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An algebraic cycle relating to a section with high penetration

By

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§0. Introduction

Let X be a complex projective manifold of dimension n, E a holomorphic vector bundle (= locally free sheaf) of rank r over X, and h a hyperplane class in the Chow ring $A^{\cdot}(X)$ of X. Consider the Lefschetz operator $L: H^t(X, \Omega_X^s(E)) \to H^{t+1}(X, \Omega_X^{s+1}(E))$ defined by the Hodge-Kähler class $\omega \in H^1(X, \Omega_X^1)$ corresponding to the hyperplane class h.

In the case where $E = \mathcal{O}_X$, there is a well-known theorem on the Lefschetz operator as follows.

Hard Lefschetz theorem. *Under the circumstances, the map:* $L^k: H^t(X, \Omega_X^s) \to H^{t+k}(X, \Omega_X^{s+k})$ *is bijective for non-negative integers* s, t *and* k *with* $k = n - s - t$.

For an arbitrary holomorphic vector bundle E, the bijectivity or even the injectivity of the map $L^k: H^t(X, \Omega_X^s(E)) \to H^{t+k}(X, \Omega_X^{s+k}(E))$ does not hold in general.

From the viewpoint of differential geometry, this can be understood as an effect of the "curvature" of E endowed with a Hermitian metric H, because the equality $[\bar{\square}_E, L] = \sqrt{-1}\,\Theta_E$ holds for the Laplacian $\bar{\square}_E$ and the curvature operator Θ_E induced by the Hermitian metric H. Moreover, if E admits a *flat* Hermitian structure, we can show that the bijectivity of the map L^k holds (cf. [C-L]).

On the other hand, from the viewpoint of algebraic geometry, it is desirable to understand the degeneracy of the map L^k in the general case without using the terms "metric" or "curvature". Thus we are interested in the following problem, which is important also in our investigation on the relationship between arithmetically normal embeddings and their normal bundles (cf. [U]).

(0.0) Problem. *Generalize the Hard Lefschetz theorem to the case of cohomologies with coefficients in a vector bundle, and describe the kernel and the cokernel of the map* $L^q: H^s(X, \Omega_X^t(E)) \to H^{s+q}(X, \Omega_X^{t+q}(E))$ $(s, t, q \in \mathbf{N} \cup \{0\})$ *in terms of some suitable algebraic objects such as algebraic cycles.*

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In the $s = t = 0$ case, which is the most important case for our application, we can obtain a result by which we have good information on $\operatorname{Ker}(L^q\colon H^s(X, \Omega_X^t(E)) \to H^{s+q}(X, \Omega_X^{t+q}(E)))$. Our result is stated as follows.

Take a non-negative integer p and a global section $\tau \in H^0(X, E)$ which is *transversal to the zero section.* Then the zero locus $Z(\tau)$ of the section τ defines a $(n-r)$-cycle Z, which represents the r-th Chern class $c_r(E) \in A^r(X)$. *Assume that the order of penetration of the section* τ *with respect to the hyperplane class* h *is greater than or equal to* p (namely, the cohomology class $L^p(\tau) = \tau \otimes \omega^p \in H^p(X, \Omega_X^p(E))$ is not zero, or equivalently $\tau \notin \operatorname{Ker}(L^p\colon H^0(X, E) \to H^p(X, \Omega_X^p(E)))$.

(0.1) Theorem. *In addition to the assumption above, we assume the following. Assumption* $(+)$: *the numerical equivalence on* $A^p(X)$ *coincides with its* **Q**-*homological equivalence* (cf. (0.2) Remark (ii)).
Then, there exists an algebraic p*-cycle* $\zeta \in A^{n-p}(X)$ *such that* $h^p \cdot \zeta > 0$ *and the intersection cycle* $Z \cdot \zeta \in A^{n-p+r}(X)$ *is numerically equivalent to zero.*

In a sense, this theorem suggests that a section with high penetration, namely, a section which can be mapped to a non-zero element of a higher cohomology group by some power of the Lefschetz operator has many directions where the section is nearly "flat" in E (cf. Remark (3.1) in §3).

(0.2) Remark. (i) In the case where $r > p$, the assertion above is affirmative for ζ to be h^{n-p}.

(ii) Relating to the assumption $(+)$, we shall enumerate several known facts in the sequel. The assumption $(+)$ is verified for complete intersections, Grassmannians and abelian varieties (cf. [L], [Gd]). By using the Hodge-Riemann bilinear relation, it is easily proved that the following assumption $(++)$ implies the assumption $(+)$ (cf. [F] Example 19.3.14).

Assumption $(++)$: *the* **Q**-*vector space* $H^{2n-2p}(X, \mathbf{Q}) \cap H^{n-p,n-p}(X)$ *is generated by the classes of algebraic* p*-cycles.*

If $p = 0, 1, n-1, n$, then the assumption $(+)$ and $(++)$ are always satisfied. Also in the $p = 2$ case, the assumption $(+)$ is verified (cf. [L]). As is well-known, it was conjectured by W. V. D. Hodge that the assumption $(++)$ is affirmative for every complex projective manifold.

(iii) The existence of an algebraic p-cycle ζ as in Theorem (0.1) does not always imply that $L^p(\tau) \neq 0$ (cf. for example Remark (3.1) (i)).

§1. Non-vanishing lemma

(1.1) Lemma. *Let* X, E, h, ω, *and* p *be the same as above. Take a global section* τ *of* E *with which is transversal to the zero section* (i.e. $Z(\tau)$ *is smooth and of pure codimension* r). *Assume that the cohomology class* $\tau \otimes \omega^p \in H^p(X, \Omega_X^p(E))$ *is* *not zero. Consider the cycle map $cl\colon A^{\cdot}(X) \to H^{2\cdot}(X, \mathbf{Q})$ and the pull-back $j^*\colon H^{\cdot}(X, \mathbf{Q}) \to H^{\cdot}(U, \mathbf{Q})$ induced by the inclusion map $j\colon U := X - Z(\tau) \hookrightarrow X$. Then, the cohomology class $j^*cl(h^p) \in H^{2p}(U, \mathbf{Q})$ does not vanish.*

Proof. Let us consider the projective bundle $f\colon B = P(E) \to X$. The section τ canonically defines an effective divisor D of B. For every point y of $Z(\tau)$, D includes the whole fibre $f^{-1}(y)$. Hence, we obtain a commutative diagram of morphisms:

$$\begin{tikzcd} B = P(E) \arrow[d, "f"'] & V := B - D \arrow[l, hook', "j_0"'] \arrow[d, "f_0 = f|_V"] \\ X & U = X - Z(\tau). \arrow[l, hook', "j"] \end{tikzcd}$$

Thus, we have only to show that $j_0^* f^* cl(h^p)\ (= f_0^* j^* cl(h^p)) \in H^{2p}(V, \mathbf{C})$ is a non-zero class.

Now we shall prove that the effective divisor D has no singularity. For each point of the open set U, the section τ can be taken as a part of a local frame of E on a suitable neighborhood of the point, which implies that D has no sigularity on the open set $f^{-1}(U)$. Hence it is enough to see that D is smooth at each point of $f^{-1}(Z(\tau)) \cap D$ in the case where $r \leqq n$. Taking a point x of $f^{-1}(Z(\tau)) \cap D$, we put y to be $f(x)$. Because our problem is local, we may assume the five condition: (a) X has a system of coordinates $(z_1, \dots, z_n)$; (*b*) $E \cong \bigoplus_{a=1}^{r} \mathcal{O}_X e_a$; (c) $\tau = f_1 e_1 + \cdots + f_r e_r$ $(f_a \in \Gamma(X, \mathcal{O}_X))$; (d) $(z_1, \dots, z_n; [\Xi_1 : , \dots , : \Xi_r])$ is a system of coordinates of $\mathbf{P}(E) \cong X \times \mathbf{P}^{r-1}$ which corresponds to the system of coordinates of X and the frame $(e_1, \dots, e_r)$; (e) $\Xi_r(x) \neq 0$. Since $f_1 = , \dots , = f_r = 0$ gives a system of defining equations for $Z(\tau)$, the assumption that the section τ is transversal to the zero section means

$$\text{(1.1.1)} \qquad \operatorname{rank}\,(\partial f_a / \partial z_b(y)) = r.$$

Let us denote the local fibre coordinates $(\Xi_1/\Xi_r, \dots, \Xi_{r-1}/\Xi_r)$ of the open set $D_+(\Xi_r)$ by $(\xi_1, \dots, \xi_s)$, where we put s to be $r-1$. Then the local equation τ_x of D on the open set $D_+(\Xi_r)$ is given by $\tau_x = f_1\xi_1 + \cdots + f_s\xi_s + f_r$, and therefore

$$\text{(1.1.2)} \qquad \begin{aligned} &(\partial\tau_x/\partial z_1(x), \dots, \partial\tau_x/\partial z_n(x), \partial\tau_x/\partial\xi_1(x), \dots, \partial\tau_x/\partial\xi_s(x)) \\ &= \Big(\cdots, \sum_{a=1}^{r} \xi_a(x)\, (\partial f_a/\partial z_b(y)), \cdots, \cdots, f_c(y), \cdots\Big), \end{aligned}$$

where we set $\xi_r(x) = 1$ for the sake of convenience, $1 \leqq b \leqq n$, and $1 \leqq c \leqq s$. If the right-hand side of (1.1.2) is equal to $(0, \dots, 0)$, then the condition (1.1.1) implies $(\xi_1(x), \dots, \xi_s(x), \xi_r(x)) = (0, \dots, 0)$ by the usual argument of linear algebra, which contradicts $\xi_r(x) = 1$. Thus we see the effective divisor D is smooth at the point x.

Then, Deligne's mixed Hodge theory (cf. [D] Corollaire (3.2.13)) tells us that the spectral sequence:

$$E_1^{t,s} = H^s(B, \Omega_B^t(\log D)) \Longrightarrow H^{t+s}(V, \mathbf{C})$$

degenerates in E_1-level. The inclusion maps $\rho^\cdot\colon \Omega_B^\cdot \to \Omega_B^\cdot(\log D)$ give rise to a homomorphism of sheaves of complex, which induces a homomorphism of spectral sequences for stupid filtrations:

$$\begin{array}{ccc} E_1^{t,s} = H^s(B, \Omega_B^t) & \Longrightarrow & H^{t+s}(B, \mathbf{C}) \\ {\scriptstyle E_1^{t,s}(\rho^\cdot)} \downarrow & & \downarrow {\scriptstyle j_0^* = E^{t+s}(\rho^\cdot)} \\ E_1^{t,s} = H^s(B, \Omega_B^t(\log D)) & \Longrightarrow & H^{t+s}(V, \mathbf{C}). \end{array}$$

Since Hodge spectral sequence on B also degenerates in E_1-level, it is enough to see that the class $E_1^{p,p}(\rho^\cdot)(f^*(\omega^p))$ corresponding to $\mathrm{Gr}^p(j_0^* f^* cl(h^p))$ is not zero. Using Le Potier's isomorphism (cf. [S-S] Theorem (5.16)):

$$\alpha\colon H^s(B, \Omega_B^t \otimes \mathcal{O}_{P(E)}(1)) \simeq H^s(X, \Omega_X^t(E))$$

(where $\mathcal{O}_{P(E)}(1)$ denotes the tautological line bundle of $P(E)$), we get a commutative diagram:

$$\begin{array}{ccccc} H^p(X, \Omega_X^p(E)) & \overset{\alpha}{\simeq} & H^p(B, \Omega_B^p \otimes \mathcal{O}_{P(E)}(1)) & \overset{\beta}{\simeq} & H^p(B, \Omega_B^p(D)) \\ {\scriptstyle \otimes\tau}\uparrow & & {\scriptstyle \otimes\tau}\uparrow & & {\scriptstyle \gamma}\uparrow \\ H^p(X, \Omega_X^p) & \xrightarrow[f^*]{} & H^p(B, \Omega_B^p) & \xrightarrow[E_1^{p,p}(\rho^\cdot)]{} & H^p(B, \Omega_B^p(\log D)) \end{array}$$

where the isomorphism β and the map γ are induced by the canonical isomorphism: $\Omega_B^p \otimes \mathcal{O}_{P(E)}(1) \simeq \Omega_B^p(D)$ and the inclusion map: $\Omega_B^p(\log D) \hookrightarrow \Omega_B^p(D)$ respectively. Then, the assumption that the class $\tau \otimes \omega^p \in H^p(X, \Omega_X^p(E))$ does not vanish obviously implies that the class $E_1^{p,p}(\rho^\cdot)(f^*(\omega^p)) \in H^p(B, \Omega_B^p(\log D))$ is not zero.

§2. Proof of Theorem (0.1)

Since $Z(\tau)$ is smooth, $Z(\tau)$ has a tubular neighborhood T in X. By using Thom's isomorphism theorem and the excision theorem, we see that $H^{2p-2r}(Z(\tau), \mathbf{Q}) \simeq H^{2p}(T, T - Z(\tau); \mathbf{Q}) \simeq H^{2p}(X, U; \mathbf{Q})$. Hence we obtain an exact sequence:

$$(*) \qquad H^{2p-2r}(Z(\tau), \mathbf{Q}) \xrightarrow{i_!} H^{2p}(X, \mathbf{Q}) \xrightarrow{j^*} H^{2p}(U, \mathbf{Q}),$$

where $i_!$ denotes the Gysin map induced by the inclusion map $i\colon Z(\tau) \hookrightarrow X$. On the other hand, by virtue of Lemma (1.1), the assumption of $L^p(\tau) \neq 0$ implies that the class $j^* cl(h^p) \in H^{2p}(U, \mathbf{Q})$ is not zero. Then, no element of the image of the push-forward map $i_*\colon A^{p-r}(Z(\tau)) \otimes \mathbf{Q} \to A^p(X) \otimes \mathbf{Q}$ is numericaly equivalent to the cycle $h^p \in A^p(X) \otimes \mathbf{Q}$. In fact, if there exists a $(n-p)$-cycle $\lambda \in A^{p-r}(Z(\tau)) \otimes \mathbf{Q}$

such that the cycle h^p is numerically equivalent to the cycle $i_*\lambda$, then our assumption $(+)$ implies that $cl(h^p) = i_!cl(\lambda)$. Hence, the exact sequence $(*)$ shows us that $j^*cl(h^p) = 0$, which contradicts the fact obtained above. Now we take a quotient group $N^\cdot(X)$ of $A^\cdot(X) \otimes \mathbf{Q}$ by the numerical equivalence. Then, the intersection product induces a perfect pairing $N^p(X) \otimes N^{n-p}(X) \to \mathbf{Q}$ of finite dimensional $\mathbf{Q}$-vector spaces. Hence we can find a p-cycle $\zeta \in A^{n-p}(X)$ such that $h^p \cdot \zeta > 0$ and $i_*\lambda \cdot \zeta = 0$ for every $(n-p)$-cycle $\lambda \in A^{p-r}(Z(\tau))$ on $Z(\tau)$. Through the projection formula, we see that $i^*\zeta \in A^{n-p}(Z(\tau))$ is numerically equivalent to zero, which means that $Z \cdot \zeta = i_*i^*\zeta \in A^{n-p+r}(X)$ is numerically equivalent to zero.

§3. Several remarks

In connection with "the order of penetration", we shall make three more remarks under the circumstances of §0. For further information, we refer to [U].

(3.1) Remark. (i) It is easy to see that $L^n(\tau) \neq 0$ is equivalent to the splitting of the exact sequence: $0 \to \mathcal{O}_X \xrightarrow{\tau} E \to E/\tau\mathcal{O}_X \to 0$, which implies $Z(\tau) = \phi$. In the $p = n$ case, Theorem (0.1) implies the latter fact $Z(\tau) = \phi$, because we can take the fundamental cycle $[X] \in A^0(X)$ as the cycle ζ in Theorem (0.1). Then the section τ gives a trivial *subbundle* of E. This can be understood that the section τ is nearly "flat" in E at each point of X in all directions. Moreover, if (E, H) is a Hermitian holomorphic vector bundle, $L^n(\tau) \neq 0$ really implies the "flatness" (on the cohomology level) of the subbundle $S = \tau\mathcal{O}_X$ in (E, H) in the following sense. Let $A \in \mathscr{E}^{1,0}(\mathrm{Hom}(S, Q))$ be the second fundamental form of the subbundle S in (E, H) and $B \in \mathscr{E}^{0,1}(\mathrm{Hom}(Q, S))$ the adjoint of $-A$, where Q denotes the quotient bundle $E/\tau\mathcal{O}_X$ of E. Then B satisfies the equation $\bar{\partial}B = 0$ in $\mathscr{E}^{0,2}(\mathrm{Hom}(Q, S))$ and defines a cohomology class $[B] \in H^1(X, \mathrm{Hom}(Q, S))$, which coincides with the extension class of the exact sequence $0 \to S \to E \to Q \to 0$ (cf. [Gf]). Hence $L^n(\tau) \neq 0$ implies $[B] = 0$. Since the second fundamental form A or $B = -A^*$ expresses the degree or twisting of the subbundle S in (E, H), we may say that $[B] = 0$ means the "flatness" of the subbundle S in (E, H) (on the cohomology level).

(ii) The condition $L^p(\tau) \neq 0$ follows if there exists a subvariety Y of dimension p in X such that the restriction of the exact sequence: $0 \to \mathcal{O}_X \xrightarrow{\tau} E \to E/\tau\mathcal{O}_X \to 0$ to Y splits. In this case, we can take the fundamental cycle $[Y] \in A^{n-p}(X)$ as the cycle ζ in Theorem (0.1). Then we may say that the section τ is nearly "flat" at each point of Y in the directions of the tangents of Y.

(iii) Even in the situation of Theorem (0.1), $L^p(\tau) \neq 0$ does not imply the existence of an *irreducible* p-cycle ζ in general. Hence, the converse of (ii) is not always affirmative (cf. [U]).

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