r_3)$, where $r_1=0$, $r_2=z$, and $r_3=z^3$. Then the extensions of $v_0{}^t$ to $K_0(x)$ are $v_1=v_0{}^{(x-r_1,2)}$, $v_2=v_0{}^{(x-r_2,3)}$, and $v_3=v_0{}^{(x-r_3,2)}$. By 3.6.1, $v_1=v_3\neq v_2$; and, since the indices are all 1, the residue deg of $v_1/v_0{}^t$ is 2 and that of $v_2/v_0{}^t$ is 1.

U. E. R. DE MATHÉMATIQUES ET D'INFORMATIQUE
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BATON ROUGE, LOUISIANA 70803
U. S. A.

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Added in Proof:

Answers to conjectures made in 2.7 are given in

B. W. Green, M. Matignon, F. Pop, On valued function fieds I, Manuscr. Math. **65** (1989), 257-276.

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