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On 2-Buchsbaum complexes

By

Mitsuhiro MIYAZAKI

0. Introduction

Let K be a field, fixed throughout this paper. By using the Stanley-Reisner ring over K the concept of commutative algebra such as Cohen-Macaulay or Buchsbaum are immediately transferred to the concept of simplicial complexes. Some of them such as Cohen-Macaulay or Buchsbaum are well behaving concept and it is known for example Cohen-Macaulayness and Buchsbaumness are topological properties. (i.e. if Δ_1 and Δ_2 are simplicial complexes whose geometric realizations are homeomorphic, then Δ_1 is Cohen-Macaulay (or Buchsbaum) if and only if Δ_2 is Cohen-Macaulay (or Buchsbaum resp.)) and characterized by the reduced oriented homology groups of Δ. (See [10], [11] and [12].) And by the result of Reisner [11] (see Theorem 3.2 of this paper) we know that if a simplicial complex Δ is Cohen-Macaulay of $\dim \Delta \geq 1$ then Δ is connected. So one can consider the Cohen-Macaulay property as a specialization of the connectedness. Baclawski [1] called the Cohen-Macaulayness as Cohen-Macaulay connectivity and defined the k-Cohen-Macaulayness by the similar way as the k-connectivity. (See §1 for definition.) Then the 2-Cohen-Macaulayness is a well behaving concept and the following facts are known.

(i) 2-Cohen-Macaulayness is a topological property. ([17])

(ii) If Δ is a Cohen-Macaulay complex of dimension r then $(r-1)$-skeleton of Δ is 2-Cohen-Macaulay. ([6])

It is natural to ask if the similar results are valid for Buchsbaumness. Since a simplicial complex Δ is Buchsbaum if and only if Δ is pure and every non-trivial link of Δ is Cohen-Macaulay (see Theorem 4.1 of this paper), it is also natural to ask if the results similar to (i) and (ii) above are valid for pureness. The purpose of this paper is to give affirmative answers to these questions. i.e.

(i) 2-Buchsbaumness (or 2-pureness) is a topological property. (See Theorems 4.3 and 5.3.)

(ii) If Δ is a Buchsbaum (or pure) complex of dimension r then the $(r-1)$-skeleton of Δ is 2-Buchsbaum (or 2-pure resp.). (See Theorems 7.4 and 7.3.)

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1. Preliminaries

We denote the number of elements of a finite set X by $\# X$ and for two sets X and Y we denote by $X - Y$ the set $\{x \in X \mid x \notin Y\}$.

We make a convention in this paper that all the simplicial complexes are finite. A simplicial complex Δ with vertex set V is a set of subsets of V such that (i) $\phi \in \Delta$ and (ii) if $\sigma \in \Delta$ and $\tau \subseteq \sigma$ then $\tau \in \Delta$. Note that we do not require that $\{x\} \in \Delta$ for any $x \in V$. An element of Δ is called a face of Δ and a maximal (with respect to the inclusion relation) face is called a facet of Δ. For Δ and a face σ of Δ, we define the dimension of σ written $\dim \sigma$ by $\dim \sigma = \# \sigma - 1$ and dimension of Δ written $\dim \Delta$ by $\dim \Delta = \max_{\sigma \in \Delta} \dim \sigma$. In particular $\dim \{\phi\} = -1$. If all the facets of Δ has the same dimension we say Δ is pure.

Now we define two subcomplexes $\mathrm{star}_\Delta(\sigma)$ and $\mathrm{link}_\Delta(\sigma)$ of Δ decided by σ by

$$\mathrm{star}_\Delta(\sigma) = \{\tau \in \Delta \mid \tau \cup \sigma \in \Delta\}$$
$$\mathrm{link}_\Delta(\sigma) = \{\tau \in \Delta \mid \tau \cup \sigma \in \Delta,\ \tau \cap \sigma = \phi\}.$$

Note $\dim \mathrm{star}_\Delta(\sigma) = \max_{\tau:\text{facet of }\Delta,\, \tau \supseteq \sigma} \dim \tau$ and $\dim \mathrm{link}_\Delta(\sigma) = \dim \mathrm{star}_\Delta(\sigma) - \# \sigma$. Moreover if Δ is pure then $\dim \mathrm{star}_\Delta(\sigma) = \dim \Delta$ and $\dim \mathrm{link}_\Delta(\sigma) = \dim \Delta - \# \sigma$. We also define a subcomplex $\Delta \backslash \sigma$ of Δ for $\sigma \in \Delta - \{\phi\}$ by

$$\Delta \backslash \sigma = \{\tau \in \Delta \mid \tau \not\supseteq \sigma\}.$$

Let W be a subset of the vertex set V of Δ. We define a subcomplex Δ_W of Δ by

$$\Delta_W = \{\sigma \in \Delta \mid \sigma \subseteq W\}.$$

Note that if $x \in V$ then $\Delta_{V - \{x\}} = \Delta \backslash x$.

Let Δ_1 and Δ_2 be simplicial complexes with vertex set V_1 and V_2 respectively. If $V_1 \cap V_2 = \phi$ then we define a complex $\Delta_1 * \Delta_2$ with vertex set $V_1 \cup V_2$ by

$$\Delta_1 * \Delta_2 = \{\sigma \cup \tau \mid \sigma \in \Delta_1,\ \tau \in \Delta_2\}.$$

One immediately verifies that $\dim(\Delta_1 * \Delta_2) = \dim \Delta_1 + \dim \Delta_2 + 1$ and a facet of $\Delta_1 * \Delta_2$ is a union of a facet of Δ_1 and a facet of Δ_2.

Next we recall some general facts about commutative algebras. General references are [8], [16] and [4]. Let K be a field and we fix the field K throughout this paper. Let $A = K[x_1, \ldots, x_n]$ be a polynomial ring over K. Then A is a $\mathbf{Z}^n$-graded ring in the natural way and if M and N are finitely generated A-modules then we can define the $\mathbf{Z}^n$-graded structure to $\mathrm{Hom}_A(M, N)$ by $[\mathrm{Hom}_A(M, N)]_\alpha = \{f \in \mathrm{Hom}_A(M, N) \mid f(M_\beta) \subseteq N_{\alpha+\beta}$ for any $\beta \in \mathbf{Z}^n\}$ and to $M \otimes_A N$ by $(M \otimes_A N)_\alpha =$ (The submodule of $M \otimes_A N$ generated by the elements $a \otimes_A b$ such that $a \in M_\beta$, $b \in N_\gamma$ and $\beta + \gamma = \alpha$.). So we can also define the $\mathbf{Z}^n$-graded structure of $\mathrm{Ext}^i_A(M, N)$ and $\mathrm{Tor}^A_i(M, N)$ for any i. Moreover if I is a homogeneous ideal (in this grading) of A then we can also define the $\mathbf{Z}^n$-graded structure to the local cohomology modules $H_I^i(M)$. See [2] and [3] for the details.

We define the dimension (Krull dimension) of M written by $\dim M$ to be the maximal length of prime ideal chains in the ring $A/\mathrm{ann}(M)$ i.e. $\dim M = \max\{d \mid \text{There exist prime ideals } P_0,\dots,P_d \text{ in } A \text{ such that } \mathrm{ann}(M) \subseteq P_0 \subsetneq \cdots \subsetneq P_d\}$. And the depth of M written by $\operatorname{depth} M$ is defined by the following three identical numbers

(i) The length of a maximal M-regular sequence in m.
(ii) $\min\{i \mid \mathrm{Ext}_A^i(K, M) \neq 0\}$
(iii) $\min\{i \mid H_m^i(M) \neq 0\}$

where $m = (x_1,\dots,x_n)A$. (All maximal M-regular sequences in m are known to have the same length.)

It is known that $\operatorname{depth} M \leq \dim M$ for arbitrary $M \neq 0$ and we say M is a Cohen-Macaulay module if $\operatorname{depth} M = \dim M$. And we say M is a Buchsbaum module if the canonical map

$$\mathrm{Ext}_A^i(K, M) \longrightarrow H_m^i(M)$$

is surjective for any $i < \dim M$. It is clear from the definition that every Cohen-Macaulay module is Buchsbaum. A residue class ring A/I, where I is a homogeneous ideal, is called a Cohen-Macaulay (or Buchsbaum) ring if A/I is a Cohen-Macaulay (or Buchsbaum resp.) A-module. Moreover we say A/I is a Gorenstein ring if

$$\mathrm{Ext}_A^i(K, A/I) \cong \begin{cases} K & \text{if } i = \dim A/I \\ 0 & \text{if } i < \dim A/I. \end{cases}$$

So if A/I is a Gorenstein ring then A/I is Cohen-Macaulay.

Next we define the shift of grading. For $\alpha \in \mathbf{Z}^n$ and a $\mathbf{Z}^n$-graded module M, we define the module $M(\alpha)$ with shifted grade by $[M(\alpha)]_\beta = M_{\alpha+\beta}$ for any $\beta \in \mathbf{Z}^n$. For example the element of $A(-1,\dots,-1)$ corresponding to $1 \in A$ has degree $(1,\dots,1)$. So if we write the fact $\forall j; \alpha_j \leq \beta_j$ by $\alpha \leq \beta$ or $\beta \geq \alpha$ for $\alpha = (\alpha_1,\dots,\alpha_n)$, $\beta = (\beta_1,\dots,\beta_n) \in \mathbf{Z}^n$ we see

$$A(-1,\dots,-1)_\alpha = \begin{cases} K & \text{if } \alpha \geq (1,\dots,1) \\ 0 & \text{if } \alpha \ngeq (1,\dots,1). \end{cases}$$

Now we define the Stanley-Reisner ring (or face ring) $K[\Delta]$ of Δ(over K) for a simplicial complex Δ with vertex set $V = \{x_1,\dots,x_n\}$. Take a polynomial ring over K whose indeterminates are in one to one correspondence with the elements of V. We denote this polynomial ring by $K[x_1,\dots,x_n](= A)$ for simplicity. Then

$$K[\Delta] = A/I_\Delta$$

where I_Δ is the ideal generated by $\{x_{j_1}\cdots x_{j_t} \mid 1 \leq j_1 < \cdots < j_t \leq n,\ \{x_{j_1},\dots,x_{j_t}\} \notin \Delta\}$.

It is easily verified that $K[\Delta]$ is a K-free module (without assuming that K is a field) and the set of all the monomials whose support is in Δ is K-free basis of $K[\Delta]$. (The support of a monomial M is $\{x \in V | M$ is divisible by $x\}$.) We can also verify that

$$I_\Delta = \bigcap_{\sigma : \text{facet of } \Delta} P_\sigma$$

where $P_\sigma = (x_i | x_i \notin \sigma)$, is the primary decomposition of I_Δ. So we see I_Δ is a radical ideal of A and $K[\Delta]$ is a reduced ring. Moreover we see

$$\begin{aligned} \dim K[\Delta] &= \operatorname{coht} I_\Delta \\ &= \max_{\sigma : \text{facet of } \Delta} \operatorname{coht} P_\sigma \\ &= \max_{\sigma : \text{facet of } \Delta} \# \sigma \\ &= \dim \Delta + 1 \end{aligned}$$

where $\operatorname{coht} I = \dim A/I$ for an ideal I of A.

If $K[\Delta]$ is a Cohen-Macaulay (Buchsbaum or Gorenstein) ring we say that Δ is a Cohen-Macaulay (Buchsbaum or Gorenstein resp.) complex (over K). Since it is known that all the minimal prime ideals of $\operatorname{ann}(M)$ has the same coheight for a Buchsbaum A-module M, we see by the above argument that if Δ is Buchsbaum then Δ is pure.

It is easy to prove that if $\{x\} \in \Delta$ then

$$K[\Delta]_x = K[\operatorname{link}_\Delta(x)][x, x^{-1}]$$

and

$$\operatorname{link}_\Delta(\tau \cup \sigma) = \operatorname{link}_{\operatorname{link}_\Delta(\tau)}(\sigma) \text{ if } \tau \cup \sigma \in \Delta \text{ and } \tau \cap \sigma = \phi.$$

So

$$K[\operatorname{link}_\Delta(\sigma)][x, x^{-1} | x \in \sigma] = K[\Delta]_{\prod_{x \in \sigma} x}$$

for any $\sigma \in \Delta$. Since it is known that if M is a Buchsbaum A-module then M_x is Cohen-Macaulay for any $x \in V$, we see if Δ is Buchsbaum then $\operatorname{link}_\Delta(\sigma)$ is Cohen-Macaulay for any $\sigma \in \Delta$ such that $\sigma \neq \phi$. Since

$$K[\operatorname{star}_\Delta(\sigma)] = K[\operatorname{link}_\Delta(\sigma)][x | x \in \sigma]$$

we can also see that every non-trivial star of a Buchsbaum complex is Cohen-Macaulay.

For a positive integer k, we define the k-Cohen-Macaulay (k-Buchsbaum and k-pure) complexes as follows. A simplicial complex Δ with vertex set V is k-Cohen-Macaulay (k-Buchsbaum or k-pure)(over K) of dimension r if for any subset W of V (including ϕ) such that $\# W < k$, Δ_{V-W} is Cohen-Macaulay (Buchsbaum or pure resp.) of dimension r and Δ is k-Cohen-Macaulay (k-

Buchsbaum or k-pure) if Δ is k-Cohen-Macaulay (k-Buchsbaum or k-pure resp.) of some dimension r.

Now we recall the notation of oriented and singular (co)chain complexes and (co)homology groups. If Δ is a simplicial complex we denote by $\tilde{C}_{\cdot}(\Delta)$(or $\tilde{C}^{\cdot}(\Delta)$) the augmented oriented chain (or cochain resp.) complex with coefficients in K and by $\tilde{H}_{\cdot}(\Delta)$(or $\tilde{H}^{\cdot}(\Delta)$) the homology (or cohomology resp.) groups of $\tilde{C}_{\cdot}(\Delta)$(or $\tilde{C}^{\cdot}(\Delta)$ resp.). If Δ_1 is a subcomplex of Δ then we denote the relative oriented chain (or cochain) complex by $C_{\cdot}(\Delta, \Delta_1)$(or $C^{\cdot}(\Delta, \Delta_1)$ resp.) and its homology (or cohomology resp.) groups by $H_{\cdot}(\Delta, \Delta_1)$(or $H^{\cdot}(\Delta, \Delta_1)$ resp.). If X is a topological space then we denote by $\tilde{C}_{\cdot}(X)$ (or $\tilde{C}^{\cdot}(X)$) the augmented singular chain (or cochain resp.) complex with coefficients in K and by $\tilde{H}_{\cdot}(X)$(or $\tilde{H}^{\cdot}(X)$) the homology (or cohomology resp.) groups of $\tilde{C}_{\cdot}(X)$(or $\tilde{C}^{\cdot}(X)$ resp.). If Y is a subspace of X then we denote the relative singular chain (or cochain) complex by $C_{\cdot}(X, Y)$(or $C^{\cdot}(X, Y)$ resp.) and its homology (or cohomology resp.) groups by $H_{\cdot}(X, Y)$(or $H^{\cdot}(X, Y)$ resp.). See, for example, [13] for the details.

For a chain (or cochain) complex $D_{\cdot}$ (or $D^{\cdot}$ resp.) we define the complex $D_{\cdot}[t]$(or $D^{\cdot}[t]$ resp.) with shift of dimension by $(D_{\cdot}[t])_i = D_{i+t}$(or$(D^{\cdot}[t])^i = D^{i+t}$ resp.).

Finally we make a convention. When considering a simplicial complex Δ with vertex set V we define the support of $\alpha \in \mathbf{Z}^{\# V}$ to be a subset of V as follows. Take and fix a bijective map of sets $\varphi: V \to \{1, \ldots, \# V\}$ and if $\alpha = (\alpha_1, \ldots, \alpha_{\# V})$ then $\operatorname{supp}\alpha = \{x \in V \mid \alpha_{\varphi(x)} \neq 0\}$. So every time we consider a simplicial complex, we assume that a map φ as above is given and fixed. Especially if we write that Δ is a simplicial complex with vertex set $V = \{x_1, \ldots, x_n\}$ then we assume φ is the map such that $\varphi(x_i) = i$ for any $i = 1, \ldots, n$, so $\operatorname{supp}\alpha = \{x_j \mid \alpha_j \neq 0\}$ for $\alpha \in \mathbf{Z}^n$.

2. Key lemmas

In this section we state some useful lemmas in the following sections.

Lemma 2.1. *Let Δ be a simplicial complex with vertex set $V = \{x_1, \ldots, x_n\}$, $[x_{i_1}, \ldots, x_{i_s}]$ an oriented face in Δ and $\sigma = \{x_{i_1}, \ldots, x_{i_s}\}$. Then there is an isomorphism of chain complexes*

$$\varphi_{\cdot}^{\Delta, [x_{i_1}, \ldots, x_{i_s}]}: \tilde{C}_{\cdot}(\operatorname{link}_{\Delta}(\sigma))[-s] \longrightarrow C_{\cdot}(\Delta, \Delta \backslash \sigma).$$

Moreover if $1 \leq t \leq s$ then we have the following commutative diagram.

$$\begin{array}{ccc}
\tilde{C}_{\cdot}(\operatorname{link}_{\Delta}(\tau))[-t] & \longrightarrow & C_{\cdot}(\operatorname{link}_{\Delta}(\tau), \operatorname{link}_{\Delta}(\tau) \backslash (\sigma - \tau))[-t] \\
\Big\downarrow \varphi_{\cdot}^{\Delta, [x_{i_{s-t+1}}, \ldots, x_{i_s}]} & & \Big\downarrow (\varphi_{\cdot}^{\operatorname{link}_{\Delta}(\tau), [x_{i_1}, \ldots, x_{i_{s-t}}]})^{-1} \\
 & & \tilde{C}_{\cdot}(\operatorname{link}_{\Delta}(\sigma))[-s] \\
 & & \Big\downarrow \varphi_{\cdot}^{\Delta, [x_{i_1}, \ldots, x_{i_s}]} \\
C_{\cdot}(\Delta, \Delta \backslash \tau) & \longrightarrow & C_{\cdot}(\Delta, \Delta \backslash \sigma)
\end{array}$$

where $\tau = \{x_{i_{s-t+1}}, \dots, x_{i_s}\}$ *and the horizontal maps are the canonical ones.*

Proof. Since $C_u(\Delta, \Delta\backslash\sigma)$ is the cokernel of the inclusion map $\tilde{C}_u(\Delta\backslash\sigma) \hookrightarrow \tilde{C}_u(\Delta)$ we see $C_u(\Delta, \Delta\backslash\sigma)$ is a free module over K with basis

$$\{F \in \Delta \mid F \notin \Delta\backslash\sigma, \# F = u\} = \{F \in \Delta \mid F \supseteq \sigma, \# F = u\}$$

for any u. By associating F with $F - \sigma = G$ we may assume that $C_u(\Delta, \Delta\backslash\sigma)$ is a free module with basis

$$\begin{aligned}&\{G \in \Delta \mid G \cap \sigma = \phi,\ G \cup \sigma \in \Delta,\ \# G = u - s\} \\ &= \{G \in \operatorname{link}_\Delta(\sigma) \mid \# G = u - s\}.\end{aligned}$$

So we can construct an isomorphism of K-free modules as follows.

$$\begin{aligned}\varphi_u^{\Delta,[x_{i_1},\dots,x_{i_s}]}\colon\ &\tilde{C}_{u-s}(\operatorname{link}_\Delta(\sigma)) \longrightarrow C_u(\Delta, \Delta\backslash\sigma) \\ &[x_{j_1}, \dots, x_{j_{u-s}}] \longmapsto [x_{j_1}, \dots, x_{j_{u-s}}, x_{i_1}, \dots, x_{i_s}] \bmod \tilde{C}_u(\Delta\backslash\sigma)\end{aligned}$$

Next we have to show that $\varphi_.^{\Delta,[x_{i_1},\dots,x_{i_s}]}$ is a chain map. Take $[x_{j_1}, \dots, x_{j_{u-s}}] \in \tilde{C}_{u-s}(\operatorname{link}_\Delta(\sigma))$. Then

$$\begin{aligned}&\partial_u \varphi_u^{\Delta,[x_{i_1},\dots,x_{i_s}]}([x_{j_1}, \dots, x_{j_{u-s}}]) \\ &= \partial_u([x_{j_1}, \dots, x_{j_{u-s}}, x_{i_1}, \dots, x_{i_s}] \bmod \tilde{C}_u(\Delta\backslash\sigma)) \\ &= \textstyle\sum_{l=1}^{u-s} (-1)^{l-1} [x_{j_1}, \dots, \hat{x}_{j_l}, \dots, x_{j_{u-s}}, x_{i_1}, \dots, x_{i_s}] \bmod \tilde{C}_{u-1}(\Delta\backslash\sigma) \\ &\quad + \textstyle\sum_{l=1}^{s} (-1)^{u-s+l-1} [x_{j_1}, \dots, x_{j_{u-s}}, x_{i_1}, \dots, \hat{x}_{i_l}, \dots, x_{i_s}] \bmod \tilde{C}_{u-1}(\Delta\backslash\sigma) \\ &= \textstyle\sum_{l=1}^{u-s} (-1)^{l-1} [x_{j_1}, \dots, \hat{x}_{j_l}, \dots, x_{j_{u-s}}, x_{i_1}, \dots, x_{i_s}] \bmod \tilde{C}_{u-1}(\Delta\backslash\sigma)\end{aligned}$$

because $\{x_{j_1}, \dots, x_{j_{u-s}}, x_{i_1}, \dots, \hat{x}_{i_l}, \dots, x_{i_s}\} \in \Delta\backslash\sigma$ where $\hat{}$ means the omission of the factor. On the other hand

$$\begin{aligned}&\varphi_{u-1}^{\Delta,[x_{i_1},\dots,x_{i_s}]} \partial_{u-s}([x_{j_1}, \dots, x_{j_{u-s}}]) \\ &= \varphi_{u-1}^{\Delta,[x_{i_1},\dots,x_{i_s}]}\left(\textstyle\sum_{l=1}^{u-s} (-1)^{l-1} [x_{j_1}, \dots, \hat{x}_{j_l}, \dots, x_{j_{u-s}}]\right) \\ &= \textstyle\sum_{l=1}^{u-s} (-1)^{l-1} [x_{j_1}, \dots, \hat{x}_{j_l}, \dots, x_{j_{u-s}}, x_{i_1}, \dots, x_{i_s}] \bmod \tilde{C}_{u-1}(\Delta\backslash\sigma).\end{aligned}$$

So we see $\varphi_.^{\Delta,[x_{i_1},\dots,x_{i_s}]}$ is a chain map and we get the first conclusion.

Next consider the following map

$$\psi_.^{\Delta,\sigma,[x_{i_{s-t+1}},\dots,x_{i_s}]}\colon C.(\operatorname{link}_\Delta(\tau), \operatorname{link}_\Delta(\tau)\backslash(\sigma - \tau))[-t] \longrightarrow C.(\Delta, \Delta\backslash\sigma)$$

where

$$\begin{aligned}&\psi_u^{\Delta,\sigma,[x_{i_{s-t+1}},\dots,x_{i_s}]}([x_{j_1}, \dots, x_{j_{u-t}}] \bmod \tilde{C}_{u-t}(\operatorname{link}_\Delta(\tau)\backslash(\sigma - \tau))) \\ &= [x_{j_1}, \dots, x_{j_{u-t}}, x_{i_{s-t+1}}, \dots, x_{i_s}] \bmod \tilde{C}_u(\Delta\backslash\sigma).\end{aligned}$$

Then we can show that $\psi_.^{\Delta,\sigma,[x_{i_{s-t+1}},\dots,x_{i_s}]}$ is a chain map by the same way as above. Also we can easily see the commutativity of the following diagram

$$
\begin{array}{ccc}
\tilde{C}.(\operatorname{link}_{\Delta}(\tau))[-t] & \longrightarrow & C.(\operatorname{link}_{\Delta}(\tau), \operatorname{link}_{\Delta}(\tau)\backslash(\sigma-\tau))[-t] \\
\downarrow{\scriptstyle \varphi.^{\Delta,[x_{i_{s-t+1}},\dots,x_{i_s}]}} & & \downarrow{\scriptstyle \psi.^{\Delta,\sigma,[x_{i_{s-t+1}},\dots,x_{i_s}]}} \\
C.(\Delta, \Delta\backslash\tau) & \longrightarrow & C.(\Delta, \Delta\backslash\sigma)
\end{array}
$$

where the horizontal maps are the canonical ones.

On the other hand the commutativity of the following diagram

$$
\begin{array}{ccc}
C.(\operatorname{link}_{\Delta}(\tau), \operatorname{link}_{\Delta}(\tau)\backslash(\sigma-\tau))[-t] & & \\
\downarrow{\scriptstyle \psi.^{\Delta,\sigma,[x_{i_{s-t+1}},\dots,x_{i_s}]}} & \nwarrow{\scriptstyle \varphi.^{\operatorname{link}_{\Delta}(\tau),[x_{i_1},\dots,x_{i_{s-t}}]}} & \tilde{C}.(\operatorname{link}_{\Delta}(\sigma))[-s] \\
 & \swarrow{\scriptstyle \varphi.^{\Delta,[x_{i_1},\dots,x_{i_s}]}} & \\
C.(\Delta, \Delta\backslash\sigma) & &
\end{array}
$$

can be easily verified. So we see the following diagram

$$
\begin{array}{ccccc}
\tilde{C}.(\operatorname{link}_{\Delta}(\tau))[-t] & \longrightarrow & C.(\operatorname{link}_{\Delta}(\tau), \operatorname{link}_{\Delta}(\tau)\backslash(\sigma-\tau))[-t] & & \\
 & & & \searrow{\scriptstyle (\varphi.^{\operatorname{link}_{\Delta}(\tau),[x_{i_1},\dots,x_{i_{s-t}}]})^{-1}} & \\
\downarrow{\scriptstyle \varphi.^{\Delta,[x_{i_{s-t+1}},\dots,x_{i_s}]}} & & \downarrow{\scriptstyle \psi.^{\Delta,\sigma,[x_{i_{s-t+1}},\dots,x_{i_s}]}} & & \tilde{C}.(\operatorname{link}_{\Delta}(\sigma))[-s] \\
 & & & \swarrow{\scriptstyle \varphi.^{\Delta,[x_{i_1},\dots,x_{i_s}]}} & \\
C.(\Delta, \Delta\backslash\tau) & \longrightarrow & C.(\Delta, \Delta\backslash\sigma) & &
\end{array}
$$

is commutative and we get the second conclusion. Q.E.D.

In later sections we use not only the statement of the following lemma but also the notation in the proof of it.

Lemma 2.2. *Let Δ be a simplicial complex with vertex set V and $X = |\Delta|$. If $p \in X$ is an interior point of $\tau \in \Delta$ then $Y = |\Delta\backslash\tau|$ is a strong deformation retract of $X - p$.*

Proof. We may consider $V = \{x_1,\dots,x_n\}$ as a set of affinely independent points of n-dimensional euclidean space. Then X is a subset of the convex hull of $\{x_1,\dots,x_n\}$ and any point $q \in X$ can be written as

$$q = \mu_1 x_1 + \cdots + \mu_n x_n, \text{ where } \mu_1 \geq 0,\dots,\mu_n \geq 0,\ \mu_1 + \cdots + \mu_n = 1$$

uniquely.

Let $p = \lambda_1 x_1 + \cdots + \lambda_n x_n$. Changing the suffix if necessary, we may assume that $\tau = \{x_1,\dots,x_t\}$ and $\lambda_1 > 0,\dots,\lambda_t > 0$, $\lambda_{t+1} = \cdots = \lambda_n = 0$. Now assume that a point $q = \mu_1 x_1 + \cdots + \mu_n x_n$ in $X - p$ is given. Since $p \neq q$, there is a number i such that $\mu_i < \lambda_i$. So if we put

$$\varepsilon(q) = 1 - \min_{1 \leq i \leq t} \left\{ \frac{\mu_i}{\lambda_i} \right\}$$

then $\varepsilon(q)$ is a continuous map from $X - p$ to $\mathbf{R}$ and $0 < \varepsilon(q) \leq 1$. Therefore we can define a continuous map $r: (X - p) \to \mathrm{aff}\{x_1, \ldots, x_n\}$ by

$$r(q) = \frac{1}{\varepsilon(q)}(q - (1 - \varepsilon(q))p).$$

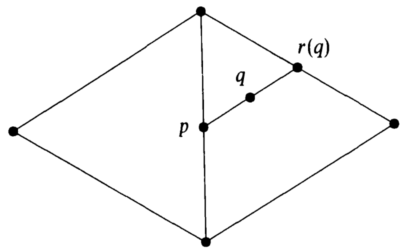


Figure 1

If we write $r(q) = v_1 x_1 + \cdots + v_n x_n$ then

(2.1) $$v_j = \frac{1}{\varepsilon(q)} \mu_j \geq 0$$

for any $j > t$. On the other hand if $j \leq t$ then

(2.2) $$\begin{aligned} v_j &= \frac{1}{\varepsilon(q)}\left(\mu_j - \min_{1 \leq i \leq t}\left\{\frac{\mu_i}{\lambda_i}\right\}\lambda_j\right) \\ &\geq \frac{1}{\varepsilon(q)}\left(\mu_j - \frac{\mu_j}{\lambda_j}\lambda_j\right) \\ &= 0. \end{aligned}$$

So $r(q) \in \mathrm{conv}\{x_1, \ldots, x_n\}$. Moreover if $\mu_j = 0$ then $v_j = 0$ by (2.1) or (2.2). So we see $\{x_j | v_j \neq 0\} \subseteq \{x_j | \mu_j \neq 0\}$ and $\{x_j | v_j \neq 0\} \in \Delta$. And if $\frac{\mu_j}{\lambda_j} = \min_{1 \leq i \leq t}\left\{\frac{\mu_i}{\lambda_i}\right\}$ then

$$v_j = \frac{1}{\varepsilon(q)}\left(\mu_j - \frac{\mu_j}{\lambda_j}\lambda_j\right) = 0$$

so $\{x_j | v_j \neq 0\} \not\supseteq \tau$ i.e. $\{x_j | v_j \neq 0\} \in \Delta \backslash \tau$. Therefore we may assume r is a map from $X - p$ to Y.

Now let $F(q, a) = (1 - a)q + ar(q)$ for $q \in X - p$ and $a \in I$, where $I = \{b \in \mathbf{R} | 0 \leq b \leq 1\}$. Then by the argument above

$$\{x_j | (1 - a)\mu_j + av_j \neq 0\} \subseteq \{x_j | \mu_j \neq 0\} \in \Delta.$$

On the other hand if $\mu_j < \lambda_j$ then

$$v_j = \mu_j + \left(\frac{1}{\varepsilon(q)} - 1\right)(\mu_j - \lambda_j) < \mu_j$$

so

$$(1-a)\mu_j + a\nu_j < \lambda_j$$

and we see $F(q, a) \neq p$. So we may consider F as a continuous map from $(X - p) \times I$ to $X - p$.

Now if $q \in Y$, then $\{x_j | \mu_j \neq 0\} \not\supseteq \tau$ and we see $\varepsilon(q) = 1$. So $r(q) = q$ and

$$F(q, a) = q \text{ for any } q \in Y \text{ and any } a \in I.$$

On the other hand

$$F(q, 1) = r(q) \in Y \text{ for any } q \in X - p.$$

These facts mean that F is a strong deformation retraction of $X - p$ to Y. So Y is a strong deformation retract of $X - p$. Q.E.D.

The final remark of this section is the following lemma whose proof is easy.

Lemma 2.3. *Let Δ be a simplicial complex with vertex set V, σ a face in Δ and W a subset of V. If $W \cap \sigma = \phi$ then* $\operatorname{link}_{\Delta_{V-W}}(\sigma) = \operatorname{link}_\Delta(\sigma)_{V-W}$.

3. A topological characterization of 2-Cohen-Macaulay complexes

In this section we give a characterization of 2-Cohen-Macaulay complexes in terms of singular homology groups of the geometric realizations of the complexes. Although the fact that the 2-Cohen-Macaulay property depends only on the topological properties of the geometric realizations of the simplicial complexes is proved by Walker [17], we state the following Theorem 3.3 and its proof because we need them in the following section.

First we state the following important theorems as references.

Theorem 3.1 (Hochster, see [15] Theorem II.4.1). *Let Δ be a simplicial complex with vertex set $V = \{x_1, \ldots, x_n\}$, $A = K[x_1, \ldots, x_n]$ and $m = (x_1, \ldots, x_n)A$. Then for any $\alpha \in \mathbf{Z}^n$,*

$$H^i_m(K[\Delta])_\alpha \cong \begin{cases} \tilde{H}_{i-\#\operatorname{supp}\alpha-1}(\operatorname{link}_\Delta(\operatorname{supp}\alpha)) & \text{if } \alpha \leq 0 \text{ and } \operatorname{supp}\alpha \in \Delta \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.2 (Reisner [11], see also [15] Corollary II.4.2). *Let Δ be as above. Then the following conditions are equivalent.*

(i) *Δ is Cohen-Macaulay.*

(ii) *For any $\sigma \in \Delta$ and any $i < \dim(\operatorname{link}_\Delta(\sigma))$, $\tilde{H}_i(\operatorname{link}_\Delta(\sigma)) = 0$.*

Now we state the following

Theorem 3.3. *Let Δ be a Cohen-Macaulay complex of dimension $(d-1)$ with vertex set $V = \{x_1, \ldots, x_n\}$ and $X = |\Delta|$. Then the following conditions are* *equivalent.*

(i) Δ *is* 2-*Cohen-Macaulay.*

(ii) *For any* $\sigma \in \Delta - \{\phi\}$, the canonical map

$$\tilde{H}_{d-1}(\Delta) \longrightarrow H_{d-1}(\Delta, \Delta \backslash \sigma)$$

is surjective.

(iii) *For any* $\sigma \in \Delta - \{\phi\}$, $\tilde{H}_{d-2}(\Delta \backslash \sigma) = 0$.

(iv) *For any* $p \in X$, $\tilde{H}_{d-2}(X - p) = 0$.

Proof. (i)$\Rightarrow$(ii): We may assume $\sigma = \{x_1, \dots, x_t\}$. Put $\sigma_j = \{x_1, \dots, x_j\}$ for any $j \in \{0, \dots, t\}$. Taking the long exact sequence of the homology groups of the following short exact sequence of chain complexes

$$0 \longrightarrow \tilde{C}.(\mathrm{link}_\Delta(\sigma_{j-1}) \backslash x_j) \longrightarrow \tilde{C}.(\mathrm{link}_\Delta(\sigma_{j-1})) \longrightarrow C.(\mathrm{link}_\Delta(\sigma_{j-1}), \mathrm{link}_\Delta(\sigma_{j-1}) \backslash x_j) \longrightarrow 0$$

we get the following exact sequence

$$\tilde{H}_{d-j}(\mathrm{link}_\Delta(\sigma_{j-1})) \longrightarrow H_{d-j}(\mathrm{link}_\Delta(\sigma_{j-1}), \mathrm{link}_\Delta(\sigma_{j-1}) \backslash x_j) \longrightarrow \tilde{H}_{d-j-1}(\mathrm{link}_\Delta(\sigma_{j-1}) \backslash x_j)$$

for any $j \in \{1, \dots, t\}$. On the other hand, since $\mathrm{link}_\Delta(\sigma_{j-1})$ is a 2-Cohen-Macaulay complex by Lemma 2.3 and $\dim \mathrm{link}_\Delta(\sigma_{j-1}) = d - j$, we see

$$\tilde{H}_{d-j-1}(\mathrm{link}_\Delta(\sigma_{j-1}) \backslash x_j) = 0$$

by Theorem 3.2. So the canonical map

$$\psi_j \colon \tilde{H}_{d-j}(\mathrm{link}_\Delta(\sigma_{j-1})) \longrightarrow H_{d-j}(\mathrm{link}_\Delta(\sigma_{j-1}), \mathrm{link}_\Delta(\sigma_{j-1}) \backslash x_j)$$

is surjective for any $j \in \{1, \dots, t\}$.

By Lemma 2.1 we see that the canonical map

$$\varphi_j \colon H_{d-1}(\Delta, \Delta \backslash \sigma_{j-1}) \longrightarrow H_{d-1}(\Delta, \Delta \backslash \sigma_j)$$

is surjective for any $j \in \{1, \dots, t\}$, where $H.(\Delta, \Delta \backslash \sigma_0) = \tilde{H}.(\Delta)$. Since the canonical map

$$\varphi \colon \tilde{H}_{d-1}(\Delta) \longrightarrow H_{d-1}(\Delta, \Delta \backslash \sigma)$$

is the composition of all the φ_j's, we see φ is surjective.

(ii)$\Rightarrow$(i): Take arbitrary $x \in V$ and $\tau \in \Delta \backslash x$. If $\tau \cup \{x\} \notin \Delta$ then $\mathrm{link}_{\Delta \backslash x}(\tau) = \mathrm{link}_\Delta(\tau)$ is a Cohen-Macaulay complex of dimension $d - \#\tau - 1$. So

$$\tilde{H}_i(\mathrm{link}_{\Delta \backslash x}(\tau)) = 0 \qquad \text{if } i < d - \#\tau - 1. \tag{3.1}$$

On the other hand if $\sigma = \tau \cup \{x\} \in \Delta$, then we have the following commutative diagram of canonical maps.

$$\begin{array}{ccc} \tilde{H}_{d-1}(\Delta) & & \\ \downarrow & \searrow^{\varphi} & \\ H_{d-1}(\Delta, \Delta\setminus\tau) & \xrightarrow[\varphi_1]{} & H_{d-1}(\Delta, \Delta\setminus\sigma) \end{array}$$

φ is surjective by the assumption so we see that φ_1 is surjective. So by Lemma 2.1 we know that the canonical map

$$\psi\colon \tilde{H}_{d-\#\tau-1}(\mathrm{link}_\Delta(\tau)) \longrightarrow \tilde{H}_{d-\#\tau-1}(\mathrm{link}_\Delta(\tau), \mathrm{link}_\Delta(\tau)\setminus x)$$

is surjective.

Now consider the following long exact sequence of homology groups.

$$\cdots \longrightarrow \tilde{H}_i(\mathrm{link}_\Delta(\tau)) \longrightarrow H_i(\mathrm{link}_\Delta(\tau), \mathrm{link}_\Delta(\tau)\setminus x)$$
$$\longrightarrow \tilde{H}_{i-1}(\mathrm{link}_\Delta(\tau)\setminus x) \longrightarrow \tilde{H}_{i-1}(\mathrm{link}_\Delta(\tau)) \longrightarrow \cdots$$

Since $\mathrm{link}_\Delta(\tau)$ and $\mathrm{link}_\Delta(\sigma)$ are Cohen-Macaulay complexes of dimension $d - \#\tau - 1$ and $d - \#\tau - 2$ respectively, we see by Theorem 3.2 and Lemma 2.1

$$\tilde{H}_i(\mathrm{link}_\Delta(\tau)) = 0$$

and

$$H_i(\mathrm{link}_\Delta(\tau), \mathrm{link}_\Delta(\tau)\setminus x) \cong \tilde{H}_{i-1}(\mathrm{link}_\Delta(\sigma)) = 0$$

if $i < d - \#\tau - 1$. So by the surjectivity of ψ and Lemma 2.3 we see

(3.2) $$\tilde{H}_i(\mathrm{link}_{\Delta\setminus x}(\tau)) = \tilde{H}_i(\mathrm{link}_\Delta(\tau)\setminus x) = 0 \qquad \text{if } i < d - \#\tau - 1.$$

Since τ is an arbitrary face in $\Delta\setminus x$, we see form (3.1) and (3.2) that $\mathrm{depth}(K[\Delta\setminus x]) \geq d$ by Theorem 3.1. So we see $\Delta\setminus x$ is a Cohen-Macaulay complex of dimension $(d - 1)$.

(ii)$\Leftrightarrow$(iii): For any $\sigma \in \Delta - \{\phi\}$ we get the following exact sequence

$$\tilde{H}_{d-1}(\Delta) \longrightarrow H_{d-1}(\Delta, \Delta\setminus\sigma) \longrightarrow \tilde{H}_{d-2}(\Delta\setminus\sigma) \longrightarrow \tilde{H}_{d-2}(\Delta).$$

Since Δ is Cohen-Macaulay of dimension $(d - 1)$, we see

$$\tilde{H}_{d-2}(\Delta) = 0$$

by Theorem 3.2. The equivalence of (ii) and (iii) easily follows from these facts.

(iii)$\Leftrightarrow$(iv): Immediately follows from Lemma 2.2. Q.E.D.

4. Topological characterizations of 2-Buchsbaum complexes

In this section we give topological characterizations of 2-Buchsbaum complexes and prove the fact that the 2-Buchsbaum property is a topological property i.e. depends only on the geometric realization. First we recall the following

Theorem 4.1 (Schenzel [12] Theorem 3.2, see also [9] Theorem 2). *Let Δ be a simplicial complex of dimension $(d-1)$ and $X=|\Delta|$. Then the following conditions are equivalent.*

(i) *Δ is Buchsbaum.*
(ii) *Δ is pure and for any $\tau \in \Delta - \{\phi\}$, $\mathrm{link}_\Delta(\tau)$ is a Cohen-Macaulay complex.*
(iii) *For any $p \in X$, $H_i(X, X-p)=0$ if $i<d-1$*

Next we remark the following

Lemma 4.2. *Let Δ be a pure complex of dimension $(d-1)$ with vertex set V. Then for any positive integer k the following conditions are equivalent.*

(i) *Δ is k-Buchsbaum.*
(ii) *For any $\tau \in \Delta - \{\phi\}$, $\mathrm{link}_\Delta(\tau)$ is a k-Cohen-Macaulay complex.*

Proof. (i)$\Rightarrow$(ii): Follows from Theorem 4.1 and Lemma 2.3.

(ii)$\Rightarrow$(i): By Theorem 4.1 and Lemma 2.3, we have only to show that Δ_{V-W} is pure of dimension $(d-1)$ for any subset W of V such that $\# W<k$. Take such W and let τ be an arbitrary facet in Δ_{V-W}. Since $\mathrm{link}_\Delta(\tau)$ is a k-Cohen-Macaulay complex by assumption, we see

$$\dim(\mathrm{link}_\Delta(\tau)_{V-W}) = \dim(\mathrm{link}_\Delta(\tau)) = d - \#\tau - 1.$$

On the other hand

$$\dim(\mathrm{link}_\Delta(\tau)_{V-W}) = \dim(\mathrm{link}_{\Delta_{V-W}}(\tau)) = \dim(\{\phi\}) = -1$$

by Lemma 2.3. So we see $\dim \tau = d-1$. Q.E.D.

Now we state the main result of this paper.

Theorem 4.3. *Let Δ be a Buchsbaum complex of dimension $(d-1)$ and $X=|\Delta|$. Then the following conditions are equivalent.*

(i) *Δ is 2-Buchsbaum.*
(ii) *For any non empty face τ in Δ and a face σ containing τ, the canonical map*

$$H_{d-1}(\Delta, \Delta\backslash\tau) \longrightarrow H_{d-1}(\Delta, \Delta\backslash\sigma)$$

is surjective.
(iii) *For any non empty face τ in Δ and a face σ containing τ,*

$$H_{d-2}(\Delta\backslash\sigma, \Delta\backslash\tau)=0.$$

(iv) *For any $p \in X$ there exists an open neighborhood U of p satisfying the following condition.*
If V is an open set satisfying (a) *and* (b) *below then V satisfies* (c) *also.*
(a) *$p \in V \subseteq U$.*
(b) *The homomorphisms of the homology groups*

$$\tilde{H}_*(X-V) \longrightarrow \tilde{H}_*(X-p)$$

induced by the inclusion map are isomorphisms.
(c) *For any* $q \in V$, $H_{d-2}(X - q, X - V) = 0$.

(v) *For any* $p \in X$ *and any open neighborhood* U *of* p, *there exists an open set* V *such that*
(a) $p \in V \subseteq U$.
(b) *The homomorphisms of the homology groups*

$$\tilde{H}_*(X - V) \longrightarrow \tilde{H}_*(X - p)$$

induced by the inclusion map are isomorphisms.
(c) *For any* $q \in V$, $H_{d-2}(X - q, X - V) = 0$.

(vi) *For any* $p \in X$, *there exists an open neighborhood* V *of* p *such that*
(b) *The homomorphisms of the homology groups*

$$\tilde{H}_*(X - V) \longrightarrow \tilde{H}_*(X - p)$$

induced by the inclusion map are isomorphisms.
(c) *For any* $q \in V$, $H_{d-2}(X - q, X - V) = 0$.

Proof. (i)$\Leftrightarrow$(ii): Follows from Theorem 3.3, Lemmas 2.1 and 4.2.
(ii)$\Leftrightarrow$(iii): From the following short exact sequence of chain complexes

$$0 \longrightarrow C.(\Delta \backslash \sigma, \Delta \backslash \tau) \longrightarrow C.(\Delta, \Delta \backslash \tau) \longrightarrow C.(\Delta, \Delta \backslash \sigma) \longrightarrow 0$$

we get the following exact sequence of homology groups.

$$\begin{aligned} H_{d-1}(\Delta, \Delta \backslash \tau) &\longrightarrow H_{d-1}(\Delta, \Delta \backslash \sigma) \\ \longrightarrow H_{d-2}(\Delta \backslash \sigma, \Delta \backslash \tau) &\longrightarrow H_{d-2}(\Delta, \Delta \backslash \tau) \end{aligned}$$

Since $H_{d-2}(\Delta, \Delta \backslash \tau) = 0$ by Theorems 3.2, 4.1 and Lemma 2.1, we see (ii)$\Leftrightarrow$(iii).

Next we note that the condition (iii) is equivalent to the following condition.

(vii) For any non empty face τ in Δ and a face σ containing τ,

$$H_{d-2}(Z, Y) = 0$$

where $Z = |\Delta \backslash \sigma|$ and $Y = |\Delta \backslash \tau|$.

(vii)$\Rightarrow$(iv): Let τ be a face in Δ scuh that p is an interior point of τ. We put $Y = |\Delta \backslash \tau|$ and $U = X - Y$, then U is an open neighborhood of p. Assume V is an open subset of X satisfying the conditions (a) and (b) and q a point in V. Since $q \notin Y$, q is an interior point of a face σ such that $\sigma \supseteq \tau$.

Now let $Z = |\Delta \backslash \sigma|$. Then the inclusion maps of topological spaces induce the following commutative diagram of chain complexes, where each horizontal line is an exact sequence.

$$\begin{array}{ccccccccc} 0 \longrightarrow & \tilde{C}.(Y) & \longrightarrow & \tilde{C}.(Z) & \longrightarrow & C.(Z, Y) & \longrightarrow 0 \\ & \downarrow{\scriptstyle \varphi.} & & \downarrow{\scriptstyle \psi.} & & \downarrow & \\ 0 \longrightarrow & \tilde{C}.(X - V) & \longrightarrow & \tilde{C}.(X - q) & \longrightarrow & C.(X - q, X - V) & \longrightarrow 0 \end{array} \tag{4.1}$$

Since Z is a deformation retract of $X-q$ by Lemma 2.2, $\psi.$ induces isomorphisms of homology groups. On the other hand there is a following commutative diagram of chain complexes induced by the inclusion maps of topological spaces.

$$\begin{array}{ccc} \tilde{C}.(Y) & \xrightarrow{\varphi.} & \tilde{C}.(X-V) \\ & {\scriptstyle \varphi'.}\searrow \quad \swarrow{\scriptstyle \varphi''.} & \\ & \tilde{C}.(X-p) & \end{array}$$

$\varphi''.$ induces isomorphisms of homology groups by the assumption and so does $\varphi'.$ by Lemma 2.2. So $\varphi.$ induces isomorphisms of homology groups.

Taking the long exact sequence of the horizontal sequences of the diagram (4.1) and using "five lemma", we see

$$H_i(Z, Y) \cong H_i(X-q, X-V) \text{ for any } i.$$

Therefore we see (vii) $\Rightarrow$ (iv).

(iv) $\Rightarrow$ (v): Let p be an arbitrary point in X and U an arbitrary open neighborhood of p. Take an open neighborhood U_1 of p satisfying the condition of U in (iv). Let τ be a face in Δ such that p is an interior point in τ. We use the same notation as in the proof of Lemma 2.2.

For any δ such that $0<\delta<1$, let

$$V_\delta = \{q \in X-p \mid \varepsilon(q) < \delta\} \cup \{p\}.$$

Since $\varepsilon(q) = 1 - \min_{1 \leq i \leq t} \left\{\frac{\mu_i}{\lambda_i}\right\}$, V_δ is an open neighborhood of p and we can take δ small enough so that

$$V_\delta \subseteq U \cap U_1.$$

Take such a δ and put $V = V_\delta$. Then we claim that V satisfies the conditions (a), (b) and (c) of (v).

(a): Clear.

(b): We put $\varepsilon_1(q) = \min\left\{\frac{\varepsilon(q)}{\delta}, 1\right\}$ and $r_1(q) = \frac{1}{\varepsilon_1(q)}(q-(1-\varepsilon_1(q))p)$ for $q \in X-p$.

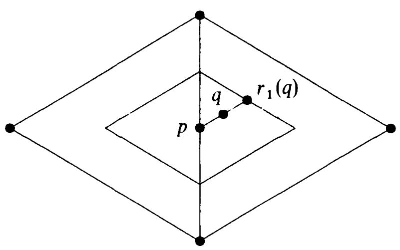


Figure 2

Then $X - V = \{q \in X - p \mid \varepsilon_1(q) = 1\} = \{q \in X - p \mid r_1(q) = q\}$. If we write

$$r_1(q) = v_1' x_1 + \cdots + v_n' x_n$$

then

$$v_j' = \frac{1}{\varepsilon_1(q)}\mu_j + \left(1 - \frac{1}{\varepsilon_1(q)}\right)\lambda_j$$

for any $j \in \{1, \ldots, n\}$. So if $\dfrac{\mu_j}{\lambda_j} = \min_{1 \le i \le t}\left\{\dfrac{\mu_i}{\lambda_i}\right\}$ and $q \in V$ then

$$\begin{aligned} \frac{v_j'}{\lambda_j} &= \frac{1}{\varepsilon_1(q)} \cdot \frac{\mu_j}{\lambda_j} + 1 - \frac{1}{\varepsilon_1(q)} \\ &= \frac{\delta}{\varepsilon(q)}(1 - \varepsilon(q)) + 1 - \frac{\delta}{\varepsilon(q)} \\ &= 1 - \delta. \end{aligned}$$

Therefore

$$\varepsilon(r_1(q)) \ge 1 - \frac{v_j'}{\lambda_j} = \delta$$

i.e. $r_1(q) \in X - V$.

So we can consider r_1 as a continuous map from $X - p$ to $X - V$ and if we put $F_1(q, a) = (1 - a)q + ar_1(q)$ for $q \in X - p$ and $a \in I$, we can show that F_1 is a strong deformation retraction of $X - p$ to $X - V$ by the same way as in the proof of Lemma 2.2. Therefore $X - V$ is a strong deformation retract of $X - p$.

(c): By the choice of U_1, (b) above and the fact $p \in V \subseteq U_1$ we see

$$H_{d-2}(X - q, X - V) = 0 \qquad \text{for any } q \in V.$$

Since U is an arbitrary neighborhood of p we see (v).

(v) $\Rightarrow$ (vi): Clear.

(vi) $\Rightarrow$ (vii): Let $U = X - Y$. Take an interior point p of τ and an open neighborhood V of p satisfying the conditions (b) and (c) of (vi). By the proof of (iv) $\Rightarrow$ (v), we see that we can take an open neighborhood V_1 of p such that $X - V_1$ is a strong deformation retract of $X - p$ and $V_1 \subseteq V \cap U$. Since V_1 is an open set, we can take an interior point q of σ such that $q \in V_1$. Then we have the following commutative diagram of chain complexes induced by the inclusion maps of topological spaces, where each horizontal line is an exact sequence.

$$\begin{array}{ccccccccc}
0 \longrightarrow & \tilde{C}.(Y) & \longrightarrow & \tilde{C}.(Z) & \longrightarrow & C.(Z, Y) & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \tilde{C}.(X - V_1) & \longrightarrow & \tilde{C}.(X - q) & \longrightarrow & C.(X - q, X - V_1) & \longrightarrow 0 \\
& \uparrow & & \| & & \uparrow & \\
0 \longrightarrow & \tilde{C}.(X - V) & \longrightarrow & \tilde{C}.(X - q) & \longrightarrow & C.(X - q, X - V) & \longrightarrow 0
\end{array}$$

By the same way as in the proof of (vii)$\Rightarrow$(iv), we see

$$H_i(Z, Y) \cong H_i(X - q, X - V_1) \cong H_i(X - q, X - V) \qquad \text{for any } i.$$

From the condition (c) of (vi) we see (vii). Q.E.D.

Since the conditions (iv), (v) and (vi) of the above theorem and (iii) of Theorem 4.1 are topological ones (i.e. depends only on $|\Delta|$), we see the following

Corollary 4.4. *2-Buchsbaum property is a topological property.*

On the other hand, the conditions (iii) of Theorem 4.1 and (v) of Theorem 4.3 are satisfied if $X = |\Delta|$ is a topological manifold without boundary. So we see that if Δ is a triangulation of a topological manifold without boundary, then Δ is 2-Buchsbaum. But in fact we can prove the following

Proposition 4.5. *If Δ is a triangulation of a homology manifold, then Δ is 2-Buchsbaum.*

Proof. From the assumption we see the validity of (iii) of Theorem 4.1, so we see that Δ is Buchsbaum (and therefore pure). Now let τ be an arbitrary non empty face in Δ. Then by the assumption and Lemmas 2.1 and 2.2 we see that

$$\tilde{H}_i(\mathrm{link}_{\mathrm{link}_\Delta(\tau)}(\sigma)) = \tilde{H}_i(\mathrm{link}_\Delta(\sigma \cup \tau)) \cong \begin{cases} K & \text{if } i = \dim(\mathrm{link}_\Delta(\sigma \cup \tau)) \\ 0 & \text{otherwise} \end{cases}$$

for any $\sigma \in \mathrm{link}_\Delta(\tau)$. So by [14] Theorem 7 (see also [15] Theorem II. 5.1) we see that $\mathrm{link}_\Delta(\tau)$ is a Gorenstein complex such that

$$\tilde{H}_{\dim(\mathrm{link}_\Delta(\tau))}(\mathrm{link}_\Delta(\tau)) \neq 0.$$

Therefore $\mathrm{link}_\Delta(\tau)$ is 2-Cohen-Macaulay by Lemma 4.6 below and we see that Δ is 2-Buchsbaum by Lemma 4.2. Q.E.D.

Lemma 4.6. *Let Δ be a Gorenstein complex of dimension $(d-1)$ such that $\tilde{H}_{d-1}(\Delta) \neq 0$. Then Δ is 2-Cohen-Macaulay.*

Proof. Since $\tilde{H}_{d-1}(\Delta) \neq 0$, we see that

$$\mathrm{Ext}_A^d(K, K[\Delta])_{(0,\dots,0)} \cong \tilde{H}_{d-1}(\Delta) \neq 0$$

where $A = K[x \mid x \in V]$ and V is the vertex set of Δ. (See [7] Theorem 5.2. See also [9] Theorem 1.) On the other hand

$$\mathrm{Ext}_A^d(K, K[\Delta]) \cong K$$

since $K[\Delta]$ is Gorenstein. So we see that

$$\mathrm{Ext}_A^d(K, K[\Delta]) = \mathrm{Ext}_A^d(K, K[\Delta])_{(0,\dots 0)}$$

and from Theorem 6.1 and Lemma 6.2 below, we get the conclusion. Q.E.D.

Next we make a remark. One might think that the condition $V \subseteq U$ in (iv) of

Theorem 4.3 is superfluous. But we see the condition is essential by the following

Example 4.7. Let D^2 be a 2-dimensional disc and a a point in ∂D^2. Let X be a subspace of $D^2 \times D^2$ such that

$$X = (\partial D^2 \times \partial D^2) \cup (\{a\} \times D^2) \cup (D^2 \times \{a\}).$$

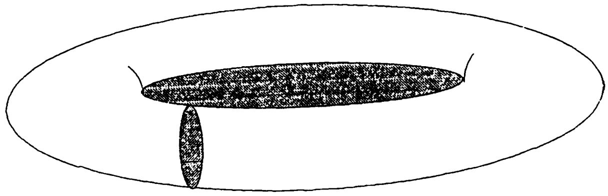


Figure 3

Then for any $p \in X$, p has an open neighborhood U such that U is homeomorphic to one of the following spaces.

$$\mathbf{R}^2,\ \{(x, y, z) \in \mathbf{R}^3 \mid z = 0 \text{ or } (y = 0, z \geq 0\},$$
$$\{(x, y, z) \in \mathbf{R}^3 \mid z = 0 \text{ or } (y = 0, z \geq 0) \text{ or } (x = 0, z \leq 0)\}.$$

So the conditions (iii) of Theorem 4.1 and (v) of Theorem 4.3 are satisfied. On the other hand, let p be a point in $(\partial D^2 \times \partial D^2) - (\{a\} \times D^2) - (D^2 \times \{a\})$, q, q_1 be different points in $(\{a\} \times D^2) - (\partial D^2 \times \partial D^2)$ and $V = X - q_1$. Then $\{q_1\} = X - V$ is a strong deformation retract of $X - p$. On the other hand by the following exact sequence

$$\tilde{H}_i(X - V) \longrightarrow \tilde{H}_i(X - q) \longrightarrow H_i(X - q, X - V) \longrightarrow \tilde{H}_{i-1}(X - V)$$

we see

$$H_i(X - q, X - V) \cong \tilde{H}_i(X - q) \qquad \text{for any } i.$$

So

$$H_1(X - q, X - V) = K \neq 0.$$

Since dim $X = 2$ and $d = 3$ in this case, we see that the condition (c) of (iv) is not satisfied.

5. A topological characterization of 2-pure complexes

In this section we prove that the 2-pureness is a topological property. First we state the following

Theorem 5.1. *Let Δ be a simplicial complex of dimension $(d - 1)$ and $X = |\Delta|$. Then the following conditions are equivalent.*

(i) *Δ is pure.*

(ii) *For any open set U in X, there exists a point p in U such that*

$$H_{d-1}(X, X-p) \neq 0.$$

Proof. (i)$\Rightarrow$(ii): Assume that an open set U in X is given. Let q be a point in U and τ a face in Δ such that q is an interior point of τ. Take a facet σ in Δ such that $\sigma \supseteq \tau$. Since U is an open set, we can take an interior point p of σ such that $p \in U$.

Since $\#\sigma = d$ by assumption we see by Lemmas 2.1 and 2.2

$$\begin{aligned} H_{d-1}(X, X-p) &\cong H_{d-1}(X, |\Delta \backslash \sigma|) \cong H_{d-1}(\Delta, \Delta \backslash \sigma) \\ &\cong \tilde{H}_{d-\#\sigma-1}(\operatorname{link}_\Delta(\sigma)) = \tilde{H}_{-1}(\{\phi\}) = K \neq 0. \end{aligned}$$

(ii)$\Rightarrow$(i): Assume that Δ is not pure. Take a facet σ in Δ such that $\#\sigma < d$. Let $Y = |\Delta \backslash \sigma|$ and $U = X - Y$. Then U is an open set in X and for any point p in U, we see by Lemmas 2.1 and 2.2

$$\begin{aligned} H_{d-1}(X, X-p) &\cong H_{d-1}(X, Y) \cong H_{d-1}(\Delta, \Delta \backslash \sigma) \\ &\cong \tilde{H}_{d-\#\sigma-1}(\operatorname{link}_\Delta(\sigma)) = \tilde{H}_{d-\#\sigma-1}(\{\phi\}) = 0 \end{aligned}$$

because $d - \#\sigma - 1 \geq 0$. So the condition (ii) does not hold. Q.E.D.

Since the condition (ii) of Theorem 5.1 is independent of a triangulation of X, we see that the pureness is a topological condition.

Before examining the topological property of 2-pure complexes, we state the following lemma which is a generalization of [1] Theorem 2.1 (a).

Lemma 5.2. *Let Δ be a pure complex of dimension $(d-1)$ with vertex set V. Then for any positive integer k the following conditions are equivalent.*

(i) *Δ is k-pure.*

(ii) *For any $(d-2)$ face τ in Δ, $\operatorname{link}_\Delta(\tau)$ consists of at least k points.*

Proof. (i)$\Rightarrow$(ii): Let τ be an arbitrary $(d-2)$ face in Δ and $W = \{x \in V \mid \{x\} \in \operatorname{link}_\Delta(\tau)\}$. Then by Lemma 2.3

$$\operatorname{link}_{\Delta_{V-W}}(\tau) = \operatorname{link}_\Delta(\tau)_{V-W} = \{\phi\}$$

and we see that τ is a facet in Δ_{V-W}. By the condition (i) we see $\# W \geq k$.

(ii)$\Rightarrow$(i): Take a minimal subset W of V such that either Δ_{V-W} is not a pure complex or $\dim \Delta_{V-W} < d-1$ and let τ be a facet in Δ_{V-W} such that $\#\tau < d$. If $x \in W$, $\Delta_{V-(W-\{x\})}$ is a pure complex of dimension $(d-1)$, so $\tau \cup \{x\}$ is a facet in $\Delta_{V-(W-\{x\})}$. So we see τ is a $(d-2)$ face in Δ, and by the assumption $\operatorname{link}_\Delta(\tau)$ has at least k points. On the other hand we see by Lemma 2.3

$$\operatorname{link}_\Delta(\tau)_{V-W} = \operatorname{link}_{\Delta_{V-W}}(\tau) = \{\phi\}.$$

Therefore $\# W \geq k$ and we see that Δ is k-pure. Q.E.D.

Now we state the following

Theorem 5.3. *Let* Δ *be a pure complex of dimension* $(d-1)$ *and* $X = |\Delta|$. *Then the following conditions are equivalent.*

(i) Δ *is* 2-*pure.*
(ii) *If* $Y = \{p \in X \mid H_{d-1}(X, X-p) = 0\}$ *then* $\dim Y \leq d-3$.

Proof. Let Δ_1 be a subset (not necessarily a subcomplex) of Δ such that

$$\Delta_1 = \{\tau \in \Delta \mid \tilde{H}_{d-\#\tau-1}(\mathrm{link}_\Delta(\tau)) = 0\}.$$

Then by Lemmas 2.1 and 2.2

$$Y = \bigcup_{\tau \in \Delta_1} (\text{interior points of } \tau).$$

(i)$\Rightarrow$(ii): If τ is a $(d-2)$ face in Δ then by Lemma 5.2 we see $\dim_K \tilde{H}_0(\mathrm{link}_\Delta(\tau)) \geq 1$. On the other hand if τ is a facet in Δ then $\tilde{H}_{-1}(\mathrm{link}_\Delta(\tau)) = K$. So we see if $\tau \in \Delta_1$ then $\dim \tau \leq d-3$. Therefore dim $Y \leq d-3$.

(ii)$\Rightarrow$(i): Let τ be a face in Δ such that $\dim \tau = d-2$. Then from the assumption and the fact stated above, $\tau \notin \Delta_1$. So $\tilde{H}_0(\mathrm{link}_\Delta(\tau)) \neq 0$ and we see that $\mathrm{link}_\Delta(\tau)$ consist of at least 2 points. By Lemma 5.2 we see (i). Q.E.D.

Since the condition (ii) of the above theorem is independent of the triangulation of X, we see the following

Corollary 5.4. 2-*pure is a topological property.*

At this point one may be tempted to believe that the k-pureness is a topological property for any positive integer k. But if $k \geq 3$ then k-pure is not a topological property, see Example 7.7.

6. A characterization of 2-Buchsbaum complexes with Tor

The next topic of this paper is the correspondence between 2-Buchsbaum property of a simplicial complex with vertex set $V = \{x_1, \dots, x_n\}$ and the structures of $\mathrm{Tor}_i^A(A/m_l, K[\Delta])$ where $A = k[x \mid x \in V]$, $m_l = (x_1^l, \dots, x_n^l)A$ and l is an integer greater than 1.

First we recall the 2-Cohen-Macaulay case.

Theorem 6.1 (Baclawski [1]). *Let* Δ *be a Cohen-Macaulay complex of dimension* $(d-1)$ *with vertex set* $V = \{x_1, \dots, x_n\}$. *Then* Δ *is* 2-*Cohen-Macaulay if and only if* $\mathrm{Tor}_{n-d}^A(K, K[\Delta]) = \mathrm{Tor}_{n-d}^A(K, K[\Delta])_{(1,\dots,1)}$.

Next we state the following

Lemma 6.2. *Let* Δ *be a simplicial complex with vertex set* $V = \{x_1, \dots, x_n\}$, $A = K[x_1, \dots, x_n]$ *and* $m_l = (x_1^l, \dots, x_n^l)A$ *where* l *is a positive integer. Then*

$$\mathrm{Tor}_i^A(A/m_l, K[\Delta])_\alpha \cong \mathrm{Ext}_A^{n-i}(A/m_l, K[\Delta])_{\alpha-(l,\dots,l)}$$

for any $\alpha\in\mathbf{Z}^n$ *and any* $i\in\mathbf{Z}$.

Proof (see the proof of [7] Theorem 5.2). Let $K. = K.(x_1^l,\dots,x_n^l; A)$ be the Koszul complex with respect to $x_1^l,\dots,x_n^l$. Then there are isomorphisms of modules

$$K_i \cong \mathrm{Hom}_A(K_{n-i}, A)((-l,\dots,-l))$$

for any $i\in\mathbf{Z}$ and these isomorphisms are compatible with boundary and coboundary maps. Since $K.$ is an A-free resolution of A/m_l, we see

$$\begin{aligned}\mathrm{Tor}_i^A(A/m_l, K[\Delta])_\alpha &= H_i(K.\otimes_A K[\Delta])_\alpha\\ &\cong H^{n-i}(\mathrm{Hom}_A(K., A)\otimes_A K[\Delta])_{\alpha-(l,\dots,l)}\\ &\cong H^{n-i}(\mathrm{Hom}_A(K., K[\Delta]))_{\alpha-(l,\dots,l)} = \mathrm{Ext}_A^{n-i}(A/m_l, K[\Delta])_{\alpha-(l,\dots,l)}\end{aligned}$$

for any $\alpha\in\mathbf{Z}^n$. Q.E.D.

By the above lemma and our former results we see the following Theorems (see [9]).

Theorem 6.3. *Let* Δ *be a simplicial complex with vertex set* $V = \{x_1,\dots,x_n\}$, $A = K[x_1,\dots,x_n]$ *and* $m_l = (x_1^l,\dots,x_n^l)$. *Then for any* $\alpha = (\alpha_1,\dots,\alpha_n)\in\mathbf{Z}^n$

$$\mathrm{Tor}_i^A(A/m_l, K[\Delta])_\alpha \cong \begin{cases}\tilde{H}^{\#E-i-1}(\mathrm{link}_\Delta(\mathrm{supp}\,\alpha - E)_E) & \text{if } \alpha\in\{0,\dots,l\}^n \text{ and } \mathrm{supp}\,\alpha - E\in\Delta\\ 0 & \text{otherwise}\end{cases}$$

where $E = \{x_j \mid \alpha_j = l\}$.

Theorem 6.4. *Let* Δ *be a simplicial complex of dimension* $(d-1)$ *with vertex set* $V = \{x_1,\dots,x_n\}$, $A = K[x_1,\dots,x_n]$ *and* $m_l = (x_1^l,\dots,x_n^l)A$. *If* $l\geq 2$ *then the following conditions are equivalent.*

(i) Δ *is Buchsbaum.*
(ii) *For any* $i > n-d$, $\mathrm{Tor}_i^A(A/m_l, K[\Delta])_\alpha = 0$ *if* $\alpha\notin\{0, l\}^n$.

Next we state the following

Lemma 6.5. *Let* Δ *be a simplicial complex with vertex set* $V = \{x_1,\dots,x_n\}$, $A = K[x_1,\dots,x_n]$ *and* $m_l = (x_1^l,\dots,x_n^l)$. *If* $l\geq 2$ *and for any* $i > n-d$, $\mathrm{Tor}_i^A(A/m_l, K[\Delta])_\alpha = 0$ *whenever* $\alpha\notin\{0, l\}^n$ *then for any facet* σ *in* Δ, $\dim\sigma\geq d-1$.

Proof. Let σ be an arbitrary facet in Δ and $\alpha = (\alpha_1,\dots,\alpha_n)$ be an element of $\mathbf{Z}^n$ such that

$$\alpha_j = \begin{cases} 1 & \text{if } x_j \in \sigma \\ l & \text{if } x_j \notin \sigma. \end{cases}$$

Then by Theorem 6.3 we see

$$\mathrm{Tor}_i^A(A/m_l, K[\Delta])_\alpha \cong \tilde{H}^{\#E-i-1}(\mathrm{link}_\Delta(\sigma)_E) = \tilde{H}^{\#E-i-1}(\{\phi\})$$

for any i where $E = V - \sigma$. On the other hand we know by assumption

$$\mathrm{Tor}_i^A(A/m_l, K[\Delta])_\alpha = 0 \qquad \text{if } i > n - d$$

since $\alpha \notin \{0, l\}^n$. So

$$\tilde{H}^{\#E-i-1}(\{\phi\}) = 0 \qquad \text{if } i > n - d$$

and we see $\# E \leq n - d$. This means $\dim \sigma \geq d - 1$. Q.E.D.

Now we state the following

Theorem 6.6. *Let Δ be a Buchsbaum complex of dimension $(d-1)$ with vertex set $V = \{x_1, \dots, x_n\}$, $A = K[x_1, \dots, x_n]$ and $m_l = (x_1^l, \dots, x_n^l)A$. If $l \geq 2$ then the following conditions are equivalent.*

(i) *Δ is 2-Buchsbaum.*
(ii) *If $\alpha \notin \{0, l\}^n \cup \{1, \dots, l\}^n$ then* $\mathrm{Tor}_{n-d}^A(A/m_l, K[\Delta])_\alpha = 0$.

Proof. Let

$$(6.1) \qquad \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow K[\Delta] \longrightarrow 0$$

be the ($\mathbf{Z}^n$-graded) minimal free resolution of $K[\Delta]$ as an A-module and x an arbitrary element in V. For a $\mathbf{Z}^n$-graded A-module M, let us call $\bigoplus_{\alpha \in \mathbf{Z}^n, \mathrm{supp}\,\alpha \not\ni x} M_\alpha$ as the degree zero part of M with respect x. Then $K[\Delta \backslash x]$ is the degree zero part of $K[\Delta]$ with respect to x.

So considering the degree zero part of the exact sequence (6.1) with respect to x we get the following exact sequence.

$$(6.2) \qquad \cdots \longrightarrow G_2 \longrightarrow G_1 \longrightarrow G_0 \longrightarrow K[\Delta \backslash x] \longrightarrow 0$$

Since G_i is the degree zero part of F_i with respect to x, it is a free module over $B = K[y \mid y \in V, y \neq x]$. So if we put $n_l = m_l \cap B$ then

$$\mathrm{Tor}_i^B(B/n_l, K[\Delta \backslash x]) = H_i(B/n_l \underset{B}{\otimes} G.).$$

On the other hand we see from the minimality of (6.1), $(F_i)_\alpha = 0$ if $\alpha \not\geq 0$ for any i. So $B/n_l \underset{B}{\otimes} G.$ is the degree zero part of $A/m_l \underset{A}{\otimes} F.$ with respect to x and we see that $\mathrm{Tor}_i^B(B/n_l, K[\Delta \backslash x])$ is the degree zero part of $\mathrm{Tor}_i^A(A/m_l, K[\Delta])$ with respect to x.

(i) $\Rightarrow$ (ii): Assume that (ii) does not hold and α be an element of $\mathbf{Z}^n - \{0, l\}^n$

$-\{1,\dots,l\}^n$ such that $\operatorname{Tor}^A_{n-d}(A/m_l, K[\Delta])_\alpha \neq 0$. Since $\alpha \notin \{1,\dots,l\}^n$, we can take $x \in V$ such that $x \notin \operatorname{supp}\alpha$ by Theorem 6.3. Let β be the element in $\mathbf{Z}^{n-1}$ such that $\operatorname{supp}\beta = \operatorname{supp}\alpha$ then we see by the above argument

$$\operatorname{Tor}^B_{n-d}(B/n_l, K[\Delta\backslash x])_\beta = \operatorname{Tor}^A_{n-d}(A/m_l, K[\Delta])_\alpha \neq 0$$

Since $\dim B = n-1$ and $\dim(\Delta\backslash x) = d-1$ we see from Theorem 6.4 and the assumption that $\beta \in \{0, l\}^{n-1}$. This means $\alpha \in \{0, l\}^n$. Contradiction.

(ii) $\Rightarrow$ (i): By the above argument and the assumption, we see

$$\operatorname{Tor}^B_{n-d}(B/n_l, K[\Delta\backslash x])_\beta = 0 \qquad \text{if } \beta \notin \{0, l\}^{n-1}$$

for any $x \in V$. Moreover since Δ is Buchsbaum we see for any $x \in V$

$$\operatorname{Tor}^B_i(B/n_l, K[\Delta\backslash x])_\beta = 0 \qquad \text{if } \beta \notin \{0, l\}^{n-1} \text{ and } i > n-d$$

by Theorem 6.4. Thanks to Theorem 6.4 and Lemma 6.5, we see $\Delta\backslash x$ is a Buchsbaum complex of dimension $(d-1)$. Q.E.D.

By the same way as above, we can prove the following theorem which is a slight generalization of Theorem 6.1.

Theorem 6.7. *Let Δ be a Cohen-Macaulay complex of dimension $(d-1)$ with vertex set $V = \{x_1,\dots,x_n\}$, $A = K[x_1,\dots,x_n]$ and $m_l = (x_1^l,\dots,x_n^l)A$ where l is a positive integer. Then the following conditions are equivalent.*

(i) *Δ is 2-Cohen-Macaulay.*
(ii) *If $\alpha \notin \{1,\dots,l\}^n$ then $\operatorname{Tor}^A_{n-d}(A/m_l, K[\Delta])_\alpha = 0$.*

7. Higher Buchsbaumness and pureness of skeletons

The final topic of this paper is a consideration of k-Buchsbaumness and k-pureness of skeletons and skeleton-like subcomplexes. First we recall the Cohen-Macaulay case.

Theorem 7.1 (Hibi [6]). *Let Δ be a Cohen-Macaulay complex, $\sigma_1,\dots,\sigma_t$ be faces of Δ such that*

(i) *$\sigma_i \cup \sigma_j \notin \Delta$ if $i \neq j$.*
(ii) *If $\Delta_1 = \{\tau \in \Delta \mid \forall i;\ \tau \not\supseteq \sigma_i\}$ then $\dim \Delta_1 < \dim \Delta$.*

Then if $\operatorname{link}_\Delta(\sigma_i)$ is 2-Cohen-Macaulay for any i, Δ_1 is 2-Cohen-Macaulay of dimension $(\dim \Delta - 1)$.

Especially by taking $\sigma_1,\dots,\sigma_t$ to be all facets of Δ, we see that the $(\dim \Delta - 1)$-skeleton of a Cohen-Macaulay complex Δ is 2-Cohen-Macaulay.

The purpose of this section is to prove the similar results for Buchsbaumness and pureness. First we note the following

Lemma 7.2. *Let Δ be a pure complex and $\sigma_1,\dots,\sigma_t$ be faces of Δ satisfying* *the conditions* (i) *and* (ii) *of Theorem* 7.1. *Then* Δ_1 *is a pure complex of dimension* $(\dim \Delta - 1)$.

Proof. Let τ be an arbitrary facet of Δ_1 and σ a facet of Δ such that $\sigma \supseteq \tau$ Since Δ is pure and $\dim \Delta_1 < \dim \Delta$, we can find σ_i such that $\sigma_i \subseteq \sigma$. By the definition of Δ_1, we can find $x \in \sigma_i$ such that $x \notin \tau$. Then $\sigma - \{x\} \supseteq \tau$ and $\sigma - \{x\} \not\supseteq \sigma_i$. On the other hand, by the condition (i) of Theorem 7.1 we see $\sigma - \{x\} \not\supseteq \sigma_j$ for any $j \neq i$. So we see $\sigma - \{x\} \in \Delta_1$. Since τ is a facet of Δ_1 we see $\sigma - \{x\} = \tau$ and $\dim \tau = \dim \Delta - 1$. Q.E.D.

Now we state the pure version of Theorem 7.1.

Theorem 7.3. *Let* Δ *be a pure complex of dimension* $(d - 1)$ *with vertex set* V *and* $\sigma_1, \ldots, \sigma_t$ *be faces of* Δ *satisfying the conditions* (i) *and* (ii) *of Theorem* 7.1. *If* $\mathrm{link}_\Delta(\sigma_i)$ *is* 2-*pure for any* i, *then* Δ_1 *is a* 2-*pure complex of dimension* $(d - 2)$.

Proof. Let x be an arbitrary element of V and τ a facet of $\Delta_1 \backslash x$. Assume $\dim \tau < d - 2$. Then by Lemma 7.2 $\tau \cup \{x\}$ is a facet in Δ_1 and we can take $y \in V$ such that $\tau \cup \{x\} \cup \{y\}$ is a facet of Δ. Since τ is a facet of $\Delta_1 \backslash x$ we see $\tau \cup \{y\} \notin \Delta_1$ and we can take σ_i such that $\tau \cup \{y\} \supseteq \sigma_i$.

Let $\Gamma = \mathrm{link}_\Delta(\sigma_i)$. Then Γ is 2-pure by assumption and $(\tau \cup \{y\}) - \sigma_i$ is not a facet of $\Gamma \backslash x$ since $\sigma_i \not\ni x$. Therefore we can take $z \in V$ such that $z \notin \tau \cup \{y\} \cup \{x\}$ and $(\tau \cup \{y\} \cup \{z\}) - \sigma_i$ is a facet of Γ.

Now since $\tau \not\supseteq \sigma_i$ we see $y \in \sigma_i$ so $\tau \cup \{z\} \not\supseteq \sigma_i$. On the other hand by the assumption (i) of Theorem 7.1 we see $\tau \cup \{z\} \cup \{y\} \not\supseteq \sigma_j$ for any $j \neq i$. Therefore $\tau \cup \{z\} \in \Delta_1$. This contradicts to the fact $\tau \cup \{z\} \not\ni x$ and τ is a facet of $\Delta_1 \backslash x$.
Q.E.D.

Unfortunately, for the Buchsbaum case we can only prove the following

Theorem 7.4. *Let* Δ *be a Buchsbaum complex and* $\sigma_1, \ldots, \sigma_t$ *be faces of* Δ *satisfying the conditions* (i) *and* (ii) *of Theorem* 7.1. *If* $\mathrm{link}_\Delta(\sigma_i)$ *is* 2-*Cohen-Macaulay for any* i, *then* Δ_1 *is a* 2-*Buchsbaum complex of dimension* $(\dim \Delta - 1)$.

Proof. Let τ be a non-empty face of Δ_1 and changing the suffix if necessary, we assume $\sigma_i \cup \tau \in \Delta$ if and only if $i \leq s$. Then

$$\begin{aligned}\mathrm{link}_{\Delta_1}(\tau) &= \{\sigma \in \Delta \mid \sigma \cup \tau \in \Delta_1,\ \sigma \cap \tau = \phi\} \\ &= \{\sigma \in \Delta \mid \sigma \cup \tau \in \Delta,\ \sigma \cap \tau = \phi,\ \sigma \cup \tau \not\supseteq \sigma_i (1 \leq i \leq t)\} \\ &= \{\sigma \in \Delta \mid \sigma \cup \tau \in \Delta,\ \sigma \cap \tau = \phi,\ \sigma \not\supseteq \tau_i (1 \leq i \leq s)\}\end{aligned}$$

where $\tau_i = \sigma_i - \tau$ for $1 \leq i \leq s$. So if we put $\Gamma = \mathrm{link}_\Delta(\tau)$ then

$$\mathrm{link}_{\Delta_1}(\tau) = \{\sigma \in \Gamma \mid \sigma \not\supseteq \tau_i (1 \leq i \leq s)\}.$$

On the other hand

$$\mathrm{link}_\Gamma(\tau_i) = \mathrm{link}_\Delta(\tau \cup \tau_i) = \mathrm{link}_{\mathrm{link}_\Delta(\sigma_i)}(\tau - \sigma_i)$$

is 2-Cohen-Macaulay for any $i \leq s$. So by Theorem 7.1 we see that $\operatorname{link}_{\Delta_1}(\tau)$ is a 2-Cohen-Macaulay complex. Since τ is an arbitrary non-empty face of Δ_1 we see by Lemmas 4.2 and 7.2 that Δ_1 is a 2-Buchsbaum complex of dimension $(\dim \Delta - 1)$. Q.E.D.

In [5] Hibi examined the following exact sequence of A-modules

$$(7.1) \qquad 0 \longrightarrow \bigoplus_{i=1}^{t} K[\operatorname{star}_{\Delta}(\sigma_i)](\alpha_i) \longrightarrow K[\Delta] \longrightarrow k[\Delta_1] \longrightarrow 0$$

where $A = K[x \mid x \in V]$, V is the vertex set of Δ, α_i the element of $\{0, -1\}^n$ such that $\operatorname{supp} \alpha_i = \sigma_i$ and $n = \# V$. Using this exact sequence and Theorems 6.6 and 6.7, we can prove Theorem 7.4 in another way.

Second proof of Theorem 7.4. Let $\dim \Delta = d - 1$. Taking the long exact sequence of Tor of (7.1) we get

$$\operatorname{Tor}_j^A(A/m_2, K[\Delta]) \longrightarrow \operatorname{Tor}_j^A(A/m_2, K[\Delta_1])$$
$$\longrightarrow \bigoplus_{i=1}^{t} \operatorname{Tor}_{j-1}^A(A/m_2, K[\operatorname{star}_{\Delta}(\sigma_i)])(\alpha_i)$$

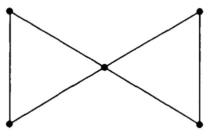
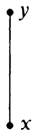
for any j where $m_2 = (x^2 \mid x \in V)A$. If $j > n - (d-1)$ $\operatorname{Tor}_{j-1}^A(A/m_2, K[\operatorname{star}_{\Delta}(\sigma_i)]) = 0$ for any i and if moreover $\alpha \notin \{0, 2\}^n$ then $\operatorname{Tor}_j^A(A/m_2, K[\Delta])_\alpha = 0$ by Theorem 6.4. So $\operatorname{Tor}_j^A(A/m_2, K[\Delta_1])_\alpha = 0$ if $\alpha \notin \{0, 2\}^n$ and $j > n - (d-1)$. So by Lemma 7.2 and Theorem 6.4, we see Δ_1 is a Buchsbaum complex of dimension $(d - 2)$. On the other hand since $K[\operatorname{star}_{\Delta}(\sigma_i)] = K[\operatorname{link}_{\Delta}(\sigma_i)][x \mid x \in \sigma_i]$ we see by Theorem 6.7 and by the assumption

$$(\operatorname{Tor}_{n-d}^A(A/m_2, K[\operatorname{star}_{\Delta}(\sigma_i)])(\alpha_i))_\alpha = 0 \qquad \text{if } \alpha \notin \{1, 2\}^n.$$

So we know that $\operatorname{Tor}_{n-d+1}^A(A/m_2, K[\Delta_1])_\alpha = 0$ if $\alpha \notin \{0, 2\}^n \cup \{1, 2\}^n$ since $\operatorname{Tor}_{n-d+1}^A(A/m_2, K[\Delta])_\alpha = 0$ if $\alpha \notin \{0, 2\}^n$. By Theorem 6.6 we see that Δ_1 is 2-Buchsbaum. Q.E.D.

Now we state an example which shows that the statement of Theorem 7.1 replaced "Cohen-Macaulay" by "Buchsbaum" is wrong.

Example 7.5. Let

$\Delta_1 =$  , $\Delta_2 =$ 

$(\dim \Delta_1 = 1)$ and $\Delta = \Delta_1 * \Delta_2$. Then Δ is a Cohen-Macaulay complex. If we put $\sigma_1 = \{x, y\}$, $t = 1$ then $\operatorname{link}_{\Delta}(\sigma_1) = \Delta_1$ is a 2-Buchsbaum complex and $\Delta \backslash \sigma_1 = \Delta_1 *$ (2 points x and y). So $\operatorname{link}_{\Delta \backslash \sigma_1}(x) = \Delta_1$ is not 2-Cohen-Macaulay and we see $\Delta \backslash \sigma_1$ is not 2-Buchsbaum by Lemma 4.2.

Finally we note the following corollary of Theorems 7.1, 7.4 and 7.3.

Corollary 7.6. *Let Δ be a k-Cohen-Macaulay (k-Buchsbaum or k-pure) complex of dimension $(d-1)$. If Δ' is the $(d-l-1)$-skeleton of Δ and $l \leq d$ then Δ' is $(k+l)$-Cohen-Macaulay ($(k+l)$-Buchsbaum or $(k+l)$-pure resp.).*

Proof. We may assume $l = 1$. Let V be the vertex set of Δ and W a subset of V such that $0 < \# W < k+1$. If we take $x \in W$ and put $W' = W - \{x\}$, then $\Delta_{V-W'}$ is Cohen-Macaulay (Buchsbaum or pure resp.) of dimension $(d-1)$ by assumption. On the other hand since

$$(\Delta')_{V-W'} = \{\sigma \in \Delta \mid \dim \sigma < d-1,\ \sigma \cap W' = \phi\}$$
$$= (\text{the } (d-2)\text{-skeleton of } \Delta_{V-W'})$$

we see by Theorem 7.1 (7.4 or 7.3 resp.) that $(\Delta')_{V-W'}$ is 2-Cohen-Macaulay (2-Buchsbaum or 2-pure resp.) of dimension $(d-2)$. So $(\Delta')_{V-W'-\{x\}} = (\Delta')_{V-W}$ is a Cohen-Macaulay (Buchsbaum or pure resp.) complex of dimension $(d-2)$.

Q.E.D.

By the above corollary we can make a k-Cohen-Macaulay complex whose barycentric subdivision is not 3-pure for any $k \geq 1$.

Example 7.7 (see [1]). Let Γ be a Cohen-Macaulay complex (e.g. a simplex) of dimension $(k+1)$ and Δ the 2-skeleton of Γ. Then by Corollary 7.6 we see Δ is k-Cohen-Macaulay. On the other hand, if $\{x, y, z\}$ is a facet of Δ then $\{\{x, y\}, \{x, y, z\}\}$ is a facet of $\mathrm{sd}(\Delta)_{V_1-\{x,y\}}$ where $\mathrm{sd}(\Delta)$ is the barycentric subdivision of Δ and V_1 is the vertex set of $\mathrm{sd}(\Delta)$. So $\mathrm{sd}(\Delta)$ is not 3-pure and we see that k-Cohen-Macaulayness, k-Buchsbaumness and k-pureness are not topological conditions if $k \geq 3$.

DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY OF EDUCATION

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