J. Math. Kyoto Univ. (JMKYAZ)
30-3 (1990) 393-401

3-folds with two P^1-bundle structures

By

Eiichi SATO

In the present paper, the author determines the structure of 3-folds which have two P^1-bundle structures.

Let X be a projective 3-fold defined over an algebraically closed field k. Then, X is said to have two P^1-bundle structures $(S, T; p, q)$ if there are two P^1-bundles $p: X \to S$ and $q: X \to T$ with projective surfaces S, T in the etale topology and moreover if (P) $\dim h(X) = 3$, where h is the morphism: $X \to S \times T$ induced by p and q.

Then we have

Theorem. *Let X be a smooth 3-fold with two P^1-bundle structures $(S, T; p, q)$. Assume that the characteristic of the ground field k is arbitrary. Then, X is one of the followings:*

1) $S \times_C T$, *where S and T are P^1-bundles over a smooth curve C.*
2) $P(T_{P^2})$, *where T_{P^2} is the tangent bundle over P^2.*

The author has already shown the above theorem in the case of characteristic zero in [Sa]. What is important for the proof is to prove that S and T are ruled, which is trivial in characteristic zero. Namely, the essential part is only that a projective surface dominated by a ruled surface is ruled in characteristic zero. (See Remark 1.3.1) But, in the case of positive characteristic, there are many unirational surfaces which are of general type [Za]. Moreover, in the case, there exists even a surface of general type which is regularly dominated by P^2 (See Proposition 2.12 and remark in [E]).

Hence, in order to prove the ruledness of S and T, we prepare two sufficient conditions about the ruledness: Proposition 2.4 and Proposition 2.7 in §2. These propositions leave us the following case: the second Betti number $\beta_2(S, l) = 2$ and K_S is numerically equivalent to zero, if S is not ruled.

Finally, in §3, we can rule out this case thanks to the fact in [Bo + Mu] (See Proposition 3.7 in this paper).

Thus, throughout this paper, the characteristic of the ground field is supposed to be positive.

Notations. We work over an algebraically closed field k of any positive

Communicated by Praf. Nagata, May 11, 1988; Revised Aug. 27, 1988

characteristic. *A* variety means an irreducible and reduced projective algebraic k-scheme. Letting $f: U \to V$ be a morphism between verieties and Y a subscheme of U, $f|_Y: Y \to V$ denotes the restricted map of f on Y. For a coherent sheaf F on a variety Y, $h^i(Y, F)$ denotes $\dim H^i(Y, F)$. For a smooth projective variety $*$, $\kappa(*)$ denotes the Kodaira dimension of $*$ (sometimes, abbreviated to κ). Moreover, Ω_* denotes the sheaf of holomorphic 1-forms on $*$ and K_* denotes the canonical bundle of $*$. For a vector bundle E, $S^m(E)$ denotes the m-th symmetric product of E.

§ 1. Preliminaries

In the present section we shall study some cohomological properties of P^1-bundles in the etale topology.

In the first place we recall a few of facts which are well-known. For the meaning of the notations, see § 5 of [Mi].

(1.1) Fact: I) Let W be a P^1-bundle over a smooth projective variety V in the etale topology. Then $\chi(W, l) = \chi(V, l)\chi(P^1, l)$ $(l \neq \text{char } k)$. Here $\chi(W, l) = \sum_i (-1)^i \beta_i(W, l)$ and $\beta_i(W, l)$ is the l-adic i-th Betti number of W $(= \dim_{Q_l} H^i(W_{et}, Q_l))$ (See Corollary 2.14 in § 5 and Corollary 4.2 in § 6 in [Mi]).
II) Particularly, if $\dim W = 2$, we have

$$\chi(\mathcal{O}_W) = (K_W^2 + \chi(W, l))/12. \qquad \text{(See Theorem 3.12 of § 5 in [Mi])}$$

III) For a smooth projective variety W, let $\mathrm{Alb}(W)$ be the Albanese variety of W. Then, we have an inequality: $\dim \mathrm{Alb}(W) = \beta_1(W, l)/2 \leq h^1(W, \mathcal{O}_W)$. Moreover, if $H^2(W, \mathcal{O}_W) = 0$, then $\dim \mathrm{Alb}(W) = h^1(W, \mathcal{O}_W)$.

Note that $H^1(W, \mathcal{O}_W)$ is canonically isomorphic to the tangent space of $\mathrm{Pic}^\circ(W)$ at the zero point where $\mathrm{Pic}^\circ(W)$ is the connected component of the Picard scheme of W containing 0. See p. 132 in [Mi] and Lecture 27 in [Mu].

Thus we have an easy

Proposition 1.2. *let Z be a geometrically ruled surface over a smooth curve C. Then we have*

1) $\beta_1(Z, l) = 2h^1(Z, \mathcal{O}_Z) = 2h^1(C, \mathcal{O}_C) = \beta_1(C, l)$.
2) $K_Z^2 = 8(1 - h^1(C, \mathcal{O}_C))$ and $\chi(Z, l) = \chi(C, l)\chi(P^1, l) = 4(1 - h^1(C, \mathcal{O}_C))$.
3) $\beta_2(Z, l) = 2$.

Now let us state the property of a surface dominated by a geometrically ruled surface.

Proposition 1.3. *Let Y be a smooth surface dominated by a geometrically ruled surface Z. Then, we have*

1) *$\beta_2(Y, l)$ is 1 or 2.*
2) *If Y is ruled, then it is a geometrically ruled surface or P^2.*

Proof. Since the surjective morphism: $Z \to Y$ induces an injection $H^2(Y_{et}, Q_l) \to H^2(Z_{et}, Q_l)$, the former is obvious. The latter is trivial. q.e.d.

Remark 1.3.1. If the above dominating morphism $f: Z \to Y$ in Proposition 1.3 is separable, then Y is ruled. For the proof, for example, see Lemma 3.1 in [Sa]. Therefore, in characteristic zero, Y dominated by a geometrically ruled surface is ruled.

Finally in this section, let us state

Proposition 1.4. *Let X be a smooth 3-fold with two P^1-bundle structures $(S, T; p, q)$. Let us assume that S and T are ruled. Then, S and T are geometrically ruled surfaces or they are P^2.*

Proof. First, recall that a smooth, projective ruled surface dominated by a geometrically ruled surface is a geometrically ruled surface or P^2. Now, $\chi(X, l) = \chi(S, l)\chi(P^1, l) = \chi(T, l)\chi(P^1, l)$ by virtue of Fact *I*. Hence, we have $\chi(S, l) = \chi(T, l)$ because of $\chi(P^1, l) = 2$. Thus we get our proof, since $\chi(P^2, l) = 3$ and for a geometrically ruled surface Z, $\chi(Z, l)$ is a multiple of 4 by 2) of Proposition 1.2. q.e.d.

§2. Two criterions on the ruledness of S and T

Let us maintain a variety X with two P^1-bundle structures $(S, T; p, q)$ in Introduction.

Then, our main goal in this section is to get two sufficient conditions for S and T to be ruled.

First, let us begin with an easy

Proposition 2.1. *Let X be a 3-fold with two P^1-bundle structures $(S, T; p, q)$. Then, for each point s in S, $qp^{-1}(s)(= C_S)$ is a curve. Similarly for each point t in T, $pq^{-1}(t)(= C_t)$ is a curve.*

By the condition P in Introduction, this proposition is easily shown (see proof of Lemma 1.5 in [Sa]).

Now, for a point s in S, X_S denotes $pq^{-1}qp^{-1}(s)$ and for t in T, X_t denotes $qp^{-1}pq^{-1}(t)$. Then we have a

Proposition 2.2. *Under the above notations, let us assume that there is a point t in T such that X_t is a curve. Then we have*

1) *$C_t(= pq^{-1}(t))$ and X_t are smooth rational curves.*
2) *$p^{-1}pq^{-1}(t)(= Y)$ is isomorphic to $P^1 \times P^1$ and two restricted maps $p|_Y$, $q|_Y$ coincide with two canonical projections from $P^1 \times P^1$ to P^1 respectively.*
3) *For every point s in C_t, $X_s = C_t$.*

Proof. Since $p|_Y: Y \to C_t$ and $q|_Y: Y \to X_t$ are P^1-bundles, we see that $\mathrm{Sing}(Y) = P^{-1}(\mathrm{Sing}(C_t)) = q^{-1}(\mathrm{Sing}(X_t))$ where Sing $*$ denotes the singular locus of a scheme $*$. Hence, Proposition 2.1 yields the smoothness of C_t, X_t and Y. Next let us prove 2). For the purpose we need

Sublemma 2.2.1. *Let $\phi: F_n \to P^1$ be a rational ruled surface with $F_n \simeq P(\mathcal{O}_{P^1} \oplus \mathcal{O}_{P^1}(n))$. Assume that C is an irreducible reduced curve of F_n, $\phi: C \to P^1$ is finite and the self-intersection number C^2 of C is non-positive. Then, we have two cases:*
1) *if $n = 0$, then C is a trivial section of ϕ.*
2) *if n is positive, then C is the minimal section in F_n.*

Proof. Let C_0 the minimal section of F_n (in case of $n = 0$, C_0 means a trivial section) and f a fiber of ϕ. Then C is linearly equivalent to $aC_0 + bf$ with integers a, b. The surjectivity of $\phi: C \to P^1$ implies that a is positive. Thus, by $C^2 \leq 0$, we have $2b \leq an$. Now, assuming that $C \neq C_0$, namely $(C, C_0) \geq 0$, we get $b \geq an$ and therefore $n = b = 0$. Thus we are done. q.e.d.

3) is obvious by virtue of 2). q.e.d.

Before stating a sufficient condition for S and T to be ruled, we recall

Lemma 2.3. *Let Z be a smooth complete surface. Assume that Z has uncountably infinitely many smooth rational curves. Then Z is ruled.*

Proof. By the assumption, we can choose an infinite subset of rational curves on Z: $W = \{C_W \simeq P^1\}$ whose Hilbert polynomial (with respect to a hyperplane section) is independent of a choice of an element in W. Letting C be a smooth curve in W, we see that the self-intersection number of $C (= C^2)$ is non-negative and, therefore, $C \cdot K_Z$ is negative by the adjunction formula. Thus we infer that $H^0(Z, K_Z^{\otimes m})$ vanishes for every positive integer m, which yields the desired result. q.e.d.

Therefore we obtain

Proposition 2.4. *Under the same conditions and notations as in Proposition* 2.1, *assume that for every point t in T, X_t is a curve with $X_* = qp^{-1}pq^{-1}(*)$, Then S and T are ruled.*

Proof. First, note that $X_t = X_{t'}$ for each point t' in X_t by Proposition 2.2. Thus, T is a disjoint union of smooth rational curves $\{C_a | a \in A\}$ and, therefore, so is S $(= \cup \{D_b | b \in B\})$. Moreover, A and B have the same cardinal number by 2) in Proposition 2.2. Now, let us consider the case that A is an uncountably infinite set. Then, by Lemma 2.3, we see that T is ruled and, therefore so is S. Next, let us consider a general case. Letting K be an algebraically closed field containing k such that $\operatorname{trans deg}_k K = \infty$, take the base extension of $X = (S, T; p, q)$ by $\operatorname{Spec} K$:

$$\bar{S} \xleftarrow{\ \bar{p}\ } \bar{X} \xrightarrow{\ \bar{q}\ } \bar{T}$$

Then, we see that the morphisms $\bar{p}$ and $\bar{q}$ induced by p and q are P^1-bundles and $(\bar{S}, \bar{T}; \bar{p}, \bar{q})$ has the same assumption as in Proposition 2.4. Hence, it follows from the above argument that $\bar{T}$ has an uncountably infinitely smooth rational curves. Thus $\bar{T}$ is ruled. Since $H^0(\bar{T}, K_{\bar{T}}^{\otimes m}) = K \times_k H^0(T, K_T^{\otimes m})$, T is ruled. Similarly S is ruled. q.e.d.

From now on we shall study another sufficient condition for S and T to be ruled.

First we prepare the following. Let X be a smooth 3-fold with two P^1-bundle structures $(S, T; p, q)$. Then, the two P^1-bundle structures of X yield the following

$$(2.5)' \qquad \begin{gathered} 0 \longrightarrow p^*\Omega_S^1 \longrightarrow \Omega_X^1 \longrightarrow \Omega_p \longrightarrow 0 \\ 0 \longrightarrow q^*\Omega_T^1 \longrightarrow \Omega_X^1 \longrightarrow \Omega_q \longrightarrow 0 \end{gathered}$$

where Ω_p and Ω_q are the relative cotangent bundles of p and q.

The above yields the following

$$(2.5) \qquad \begin{gathered} 0 \longrightarrow p^*K_S \longrightarrow \overset{2}{\Lambda}\Omega_X^1 \longrightarrow p^*\Omega_S^1 \otimes \Omega_p \longrightarrow 0 \\ 0 \longrightarrow q^*K_T \longrightarrow \overset{2}{\Lambda}\Omega_X^1 \longrightarrow q^*\Omega_T^1 \otimes \Omega_q \longrightarrow 0 \end{gathered}$$

On the other hand we have a well-known

Proposition 2.6. Let us consider the following exact sequence of vector bundles: $0 \longrightarrow E_1 \longrightarrow E \longrightarrow E_2 \longrightarrow 0$. Then, $S^m(E)(= F_{m+1})$ has a sequence of subbundles:

$$0 = F_0 \subset F_1 \subset \cdots \subset F_m \subset F_{m+1}$$

where $F_{i+1}/F_i = S^{m-i}(E_1) \otimes S^i(E_2) (1 \leq i \leq m)$.

Thus, applying Proposition 2.6 to the exact sequences 2.5,

Corollary 2.6.1. *Let X be a smooth 3-fold with two P^1-bundle structures $(S, T; p, q)$. Then, there are canonical isomorphisms*:

$$H^0(X, p^*K_S^{\otimes m}) (\simeq H^0(S, K_S^{\otimes m})) \simeq H^0(X, S^m(\overset{2}{\Lambda}\Omega_X)) \simeq H^0(X, q^*K_T^{\otimes m}).$$

Proof. By restricting the vector bundle $p^*K_S^{\otimes m-i} \otimes S^i(p^*\Omega_S^1 \otimes \Omega_p) (= G_i)$ to a fiber of p, we see that for each integer i $(1 \leq i \leq m)$ $G_{i|_{p^{-1}(s)}} = S^i(\mathcal{O}(-2) \oplus \mathcal{O}(-2))$ on $p^{-1}(s)(= P^1)$, therefore, $H^0(X, G_i)$ vanishes, and it follows that the quotient of $S^m(\overset{2}{\Lambda}\Omega_X)$ by $p^*K_S^{\otimes m}$ has only a zero section by Proposition 2.6. The quotient of $S^m(\overset{2}{\Lambda}\Omega_X)$ by $q^*K_T^{\otimes m}$ has only a zero section in the same way as above. Thus we complete our proof. q.e.d.

Thus, we have an important criterion about the ruledness of S and T.

Proposition 2.7. *Let X be a smooth 3-fold with two P^1-bundle structures.*

Assume $H^0(X, S^m(\overset{2}{\Lambda}\Omega_X)) = 0$ *for every positive integer m. Then, S and T are ruled.*

§ 3. Ruledness of S and T

Let us maintain a smooth 3-fold X with two P^1-bundle structures $(S, T; p, q)$. In the present section, we shall show that S and T are ruled by using the results in § 2.

First, taking into account of Proposition 2.4 and Proposition 2.7, we consider the following two conditions:

(3.1.1) There is a point s in S such that $pq^{-1}qp^{-1}(s) = S$. Note that S is unirational, since $q^{-1}qp^{-1}(s)$ is a rational surface.
(3.1.2) There is a positive integer m such that $H^0(X, S^m(\overset{2}{\Lambda}\Omega_X))$ has a non-zero section.

Thus, in order to prove that S and T are ruled, we have only to show

Proposition 3.2. *Let X be a smooth 3-fold with two* P^1*-bundle structures* $(S, T; p, q)$. *Then, there exists no such X enjoying two conditions* (3.1.1) *and* (3.1.2).

For the purpose, we make several preparations.
First, let us start with an easy

Proposition 3.3. *Assume that the condition* 3.1.2 *holds. Then we have* $p^*K_S^{\otimes m} = q^*K_T^{\otimes m}$.

It is trivial by Corollary 2.6.1.

In the next place, we show that Proposition 3.3 yields the fact that K_S and K_T are numerically equivalent to zero. For the purpose, let me state a proposition by Kleiman [K].

Let V be a complete algebraic scheme over k and M an invertible sheaf on V. We call M numerically trivial and write $M \equiv 0$ if $(M.C)_V = 0$ for all closed integral curves C in V. Then he shows that

Proposition 3.4. (§ 4. Corollary 1 [K])
Let $f: V' \longrightarrow V$ *be a morphism between algebraic complete schemes, M an invertible sheaf on V and* $M' = f^*M$. *Then we have*

(i) $M \equiv 0$ *implies* $M' \equiv 0$, *and conversely,*
(ii) $M' \equiv 0$ *implies* $M \equiv 0$, *if f is surjective.*

Now, we have an important

Proposition 3.5. *Under the condition in Proposition* 3.2, *let us assume the condition* 3.1.1 *and* $p^*K_S^{\otimes m} = q^*K_T^{\otimes m}$. *Then* K_S *and* K_T *are numerically equivalent to zero.*

Moreover, $\kappa(S) = \kappa(T) = 0$.

Proof. Take a point s in S such tht $pq^{-1}qp^{-1}(s) = S$ by our assumption. Then, letting $f = p^{-1}(s)$, we see that $q^*K_{T|f}^{\otimes m}$ is trivial, which implies that $K_{T|q(f)}^{\otimes m} \equiv 0$ by Proposition 3.4. Now, consider $p^*K_{S|W}^{\otimes m} = q^*K_{S|W}^{\otimes m}$ with $W = q^{-1}(q(f))$. Noting that $W \to S$ is surjective, we infer that K_S is numerically equivalent to zero and so is K_T thanks to Proposition 3.4.

The latter part is obvious. q.e.d.

By Fact I and II in 1.1, 1) of Proposition 1.3 and the above Proposition 3.5, we easily get

$$\beta_2(S, l) = 2. \tag{3.6}$$

(See also the table of possible invariants for surfaces with $\kappa = 0$ in the Introduction in [Bo + Mu])

Moreover, the following stated after theorem 6 in the Introduction in [Bo + Mu] takes an essential part of the proof of Proposition 3.2.

Proposition 3.7. *If X is a surface with $\kappa = 0$, $\beta_2 = 2$, then $\beta_1 = 2$, hence* $\mathrm{Alb}(X)$ *is an elliptic curve and the fibers of the canonical map π: $X \to \mathrm{Alb}(X)$ are either almost all non-singular elliptic curves or almost all rational curves with ordinary cusps.*

The latter is only possible if char $k = 2$ *or* 3.

Proof of Proposition 3.2. By virtue of (3.6), condition 3.1.1 contradicts Proposition 3.7. Thus we complete our proof. q.e.d.

Combining Proposition 2.4 and Proposition 3.2, we get

Theorem 3.8. *Let X be a smooth 3-fold with two P^1-bundle structures $(S, T; p, q)$. Then, S and T are geometrically ruled surfaces or they are P^2.*

§4. Proof of Theorem

In this section we shall give a proof of Thereom.

The argument in §3 [Sa] by which our Theorem is proved in characteristic zero, is still valid almost everywhere in positive characteristic. But, since we used the fact that a morphism in characteristic zero is separable in the proof of Propsition 3.8 [Sa], we shall make a slight modification as for the proposition.

Now, let us begin a proof of theorem.

By the result in Theorem 3.8, we divide into two cases:

a) S and T are geometrically ruled surfaces.

b) S and T are P^2.

Let us start with case a).

Let $\bar{q}$: $T \to C$ be the P^1-bundle over a non-singular curve C. Put $\bar{q}^{-1}(c) = l_c$ for a point c in C.

Remark 4.1. Under the above notation, let us assume that there is a point c

of C such that $p\colon q^{-1}(l_c) \to S$ is surjective. Then for every point c in C, $p\colon q^{-1}(l_c) \to S$ is surjective.

Therefore we shall consider the structure of X in two cases as follows.

(4.2) For every point c in C, $\dim pq^{-1}(l_c) = 1$.

(4.3) For every point c in C, $\dim pq^{-1}(l_c) = 2$.

First let us treat the case 4.2. Then we have

Proposition 4.4. (*Proposition* 3.7 *in* [*Sa*]), *In the case* 4.2, *X is isomorphic to* $S \times {}_C T$, *where both S and T are* P^1*-bundles over a non-singular curve C.*

The proof in Proposition 3.7 in [Sa] is available even in the case of positive characteristic.

In the next place, we observe the case (4.3).

Let $\bar{p}\colon S \to B$ be the P^1-bundle over a non-singular curve B.

Then, it is easily seen that if there is a point t_0 in T such that $\bar{p}pq^{-1}(t_0)$ is one point in B, then for every point t in T, $\bar{p}pq^{-1}(t)$ is one point.

Thus we divide into two cases. Namely, the image of every fiber of q via $\bar{p}p\colon X \to B$ is

α) a point ,or

β) B.

Let us study the case α.

By the condition, there is a point b in B such that a rational ruled surface $(\bar{p}p)^{-1}(b)$ contains infinitely many fibers of $q\colon X \to T$. Thus we see that the image of the rational ruled surface via q is a curve. Hence, by Remark 4.1, we can reduce to the first case 4.2.

Next we shall deal with case β).

Since $q^{-1}(l_c)$ is a rational ruled surface and for each point b in B, $(\bar{p}p)^{-1}(b) \cap q^{-1}(l_c)$ has an irreducible component whose self-intersection number is non-positive, we see that $q^{-1}(l_c) = P^1 \times P^1$ by sublemma 2.2.1. Since $p\colon q^{-1}(l_c) \to S$ is surjective, we infer that S is $P^1 \times P^1$. Let $p'\colon S \to P^1 (= B')$ be another canonical projection besides $\bar{p}$. Noting that each fiber of $q^{-1}(l_c) \to l_c$ goes to a point via $p'p$, we can reduce β) to the case α). Hence, we finish the observation of the case (4.3).

Thus, summarizing the above argument, we obtain

Proposition 4.5. *In the case* a), *X is isomorphic to* $S \times {}_C T$ *where S and T are geometrically ruled surfaces over a non-singular curve C.*

Finally, let us consider the case b). Then we see that p and q are P^1-bundles in the Zariski topology by Lemma 1.3, and Corollary 1.4 in [Sa]. Hence we complete a proof of the case b) by virtue of 2) in Theorem A in [Sa].

Thus we finish our proof of Theorem.

DEPARTMENT OF MATHEMATICS
COLLEGE OF GENERAL EDUCATION
KYUSHU UNIVERSITY

References

[Bo + Mu] E. Bombieri and D. Mumford, Enriques' classification of surfaces in char. p. II, Complex Analysis and Algebraic Geometry, A collection of papers dedicated to K. Kodaira, Iwanami Shoten Publishers, Tokyo, and Princeton University Press, Cambridge (1977), 23–42.

[E] T. Ekedahl, Canonical models of surfaces of general type in positive characteristic, preprint.

[Ha] R. Hartshorne. Algebraic Geometry. Springer, Heidelberg, 1977.

[K] S. Kleiman, Toward numerical theory of ampleness, Annals of Math., **84** (1966), 293–344.

[Mi] J.S. Milne, Etale cohomology, Princeton Univ. Press.

[Mu] D. Mumford, Lectures on curves on an algebraic surface. Annals of Math. Studies 59, Princeton University Press. Princeton, 1966.

[Sa] E. Sato, Varieties which have two projective space bundle structures, J. Math. Kyoto Univ., **25**-3 (1985) 445–457.

[Za] O. Zariski, On Castelnuovo's criterion of rationality $P_a = P_2 = O$, Illinois J. Math., **2** (1958), 303–315.