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A stochastic equation based on a Poisson system for a class of measure-valued diffusion processes

By

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§1. Introduction

Measure-valued diffusion processes are of a typical class of infinite dimensional diffusion processes, which arise in various fields such as mathematical biology and filtering theory. Above all, measure-valued branching diffusions in population dynamics and Fleming-Viot diffusion models in population genetics have been studied extensively by many authors from points of large time behaviors based on analysis of the distribution at fixed time $t\geqq 0$, (cf. [13], [1], [2], [6], [11], [9], [4]).

In the present paper we are concerned with probabilistic structure of sample paths for a class of measure-valued diffusion processes including measure-valued branching diffusions and Fleming-Viot diffusion models. For this purpose we will formulate a stochastic equation based on a Poisson system associated with excursion laws of one-dimensional continuous state branching diffusions, which gives an intuitive and comprehensible description of a class of measure-valued diffusion processes and makes the sample path structure clearly observed. Furthermore, by solving the stochastic equation we can provide a new interesting class of measure-valued diffusion processes.

Let S be a basic space that is a locally compact separable metric space, $\boldsymbol{B}(S)$ be the Borel field of S, $M(S)$ be the set of bounded measures on S, and $M_1(S)$ be the set of probability measures on S. $M(S)$ and $M_1(S)$ are equipped with the usual weak topology. We denote by $C_b(S)$ and $C_0(S)$ the set of bounded continuous functions on S and the set of continuous functions of S vanishing at infinity, if S is non-compact. In this paper we will discuss diffusion processes on the state spaces $M(S)$ and $M_1(S)$, which we call measure-valued diffusion processes.

Let us consider the following operator L acting on a class of function on $M(S)$:

$$LF(\mu)=\frac{1}{2}\int_S \mu(dx)\frac{\delta^2 F(\mu)}{\delta\mu(x)^2}+\int_S \mu(dx)A\Big(\frac{\delta F}{\delta\mu}\Big)(x) \tag{1.1}$$

where A is a generator of a Markov process on the state space S with the domain $\boldsymbol{D}(A)$, and $\delta F(\mu)/\delta\mu(x)=\lim_{\varepsilon\downarrow 0}(F(\mu+\varepsilon\delta_x)-F(\mu))/\varepsilon$ (if exists). For example if $F(\mu)=f(\langle\mu,\phi\rangle)$, then $\delta F(\mu)/\delta\mu(x)=\phi(x)f'(\langle\mu,\phi\rangle)$.

The domain of L is given by

$$\boldsymbol{D}(L)=\{F(\mu)=f(\langle\mu,\phi_1\rangle,\cdots,\langle\mu,\phi_k\rangle):\ k\geqq 1,\ \phi_i\in\boldsymbol{D}(A),\text{ and } f\in C_b^2(R^k)\}\,. \tag{1.2}$$

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Here δ_x stands for the Dirac measure at $x\in S$, and $C_0^2(R^k)$ denotes the set of C^2-functions on R^k that are bounded together with derivatives of order $\leqq 2$.

In particular, if S is a finite set $\{1, 2, \cdots, n\}$, then L of (1.1) turns to

$$L=\frac{1}{2}\sum_{i=1}^{n} x_i\frac{\partial^2}{\partial x_i^2}+\sum_{i=1}^{n}\Big(\sum_{j=1}^{n} x_j m_{ji}\Big)\frac{\partial}{\partial x_i} \quad (x_i\geqq 0,\ 1\leqq i\leqq n),$$

which generates an n-dimensional continuous state branching diffusion with type transition rates (m_{ji}).

For the operator $(L, \boldsymbol{D}(L))$ of (1.1) it is known that there is a unique diffusion process $(\Omega, \boldsymbol{F}, \boldsymbol{F}_t, P_\mu; X_t)$ on the state space $M(S)$ such that for every $F\in\boldsymbol{D}(L)$ and $\mu\in M(S)$

$$F(X_t)-\int_0^t LF(X_s)ds$$

is an $((\boldsymbol{F}_t), P_\mu)$-martingale. Then the distribution of X_t is determined by the following relation:

(1.3) $\qquad E_\mu(\exp(-\langle X_t, \phi\rangle))=\exp(-\langle\mu, u_t\rangle) \qquad$ for every non-negative $\phi\in C_0(S)$

where $u_t=u_t(x)$ is a mild solution of the equation

(1.4)
$$\frac{\partial u_t}{\partial t}=Au_t-\frac{1}{2}u_t^2$$
$$u_0=\phi\,.$$

The diffusion process $(\Omega, \boldsymbol{F}, \boldsymbol{F}_t, P_\mu; X_t)$ is called *a measure-valued branching diffusion (abbrev.* MBD *process) driven by* $(A, \boldsymbol{D}(A))$, (cf. [13], [6]).

We next introduce the following operator acting on a class of functions on $M_1(S)$:

(1.5)
$$\tilde{L}F(p)=\frac{1}{2}\int_{S\times S} p(dx)(\delta_x(dy)-p(dy))\frac{\delta^2 F(p)}{\delta p(x)\delta p(y)}+\int_S p(dx)A\Big(\frac{\delta F}{\delta p}\Big)(x)$$

where $(A, \boldsymbol{D}(A))$ is the same as (1.1) and we take $\boldsymbol{D}(\tilde{L})=\boldsymbol{D}(L)$ as the domain of $\tilde{L}$. Then there is a unique diffusion process $(\tilde{\Omega}, \tilde{\boldsymbol{F}}, \tilde{\boldsymbol{F}}_t, \tilde{P}_p; \tilde{X}_t)$ on the state space $M_1(S)$ such that for every $F\in\boldsymbol{D}(\tilde{L})$ and $p\in M_1(S)$

$$F(Y_t)-\int_0^t \tilde{L}F(Y_s)ds$$

is an $((\tilde{\boldsymbol{F}}_t), \tilde{P}_p)$-martingale.

The diffusion process $(\tilde{\Omega}, \tilde{\boldsymbol{F}}, \tilde{\boldsymbol{F}}_t, \tilde{P}_p; Y_t)$ is called *a Fleming-Viot diffusion model (abbrev.* FVD *process) with mutation operator* $(A, \boldsymbol{D}(A))$, which arises in the theory of population genetics, (cf. [4]).

Concerning their sample path properties the MBD processes and the FVD processes have a common picture. Suppose that $S=R^d$, and $A=-(-\Delta)^{\alpha/2}$ $(0<\alpha\leqq 2)$ (the generator of the symmetric stable process of order α). Then it is known that if $d=1$ and $1<\alpha\leqq 2$, both random measures X_t and Y_t are absolutely continuous with respect to the Lebesgue measure on R^1 for all $t>0$ and their densities $X_t(x)$ and $Y_t(x)$ are jointly continuous in $t>0$ and $x\in R^1$ almost surely w.r.t. P_μ and $\tilde{P}_p$ for every $\mu\in M(R^1)$ and $p\in M_1(R^1)$ respectively. Furthermore the density processes $X_t(x)$ and $Y_t(x)$ satisfy the following stochastic partial differential equations (abbrev. SPDE):

$$\frac{\partial X_t(x)}{\partial t}=AX_t x)+\sqrt{X_t(x)}W_t'(x) \tag{1.6}$$

$$\frac{\partial Y_t(x)}{\partial t}=AY_t(x)+\sqrt{Y_t(x)}W_t'(x)-\left(\int_{R^1}\sqrt{Y_t(y)}W_t'(y)dy\right)Y_t(x) \tag{1.7}$$

where $W_t'(x)$ is a space-time white noise on R^1 that is a centered Gaussian field on $[0, \infty]\times R^1$ with covariance

$$E(W_t'(x)W_s'(y))=\delta(t-s)\delta(x-y)$$

where δ is the Dirac δ-function, and the equations (1.6) and (1.7) should be understood as continuous processes taking values in the space of Schwartz distributions $S'(R^1)$, (see [8]). On the other hand if either $d=1$ and $0<\alpha\leqq 1$ or $d\geqq 2$, then both random measures X_t and Y_t are singular with respect to the Lebesgue measure on R^d for all $t>0$ almost surely w.r.t. P_μ and $\tilde{P}_p$ for every $\mu\in M(R^d)$ and $p\in M_1(R^d)$, (see [9]).

Also, suppose that A is a bounded generator of a Markov process on the state space S. Then the corresponding FVD process Y_t indeed takes values in the set of of atomic measures on S for all $t>0$, $\tilde{P}_p$-almost surely for every $p\in M_1(S)$, (cf. [3]), and the same fact should hold for the MBD process.

In this paper we are interested in the last case. In particular we would like to give probabilistic construction of sample paths for a class of measure-valued diffusion processes taking values in the set of atomic measures. For this we will formulate a stochastic equation based on a Poisson system associated with excursion laws of continuous state branching diffusions on $[0, \infty)$. For such a Poisson construction associated with an excursion law of a continuous state branching diffusion we mention Pitman-Yor's paper [10], where they constructed a two parameter process $X(t, c)$, $t\geqq 0$, $0\leqq c\leqq 1$ such that for each c, $X^c(t)=X(t, c)$ is a diffusion process on $[0, \infty)$ generated by

$$G=2y\frac{d^2}{dy^2}+c\frac{d}{dy} \quad \text{(0 is reflective),} \tag{1.8}$$

and further that $X(c)=\{X(t, c), t\geqq 0\}$ is a $C([0, \infty), [0, \infty))$-valued process with independent increments using a Poisson system associated with an excursion law of a continuous state branching diffusion on $[0, \infty)$ generated by

$$G=2y\frac{d^2}{dy^2} \quad \text{(0 is trap.)} \tag{1.9}$$

Noting that $X(t, c)$ is non-decreasing in c, one can regard it as a measure-valued process, in fact it is equivalent to an MBD process with immigrations governed by the following generator:

$$LF(\mu)=2\int_{[0,1]}\mu(dx)\frac{\delta^2 F(\mu)}{\delta\mu(x)^2}+\int_{[0,1]}dx\frac{\delta F(\mu)}{\delta\mu(x)}. \tag{1.10}$$

Moreover it follows immediately from the Poisson construction that the MBD process lives in the set of atomic measures for all $t>0$, even though the process starts at a non-atomic measure.

Developing this method one can formulate a stochastic equation, and by solving the equation we obtain a new class of measure-valued diffusion processes, of which essential state space coincides with $M^{a}(S)$, the set of atomic measures on S.

This paper is organized as follows. In § 2, we establish a relation between a class of $M(S)$-valued diffusions and a class of $M_1(S)$-valued diffusions, which will enable us to reduce problems for $M_1(S)$-valued diffusions to those of $M(S)$-valued diffusions.

In § 3 we construct a class of MBD processes with immigrations (abbrev. MBDI processes) by making use of a Poisson system associated with excursion laws of continuous state branching diffusions.

In § 4 and § 5 we formulate a stochastic equation based on the Poisson system. It will be shown that every solution of the stochastic equation defines an $M^{a}(S)$-valued continuous process in the sense of total variation norm in $t>0$.

Finally, in § 6 we translate these results into the context of $M_1(S)$-valued processes through the relation established in § 2, thus we obtain an interesting class of diffusion models in population genetics, which generalizes the Fleming-Viot diffusion model. We will also prove a strong ergodic theorem for a simple FVD process by constructing a coupling process base on a Poisson construction of MBDI processes.

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§ 2. $M(S)$-valued diffusions and $M_1(S)$-valued diffusions

In this section we will establish a kind of skew product relation between $M(S)$-valued diffusions and $M_1(S)$-valued diffusions, namely, $M(S)$-valued diffusion can be obtained by a skew product from $M_1(S)$-valued diffusions. This idea was first given by H. Tanaka (private communication) in the case where the $M(S)$-valued diffusions is the direct product of finitely many independent diffusions on $[0, \infty)$ generated by $x\frac{d^2}{dx^2}+(\gamma-x)\frac{d}{dx}$ with a constant $\gamma>0$, (S is therefore a finite set).

Suppose that we are given a bounded, uniformly positive and measurable function $\beta(x)$ defined on S. Generalizing the operators of (1.1) and (1.5) let us consider the following two operators L and $\tilde{L}$ on $M(S)$ and $M_1(S)$ respectively.

$$(2.1)\qquad LF(\mu)=\int_S \mu(dx)\beta(x)\frac{\delta^2F(\mu)}{\delta\mu(x)^2}+A\Big(\mu, \frac{\delta F}{\delta\mu}\Big)\quad (\mu\in M(S))$$

$$(2.2)\qquad \tilde{L}F(p)=\int_S p(dx)(\beta(x)\delta_x(dy)+(\langle p, \beta\rangle-\beta(x)-\beta(y))p(dy))\frac{\delta^2F(p)}{\delta p(x)\delta p(y)}$$
$$+\int_S p(dx)(\langle p, \beta\rangle-\beta(x))\frac{\delta F(p)}{\delta p(x)}+\tilde{A}\Big(p, \frac{\delta F}{\delta p}\Big)\quad (p\in M_1(S))$$

where $\boldsymbol{D}(L)=\boldsymbol{D}(\tilde{L})$ are defined by replacing $\boldsymbol{D}(A)$ by $\boldsymbol{D}$ in (1.2), where $\boldsymbol{D}$ is a subspace of $C_b(S)$, $A(\mu, \phi): M(S)\times\boldsymbol{D}\to R^1$ and $\tilde{A}(p, \phi): M_1(S)\times\boldsymbol{D}\to R^1$ are measurable, and for each $\mu\in M(S)$ and $p\in M_1(S)$, both $A(\mu, \cdot)$ and $\tilde{A}(p, \cdot)$ are linear functionals defined on $\boldsymbol{D}$.

Let $W=C([0, \infty), M(S))$ be the set of all $M(S)$-valued continuous paths $w: [0, \infty)\to M(S)$, which is equipped with the usual filtration $(\boldsymbol{F}_t(W))$, and denote by $X_t(w)=w(t)$ the coordinate function at time $t\geqq 0$.

Let $\mu\in M(S)$. By the $(M(S), L, \mu)$-martingale problem we mean to find a probability measure P on W such that

(i) $P(X_0=\mu)=1$,

(ii) $F(X_t)-\int_0^t LF(X_s)ds$ is an $((\boldsymbol{F}_t), P)$-martingale for every $F\in\boldsymbol{D}(L)$.

Let define $\zeta=\inf\{t\geqq 0: \langle X_t, 1\rangle=0\}$ or $=\infty$ if $\{\cdot\}$ is empty. A probability measure P on W is called a solution of the $(M(S), L, \mu)$-martingale problem up to ζ, if (i) and the following (ii)' are fulfilled,

(ii)' $F(X_{t\wedge\zeta})-\int_0^{t\wedge\zeta} LF(X_s)ds$ is an $((\boldsymbol{F}_{t\wedge\zeta}), P)$-martingale for every $F\in\boldsymbol{D}(L)$.

For $p\in M_1(S)$, $(M_1(S), \tilde{L}, p)$-martingale problem is also defined in the same fashion.

We suppose that L and $\tilde{L}$ satisfy the following conditions:

(2.3) $\quad 1\in\boldsymbol{D}$, and $A(\mu, 1)$ is bounded in $\mu\in M(S)$,

(2.4) $\quad A(\mu, \phi)-A(\mu, 1)\langle p, \phi\rangle=\tilde{A}(p, \phi)$ for every $\phi\in\boldsymbol{D}$, $\mu\in M(S)\setminus\{0\}$ and $p\in M_1(S)$ with $\mu=\langle\mu, 1\rangle p$.

Then we have

Theorem 2.1. *Assume* (2.3) *and* (2.4). *Let* $\mu\in M(S)\setminus\{0\}$. *If* P *is a solution of the* $(M(S), L, \mu)$*-martingale problem up to* ζ, *then*

(i) $\int_0^\zeta \frac{ds}{\langle X_s, 1\rangle}=+\infty$ *P-a.s., hence* $C_t=\int_0^t \frac{ds}{\langle X_s, 1\rangle}$ *defines a hommeomorph from* $[0, \zeta)$ *to* $[0, \infty)$.

(ii) *Let* $D_t: [0, \infty)\to[0, \zeta)$ *be the inverse function of* C_t, *and set* $Y_t=X_{D_t}/\langle X_{D_t}, 1\rangle$ *for* $0\leqq t<\infty$. *Then* Y_t *is an* $M_1(S)$*-valued continuous process.*

Furthermore the probability law of (Y_t, P) *is a solution of the* $(M_1(S), \tilde{L}, p)$*-martingale problem with* $p=\mu/\langle\mu, 1\rangle$.

Proof. 1°. Let $r_t=\langle X_t, 1\rangle$. Since for every $f\in C_b^2(R^1)$

$$f(\langle X_{t\wedge\zeta}, 1\rangle)-\int_0^{t\wedge\zeta}\left\{\frac{\langle X_s, \beta\rangle}{2}f''(\langle X_s, 1\rangle)+A(X_s, 1)f'(\langle X_s, 1\rangle)\right\}ds$$

is a $((\boldsymbol{F}_t), P)$-martingale, there is a $((\boldsymbol{F}_t), P)$-martingale M_t satisfying

$$r_{t\wedge\zeta}-r_0=M_{t\wedge\zeta}+\int_0^{t\wedge\zeta}A(X_s, 1)ds \qquad)2.5)$$

and the quadratic variarion process of M_t is

(2.6) $$\langle M\rangle_{t\wedge\zeta}=\int_0^{t\wedge\zeta}\langle X_s, \beta\rangle ds.$$

Set

$$\xi_t=r_{D_t} \quad\text{and}\quad N_t=\int_0^{D_t}\frac{dM_s}{r_s} \quad\text{for } 0\leqq t<C_\zeta$$

Then N_t is a local martingale with the quadratic variation process

(2.7) $$\langle N\rangle_t=\int_0^t\frac{\langle X_{D_s}, \beta\rangle}{r_{D_s}}ds$$

and ξ_t satisfies that for $0\leqq t<C_\zeta$,

(2.8) $$\xi_t-\xi_0=\int_0^t\xi_s dN_s+\int_0^t\xi_s A(X_{D_s}, 1)ds,$$

hence it holds that for $0\leqq t<C_\zeta$

(2.9) $$\xi_t=\xi_0\exp\Bigl(N_t-\frac{1}{2}\langle N\rangle_t+\int_0^t A(X_{D_s}, 1)ds\Bigr).$$

Note that $\langle N\rangle_t\leqq(\sup\beta(x))t$ by (2.7) and for some Browmian motion B_t, $N_t=B_{D_t}$, which implies that $\lim_{t\to C_\zeta}\xi_t$ exists and is positive almost surely on the event $[C_\zeta<+\infty]$, so that $\lim_{t\to\zeta}r_t$ also exists and is positive a.e. on $[C_\zeta<\infty]$. Hence by the definition of ζ we see that $\zeta=\infty$, $\lim_{t\to\infty}r_t$ exists and is positive a.e. on $[C_\zeta<\infty]$, which implies $P(C_\zeta=\infty)=1$.

2°. By the first step Y_t is well-defined for all $t\geqq 0$ as a continuous $M_1(S)$-valued process. For an $F(\mu)=f(\langle\mu, \phi_1\rangle, \cdots, \langle\mu, \phi_k\rangle)$ with $\phi_i\in\boldsymbol{D}$ and $f\in C_0^2(R^k)$, set $\tilde{F}(\mu)=F(p)$ with $p=\mu/\langle\mu, 1\rangle$ if $\mu\in M(S)\setminus\{0\}$. Then a straight-forward calculation yields

$$\begin{aligned}
&L\tilde{F}(\mu)\\
&=\frac{1}{\langle\mu, 1\rangle}\sum_{i=1}^k\sum_{j=1}^k D_iD_jf(\langle p, \phi_1\rangle, \cdots, \langle p, \phi_k\rangle)\int_S p(dx)\beta(x)(\phi_i(x)-\langle p, \phi_i\rangle)(\phi_j(x)-\langle p, \phi_j\rangle)\\
&\quad-\frac{1}{\langle\mu, 1\rangle}\sum_{i=1}^k D_if(\langle p, \phi_1\rangle, \cdots, \langle p, \phi_k\rangle)(\langle p, \beta\phi_i\rangle-\langle p, \beta\rangle\langle p, \phi_i\rangle)\\
&\quad+\frac{1}{\langle\mu, 1\rangle}\sum_{i=1}^k D_if(\langle p, \phi_1\rangle, \cdots, \langle p, \phi_k\rangle)(A(\mu, p_i)-A(\mu, 1)\langle p, \phi_i\rangle)\\
&=\frac{\tilde{L}F(p)}{\langle\mu, 1\rangle}.
\end{aligned}$$

Thus one can easily see that for every $F\in\boldsymbol{D}(\tilde{L})$

$$F(Y_t)-\int_0^t\tilde{L}F(Y_s)ds$$

is a martingale, hence the probability law of (Y_t, P) is a solution of the $(M_1(S), \tilde{L}, \mu)$-martingale problem.

Theorem 2.1 can be rephrased in the following way.

Corollary 2.2. *Let* $(\Omega, \boldsymbol{F}, \boldsymbol{F}_t, P_\mu; X_t, 0\leqq t<\zeta)$ *be an* $M(S)$*-valued diffusion process satisfying that*

(i) $\zeta=\inf\{t\geqq 0: \langle X_t, 1\rangle=0\}$ *or* $=\infty$ *if* $\{\cdot\}$ *is empty,*

(ii) $F(X_{t\wedge\zeta})-\int_0^{t\wedge\zeta}LF(X_s)ds$ *is a martingale for every* $F\in\boldsymbol{D}(L)$.

Then the process $Y_t, 0\leqq t<\infty$, *defined in Theorem* 2.1 *for* $\mu\neq 0$, *is an* $M_1(S)$*-valued diffusion process starting at* $Y_0=\mu/\langle\mu, 1\rangle$ *satisfying that*

$$F(Y_t)-\int_0^t\tilde{L}F(Y_s)ds \text{ is a martingale for every } F\in\boldsymbol{D}(\tilde{L}).$$

Thus we have obtained an $M_1(S)$-valued diffusion process governed by $\tilde{L}$ of (2.2) from an $M(S)$-valued diffusion process governed by L of (2.1) by way of a normalization and a random time change. Accordingly most of sample path properties of the $M(S)$-valued diffusion inherit those of the $M_1(S)$-valued diffusion.

Next we would like to assert that the uniqueness of solutions for the $(M_1, \tilde{L}, p)$-martingale problem is also reduced to that for the $(M(S), L, \mu)$-martingale problem up to ζ for some $\mu\in M(S)\setminus\{0\}$ with $\mu=\langle\mu, 1\rangle p$ under additional mild assumptions:

(2.10) $A(\mu, 1)$ is continuous in $\mu\in M(S)$ and for some $C>0$

$|A(\mu, 1)|\leqq C\langle\mu, 1\rangle$ for every $\mu\in M(S)$.

(2.11) $\boldsymbol{D}$ contains a countable subset $\boldsymbol{C}$ such that for every $\phi\in\boldsymbol{D}$ there is a sequence $\{\phi_n\}$ from $\boldsymbol{C}$ satisfying that ϕ_n converges to ϕ uniformly and $\lim_{n\to\infty}A(\mu, \phi_n)=A(\mu, \phi)$ for every $\mu\in M(S)$.

Theorem 2.3. *Assume* (2.10) *and* (2.11) *in addition to* (2.3) *and* (2.4). *If the uniqueness of solutions holds for the* $(M(S), L, \mu)$*-martingale problem up to* ζ *for some* $\mu\neq 0\in M(S)$, *then so does it for the* $(M_1(S), \tilde{L}, p)$*-martingale problem* $p=\mu/\langle\mu, 1\rangle$.

For the proof we first prepare several lemmas.

Lemma 2.4. *Let* $M_i(t), 1\leqq i\leqq n$, *be continuous martingales defined on a probability space* $(\Omega, \boldsymbol{F}, \boldsymbol{F}_t, P)$ *with filtration, and let* $m_{ij}(t), 0\leqq i, j\leqq n$ *be* $(\boldsymbol{F}_t)$*-adapted bounded functionals defined on* $(\Omega, \boldsymbol{F}, \boldsymbol{F}_t, P)$. *Suppose that* $\{m_{ij}(t)\}, 0\leqq i, j\leqq n$ *is a symmetric and non-negative definite* $(n+1)\times(n+1)$ *matrix for every* $t\geqq 0$ *a.s. and the quadratic variation processes of* $M_i(t), 1\leqq i\leqq n$ *are given by*

$$\langle M_i, M_j\rangle(t)=\int_0^t m_{ij}(s)ds \qquad \text{for} \quad 1\leqq i, j\leqq n.$$

Then there exist stochastic processes $M_i'(t), 0\leqq i\leqq n$ *and* $m_{ij}'(t), 0\leqq i, j\leqq n$ *defined on a probability space* $(\Omega', \boldsymbol{F}', \boldsymbol{F}_t', P')$ *with filtration such that* $M_i'(t), 0\leqq i\leqq n$ *are* $(\boldsymbol{F}_t')$*-martingales with quadratic variation processes*

$$\langle M_i', M_j'\rangle(t)=\int_0^t m_{ij}'(s)ds \qquad for \quad 0\leqq i,\ j\leqq n\,,$$

and the probability law of $\{M_i'(t),\ 1\leqq i\leqq n,\ m_{ij}'(t),\ 0\leqq i,\ j\leqq n\}$ *coincides with that of* $\{M_i(t),\ 1\leqq i\leqq n,\ m_{ij}(t),\ 0\leqq i,\ j\leqq n\}$.

Proof. 1°. Let $B_i(t)$, $0\leqq i\leqq n$ be an $(n+1)$-dimensional Brownian motion independent of $M_i(t)$, $1\leqq i\leqq n$ and $m_{ij}(t)$, $0\leqq i,\ j\leqq n$. (We may assume $B_i(t)$, $1\leqq i\leqq n$ is an $(\boldsymbol{F}_t)$-Brownian motion defined on $(\Omega, \boldsymbol{F}, \boldsymbol{F}_t, P)$.) For $\varepsilon>0$ define $a_{i,\varepsilon}(t)$, $0\leqq i\leqq n$ by

$$\sum_{j=1}^n (m_{ij}(t)+\varepsilon\delta_{ij})a_{j,\varepsilon}(t)=m_{i0}(t) \qquad \text{for} \quad 1\leqq i\leqq n\,, \tag{2.12}$$

$$\sum_{i=1}^n \sum_{j=1}^n a_{i,\varepsilon}(t)a_{j,\varepsilon}(t)(m_{ij}(t)+\varepsilon\delta_{ij})+(a_{0,\varepsilon}(t))^2=m_{00}(t)\,. \tag{2.13}$$

Since $\{m_{ij}(t)+\varepsilon\delta_{ij}\}$, $1\leqq i,\ j\leqq n$ is an invertible matrix, $a_{i,\varepsilon}(t)$, $1\leqq i\leqq n$ are uniquely determined and bounded $(\boldsymbol{F}_t)$-adapted functionals. Since $\{m_{ij}(t)\}$, $0\leqq i,\ j\leqq n$ is non-negative definite it is easy to check

$$m_{00}(t)\geqq \sum_{i=1}^n \sum_{j=1}^n a_{i,\varepsilon}(t)a_{j,\varepsilon}(t)(m_{ij}(t)+\varepsilon\delta_{ij})\,,$$

hence $a_{0,\varepsilon}(t)$ also is well-defined as an $(\boldsymbol{F}_t)$-adapted, bounded and non-negative functionals. Let define

$$M_{i,\varepsilon}(t)=M_i(t)+\sqrt{\varepsilon}B_i(t) \qquad \text{for } 1\leqq i\leqq n\,, \text{ and}$$

$$M_{0,\varepsilon}(t)=\sum_{i=1}^n \int_0^t a_{i,\varepsilon}(s)dM_{i,\varepsilon}(s)+\int_0^t a_{0,\varepsilon}(s)dB_0(s)\,.$$

By (2.12) and (2.13) $M_{i,\varepsilon}(t)$, $0\leqq i\leqq n$ are continuous martingales with quadratic variation processes

$$\langle M_{i,\varepsilon}, M_{j,\varepsilon}\rangle(t)=\int_0^t (m_{ij}(s)+\varepsilon\delta_{ij})ds \qquad \text{for} \quad 1\leqq i,\ j\leqq n$$

$$\langle M_{i,\varepsilon}, M_{0,\varepsilon}\rangle(t)=\int_0^t m_{i0}(s)ds \qquad \text{for} \quad 0\leqq i\leqq n$$

Since it is obvious that $M_{i,\varepsilon}(t)$, $1\leqq i\leqq n$ and $\langle M_{i,\varepsilon}, M_{j,\varepsilon}\rangle(t)$, $1\leqq i,\ j\leqq n$ converges to $M_i(t)$, $1\leqq i\leqq n$ and $\langle M_i, M_j\rangle(t)$, $1\leqq i,\ j\leqq n$ almost surely and that the probability laws of a family of continuous processes $\{M_{i,\varepsilon}, \langle M_{i,\varepsilon}, M_{j,\varepsilon}\rangle, 0\leqq i,\ j\leqq n\}$, $\varepsilon>0$ are tight, accordingly there exist $(\boldsymbol{F}_t')$-adapted continuous processes $\{M_i'(t), m_{ij}'(t), 0\leqq i,\ j\leqq n\}$ on a probability space $(\Omega', \boldsymbol{F}', \boldsymbol{F}_t', P')$ with filtration such that $M_i'(t)$, $0\leqq i\leqq n$ are martingales with quadratic variation processes

$$\langle M_i', M_j'\rangle(t)=\int_0^t m_{ij}'(s)ds \qquad \text{for} \quad 0\leqq i,\ j\leqq n$$

and the probability law of $\{M_i'(t),\ 1\leqq i\leqq n,\ m_{ij}'(t),\ 0\leqq i,\ j\leqq n\}$ coincides with that of $\{M_i(t),\ 1\leqq i\leqq n,\ m_{ij}(t),\ 0\leqq i,\ j\leqq n\}$, completing the proof of Lemma 2.4.

Lemma 2.5. *Let $\tilde{P}_p$ be a solution of the $(M_1(S), \tilde{L}, p)$-martingale problem for a $p\in M_1(S)$. Then there exist an $M_1(S)$-valued continuous $(\boldsymbol{F}_t)$-adapted process Y'_t and a continuous martingale M'_t on a probablity space with filtration $(\Omega', \boldsymbol{F}', \boldsymbol{F}'_t, P')$ such that the probability law of the process Y'_t coincides with $\tilde{P}_p$ and that for every $\phi\in\boldsymbol{D}$, setting*

$$(2.14)\qquad M_\phi(t)=\langle Y'_t, \phi\rangle-\langle Y'_0, \phi\rangle-\int_0^t(\tilde{A}(Y'_s, \phi)-\langle Y'_s, \beta\phi\rangle+\langle Y_s, \beta\rangle\langle Y'_s, \phi\rangle)ds,$$

$M_\phi(t)$, $\phi\in\boldsymbol{D}$ are martingales and their quadratic variation processes are

(2.15) $\langle M_\phi, M_\varphi\rangle(t)$

$$=\int_0^t(\langle Y'_s, \beta\phi\varphi\rangle-\langle Y'_s, \beta\phi\rangle\langle Y'_s, \varphi\rangle-\langle Y'_s, \phi\rangle\langle Y'_s, \beta\varphi\rangle+\langle Y'_s, \beta\rangle\langle Y'_s, \phi\rangle\langle Y'_s, \varphi\rangle)ds,$$

$$\langle M', M_\phi\rangle(t)=\int_0^t(\langle Y'_s, \beta\phi\rangle-\langle Y'_s, \beta\rangle\langle Y'_s, \phi\rangle)ds,$$

$$\langle M'\rangle(t)=\int_0^t\langle Y'_s, \beta\rangle ds.$$

Proof. For $\boldsymbol{C}=\{\phi_n, n\geqq 1\}$ of (2.11) set

$$M_n(t)=\langle Y_t, \phi_n\rangle-\langle Y_0, \phi_n\rangle-\int_0^t(\tilde{A}(Y_s, \phi_n)-\langle Y_s, \beta\phi_n\rangle+\langle Y_s, \beta\rangle\langle Y_s, \phi_n\rangle)ds.$$

Since $\tilde{P}_p$ is a solution of the $(M_1(S), \tilde{L}, p)$-martingale problem, $M_n(t)$, $n\geqq 1$ are P_p-martingales with quadratic variation processes

$\langle M_n, M_m\rangle(t)$

$$=\int_0^t(\langle Y_s, \beta\phi_n\phi_m\rangle-\langle Y_s, \beta\phi_n\rangle\langle Y_s, \phi_m\rangle-\langle Y_s, \beta\phi_m\rangle\langle Y_s, \phi_n\rangle+\langle Y_s, \beta\rangle\langle Y_s, \phi_n\rangle\langle Y_s, \phi_m\rangle)ds$$

for $n\geqq 1$ and $m\geqq 1$. Define $m_{nm}(t)$ for $n\geqq 1$, $m\geqq 1$ by

$$\langle M_n, M_m\rangle(t)=\int_0^t m_{nm}(s)ds,$$

$$m_{0n}(t)=m_{n0}(t)=\langle Y_t, \beta\phi_n\rangle-\langle Y_t, \beta\rangle\langle Y_t, \phi_n\rangle,$$

$$m_{00}(t)=\langle Y_t, \beta\rangle.$$

Then for each $N\geqq 1$, $\{M_n(t), 1\leqq n\leqq N, m_{nm}(t), 0\leqq n, m\leqq N\}$ satisfies the assumption of Lemma 2.4. Repeating the same argument as the proof of Lemma 2.4 one can show that there exists an $M_1(S)$-valued, $(\boldsymbol{F}'_t)$-adapted continuous process Y'_t and a sequence of martingales $M'_n(t)$, $n\geqq 1$ defined on a probability space $(\Omega', \boldsymbol{F}', \boldsymbol{F}'_t$ '$P')$ with filtration such that the probability law of $\{Y'_t, M'_n(t), n\geqq 1\}$ coincides with that of $\{Y_t, M_n(t)$, $n\geqq 1$ and that

(2.16)

$$\langle M'_n, M'_m\rangle(t)$$

$$=\int_0^t(\langle Y'_s, \beta\phi_n\phi_m\rangle-\langle Y'_s, \beta\phi_n\rangle\langle Y'_s, \phi_m\rangle-\langle Y'_s, \beta\phi_m\rangle\langle Y'_s, \phi_n\rangle+\langle Y'_s, \beta\rangle\langle Y'_s, \phi_n\rangle\langle Y'_s, \phi_m\rangle)ds,$$

for $n\geqq 1$ and $m\geqq 1$,

$$\langle M'_n, M'_0\rangle(t)=\int_0^t(\langle Y'_s, \beta\phi_n\rangle-\langle Y'_s, \beta\rangle\langle Y'_s, \phi_n\rangle)ds,$$

$$\langle M'_0\rangle(t)=\int_0^t\langle Y'_s, \beta\rangle ds,$$

where

$$M'_n(t)=\langle Y'_t, \phi_n\rangle-\langle Y'_0, \phi_n\rangle-\int_0^t(\tilde{A}(Y'_s, \phi_n)-\langle Y'_s, \beta\phi_n\rangle+\langle Y'_s, \beta\rangle\langle Y'_s, \phi_n\rangle)ds.$$

Denote $M'(t)=M'_0(t)$, and for $\phi\in \boldsymbol{D}$ set

$$M'_\phi(t)=\langle Y'_t, \phi\rangle-\langle Y'_0, \phi\rangle-\int_0^t(\tilde{A}(Y'_s, \phi)-\langle Y'_s, \beta\phi\rangle+\langle Y'_s, \beta\rangle\langle Y'_s, \phi\rangle)ds.$$

Then using the condition (2.11) and (2.16) we see that $M'_\phi(t)$, $\phi\in\boldsymbol{D}$ are martingales and their quadratic variation processes are the desired ones.

Lemma 2.6. *Let M_t be a continuous square-integrable martingales on a probability space $(\Omega, \boldsymbol{F}, \boldsymbol{F}_t, P)$ with filtration and let $b(x, t)=(b(x, t:\omega))$ be a $(\boldsymbol{F}_t)$-adapted precesses that are bounded and continuous functional in $(x, t)\in R^1\times[0, \infty)$ a.s.. Consider the following stochastic differential equation*:

$$dx_t=x_t dM_t+b(x_t, t)dt. \tag{2.17}$$

Then for every $x\in R^1$ there are stochastic processes x'_t, M'_t and $b'(x, t)$ on a probability space $(\Omega', \boldsymbol{F}', \boldsymbol{F}'_t, P')$ with filtration such that

(i) *$x'_t(\omega')$ is an $(\boldsymbol{F}'_t)$-adapted continuous process,*

(ii) *$M'_t(\omega')$ and $b'(x, t, \omega')$ are $(\boldsymbol{F}'_t)$-adapted, and the probability law of $(M'_t(\omega'), b'(x, t, \omega'))$ coincides with that of $(M_t(\omega), b(x, t, \omega))$, and*

(iii) $x'_t=x+\int_0^t x'_s dM'_s+\int_0^t b'(x'_s, s)ds$

holds almost surely w.r.t. P'.

Since the proof is essentially the same as in the classical case of Ito's stochastic differential equations, we will omit it.

Proof of Theorem 2.3. Let $\tilde{P}_p$ be a solution of the $(M_1(S), \tilde{L}, p)$-martingale problem. By Lemma 2.5 we have an $M_1(S)$-valued continuous and $(\boldsymbol{F}'_t)$-adapted process Y'_t and a continuous martingale M'_t defined on a probability space $(\Omega', \boldsymbol{F}', \boldsymbol{F}'_t, P)$ with filtration such that the probability law of Y'_t coincides with $\tilde{P}_p$, and (2.14) and (2.15) hold. Furthermore, by Lemma 2.6 we may assume that there is a continuous $(\boldsymbol{F}'_t)$-adapted process z_t defined on $(\Omega', \boldsymbol{F}', \boldsymbol{F}'_t, P')$ satisfying

$$dz_t = z_t dM'_t + z_t A(z_t Y'_t, 1)dt \tag{2.18}$$

$$z_0 = \langle \mu, 1 \rangle .$$

We note that $z_t > 0$ for all $t \geqq 0$ almost surely, because z_t satisfies

$$z_t = z_0 \exp\left\{M'_t - \frac{1}{2}\langle M' \rangle_t + \int_0^t A(z_s Y'_s, 1)ds\right\}. \tag{2.19}$$

Noting that $D_t = \int_0^t z_s ds$ is a hommeomorph from $[0, \infty)$ to $[0, D_\infty)$, we define

$$X_t = z_{E_t} Y'_{E_t} \quad \text{for} \quad 0 \leqq t < D_\infty . \tag{2.20}$$

where E_t denotes the inverse function of D_t. Using the Ito formula together with (2.18), (2.14), (2.15) and (2.4) we see

$$\begin{aligned} &z_t \langle Y'_t, \phi \rangle \\ &= z_0 \langle Y'_0, \phi \rangle + \int_0^t z_s \langle Y'_s, \phi \rangle dM'_s + \int_0^t z_s dM_\phi(s) + \int_0^t z_s A(z_s Y'_s, \phi) ds . \end{aligned} \tag{2.21}$$

Here recall that $M_\phi(t)$ is defined by (2.14). Hence it follows from (2.20) and (2.21) that for every $F \in \boldsymbol{D}(L)$ and $r > 0$

$$F(X_{t \wedge D_r}) - \int_0^{t \wedge D_r} LF(X_s) ds$$

is a martingale, in particular for every $f \in C_0^2(R^1)$

$$f(\langle X_{t \wedge D_r}, 1 \rangle) - \int_0^{t \wedge D_r} (f'(\langle X_s, 1 \rangle) A(X_s, 1) + \frac{1}{2} f''(\langle X_s, 1 \rangle) \langle X_s, \beta \rangle) ds$$

is a martingale. Accordingly one can construct a standard Brownian motion B_t such that for every $t > 0$ and $r > 0$

$$\langle X_{t \wedge D_r}, 1 \rangle - \langle \mu, 1 \rangle = \int_0^{t \wedge D_r} \sqrt{\langle X_s, \beta \rangle} dB_s + \int_0^{t \wedge D_r} A(X_s, 1) ds . \tag{2.22}$$

Using (2.10) we have

$$E'(\langle X_{t \wedge D_r}, 1 \rangle) \leqq \langle \mu, 1 \rangle e^{Ct} \quad \text{for every } t \geqq 0 \text{ and } r \geqq 0,$$

hence for any fixed $t > 0$, $M_r = \int_0^{t \wedge D_r} \sqrt{\langle X_s, \beta \rangle} dB_s$ is a uniformly integrable, so that $\lim_{r \to \infty} M_r$ exists a.s. for all $t \geqq 0$. This implies that $\lim_{t \to D_\infty} \langle X_t, 1 \rangle$ exists a.s. on the event $[D_\infty < \infty]$. Since

$$\lim_{t \to D_\infty} \langle X_t, 1 \rangle = \lim_{t \to \infty} z_t \quad \text{and} \quad D_\infty = \int_0^\infty z_s ds ,$$

$\lim_{t \to D_\infty} \langle X_t, 1 \rangle = 0$ holds a.s. on $[D_\infty < \infty]$.

Hence $D_\infty=\zeta$ holds a.s., therefore the probability law of $(X_t, 0\leqq t<\zeta)$ is a solution of the $(M(S), L, \mu)$-martingale problem up to ζ, which is uniquely determined by the assumption. Let define C_t by

$$C_t=\int_0^t \frac{ds}{\langle X_s, 1\rangle} \quad \text{for} \quad 0\leqq t<\zeta.$$

By Theorem 2.1, C_t is a hommeomorph from $[0, \zeta)$ to $[0, \infty)$, and as easily seen, the inverse function coincides with D_t. Thus we see

$$Y'_t=\frac{X_{D_t}}{\langle X_{D_t}, 1\rangle} \quad \text{for all} \quad t\geqq 0.$$

Therefore the probability law of $(Y'_t, 0\leqq t<\infty)$ is uniquely determined, which completes the proof of Theorem 2.3.

§3. Poisson construction of a class of MBDI processes

In this section we will construct a class of measure-valued branching diffusions with immigrations generated by L of (3.1) by making use of a Poisson system associated with excursion laws of continuous state braching diffusions on $[0, \infty)$ following the method in [10]:

$$LF(\mu)=\frac{1}{2}\int_S \mu(dx)\beta(x)\frac{\delta^2 F(\mu)}{\delta\mu(x)^2}+\int_S(\mu(dx)\gamma(x)+V(dx))\frac{\delta F(\mu)}{\delta\mu(x)} \tag{3.1}$$

where $\beta(x)$ is a bounded, uniformly positive and measurable function defined on S, $\gamma(x)$ is a bounded measurable function defined on S, and V is a bunded measure on S.

We first introduce excursion laws of continuous state branching diffusions on $[0, \infty)$. For each $\beta>0$ and $\gamma\in R$, $p_t^{\beta,\gamma}(y, dz)$ denotes the transition probability of a diffusion process $(\Omega, \boldsymbol{F}, \boldsymbol{F}_t, p_y^{\beta,\gamma}, y_t)$ on $[0, \infty)$ generated by

$$A^{\beta,\gamma}=\frac{\beta y}{2}\frac{d^2}{dy^2}+\gamma y\frac{d}{dy} \quad \text{with } 0 \text{ as a trap.} \tag{3.2}$$

Such a diffusion process is called a continuous state branching diffusion (abbrev. CB-diffusion).

An entrance law for the diffusion process is given by

$$\lambda_t^{\beta,\gamma}(dz)=e^{\gamma t}C_t^2\exp(-zC_t) \quad \text{for} \quad z>0 \tag{3.3}$$

$$\lambda_t^{\beta,\gamma}(\{0\})=+\infty.$$

$$C_t=2\gamma/\beta(e^{\gamma t}-1) \quad \text{if} \quad \gamma\neq 0 \tag{3.4}$$

$$=2/\beta t \quad \text{if} \quad \gamma=0.$$

Then $(\lambda_t^{\beta,\gamma})_{t>0}$ satisfies

$$\int_{[0,\infty)}\lambda_s^{\beta,\gamma}(dy)p_t^{\beta,\gamma}(y, dz)=\lambda_{t+s}^{\beta,\gamma}(dz) \quad \text{for} \quad t>0 \text{ and } s>0. \tag{3.5}$$

Such facts can be easily verified using the following expressions:

(3.6) $$\int_{[0,\infty)} e^{-az} p_t^{\beta,\gamma}(y, dz) = \exp(-y\Psi^{\beta,\gamma}(t, a)) \quad \text{for} \quad a \geqq 0,$$

(3.7) $$\Psi^{\beta,\gamma}(t, a) = e^{\gamma t} a C_t/(a + C_t).$$

(See [10] for the details.)

Denoting by W_+ the space of continuous fnuctions $w: [0, \infty) \to [0, \infty)$ satisfying

(i) $w(0)=0$, $\sigma(w)=\inf\{t>0: w(t)=0\}>0$,

(ii) $w(t)=0$ for $t \geqq \sigma(w)$, if $\sigma(w)<\infty$.

We denote by $\boldsymbol{B}(W_+)(\boldsymbol{B}_t(W_+))$ the σ-field generated by cylindrical subsets of W_+ (up to time t). (For a general topological space X, we also denote by $\boldsymbol{B}(X)$ the topological Borel field.) Then there is a unique σ-finite measure $Q^{\beta,\gamma}$ on W_+ such that for every $n \geqq 1$, $0<t_1<t_2<\cdots<t_n$, and $E_1 \in \boldsymbol{B}[0, \infty)$, $\cdots$, $E_n \in \boldsymbol{B}[0, \infty)$

$$Q^{\beta,\gamma}(w: w(t_1) \in E_1, w(t_2) \in E_2, \cdots, w(t_n) \in E_n)$$
$$= \int_{E_1 \times \cdots \times E_n} \lambda_{t_1}^{\beta,\gamma}(dy_1) p_{t_2-t_1}^{\beta,\gamma}(y_1, dy_2) \cdots p_{t_n-t_{n-1}}^{\beta,\gamma}(y_{n-1}, dy_n).$$

The following is a direct consequence from (3.4) and (3.6).

Lemma 3.1. *Let* $0<s<t$ *and let* $\Phi_s(w)$ *be a bounded* $\boldsymbol{B}_s(W_+)$*-measurable function on* W_+. *Then*

(i) $\int_{W_+} Q^{\beta,\gamma}(dw) w(t)^n = e^{\gamma t} n! C_t^{-n+1}$ *for* $n \geqq 1$,

(ii) $\int_{W_+} Q^{\beta,\gamma}(dw)\Big(w(t) - \gamma\int_0^t w(r)dr\Big) = 1$,

(iii) $\int_{W_+} Q^{\beta,\gamma}(dw)\Big(w(t) - w(s) - \gamma\int_s^t w(r)dr\Big)\Phi_s(w) = 0$,

(iv) $\int_{W_+} Q^{\beta,\gamma}(dw)\Big(w(t) - w(s) - \gamma\int_s^t w(r)dr\Big)^2 \Phi_s(w) = \int_{W_+} Q^{\beta,\gamma}(dw)\Big(\beta\int_s^t w(r)dr\Big)\Phi_s(w)$.

In order to construct an MBDI process generated by L of (3.1) we fix an initial point $X_0 = \mu \in M(S)$. Let $N^\mu(dxdw)$ be a Poisson random measure on $S \times W_+$ with intensity measure

(3.8) $$\mu Q(dxdw) = \mu(dx) Q^{\beta(x),\gamma(x)}(dw).$$

(See e.g. [5] p. 42 for the definition of the Poisson random measure.)

Let define an $M(S)$-valued process X_t^0 by

(3.9) $$X_t^0(dx) = \int_{W_+} w(t) N^\mu(dxdw) \quad \text{for} \quad t>0.$$

We will first show that X_t^0 is an $M(S)$-valued diffusion process starting at μ generated by

$$(3.10)\qquad L^0F(\mu)=\frac{1}{2}\int_S\mu(dx)\beta(x)\frac{\delta^2F(\mu)}{\delta\mu(x)^2}+\int_S\mu(dx)\gamma(x)\frac{\delta F(\mu)}{\delta\mu(x)}.$$

We will often use the following basic properties of Poisson random measures.

Lemma 3.2. *Let $(S, \boldsymbol{B}(S), \Lambda)$ be a σ-finite meausre space, and let N be a Poisson random measure on $(S, \boldsymbol{B}(S))$ with intensity measure Λ defined on a probability space $(\Omega, \boldsymbol{F}, P)$. We denote $\tilde{N}=N-\Lambda$ Then for non-negative measurable functions Φ and Ψ defined on S*

(i) $E(\exp(-\langle N, \Phi\rangle))=\exp(-\langle\Lambda, 1-e^{-\Phi}\rangle)$,

(ii) $E(\langle N, \Psi\rangle\exp(-\langle N, \Phi\rangle))=\langle\Lambda, \Psi e^{-\Phi}\rangle E(\exp(-\langle N, \Phi\rangle))$,

(iii) $E(\langle N, \Psi\rangle^2\exp(-\langle N, \Phi\rangle))=(\langle\Lambda, \Psi^2e^{-\Phi}\rangle+\langle\Lambda, \Psi e^{-\Phi}\rangle)E(\exp(-\langle N, \Phi\rangle))$,

(iv) $E\langle\tilde{N}, \Phi\rangle^4=\langle\Lambda, \Phi^4\rangle+3\langle\Lambda, \Phi^2\rangle^2$.

Denote by $\boldsymbol{F}_t^0$ the σ-field generated by $\{N^\mu(E\times F): E\in\boldsymbol{B}(S), F\in\boldsymbol{B}_t(W_+)\}$ for each $t\geqq 0$. Then

Lemma 3.3. (i) *The support of X_t^0 is a finite set for all $t>0$, X_t^0 is continuous in $t>0$ in the total variation norm, and X_t^0 converges to μ weakly as $t\to 0$ P-a.s.*

(ii) *For $\phi\in C_b(S)$ define $M_t^0(\phi)$ by*

$$\langle X_t^0, \phi\rangle=\langle\mu, \phi\rangle+M_t^0(\phi)+\int_0^t\langle X_s^0, \gamma\phi\rangle ds.$$

Then $M_t^0(\phi)$ is an $(\boldsymbol{F}_t^0)$-martingale with quadratic variation process

$$\langle M^0(\phi)\rangle_t=\int_0^t\langle X_s^0, \beta\phi^2\rangle ds.$$

Proof. By (3.3) and Lemma 3.2

$$E(N\{(x. w): w(t)>0\})=\int_S\mu(dx)\lambda_t^{\beta(x),\gamma(x)}((0, \infty))<\infty,$$

hence the support of X_t^0 is a finite set and is non-increasing in $t>0$ P-a.s. Also, for $\phi\geqq 0\in C_b(S)$

$$E(\exp(-\langle X_t^0, \phi\rangle))=\exp\Big(-\int_S\mu(dx)Q^{\beta(x),\gamma(x)}(dw)(1-e^{-\phi(x)w(t)})\Big)$$

$$\longrightarrow\exp(-\langle\mu, \phi\rangle)\qquad\text{as}\quad t\to 0,$$

which yields (i).

For (ii) let $0<r<t$. Note that $\boldsymbol{F}_t^0$ is generated by the following form of functionals;

$$H_r(w)=\exp\Big(-\int_S\phi_r(x, w)N^\mu(dxdw)\Big)$$

where $\phi_r(x, w): S\times W_+\to[0, \infty)$ is $\boldsymbol{B}(S)\times\boldsymbol{B}_r(W_+)$-measurable for each $r\geqq 0$. Using Lemma 3.2, Lemma 3.1 and

$$(3.11)\qquad M_t^0(\phi)-M_r^0(\phi)=\int_{S\times W_+}(w(t)-w(r)-\gamma(x)\int_r^t w(s)ds)\phi(x)N^\mu(dxdw),$$

we have

$$E((M_t^0(\phi)-M_r^0(\phi))H_r)$$
$$=\langle\mu Q,\ (w(t)-w(r)-\int_r^t\gamma(x)w(s)ds)\phi(x)\exp(-\Phi_r(x,\ w))\rangle E(H_r)$$

and

$$E((M_t^0(\phi)-M_r^0(\phi))^2H_r)$$
$$=\langle\mu Q,\ (w(t)-w(r)-\int_r^t\gamma(x)w(s)ds)^2\phi(x)^2\exp(-\Phi_r(x,\ w))\rangle E(H_r)$$
$$+\langle\mu Q,\ (w(t)-w(r)-\int_r^t\gamma(x)w(s)ds)\phi(x)\exp(-\Phi_r(x,\ w))\rangle^2 E(H_r)$$
$$=\langle\mu Q,\ \int_r^t\beta(x)w(s)ds\phi(x)^2\exp(-\Phi_r(x,\ w))\rangle E(H_r)$$
$$=E\Big(\int_r^t\langle X_s^0,\ \beta\phi^2\rangle dsH_r\Big)$$

Therefore, $M_t^0(\phi)$ is an $(\boldsymbol{F}_t^0)$-martingale and its quadratic variation process is of the desired form.

Corollary 3.4. *X_t^0 is an $M(S)$-valued diffusion process starting at μ such that for every $F\in\boldsymbol{D}(L^0)=\boldsymbol{D}(L)$,*

$$F(X_t^0)-\int_0^t L^0F(X_s^0)ds$$

is an $(\boldsymbol{F}_t^0)$-martingale.

Proof. The martingale property follows from Lemma 3.3. Also, it is known that the uniqueness of solutions holds for the $(M(S),\ L^0,\ \mu)$-martingale problem (see e.g. [8], Appendix.) Hence X_t^0 is a diffusion process.

In order to incorporate an immigration factor to the MBD process we prepare another Poisson system independent of $N^\mu(dxdw)$.

Let $N_p(dtdxdw)$ be a Poisson point process on $S\times W_+$ with characteristic measure

$$VQ(dxdw)=V(dx)Q^{\beta(x),\gamma(x)}(dw),$$

(see [5] for the definition of Poisson point process), in other words, it is a Poisson random measure on $[0,\ \infty)\times S\times W_+$ with intensity measure $dtVQ(dxdw)$. Note that we are assuming the independence of $N^\mu(dxdw)$ and $N_p(dtdxdw)$.

Let define an $M(S)$-valued process X_t by

$$(3.12)\quad X_t(dx)=\int_{W_+}w(t)N^\mu(dxdw)+\int_{(0,\,t]\times W_+}w(t-s)N_p(dsdxdw)\qquad\text{for every } t>0.$$

The meaning of this expression would be intuitively clear. In the MBD process generated by L^0 of (3.10) each mass at a point x of S fluctuates according to an excursion path of the CB-diffusion with coefficients $(\beta(x), \gamma(x))$. Moreover immigrants enter the space S at random times. Then the spatial distribution of immigrants follows $V(dx)$. After that the mass of each immigrant fluctuates according to an excursion path of a CB-diffusion with coefficients depending on the place where it immigrates.

Denote by X_t^0 and X_t^1 the first and the second term of the right hand side of (3.12) respectively. It is easy to see that the support of X_t^1 is a countable set for all $t>0$ a.s., and $X_t^1(E)>0$ a.e. holds whenever $V(E)>0$ for every fixed $t>0$.

Let $\boldsymbol{F}_t^1$ be the σ-field generated by such events

$$\{N_p(I\times E\times F): I\in\boldsymbol{B}[0, r],\ E\in\boldsymbol{B}(S),\ F\in\boldsymbol{B}_{t-r}(W_+),\ 0\leqq r\leqq t\}.$$

Then we have

Lemma 3.5. *For $\phi\in C_b(S)$ define $M_t^1(\phi)$ by*

$$\langle X_t^1, \phi\rangle=M_t^1(\phi)+\int_0^t(\langle X_s^1, \gamma\phi\rangle+\langle V, \phi\rangle)ds.$$

Then $M_t^1(\phi)$ is an $(\boldsymbol{F}_t^1)$-martingale with quadratic variation process

$$\langle M^1(\phi)\rangle_t=\int_0^t(\langle X_s^1, \beta\phi^2\rangle ds.$$

Proof. By Lemma 3.1

$$M_t^1(\phi)=\int_{(0,t]\times S\times W_+}(w(t-s)-\int_0^{t-s}\gamma(x)w(r)dr)\phi(x)\tilde{N}_p(dsdxdw)$$

where

$$\tilde{N}_p(dsdxdw)=N_p(dsdxdw)-dsVQ(dxdw).$$

Let $0<r<t$, and let

$$H_r(w)=\exp\Big(-\int_{(0,r]\times S\times W_+}w(r-s)\phi(x)N_p(dsdxdw)\Big).$$

For covenience we use the convention: $w(t)=0$ for $t\leqq 0$ for $w\in W_+$. Using Lemma 3.2 and Lemma 3.1 one can easily see

$$\begin{aligned}&E((M_t^1(\phi)-M_r^1(\phi))H_r)\\&=E\Big(\int_{(0,t]\times S\times W_+}(w(t-s)-w(r-s)-\int_r^t\gamma(x)w(v-s)dv)\phi(x)\tilde{N}_p(dsdxdw)H_r)\\&=\int_{(0,t]\times S\times W_+}dsV(dx)Q^{\beta(x),\gamma(x)}(dw)(w(t-s)-w(r-s)\\&\quad-\int_r^t\gamma(x)w(v-s)dv)\phi(x)(e^{-w(r-s)\phi(x)}-1)E(H_r)\\&=0.\end{aligned}$$

Since the quadratic variation process can be calculated in the same way, it is omitted.

Denote by $\boldsymbol{F}_t$ the σ-field generated by $\boldsymbol{F}_t^0$ and $\boldsymbol{F}_t^1$, i.e. $\boldsymbol{F}_t=\boldsymbol{F}_t^0\vee\boldsymbol{F}_t^1$.

Theorem 3.6. (i) *X_t is an $M(S)$-valued diffusion process starting at μ satisfying that for every $F\in\boldsymbol{D}(L)$*

$$F(X_t)-\int_0^t LF(X_s)ds$$

is an $(\boldsymbol{F}_t)$-martingale, hence X_t is equivalent to an MBDI *process starting at μ, which is uniquely determined by $(L, \boldsymbol{D}(L))$ of* (3.1)

(ii) *X_t is continuous in $t>0$ in the total variation norm, and $X_t\to\mu$ weakly as $t\to 0$ P-almost surely.*

Proof. (i) The martingale property follows from Lemma 3.3 and Lemma 3.5. Using the function $\varphi^{\beta,\gamma}(t, a)$ of (3.7) we define $u_t(x)=\varphi^{\beta(x),\gamma(x)}(t, \phi(x))$. Then $u_t(x)$ satisfies

$$\frac{\partial u_t(x)}{\partial t}=\gamma(x)u_t(x)-\frac{\beta(x)}{2}u_t(x)^2 \quad\text{and}\quad u_0(x)=\phi(x).$$

Then it is easy to see that if $0\leq r<t$

$$E(\exp(-\langle X_t, \phi\rangle)|\boldsymbol{F}_r)=\exp(-\langle X_r, u_{t-r}\rangle-\int_0^{t-r}\langle V, u_s\rangle ds),$$

which implies that the probability law of X_t is the unique solution for the $(M(S), L, \mu)$-martingale problem. Hence X_t is an $M(S)$-valued diffusion process. (ii) will be proved in Theorem 4.1 of §4.

4. A stochastic equation, I

Let us consider the following operator L acting on $\boldsymbol{D}(L)$ of (1.2):

$$LF(\mu)=\frac{1}{2}\int_S\mu(dx)\beta(x)\frac{\delta^2F(\mu)}{\delta\mu(x)^2}+\int_S(\mu(dx)\gamma(x)+V(\mu, dx))\frac{\delta F(\mu)}{\delta\mu(x)} \tag{4.1}$$

where $\beta(x)$ and $\gamma(x)$ are the same as in (3.1), and $V(\mu, dx)$ is a measure kernel on $M(S)\times\boldsymbol{B}(S)$, that is

(i) for each $\mu\in M(S)$, $V(\mu, \cdot)\in M(S)$, and

(ii) for each $E\in\boldsymbol{B}(S)$, $V(\cdot, E)$ is measurable in μ.

When $V(\mu, dx)$ is independent of $\mu\in M(S)$, we in §3 constructed an $M(S)$-valued diffusion process governed by L by making use of a Poisson system associated with excursion laws of CB-diffusions. Developing this idea we would like to formulate a stochastic equation describing an $M(S)$-valued diffusion process governed by L of (4.1).

Biologically, the resultant diffusion process should be called *a generalized MBDI process*, in which the immigration distribution depends on the present configuration of population.

In this section we restrict ourselves into the following case:

(4.2) $\beta(x)=\beta$ *and* $\gamma(x)=\gamma$ *are constant functions.*

For simplicity we also assume

(4.3) $V(\mu, S)$ *is bounded in* $\mu\in M(S)$.

So, we will drop the superscripts β and γ, i.e. $Q^{\beta,\gamma}=Q$, $\cdots$ etc.

Since $V(\mu, dx)$ corresponds to a spatial distribution of new immigrants and it depends on the present state μ, we prepare an auxiliary function $A(\mu, u)$ with distribution $V(\mu, \cdot)$ just as in the theory of stochastic differential equations for jump type Markov processes as follows.

Let $(U, \boldsymbol{B}(U), m)$ be a σ-finite measure space, and we add an isolated point Δ to S. Let $A(\mu, u)$: $M(S)\times U\rightarrow S\cup\{\Delta\}$ be a measurable mapping satisfying that for $\mu\in M(S)$

$$\int_U \phi(A(\mu, u))m(du)=\int_S V(\mu, dx)\phi(x) \tag{4.4}$$

for every bounded measurable function ϕ with $\phi(\Delta)=0$.

Suppose that we are now given an MBD process X_t^0 generated by L^0 of (3.10) under the assumption (4.2), and a Poisson point process $N_p(dtdudw)$ on $U\times W_+$ with characteristic measure $m\times Q$ on a common probability space $(\Omega, \boldsymbol{F}, P)$ such that X_t^0 and N_p are independent. Here the initial state may be chosen arbitrarily unless specified.

Let $\boldsymbol{F}^0$ and $\boldsymbol{F}_t^0$ be the σ-fields generated by $\{X_t^0: t\geqq 0\}$ and $\{X_s^0: 0\leqq s\leqq t\}$ respectively, and let $\boldsymbol{F}^1$ and $\boldsymbol{F}_t^1$ be the σ-fields generated by N_p and $\{N_p(I\times A\times F): I\in\boldsymbol{B}[0, r], A\in\boldsymbol{B}(U), F\in\boldsymbol{B}_{t-r}(W_+), 0\leqq r\leqq t\}$ respectively. We denote $\boldsymbol{F}_t=\boldsymbol{F}_t^0\wedge\boldsymbol{F}_t^1$, the σ-field generated by $\boldsymbol{F}_t^0$ and $\boldsymbol{F}_t^1$.

Based on $\{X_t^0, N_p\}$ let us consider the following stochastic equation:

$$X_t=X_t^0+\int_{(0,t]\times U\times W_+} w(t-s)I_S\delta_{A(X_s,u)}N_p(dsdudw). \tag{4.5}$$

Here δ_x stands for the Dirac measure at $x\in S$, I_A is the characteristic function the set A, and for a funcion ϕ on $S\cup\{\Delta\}$ we use the notation $(\phi\delta_a)(dx)=\phi(x)\delta_a(dx)$, so that for $\phi\in C_b(S\cup\{\Delta\})$ with $\phi\{\Delta\}=0$,

$$\langle X_t, \phi\rangle=\langle X_t^0, \phi\rangle+\int_{(0,t]\times U\times W_+} w(t-s)\phi(A(X_s, u))N_p(dsdudw).$$

The equation (4.5) clearly is a generalization of (3.12) under the condition (4.2), in which $A(X_s, u)$ shows the location where a new immigrant enters depending on the present situation X_s.

An $M(S)$-valued stochastic process X_t defined on $(\Omega, \boldsymbol{F}, P)$ is a solution of the equation (4.5) if

(i) $X_t(\omega)$ is jointly measurable in (t, ω) and $(\boldsymbol{F}_t)$-adapted, and

(ii) the equation (4.5) holds for all $t\geqq 0$ P-a.s.

We first prove some regularity of solutions of the equation (4.5). Recall that $M^a(S)$ is the set of atomic measure on S, namely, every element μ of $M^a(S)$ has a countable subset of S as its support. Then we have

Theorem 4.1. *Let X_t be a solution of the equation* (4.5) *under the conditions* (4.2)-(4.4). *Then*

(i) $X_t \in M^{\alpha}(S)$ *for all* $t>0$, *P-a.s.*

(ii) X_t *is continuous in* $t>0$ *in the total variation norm P-a.s.*

(iii) X_t *converges to* X_0 *weakly as* $t \to 0$ *P-a.s.*

Proof. 1°. Recall $Q=Q^{\beta,\gamma}$. We first claim that

$$Y_t=\int_{(0,t]\times W_+} w(t-s)N_p(ds, U, dw)$$

is continuous in $t \geqq 0$ with $Y_0=0$, P-a.s. Recalling that $w(t)=0$ for $t \leqq 0$ and using Lemma 3.1 one can show that for every $k \geqq 1$

$$\left|\int_{[0,\infty)\times W_+} (w(t-r)-w(s-r))^k dr Q(dw)\right| \leqq C_k(|t-s|^k+|t-s|^{k/2}) \tag{4.6}$$

for every $t \geqq 0$ and $s \geqq 0$, where C_k is a constant. Since

$$E(Y_t-Y_s)^4 \leqq 8E\left(\int (w(t-r)-w(s-r))\tilde{N}_p(dr, U, dw)\right)^4$$
$$+8\left(\int (w(t-r)-w(s-r))dr m(du)Q(dw)\right)^4,$$

by Lemma 3.2 and (4.6) we have a constant $C>0$ such that

$$E(Y_t-Y_S)^4 \leqq C(|t-s|^2+|t-s|^4) \quad \text{for every } t \geqq 0 \text{ and } s \geqq 0. \tag{4.7}$$

Accordingly, by Kolmogorov's theorem there is a continuous process Y'_t such that $Y_t=Y'_t$, P-a.s. for all $t \geqq 0$, hence it holds for every rational $t \geqq 0$, P-a.s. Noting further that Y_t is lower semi-continuous, we see

$$Y'_t \geqq Y_t \text{ holds for all } t \geqq 0. \tag{4.8}$$

Let F_n be an increasing sequence of subsets of W_+ satisfying $Q(F_n)<\infty$ and $\bigcup_{n \geqq 1} F_n = W_+$, and set

$$Y_t^n=\int_{(0,t]\times F_n} w(t-s)N_p(ds, U, dw).$$

Clearly Y_t^n is a continuous process, and the above calculation yields that for some constant $C>0$

$$E(Y_t^n-Y_s^n)^4 \leqq C(|t-s|^2+|t-s|^4) \tag{4.9}$$
$$\text{for every } t \geqq 0,\ s \geqq 0 \text{ and } n \geqq 1.$$

Accordingly, by Kolmogorov's theorem the distributions of $(Y_s^n, t \geqq 0)$ are tight, hence so are those of $(Y'_t-Y_t^n, t \geqq 0)$. Furthermore, it is obvious that any finite dimensional distribution of $(Y'_t-Y_t^n, t \geqq 0)$ converges to the Dirac measure at the origin, from which it follows

$$\lim_{n\to\infty} P(\sup_{0\leq t\leq T} |Y'_t - Y^n_t| \geqq \varepsilon) = 0 \quad \text{for every } \varepsilon>0 \text{ and } T>0. \tag{4.10}$$

By (4.8) and the definition of Y^n_t

$$\sup_{0\leq t\leq T} |Y'_t - Y_t| \leqq \sup_{0\leq t\leq T} |Y'_t - Y^n_t|,$$

therefore from (4.10) it follows that $Y'_t = Y_t$ for all $t\geqq 0$ P-a. s.

2°. Denote

$$X^1_t = \int_{(0,t]\times U\times W_+} w(t-s) I_S \delta_{A(X_s,u)} N_p(dsdudw). \tag{4.11}$$

Clearly

$$\|X^1_t - X^1_r\|_{var} \leqq \int_{(0,t]\times W_+} |w(t-s) - w(r-s)| N_p(ds, U, dw),$$

where $\|\cdot\|_{var}$ stands for the total variation norm. Let us denote $f_t(s, w) = w(t-s)$. Since Y_t is continuous in $t\geqq 0$ P-a. s.,

$$\lim_{t\to r} \int_{(0,\infty]\times W_+} f_t(s, w) N_p(ds, U, dw) = \int_{(0,\infty]\times W_+} f_r(s, w) N_p(ds, U, dw).$$

Also, $f_t(s, w)\geqq 0$ and $\lim_{t\to r} f_t(s, w) = f_r(s, w)$ for every (s, w), which implies that $\{f_t(s, w)\}_{t\to r}$ is uniformly integrable with respect to $N_p(ds, U, dw)$. Hence it holds

$$\lim_{t\to r} \int_{(0,\infty]\times W_+} |f_t(s, w) - f_r(s, w)| N_p(ds, U, dw) = 0.$$

Thus we have shown $\lim_{t\to r} \|X^1_t - X^1_r\|_{var} = 0$ for all $r\geqq 0$, P-a. s.

3°. By Lemma 3.3 and Corollary 3.4 we know that X^0_t satisfies the desired properties. Therefore, $X_t = X^0_t + X^1_t$ also satisfies (i), (ii) and (iii).

We next assert that every solution of the stochastic equation (4.5) solves the $(M(S), L)$-martingale problem.

Let $\Phi(t, u, \omega)$: $[0, \infty)\times U\times\Omega\to R$ be jointly measurable. Φ is called $(\boldsymbol{F}_t)$-predictable if for every $u\in U$, $\Phi(\cdot, u, \cdot)$ is an $(\boldsymbol{F}_t)$-predictable process, (see [5], p. 21). Note that any $(\boldsymbol{F}_t)$-predictable functional $\Phi(t, u, \omega)$ can be approximated by such functionals

$$\sum_{i=1}^{k} \Phi_{s_i}(\omega) I_{A_i}(u) I_{(s_i, s_{i+1}]}(t)$$

where $A_i\in \boldsymbol{B}(U)$ and $\Phi_{s_i}(\omega)$ is an $\boldsymbol{F}_{s_i}$-measurable random variable.

Lemma 4.2. *For a bounded $(\boldsymbol{F}_t)$-predictable funcional $\Phi(t, u, \omega)$ set*

$$Y_t = \int_{(0,t]\times U\times W_+} w(t-s) \Phi(s, u, \cdot) N_p(dsdudw). \tag{4.12}$$

Define M_t by

(4.13)
$$Y_t=M_t+\gamma\int_0^t Y_s ds+\int_0^t\Big(\int_U m(du)\Phi(s, u, \cdot)\Big)ds .$$

Then M_t is an $(\boldsymbol{F}_t)$-martingale with quadratic variation process

(4.14)
$$\langle M_t\rangle=\beta\int_0^t\Big(\int_{(0, r]\times U\times W_+} w(r-s)\Phi(s, u, \cdot)^2\mathbf{N}_p(dsdudw)\Big)dr .$$

Proof. 1°. We will first show this assuming $\Phi(t, u, \omega)=I_A(u)I_{(a,b]}(s)$ with $A\in \boldsymbol{B}_a(U)$ and $0\leqq a<b$. Using Lemma 3.1(ii), (4.12) and (4.13) we see

(4.15)
$$M_t=\int_{(0, t]\times U\times W_+}\Big(w(t-s)-\gamma\int_s^t w(r-s)dr\Big)\tilde{N}_p(dsdudw).$$

For $0<r<t$ let

$$H_r=\exp\Big(-\int_{(0, r]\times U\times W_+} f(r-s, u, w)\tilde{N}_p(dsdudw)\Big),$$

where $f(t, u, w)$: $[0, \infty)\times U\times W_+\to[0, \infty)$ is a jointly measurable function and for each $t\geqq 0$, $f(t, \cdot, \cdot)$ is $\boldsymbol{B}(U)\times\boldsymbol{B}_t(W_+)$-measurable. Note that such random variables generates the σ-field $\boldsymbol{F}_t^1$. Then by Lemma 3.2 and 3.1 we see

$$\begin{aligned}
&E((M_t-M_r)H_r)\\
&=E\Big(\int_{(0, t]\times U\times W_+}\Big(w(t-s)-w(r-s)-\gamma\int_r^t w(v-s)dv\Big)I_A(u)I_{(a,b]}(s)\tilde{N}_p(dsdudw)\\
&\quad\cdot\exp\Big(-\int_{(0, r]\times U\times W_+} f(r-s, u, w)N_p(dsdudw)\Big)\Big)\\
&=\int_{(0, t]\times U\times W_+} dsm(du)Q(dw)\Big(w(t-s)-w(r-s)-\gamma\int_r^t w(v-s)dv\Big)I_A(u)I_{(a,b]}(s)\\
&\quad\cdot(e^{-f(r-s, u, w)}-1)E(H_r)\\
&=0.
\end{aligned}$$

and

$$\begin{aligned}
&E((M_t-M_r)^2H_r)\\
&=E\Big(\Big(\int_{(0, t]\times U\times W_+}\Big(w(t-s)-w(r-s)-\gamma\int_r^t w(v-s)dv\Big)I_A(u)I_{(a,b]}(s)\tilde{N}_p(dsdudw)\Big)^2\\
&\quad\cdot\exp\Big(-\int_{(0, r]\times U\times W_+} f(r-s, u, w)N_p(dsdudw)\Big)\Big)\\
&=\int_{(0, t]\times U\times W_+} dsm(du)Q(dw)\Big((wt-s)-w(r-s)-\gamma\int_r^t w(v-s)dv\Big)^2 I_A(u)I_{(a,b]}(s)\\
&\quad\cdot e^{-f(r-s, u, w)}E(H_r)\\
&=\Big(\int_{(0, t]\times U\times W_+} dsm(du)Q(dw)\Big(\beta\int_r^t w(v-s)dv\Big)I_A(u)I_{(a,b]}(s)e^{-f(r-s, u, w)}\Big)E(H_r)
\end{aligned}$$

$$=E\Big(\int_{(0,t]\times U\times W_+}\Big(\beta\int_r^t w(v-s)dv\Big)I_A(u)I_{(a,b]}(s)N_p(dsdudw)H_r\Big)$$

Hence it follows that M_t is an $(\boldsymbol{F}_t^1)$-martingale with quadratic variation process

$$\langle M\rangle_t=\beta\int_0^t\Big(\int_{(0,r]\times U\times W_+}w(r-s)I_A(u)I_{(a,b]}(s)N_p(dsdudw)\Big)dr.$$

However since $\boldsymbol{F}^0$ and $\boldsymbol{F}_t^1$ are independent and $\boldsymbol{F}_t=\boldsymbol{F}_t^0\vee\boldsymbol{F}_t^1$, M_t is an $(\boldsymbol{F}_t)$-martingale.

2°. We next assume $\Phi(s, u, \omega)=\Phi_s(\omega)I_A(u)I_{(a,b]}(s)$ with an $\boldsymbol{F}_a$-measurable random variable Φ_a, $A\in\boldsymbol{B}(U)$ and $0<a<b$. Denote by M_t' the martingale discussed in 1°, then

$$M_t=\Phi_a M_t' \text{ if } t<a, \text{ and } M_t=0 \text{ otherwise},$$

hence M_t is an $(\boldsymbol{F}_t)$-martingale with quadratic variation process

$$\langle M\rangle_t=\Phi_a^2\langle M'\rangle_t \text{ if } t>a, \text{ and } \langle M\rangle_t=0 \text{ otherwise},$$

which shows (4.14).

3°. Finally for a general $(\boldsymbol{F}_t)$-predictable functional $\Phi(s, u, \cdot)$ it is a routine task to show it by approximating the functional by linear combinations of such functionals treated in 2°. Therefore the proof of Lemma 4.2 is complete.

Theorem 4.3. *Let X_t be a solution of the stochastic equation* (4.5) *based on* $\{X_t^0, N_p\}$ *under the assumption* (4.2)-(4.4). *Then for every $F\in\boldsymbol{D}(L)$*

$$F(X_t)-\int_0^t LF(X_s)ds$$

is an $(\boldsymbol{F}_t)$-martingale.

Proof. For $\phi\in C_b(S\cup\{\Delta\})$ with $\phi(\Delta)=0$, set

$$\langle X_t^1, \phi\rangle=\int_{(0,t]\times U\times W_+}w(t-s)\phi(A(X_s, u))N_p(dsdudw).$$

Note that $\phi(A(X_s, u))$ is $(\boldsymbol{F}_t)$-predictable, because X_t is a continuous process by Theorem 4.1. Accordingly, by (4.4) and Lemma 4.2

$$\langle X_t^1, \phi\rangle=M_t^1(\phi)+\gamma\int_0^t\langle X_s^1, \phi\rangle ds+\int_0^t\Big(\int_S V(X_s, dx)\phi(x)\Big)ds.$$

where $M_t^1(\phi)$ is an $(\boldsymbol{F}_t)$-martingale with quadratic variation process

$$\langle M^1(\phi)\rangle_t=\beta\int_0^t\langle X_s^1, \phi^2\rangle ds.$$

On the other hand it is easy to see

$$\langle X_t^0, \phi\rangle=\langle X_0^0, \phi\rangle+M_t^0(\phi)+\gamma\int_0^t\langle X_s^0, \phi\rangle ds,$$

where $M_t^0(\phi)$ is an $(\boldsymbol{F}_t^0)$-martingale with quadratic variation process

$$\langle M^0(\phi)\rangle_t=\beta\int_0^t\langle X_s^0,\ \phi^2\rangle ds\,,$$

and using the independence of X_t^0 and N_p one can see

$$\langle M^0(\phi),\ M^1(\phi)\rangle_t=0\,,$$

hence $X_t=X_t^0+X_t^1$ satisfies

$$\langle X_t,\ \phi\rangle=\langle X_0^0,\ \phi\rangle+M_t(\phi)+\int_0^t\Big(\gamma\langle X_s,\ \phi\rangle+\int_S V(X_s,\ dx)\phi(x)\Big)ds\,,$$

and $M_t(\phi)$ is an $(\boldsymbol{F}_t)$-martingale with quadratic variation process

$$\langle M(\phi)\rangle_t=\int_0^t\beta\langle X_s,\ \phi^2\rangle ds\,.$$

Therefore, using the Ito formula we have the desired martingale property.

Theorem 4.4. *Assume the conditions* (4.2)-(4.4). *Suppose that for every* $\mu\in M(S)$ *and* $\{X_t^0,\ N_p\}$ *with* $X_0^0=\mu$, *the stochastic equation* (4.5) *has a unique solution. Then the solutions* X_t *defines an* $M(S)$*-valued diffusion process* $(\Omega,\ \boldsymbol{F},\ \boldsymbol{F}_t,\ P;\ X_t)$.

Proof. It suffices to show the strong Markov property for $(\Omega,\ \boldsymbol{F},\ \boldsymbol{F}_t,\ P;\ X_t)$.

1°. Let τ be any $(\boldsymbol{F}_t)$-stopping time satisfying $\tau<\infty$ P-a.s., and set

$$Z_t=\int_{(0,t]\times U\times W_+}w(t-s)I_{(0,\tau]}(s)\delta_{A(X_s,u)}N_p(dsdudw)\,.$$

For every bounded measurable function ϕ on $S\cup\{\Delta\}$ with $\phi(\Delta)=0$, set

$$M_t^0(\phi)=\langle X_t^0,\ \phi\rangle-\langle X_0^0,\ \phi\rangle-\gamma\int_0^t\langle X_s^0,\ \phi\rangle ds\,, \tag{4.16}$$

and

$$M_t^1(\phi)=\langle Z_t,\ \phi\rangle-\gamma\int_0^t\langle Z_s,\ \phi\rangle ds-\int_0^t I_{(0,\tau]}(s)\Big(\int_S V(X_s,\ dx)\phi(x)\Big)ds\,. \tag{4.17}$$

Since $I_{(0,\tau]}(t)\phi(A(X_t,\ u))$ is $(\boldsymbol{F}_t)$-predictable, by Lemma 4.2, $M_t^0(\phi)$ and $M_t^1(\phi)$ are $(\boldsymbol{F}_t)$-martingales with quadratic variation processes

$$\langle M^0(\phi)\rangle_t=\beta\int_0^t\langle X_s^0,\ \phi^2\rangle ds\,, \tag{4.18}$$

$$\langle M^1(\phi)\rangle_t=\beta\int_0^t\langle Z_s,\ \phi^2\rangle ds\,,$$

and using the independence of $(X_t^0)_{t\ge0}$ and N_p we have

$$\langle M^0,\ M^1\rangle_t=0\,. \tag{4.15}$$

Let $Y_t^0=X_{t+\tau}^0+Z_{t+\tau}$, then by (4.16)-(4.19)

(4.20) $$\langle Y_t^0, \phi\rangle=\langle X_t, \phi\rangle+\gamma\int_0^t\langle Y_s^0, \phi\rangle ds+M_t^{0*}(\phi),$$

where $M_t^{0*}(\phi)$ is an $(\boldsymbol{F}_{t+\tau})$-martingale with quadratic variation process

$$\langle M^{0*}(\phi)\rangle_t=\beta\int_0^t\langle Y_s^0, \phi^2\rangle ds.$$

Hence $(Y_t^0)_{t\geq 0}$ is an MBD process generated by L^0 of (3.10) starting at X_τ under the condition (4.2). In particular, $(Y_t^0, P(\cdot|\boldsymbol{F}_\tau))$ also is an MBD process generated by L^0 of (3.10) starting at a non-random point X_τ w.r.t. $P(\cdot|\boldsymbol{F}_\tau)$.

2°. Let $Y_t=X_{\tau+t}$ for $t\geqq 0$. Clearly we have

(4.21) $$Y_t=Y_t^0+\int_{(0,t]\times U\times W_+}w(t-s)I_S\delta_{A(Y_s,u)}N_p^\tau(dsdudw),$$

where $N_p^\tau(dtdudw)=N_p(dt+\tau, dudw)$.

3°. We next claim that N_p^τ is a Poisson point process on $U\times W_+$ with the same characteristic measure $m\times Q$ which is independent of $(Y_t^0)_{t\geq 0}$. Denote by $\boldsymbol{F}_t^2$ be the σ-field generated by $\{N(I\times A\times F): I\in\boldsymbol{B}[0, t], A\in\boldsymbol{B}(U), F\in\boldsymbol{B}(W_+)\}$, and set $\boldsymbol{F}_t^*=\boldsymbol{F}^0\vee\boldsymbol{F}_t^2$. Since τ is a $(\boldsymbol{F}_t^*)$-stopping time and N_p is a stationary $(\boldsymbol{F}_t^*)$-Poisson point process in the sense of [5], p. 60. N_p^τ also is a stationary $(\boldsymbol{F}_{t+\tau}^*)$-Poisson point process on $U\times W_+$ with characteristic measure $m\times Q$ which is independent of $\boldsymbol{F}_\tau^*$, (see [7], Theorem 5.1). Note that $(X_t^0)_{t\geq 0}$ is $\boldsymbol{F}^0$-measurable and τ is $\boldsymbol{F}_\tau^*$-measurable, hence $(Y_t^0)_{t\geq 0}$ is $\boldsymbol{F}_\tau^*$-measurable. Therefore we have shown the independence of N_p^τ and $(Y_t^0)_{t\geq 0}$.

4°. Since by (4.21) Y_t is a solution of the equation (4.5) associated with $\{Y_t^0, N_p^\tau\}$, the uniqueness assumption implies that the probability law of $(Y_t)_{t\geq 0}$ under $P(\cdot|\boldsymbol{F}_\tau)$ is uniquely determined by X_τ, from which it follows the strong Markov property of $(\Omega, \boldsymbol{F}, \boldsymbol{F}_t, P; X_t)$.

Now we give a sufficient condition for the uniquely existence of solutions for the equation (4.5).

Theorem 4.5. *In addition to* (4.2)-(4.4) *suppose that there is a constant* $K>0$ *such that*

(4.22) $$\int_U I(A(\mu, u)\neq A(\mu', u))m(du)\leqq K\|\mu-\mu'\|_{var},$$

for every μ *and* μ' *of* $M(S)$. *Then for every* $\{X_t^0, N_p\}$ *there exists a unique solution for the equation* (4.5).

Proof. Define a sequence of approximating solutions of (4.5) by

$$X_t^{n+1}=X_t^0+\int_{(0,t]\times U\times W_+}w(t-s)I_S\delta_{A(X_s^n,u)}N_p(dsdudw) \quad \text{for} \quad n\geqq 0.$$

Denote

$$Y_t^n=\int_{(0,t]\times U\times W_+} w(t-s)I(A(X_s^n, u)\neq A(X_s^{n-1}, u))N_p(dsdudw).$$

Clearly

(4.23) $$\|X_t^{n+1}-X_t^n\|_{var}\leqq 2Y_t^n.$$

By Lemma 4.2

$$E(Y_t^n)=\gamma\int_0^t E(Y_s^n)ds+\int_0^t E\left(\int_U I(A(X_s^n, u)\neq A(X_s^{n-1}, u))m(du)\right)ds,$$

hence by (4.22)

(4.24) $$E(Y_t^n)\leqq K\int_0^t e^{\gamma(t-s)}E(\|X_s^n-X_s^{n-1}\|_{var})ds.$$

Thus it follows from (4.23) and (4.24) that

(4.25) $$E\|X_t^{n+1}-X_t^n\|_{var}\leqq 2Ke^{\gamma t}\int_0^t E\|X_s^n-X_s^{n-1}\|_{var}ds,$$

which implies that there exists an $M(S)$-valued measurable and $(\boldsymbol{F}_t)$-predictable process X_t such that

$$\lim_{n\to\infty} E\|X_t^n-X_t\|_{var}=0.$$

Obviously X_t is a solution of the equation (4.5) for given $\{X_t^0, N_p\}$. Also the uniqueness can be proved in a standard manner.

Corollary 4.6. *In addition to* (4.2) *and* (4.3) *suppose that there exists a σ-finite measure V on S such that for every $\mu\in M(S)$, $V(\mu, dx)$ is absolutely continuous with respect to $V(dx)$ and its density $v(\mu, x)$ satisfies that for some $K>0$,*

(4.26) $$\int_S |v(\mu, x)-v(\mu', x)|V(dx)\leqq K\|\mu-\mu'\|_{var}$$

holds for every μ and μ' of $M(S)$. Then there exist a σ-finite measure space $(U, \boldsymbol{B}(U), m)$ and a jointly measurable map $A(\mu, u)\colon M(S)\times U\to S\cup\{\Delta\}$ such that the conditions (4.4) *and* (4.22) *are fulfilled. Accordingly for every $\{X_t^0, N_p\}$ there exists a unique solution for the equation* (4.5), *which defines an $M(S)$-valued diffusion process.*

Proof. Let $U=[0, \infty)\times S$ and $m(du)=dr\,V(dx)$ for $u=(r, x)\in[0, \infty)\times S$, and define $A(\mu, u)$ by

$$A(\mu, u)=x \text{ if } u=(r, x) \text{ and } 0\leqq r\leqq v(\mu, x), \text{ and } A(\mu, u)=\Delta \text{ otherwise}.$$

Then it is obvious that (4.4) and (4.21) hold. Thus Theorem 4.5 is applicable. Furthermore, the solution defines an $M(S)$-valued diffusion process since it satisfies the strong Markov property by Theorem 4.4.

§5. Stochastic Equation, II

In this section we will discuss a spatially inhomogeneous case:

(5.1) $$LF(\mu)=\frac{1}{2}\int_{S}\mu(dx)\beta(x)\frac{\delta^2F(\mu)}{\delta\mu(x)^2}+\int_{S}(\mu(dx)\gamma(x)+V(\mu,dx))\frac{\delta F(\mu)}{\delta\mu(x)},$$

where $\beta(x): S\to R$ is bounded, uniformly positive and measurable, $\gamma(x): S\to R$ is bounded measurable, and $V(\mu, dx)$ is a measure kernel on $M(S)\times\boldsymbol{B}(S)$ satisfying the conditions (4.3) and (4.4).

Intuitively the corresponding stochastic equation should be formulated in the following way. Let X_t^0 be an MBD process generated by L^0 of (3.10). We prepare an independent system of Poisson point processes $\{N_p^{\beta,\gamma}: \beta>0, \gamma\in R\}$ on $U\times W_+$, where $(U, \boldsymbol{B}(U), m)$ be a σ-finite measure space and $N_p^{\beta,\gamma}$ is a Poisson point system on $U\times W_+$ with characteristic measure $m(du)Q^{\beta,\gamma}(dw)$. We also assume the independence of X_t^0 and $\{N_p^{\beta,\gamma}: \beta>0, \gamma\in R^1\}$. Then the desired stochastic equation would be

(5.2) $$X_t(dx)=X_t^0(dx)+\int_{(0,t]\times U\times W_+}w(t-s)I_S(x)\delta_{A(X_s,u)}(dx)N_p^x(dsdudw)$$

where $N_p^x(dsdudw)=N_p^{\beta(x),\gamma(x)}(dsdudw)$, δ_x stands for the Dirac measure at

$x\in S$, and $I_A(x)=1$ if $x\in A$, $I_A(x)=0$ otherwise.

However we do not know how to give a precise meaning to the second term of the right hand side of (5.2) because of the independence of $\{N_p^x: x\in S\}$.

If we impose the following restrictive assumption:

(5.3) $\beta(x)$ are $\gamma(x)$ are of the form

$$\beta(x)=\sum_{n=1}^{\infty}\beta_n I_{S_n}(x),\quad\text{and}\quad \gamma(x)=\sum_{n=1}^{\infty}\gamma_n I_{S_n}(x)$$

where $(S_n)_{n\geq 1}$ is a measurable partition of S, (β_n) is a bounded and uniformly positive sequence, and (γ_n) is a bounded sequence, then the equation (5.2) clearly makes sense, which turns to

(5.4) $$X_t(dx)=X_t^0(dx)+\sum_{n=1}^{\infty}\int_{(0,t]\times U\times W_+}w(t-s)I_{S_n}(x)\delta_{A(X_s,u)}(dx)N_p^n(dsdudw)$$

where $N_p^n(dtdudw)=N_p^{\beta_n,\gamma_n}(dtdudw)$.

In fact, we can prove the corresponding results to Theorems 4.3, 4.4, and 4.5 for the stochastic equation (5.4).

In order to treat the $M(S)$-valued diffusion process governed by (5.1) with general $\beta(x)$ and $\gamma(x)$ we here take another strategy. We first consider the following case:

(5.5) $$\gamma(x)\equiv 0.$$

For a general $\gamma(x)$ one can reduce it to this case by using a suitable drift transformation.

In order to formulate a stochastic equation we assume that there is a measurable mapping $B(\mu, u): M(S)\times U\to S\cup\{\Delta\}$ satisfying that for $\mu\in M(S)$

(5.6) $$\int_U\phi(B(\mu,u))m(du)=\int_S V(\mu,dx)(\phi/\beta)(x)$$

for every bounded measurable function ϕ defined on $S\cup\{\Delta\}$ with $\phi(\Delta)=0$.

Suppose that we are given an MBD process X_t^0 generated by L^0 of (3.10) and a Poisson point process N_p on $U\times W_+$ with characteristic measure $m\times Q^{1,0}$ on a common probability space $(\Omega, \boldsymbol{F}, P)$ such that X_t^0 and N_p are mutually independent. The initial state $X_0^0=X_0$ may be chosen arbitrarily unless specified.

Based on $\{X_t^0, N_p\}$ let us consider the following stochastic equation:

$$X_t=X_t^0+\int_{(0,t]\times U\times W_+} w(t-s)\beta I_S\delta_{B(X_s,u)}N_p(dsdudw), \tag{5.7}$$

so that for every bounded measurable function ϕ defined on $S\cup\{\Delta\}$ with $\phi(\Delta)=0$,

$$\langle X_t, \phi\rangle=\langle X_t^0, \phi\rangle+\int_{(0,t]\times U\times W_+} w(t-s)(\beta\phi)(B(X_s, u))N_p(dsdudw).$$

The σ-fields $\boldsymbol{F}_t^0$, $\boldsymbol{F}_t^1$, $\boldsymbol{F}_t$ and the notion of a solution for the equation (5.7) are defined in the same manner as the equation (4.5). Then by the same proof of Theorem 4.1 one can see that every solution X_t of the equation (5.7) satisfies the following property:

(5.8) $P(X_t\in M^a(S)$ for all $t>0$, X_t is continuous in $t>0$ in the total variation norm, and weakly continuous at $t=0)=1$.

Theorem 5.1. *For given* $\{X_t^0, N_p\}$, *let* X_t *be a solution of the equation* (5.7). *Then for every* $F\in\boldsymbol{D}(L)$,

$$F(X_t)-\int_0^t LF(X_s)ds$$

is an $(\boldsymbol{F}_t)$*-martingale.*

Proof. For $\phi\in C_b(S\cup\{\Delta\})$ with $\phi(\Delta)=0$, set

$$\langle X_t^1, \phi\rangle=\int_{(0,t]\times U\times W_+} w(t-s)(\beta\phi)(B(X_s, u))N_p(dsdudw),$$

and

$$M_t^1(\phi)=\langle X_t^1, \phi\rangle-\int_0^t\Big(\int_S V(X_s, dx)\phi(x)\Big)ds.$$

Then, by Lemma 4.2 together with (5.6), $M_t^1(\phi)$ is an $(\boldsymbol{F}_t)$-martingale with quadratic variation process

$$\begin{aligned}\langle M^1(\phi)\rangle_t&=\int_0^t\Big(\int_{(0,r]\times U\times W_+} w(r-s)(\beta\phi)^2(B(X_s, u))N_p(dsdudw)\Big)dr\\&=\int_0^t\langle X_s^1, \beta\phi^2\rangle ds.\end{aligned}$$

Since it is easily seen that $M_t^0(\phi)=\langle X_t^0, \phi\rangle-\langle X_0^0, \phi\rangle$ also is an $(\boldsymbol{F}_t)$-martingale such that their quadratic variation process satisfy

$$\langle M^0(\phi), M^1(\phi)\rangle_t=0, \quad\text{and}\quad \langle M^0(\phi)\rangle_t=\int_0^t\langle X_s^0, \beta\phi^2\rangle ds,$$

we obtain

$$\langle X_t, \phi\rangle=\langle X_t^0, \phi\rangle+\int_0^t\Big(\int_S V(X_s, dx)\phi(x)\Big)ds+M_t(\phi), \tag{5.9}$$

where $M_t(\phi)$ is an $(\boldsymbol{F}_t)$-martingale with quadratic variation process

$$\langle M(\phi)\rangle_t=\int\langle X_s, \beta\phi^2\rangle ds,$$

hence the desired martingale property follows from (5.9) together with the Ito formula.

Furthermore, by the same arguments as Theorems 4.4 and 4.5 we have

Theorem 5.2. *In addition to the conditions* (5.5) *and* (5.6), *suppose that there is a constant* $K>0$ *such that*

$$\int_U I(B(\mu, u)\neq B(\mu', u))m(du)\leqq K\|\mu-u'\|_{var} \tag{5.10}$$

for every μ *and* μ' *of* $M(S)$. *Then for every* $\{X_t^0, N_p\}$ *there exists a unique solution* X_t *for the equation* (5.7), *which defines a diffusion process taking values in* $M(S)$ *such that for every* $F\in\boldsymbol{D}(L)$,

$$F(X_t)-\int_0^t LF(X_s)ds$$

is an $(\boldsymbol{F}_t)$*-martingale, and further that the property* (5.8) *holds.*

Corollary 5.3. *In addition to* (4.3) *and* (5.5), *suppose that there is a* σ*-finite measure* $V(dx)$ *on* S *such that for every* $\mu\in M(S)$, $V(\mu, dx)$ *is absolutely continuous with respect to* $V(dx)$, *and its density* $v(\mu, x)$ *satisfies*

$$\int_S|v(\mu, x)-v(\mu', x)|V(dx)\leqq K\|\mu-\mu'\|_{var} \tag{5.11}$$

for every μ *and* μ' *of* $M(S)$. *Then there are* σ*-finite measure space* $(U, \boldsymbol{B}(U), m)$ *and a measurable map* $B(\mu, u): M(S)\times U\to S\cup\{\Delta\}$ *such that the conditions* (5.6) *and* (5.10).

Proof. It is clear that $V'(\mu, dx)=(1/\beta(x))V(\mu, dx)$ satisfies the condition (4.26), hence Corollary 4.6 and Theorem 5.2 imply Corollary 5.3.

In order to treat an L of (5.1) with a general $\gamma(x)$ we consider a drift transformation. Let $(\Omega, \boldsymbol{F}, \boldsymbol{F}_t, P_\mu; X_t)$ be an $M(S)$-valued diffusion process governed by $(L, \boldsymbol{D}(L))$ of (5.1) with $\gamma(x)\equiv 0$, which has been obtained in Theorem 5.2. Then,

$$M_t=\langle X_t, \gamma/\beta\rangle-\langle X_0, \gamma/\beta\rangle-\int_0^t\Big(\int_S V(X_s, dx)(\gamma/\beta)(x)\Big)ds$$

is an $((\boldsymbol{F}_t), P_\mu)$-martingale with quadratic variation process

$$\langle M\rangle_t=\int_0^t\langle X_s, \gamma^2/\beta\rangle ds.$$

Let N_t be a multiplicative functional defined by

$$N_t=\exp\Big(M_t-\frac{1}{2}\langle M\rangle_t\Big).$$

As easily seen, $E_\mu(N_t)=1$ for every $t>0$ and $\mu\in M(S)$, and N_t satisfies

$$N_t=1+\int_0^t N_s dM_s.$$

Then a new probability measure $\tilde{P}_\mu$ on the measurable space $(\Omega, \boldsymbol{F}, \boldsymbol{F}_t)$ is well defined by the following formula: for every $t\geqq 0$,

$$\tilde{P}_\mu(A)=E_\mu(N_t: A) \qquad \text{for} \quad A\in\boldsymbol{F}_t.$$

Then we have

Theorem 5.4. *$(\Omega, \boldsymbol{F}, \boldsymbol{F}_t, \tilde{P}_\mu; X_t)$ is an $M(S)$-valued diffusion process such that for every $F\in\boldsymbol{D}(L)$,*

$$F(X_t)-\int_0^t LF(X_s)ds$$

is an $((\boldsymbol{F}_t), \tilde{P}_\mu)$-martingale, and further that

(5.12) *$\tilde{P}_\mu(X_t\in M^a(S)$ for all $t>0$, X_t in continuous it $t>0$ in the total variation norm, and X_t is weakly continuous at $t=0)=1$.*

Proof. Note that for a bounded measurable function ϕ defined on $S\cup\{\Delta\}$ with $\phi(\Delta)=0$,

$$M_t(\phi)=\langle X_t, \phi\rangle-\langle \mu, \phi\rangle-\int_0^t\Big(\int_S V(X_s, dx)\phi(x)\Big)ds$$

is an $((\boldsymbol{F}_t), P_\mu)$-martingale with quadratic variation process

$$\langle M(\phi)\rangle_t=\int_0^t\langle X_s, \beta\phi^2\rangle ds.$$

This implies

$$\langle M(\phi), N\rangle_t=\int_0^t N_s d\langle M(\phi), M\rangle_s=\int_0^t N_s\langle X_s, \gamma\phi\rangle ds, \tag{5.13}$$

because of $M_t=M_t(\gamma/\beta)$, hence using the Ito formula together with (5.13) we see that

$$\tilde{M}_t(\phi)=\langle X_t, \phi\rangle-\langle \mu, \phi\rangle-\int_0^t\Big(\langle X_s, \gamma\phi\rangle+\int_S V(X_s, dx)\phi(x)\Big)ds$$

is an $((\boldsymbol{F}_t), \tilde{P}_\mu)$-martingale with quadratic variation process

$$\langle \tilde{M}(\phi)\rangle_t=\int_0^t\langle X_s, \beta\phi^2\rangle ds.$$

Therefore the desired martingale property follows. The property (5.12) and the strong

Markov property of $(\Omega, \boldsymbol{F}, \boldsymbol{F}_t, \tilde{P}_\mu; X_t)$ are inherited from the diffusion process $(\Omega, \boldsymbol{F}, \boldsymbol{F}_t, P_\mu; X_t)$ and (5.8) by the general theory of Markov processes.

As a conclusion of this section we have obtained

Corollary 5.5. *Consider the operator* $(L, \boldsymbol{D}(L))$ *of* (5.1) *with general coefficiets* $\beta(x)$ *and* $\gamma(x)$. *Suppose that* $V(\mu, dx)$ *satisfies the assumption of Corollary* 5.3. *Then there exists a diffusion process* $(\Omega, \boldsymbol{F}, \boldsymbol{F}_t, P_\mu; X_t)$ *taking values in* $M(S)$ *such that for every* $F\in\boldsymbol{D}(L)$,

$$F(X_t)-\int_0^t LF(X_s)ds$$

is an $(\boldsymbol{F}_t)$*-martingale, and further that the property* (5.8) *holds.*

§6. Construction of a generalized Fleming-Viot diffusion model

In this section we will construct a class of $M_1(S)$-valued diffusion processes as an application of the stochastic equation discussed in the previous two sections.

Let us consider the following operator acting on a class of functions on $M_1(S)$:

$$\begin{aligned}(6.1)\qquad \tilde{L}F(p)=&\frac{1}{2}\int_S p(dx)(\beta(x)\delta_x(dy)+(\langle p, \beta\rangle-\beta(x)-\beta(y))p(dy))\frac{\delta^2F(p)}{\delta p(x)\delta p(y)}\\ &+\int_S p(dx)\Big(\int_S M(p, x, dy)\Big(\frac{\delta F(p)}{\delta p(y)}-\frac{\delta F(p)}{\delta p(x)}\Big)\Big)\\ &+\int_S p(dx)(\alpha(p, x)-\langle p, \alpha(p, \cdot)\rangle)\frac{\delta F(p)}{\delta p(x)}\end{aligned}$$

where the domain $\boldsymbol{D}(\tilde{L})=\boldsymbol{D}(L)$ which is defined in (1.2) with $\boldsymbol{D}(A)=C_b(S)$, $\beta(x): S\to(0, \infty)$ is bounded measurable and uniformly positive, $M(p, x, dy): M_1(S)\times S\times\boldsymbol{B}(S)\to[0, \infty)$ is a bounded measure kernel, and $\alpha(p, x): M_1(S)\times S\to R$ is bounded measurable.

We interpret an $M_1(S)$-valued diffusion process governed by $\tilde{L}$ of (6.1) as an infinite-allelic diffusion model in the theory of population genetics. Genetically, S is regarded as the set of alleles, each $p\in M_1(S)$ means a gene frequency of alleles, $\beta(x)$ corresponds to variance of the number of offsprings that depends on the allele $x\in S$, $M(p, x, dy)$ is a mutation transition kernel depending on the gene frequency and $\alpha(p, x)$ is a haploid selective intensity of the allele $x\in S$ which is also depends on the gene frequency.

In particular, if $M(p, x, dy)$ and $\alpha(p, x)$ are independent of $p\in M_1(S)$, and $\beta(x)$ is constant, then $\tilde{L}$ of (6.1) is the generator of the Fleming-Viot diffusion model. In this case it is known that $(\tilde{L}, \boldsymbol{D}(\tilde{L}))$ generates a unique $M_1(S)$-valued diffusion process $(\Omega, \boldsymbol{F}, \boldsymbol{F}_t, P_p; Y_t)$ and further that for every $p\in M_1(S)$, and

(6.2) $P_p(Y_t\in M_1^a(S)$ for all $t>0$, and Y_t is continuous in $t>0$ in the total variation norm and weakly continuous at $t=0)=1$, (cf. [3]).

We will here construct an $M_1(S)$-valued diffusion process governed by $(\tilde{L}, \boldsymbol{D}(\tilde{L}))$ and to show the property (6.2) as an application of the stochastic equations. It should be emphasized that the mutation kernel $M(p, x, dy)$ and the selective intensity $\alpha(x, p)$ in $\tilde{L}$ of (6.1) depend on $p \in M_1(S)$.

Theorem 6.1. *Suppose that the following conditions are fulfilled:*

(i) $\alpha(p, x)=M(p, x, S)+c\beta(x)$ *with some* $c \in R^1$,

(ii) *there exists a σ-finite measure $V(dr)$ on S such that for every $(p, x) \in M_1(S) \times S$, $M(p, x, dy)$ is absolutely continuous with respect to $V(dx)$ and its density $m(p, x, y)$ satisfies that for some constant $K>0$,*

$$\int_S (\sup_{x \in S} m(p, x, y)) V(dy) \leqq K \quad \text{for every} \quad p \in M_1(S), \tag{6.3}$$

$$\int_S |m(p, x, y)-m(p', x, y)| V(dy) \leqq K \|p-p'\|_{var} \tag{6.4}$$

for every $p \in M_1(S)$, $p' \in M_1(S)$ and $x \in S$.

Then there is an $M_1(S)$-valued diffusion process $(\Omega, \boldsymbol{F}, \boldsymbol{F}_t, P_p ; Y_t)$ such that for every $F \in \boldsymbol{D}(\tilde{L})$

$$F(Y_t)-\int_0^t \tilde{L} F(Y_s) ds$$

is a P_p-martingale for every $p \in M_1(S)$ and further that the diffusion process $(\Omega, \boldsymbol{F}, \boldsymbol{F}_t, P_p ; Y_t)$ satisfies (6.2).

Proof. For simplicity we will prove the theorem assuming $c=-1$. Otherwise, it can be easily reduced to this case by using a drift transformation by a natural multiplicative functional as in the proof of Theorem 5.4.

Define a measure kernel $V(\mu, dx): M(S) \setminus \{0\} \times \boldsymbol{B}(S) \rightarrow R_+$ by

$$v(\mu, u)=\int_S p(dy) m(p, y, x) \quad \text{and} \quad V(\mu, dx)=v(\mu, x) V(dx) \quad \text{with } p=\mu/\langle\mu, 1\rangle.$$

As in the proof of Corollary 4.6 let $U=[0, \infty) \times S$, $\boldsymbol{B}(U)=\boldsymbol{B}[0, \infty) \times \boldsymbol{B}(S)$, and $m(du)= dr V(dx)$ for $u=(r, x) \in [0, \infty) \times S$, and set $B(\mu, u)=x$ if $0 \leqq r \leqq v(\mu, x)/\beta(x)$ and $B(\mu, u)=\Delta$ otherwise. Then $B(\mu, u): M_1(S) \setminus \{0\} \times U \rightarrow S \cup \{\Delta\}$ is a measurable map satisfying (5.6). Moreover by (6.4)

$$\int_U m(du) I(B(\mu, u) \neq B(\mu', u)) \leqq K \|p-p'\|_{var} \quad \text{with } p=\mu/\langle\mu, 1\rangle \tag{6.5}$$

and $p'=\mu'/\langle\mu', 1\rangle$.

For each $\delta>0$ define

$$B_\delta(\mu, u)=x \text{ if } u=(r, x) \text{ and } 0 \leqq r \leqq \min\{(\langle\mu, 1\rangle/\delta)^2, 1\} v(\mu, x)/\beta(x),$$

$$B_\delta(\mu, u)=\Delta \quad \text{otherwise}.$$

Clearly $B_\delta(\mu, u)=B(\mu, u)$ if $\langle\mu, 1\rangle \geqq \delta$, and using (6.3) and (6.4) one can easily check that there is a constant $K_\delta>0$ satisfying that for every μ and μ' of $M(S)$,

$$\int_U m(du)I(B_\delta(\mu, u)\neq B_\delta(\mu', u))\leqq K_\delta\|\mu-\mu'\|_{var}. \tag{6.6}$$

Let $p\in M_1(S)$ be fixed, and let X_t^0 be an MBD process starting at $X_0^0=p$ generated by

$$L^0F(\mu)=\frac{1}{2}\int_S\mu(dx)\beta(x)\frac{\delta^2F(\mu)}{\delta\mu(x)^2}. \tag{6.7}$$

Let N_p be a Poisson point process on $U\times W_+$ with characteristic measure $m\times Q^{1,0}$, that is independent of X_t^0 system. Since B_δ satisfies (5.10), by Theorem 5.2 there exists a unique $M(S)$-valued solution $X_t^{(\delta)}$ of the following stochastic equation:

$$X_t^{(\delta)}=X_t^0+\int_{(0,t]\times U\times W_+}\beta(B(X_s^{(\delta)}, u))w(t-s)I_S\delta_{B(X_s^{(\delta)},u)}N_p(dsdudw). \tag{6.8}$$

Let $\zeta_\delta=\inf\{t\geqq 0: \langle X_t^{(\delta)}, 1\rangle\leqq\delta\}$. Since $B_\delta(\mu, u)=B(\mu, u)$ if $\langle\mu, 1\rangle\geqq\delta$, Theorem 5.2 implies that $X_t^{(\delta)}=X_t^{(\delta')}$ holds for $0\leqq t\leqq\zeta_\delta\wedge\zeta_{\delta'}$, hence there exists a unique solution X_t for the equation (6.9) up to time $\zeta=\lim_{\delta\downarrow 0}\zeta_\delta$,

$$X_t=X_t^0+\int_{(0,t]\times U\times W_+}w(t-s)\beta I_S\delta_{B(X_s,u)})N_p(dsdudw). \tag{6.9}$$

Furthermore, it is easy to see that $(X_t)_{0\leq t<\zeta}$ has the strong Markov property since so does $(X_t^{(\delta)})_{0\leq t\leq\zeta^{(\delta)}}$ for each $\delta\geqq 0$ by Theorem 5.3. Accordingly, by virtue of Corollary 2.2 we have an $M_1(S)$-valued diffusion process $(\Omega, \boldsymbol{F}, \boldsymbol{F}_t, P_p; Y_t)$ such that for every $F\in\boldsymbol{D}(\tilde{L})$

$$F(Y_t)-\int_0^t\tilde{L}F(Y_s)ds$$

is a P_p-martingale for every $p\in M_1(S)$. Also, the property (6.2) is obvious by (5.8) and Theorem 2.1.

Finally we apply the Poisson construction for MBDI processes to prove an ergodic theorem for a simple FVD process. Let us consider the following operator on $\boldsymbol{D}(\tilde{L})=\boldsymbol{D}(L)$:

$$\tilde{L}F(p)=\frac{1}{2}\int_{S\times S}p(dx)(\delta_x(dy)-p(dy))\frac{\delta^2F(p)}{\delta p(x)\delta p(y)}+\int_{S\times S}p(dx)M(dy)\Big(\frac{\delta F(p)}{\delta p(y)}-\frac{\delta F(p)}{\delta p(x)}\Big), \tag{6.10}$$

where $M(dx)$ is a finite measure on S.

We denote by $(\Omega, \boldsymbol{F}, \boldsymbol{F}_t, P_p; Y_t)$ the $M_1(S)$-valued diffusion process governed by $(\tilde{L}, \boldsymbol{D}(\tilde{L}))$ of (6.9), that is a simple Fleming-Viot model in which the mutation kernel $M(p, x, dy)$ is independent of $p\in M_1(S)$ and $x\in S$, i.e. $M(p, x, dy)=M(dy)$. Then it is easy to see that the diffusion process $(\Omega, \boldsymbol{F}, \boldsymbol{F}_t, P_p; Y_t)$ has a unique reversible stationary measure $R(dp)$ such that for every finite measurable partition $(S_1, S_2, \cdots, S_n)$

of S, the distribution of $(p(S_1), p(S_2), \cdots, p(S_n))$ under R is a Dirichlet distribution (or a beta distribution) with parameters $(2M(S_1), 2M(S_2), \cdots, 2M(S_n))$, namely its density function is

$$\frac{\Gamma\left(\sum_{i=1}^{n} 2M(S_i)\right)}{\prod_{i=1}^{n} \Gamma(2M(S_i))} I\left((y_1 \geqq 0, \cdots y_{n-1} \geqq 0, \sum_{i=1}^{n-1} y_i \leqq 1\right) \prod_{i=1}^{n} y_i^{2M(S_i)-1}$$

where $y_n = 1 - y_1 - \cdots - y_{n-1}$, and $\Gamma(\alpha)$ is Gamma function.

For the FVD process $(\Omega, \boldsymbol{F}, \boldsymbol{F}_t, P_p; Y_t)$ we can construct a nice coupling process by making use of the Poisson construction in § 3 and Theorem 2.1.

Lemma 6.2.[1] *There exists a diffusion process $(\Omega^*, \boldsymbol{F}^*, \boldsymbol{F}^*_t, P_{(p^1, p^2)}; (Y^1_t, Y^2_t))$ taking values in $M_1(S) \times M_1(S)$ such that for every $(p^1, p^2) \in M_1(S) \times M_1(S)$*

(i) *the probability law of the process Y^1_t under $P_{(p^1, p^2)}$ coincides with that of Y_t under P_{p^1},*

(ii) *the probability law of the process Y^2_t under $P_{(p^1, p^2)}$ coincides with that of Y_t under P_{p^2}, and*

(iii) *$Y^1_t = Y^2_t$ holds eventually $P_{(p^1, p^2)}$-almost surely.*

Proof. Let any $(p^1, p^2) \in M_1(S) \times M_1(S)$ be fixed. We prepare a Poisson system $\{N^0, N_p\}$ satisfying that

(a) N^0 is a Poisson random measure on $S \times S \times W_+$ with intensity measure $p^1 \times p^2 \times Q^{1,0}$,

(b) N_p is a Poisson point process on $S \times W_+$ with characteristic measure $M \times Q^{1,0}$, and

(c) N^0 and N_p are mutually independent.

Let define three $M(S)$-valued processe by

$$U^1_t = \int_{S \times S \times W_+} w(t) \delta_x N^0(dx\,dy\,dw),$$

$$U^2_t = \int_{S \times S \times W_+} w(t) \delta_y N^0(dx\,dy\,dw),$$

$$V_t = \int_{(0,t] \times S \times W_+} w(t-s) \delta_x N_p(ds\,dx\,dw),$$

and set

$$X^1_t = U^1_t + V_t \quad \text{and} \quad X^2_t = U^2_t + V_t.$$

Then by Theorem 3.6, both processes X^1_t and X^2_t are MBDI processes starting at p^1 and p^2 generated by

$$LF(\mu) = \frac{1}{2} \int_S \mu(dx) \frac{\delta^2 F(\mu)}{\delta \mu(x)^2} + \int_S M(dx) \frac{\delta F(\mu)}{\delta \mu(x)}. \tag{6.10}$$

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1) T. G. Kurtz also constructed a similar coupling process by a different method. (Oral communication).

Clearly, $\langle U_t^1, 1\rangle=\langle U_t^2, 1\rangle$ holds for all $t\geqq 0$ a.s., and the process $r_t^0=\langle U_t^1, 1\rangle$ is equivalent to a CB-diffusion $(\Omega, \boldsymbol{F}, \boldsymbol{F}_t, P_y^{1,0}, y_t)$ generated by

$$\frac{y}{2}\frac{d^2}{dy^2} \quad \text{(0 is a trap.)}$$

hence $\zeta^0=\inf\{t\geqq 0: r_t^0=0\}<\infty$ a.s., which implies that $X_t^1=X_t^2$ holds eventually a.s. Also, noting $\langle X_t^1, 1\rangle=\langle X_t^2, 1\rangle$ for all $t\geqq 0$ a.s., we set $r_t=\langle X_t^1, 1\rangle$ and

$$C_t=\int_0^t \frac{ds}{r_s}.$$

Then, by Theorem 2.1, its inverse function D_t is well defined on $[0, \infty)$, and both $M_1(S)$-valued processes

$$Y_t^1=\frac{X_{D_t}^1}{r_{D_t}} \quad \text{and} \quad Y_t^2=\frac{X_{D_t}^2}{r_{D_t}}$$

are equivalent to the FVD processes starting at p^1 and p^2 respectively, and it is obvious that the property (c) holds. Thus the proof of Lemma 6.2 is complete.

The following theorem is a direct consequence of Lemma 6.2.

Theorem 6.3. *Let* $(\Omega, \boldsymbol{F}, \boldsymbol{F}_t, P_p; Y_t)$ *be the FVD process generated by* $(\tilde{L}, \boldsymbol{D}(\tilde{L}))$ *of* (6.9). *Then for every* $p\in M_1(S)$,

$$\lim_{t\to\infty}\|P_p(Y_t\in\cdot)-R\|_{var}=0.$$

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