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Erdös-Rényi law for stationary Gaussian sequences

By

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§1. Introduction

Erdös and Rényi [11] discovered a new law of large numbers, nowadays called the Erdös-Rényi law. This law states that for an i.i.d. sequence $\{\xi_j;\ j=1, 2, \cdots\}$ with partial sums $S_0=0$ and $S_n=\sum_{j=1}^{n}\xi_j$, if the moment generating function $M(t)=E\exp(t\xi_1)$ exists for all $t\in(0, t_1)$, then for each $\alpha\in\{M'(t)/M(t);\ t\in(0, t_1)\}$ and $c=c(\alpha)$ such that

$$\exp(-1/c)=I(\alpha):=\inf_t M(t)\exp(-t\alpha),$$

we have

$$\lim_{n\to\infty} \boldsymbol{D}(n, [c\log n])=\alpha, \quad \text{a.s.},$$

where

$$\boldsymbol{D}(n, k)=\max_{0\leq j\leq n-k}\frac{S_{j+k}-S_j}{k}, \qquad 1\leqq k\leqq n,$$

and $[\cdot]$ denotes the integral part.

Many general versions of the Erdös-Rényi law for i.i.d. sequences have been developed by Book [1]~[2], M. Csörgo [5]~[6], S. Csörgo [7], Deheuvels [8]~[9] and Steinebach [17]~[20] and others.

However, Deo [10] initially developed the original Erdös-Rényi law to a stationary Gaussian sequence under a condition on the correlation function. More precisely, suppose $\{\xi_j;\ j=1, 2, \cdots\}$ is a stationary Gaussian sequence with $E\xi_1=0$, $E\xi_1^2=1$ and $r_n=E\xi_1\xi_{1+n}$, $n=1, 2, \cdots$, such that

$$\lim_{n\to\infty} n^{1+\beta}r_n=0 \qquad \text{for some } \beta>0$$

and

$$0<\sigma^2=1+2\sum_{j=1}^{\infty}r_j,$$

then for each $0<c<\infty$

$$\lim_{n\to\infty} \boldsymbol{D}(n, [c\log n])=\sigma\sqrt{2/c}, \text{ a.s.}.$$

Our object of this paper is to improve Deo's result and obtain a general form of the Erdös-Rényi law for stationary Gaussian sequences. Our result is as follows: Let $\{\xi_j;\ j=1, 2, \cdots\}$ be a stationary Gaussian sequence with $E\xi_1=0$, $E\xi_1^2=1$ and $r_n=E\xi_1\xi_{1+n}$

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for all $n \geqq 1$. Define $S_n = \sum_{j=1}^{n} \xi_j$ and $ES_n^2 = \sigma_n^2 \to \infty$ as $n \to \infty$. For $1 \leqq k \leqq n$, set

$$C(n, k) = \max_{0 \le j \le n-k} \frac{S_{j+k} - S_j}{\sqrt{k}\, \sigma_k}. \tag{1}$$

We assume that the correlation function is either

$$r_n \leqq 0, \quad n = 1, 2, \cdots, \tag{2}$$

or

$$\lim_{n \to \infty} n^{\nu} r_n = 0 \qquad \text{for some} \quad \nu > 0. \tag{3}$$

Then we have for each $0 < c < \infty$

$$\lim_{n \to \infty} C(n, [c \log n]) = \sqrt{2/c}, \quad \text{a.s.}. \tag{4}$$

In §4 we obtain an extension of the above result: Suppose that $\{a_n : n = 1, 2, \cdots\}$ is a sequence of positive integers such that

(i) a_n is increasing, (5)

(ii) $\lim_{n \to \infty} \frac{a_n}{n^{\zeta}} = 0$ for all $\zeta > 0$,

(iii) $\frac{a_n}{n^{\zeta_0}}$ is decreasing for some $\zeta_0 > 0$. (7)

Define for $n \geqq 2$

$$C^*(n, a_n) = \max_{0 \le j \le n-a_n} \frac{S_{j+a_n} - S_j}{\sqrt{2 \log n}\, \sigma_{a_n}}. \tag{8}$$

Then under the condition (2) we have

$$\lim_{n \to \infty} C^*(n, a_n) = 1, \quad \text{a.s.}. \tag{9}$$

If σ_n is a regularly varying function, then under the condition (3) we shall also obtain the result (9). Taking $a_n = [c \log n]$ in (9), we get (4) from (9) (cf. Remarks 1 and 2).

To obtain these results we shall investigate its upper and lower bound separately.

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§ 2. Upper bound

Theorem 2.1. *Let $\{\xi_j ; j = 1, 2, \cdots\}$ be a stationary Gaussian sequence with $E\xi_1 = 0$, $E\xi_1^2 = 1$ and $ES_n^2 = \sigma_n^2$. Then we have for each $0 < c < \infty$*

$$\limsup_{n \to \infty} C(n, [c \log n]) \leqq \sqrt{2/c}, \quad \text{a.s.}. \tag{10}$$

Proof. Let $\boldsymbol{N}$ be the set of all positive integers. For $k \in \boldsymbol{N}$ and $c > 0$, let $n_k = \max\{n \in \boldsymbol{N}; k = [c \log n]\}$ so that $k = [c \log n]$ if and only if $n_{k-1} < n \leqq n_k$. Then we have, for large k and any $\varepsilon > 0$

$$P\{C(n_k, [c\log n_k])>\sqrt{2/c}+\varepsilon\}$$
$$=P\Big\{\max_{0\le j\le n_k-[c\log n_k]}\frac{S_{j+[c\log n_k]}-S_j}{\sqrt{[c\log n_k]}\sigma_{[c\log n_k]}}>\sqrt{2/c}+\varepsilon\Big\}$$
$$\leqq n_k P\Big\{\frac{S_{[c\log n_k]}}{\sigma_{[c\log n_k]}}>(\sqrt{2/c}+\varepsilon)\sqrt{[c\log n_k]}\Big\}$$
$$\leqq n_k^{1-(1/2)(\sqrt{2}+\varepsilon\sqrt{c})^2}\leqq n_k^{-\delta_1}\leqq\exp\{-(\delta_1/c)k\}$$

where $\delta_1=\delta_1(\varepsilon)>0$. Thus

$$\sum_k P\{C(n_k, [c\log n_k])>\sqrt{2/c}+\varepsilon\}<\infty .$$

By the Borel-Cantelli lemma,

$$\limsup_{k\to\infty} C(n_k, [c\log n_k])\leqq\sqrt{2/c}+\varepsilon, \quad \text{a. s.},$$

but we see that from the definition of n_k

$$C(n, [c\log n])\leqq C(n_k, [c\log n_k]).$$

Since ε is arbitrary, we have

$$\limsup_{n\to\infty} C(n, [c\log n])\leqq\sqrt{2/c}, \quad \text{a. s.}.$$

§ 3. Lower bound

First we shall consider the case in which the assumption (2) holds. In this case we need a preliminary lemma due to Slepian [16].

Lemma 3.1. *Let $\{\xi_j; j=1, 2, \cdots\}$ and $\{\eta_j; j=1, 2, \cdots\}$ be standard normal random vaiables with* $\mathrm{cov}(\xi_i, \xi_j)\leqq\mathrm{cov}(\eta_i, \eta_j)$ *for each $i\neq j=1, \cdots, n$. Then for any real $u_1, \cdots, u_n$.*

$$P\{\xi_j\leqq u_j \text{ for } j=1, \cdots, n\}\leqq P\{\eta_j\leqq u_j \text{ for } j=1, \cdots, n\}.$$

We note that if $\mathrm{cov}(\xi_j, \xi_j)\leqq 0$ for $i\neq j=1, \cdots, n$.

$$P\{\max_{1\le j\le n}\xi_j\leqq u_n\}\leqq\Phi(u_n)^n, \tag{11}$$

where $\Phi(\cdot)$ denotes the standard normal distribution function.

Theorem 3.1. *Let $\{\xi_j; j=1, 2, \cdots\}$ be a stationary Gaussian sequence with $E\xi_1=0$ and $E\xi_1^2=1$ such that $ES_n^2=\sigma_n^2$ and $r_n:=E\xi_1\xi_{1+n}\leqq 0$, $n\geqq 1$. Then we have for each $0<c<\infty$*

$$\liminf_{n\to\infty} C(n, [c\log n])\geqq\sqrt{2/c}, \quad \text{a. s.}.$$

Proof. For each $0<c<\infty$ and $n\in N$, we define the positive integer h_n by

$$h_n=\Big[\frac{n}{c\log n}\Big].$$

For $i=1, 2, \cdots, h_n$, we define the partial sum

$$Z_{n,i}=\xi_{(i-1)[c\log n]+1}+\cdots+\xi_{i[c\log n]}.$$

Since $E\xi_n\xi_m\leqq 0$ for $n\neq m$,

$$EZ_{n,i}Z_{n,j}\leqq 0,\qquad i\neq j. \tag{12}$$

Note also that

$$\mathrm{Var}(Z_{n,i})=\sigma^2_{[c\log n]}$$

for $i=1,\cdots,h_n$, and

$$\frac{Z_{n,i}}{\sigma_{[c\log n]}}\overset{d}{\sim}N(0,1). \tag{13}$$

From (12) and (13), we can apply the inequalty (11) to the sequence $\{Z_{n,i}\,;\,i=1,\cdots,h_n\}$. Then we get for $0<\varepsilon<\sqrt{2/c}$ and large n

$$\begin{aligned}
&P\{C(n,[c\log n])\leqq\sqrt{2/c}-\varepsilon\}\\
&=P\left\{\max_{0\leq j\leq n-[c\log n]}\frac{S_{j+[c\log n]}-S_j}{\sqrt{[c\log n]}\sigma_{[c\log n]}}\leqq\sqrt{2/c}-\varepsilon\right\}\\
&=P\left\{\max_{1\leq j\leq h_n}\frac{Z_{n,j}}{\sigma_{[c\log n]}}\leqq(\sqrt{2}-\varepsilon\sqrt{c})\sqrt{\log n}\right\}\\
&\leqq\{\Phi((\sqrt{2}-\varepsilon\sqrt{c})\sqrt{\log n})\}^{h_n}.
\end{aligned}$$

If $u_n=(\sqrt{2}-\varepsilon\sqrt{c})\sqrt{\log n}$ and $Z\overset{d}{\sim}N(0,1)$, then

$$\begin{aligned}
\Phi(u_n)&=1-P(Z\geqq u_n)\\
&\leqq\exp\{-P(Z\geqq u_n)\}.
\end{aligned}$$

Clearly there exist $\delta''>0$ and $K>0$ such that for all sufficiently large n, we have

$$P(Z\geqq u_n)\geqq K\left(\frac{1}{u_n}-\frac{1}{u_n^3}\right)\exp\left(-\frac{1}{2}u_n^2\right)\geqq Kn^{-1+\delta''}$$

and

$$h_n\geqq n^{1-(\delta''/2)}.$$

Thus we get

$$\Phi(u_n)^{h_n}\leqq\exp\{-h_nP(Z\geqq u_n)\}\leqq\exp\{-Kn^{\delta'}\} \tag{14}$$

where $\delta'=\delta''/2>0$. Then for large n and some $\delta'>0$

$$P\{C(n,[c\log n])\leqq\sqrt{2/c}-\varepsilon\}\leqq\exp\{-Kn^{\delta'}\}$$

Therefore, the series

$$\sum_{n=1}^{\infty}P\{C(n,[c\log n])\leqq\sqrt{2/c}-\varepsilon\}$$

is convergent and using the Borel-Cantelli lemma we get

$$\liminf_{n\to\infty}C(n,[c\log n])\geqq\sqrt{2/c},\quad \text{a.s.}.$$

From Theorems 2.1 and 3.1 we can conclude

Theorem 3.2. *Under the assumptions of Theorem* 3.1, *we have for each* $0<c<\infty$

$$\lim_{n\to\infty} C(n, [c\log n])=\sqrt{2/c}, \quad \text{a.s.}.$$

Next we shall consider the case in which the condition (3) holds.

Theorem 3.3. *Let* $\{\xi_j;\ j=1, 2, \cdots\}$ *be a stationary Gaussian sequence with* $E\xi_1=0$, $E\xi_1^2=1$ *and* $ES_n^2=\sigma_n^2\to\infty$ *as* $n\to\infty$. *Let the correlation* r_n *be such that*

$$\lim_{n\to\infty} n^{\nu} r_n=0 \tag{15}$$

for some $\nu>0$. *Then for each* $0<c<\infty$

$$\liminf_{n\to\infty} C(n, [c\log n])\geqq\sqrt{2/c}, \quad \text{a.s.}.$$

For the proof of Theorem 3.3, we shall need the following lemmas.

Lemma 3.2. (Leadbetter et al. [15]) *Let* $\{\xi_j;\ j=1, 2, \cdots n\}$ *be* $N(0, 1)$*-random variables with* $\mathrm{cov}(\xi_i, \xi_j)=\Lambda_{ij}$ *such that*

$$\delta=\max_{i\neq j}|\Lambda_{ij}|<1.$$

Then for any real numbers u_n *and integers* $1\leqq l_1<l_2<\cdots<l_{k_n}<n$ *with* $k_n<n$,

$$P\Big\{\max_{1\leq j\leq k_n}\xi_{l_j}\leqq u_n\Big\}\leqq\Phi(u_u)^{k_n}+K\sum_{1\leq i<j\leq k_n}|r_{ij}|\exp\Big(-\frac{u_n^2}{1+|r_{ij}|}\Big),$$

where $r_{ij}=\Lambda_{l_il_j}$ *and* $K=K(\delta)$ *is a constant independent of* n.

Lemma 3.3. *Let* $\xi_j(j=1, 2, \cdots, n)$, δ, k_n *and* r_{ij} *be as in Lemma* 3.2. *Assume that* $|r_{ij}|\leqq\rho_{|i-j|}<1(i\neq j)$ *and, for some* $\nu>0$

$$\rho_m<m^{-\nu} \qquad \text{for all} \quad m=1, 2, \cdots, k_n-1.$$

Then, there exist $\delta_0>0$, $2>\eta_0>0$ *and* $K>0$ *depending on* δ *and* ν *only such that for all* $0<\eta<\eta_0$

$$\begin{aligned}\Sigma_n:=&\sum_{1\leq i<j\leq k_n}|r_{ij}|\exp\Big(-\frac{u_n^2}{1+|r_{ij}|}\Big)\\ \leqq& Kn^{-\delta_0},\end{aligned}$$

where $u_n=\sqrt{(2-\eta)\log n}$, $0<\eta<2$.

Proof. For the δ given in Lemma 3.2, we take δ' such that $0<\delta<\delta'<1$, and choose a such that

$$0<a<\frac{1-\delta'}{1+\delta}<\frac{1-\delta}{1+\delta}.$$

Let $0<\eta\leqq\delta'-\delta$. We split Σ_n into two sums:

$$\Sigma_n=\sum_{\substack{1\leq i<j\leq k_n\\ |i-j|<[n^a]}}|r_{ij}|\exp\Big(-\frac{u_n^2}{1+|r_{ij}|}\Big)$$

$$+\sum_{\substack{1\le i<j\le k_n\\|i-j|\ge[n^a]}}|r_{ij}|\exp\left(-\frac{u_n^2}{1+|r_{ij}|}\right)$$

$$=\Sigma_n^{(1)}+\Sigma_n^{(2)},\quad \text{say}.$$

Then,

$$\Sigma_n^{(1)}\leqq\delta\sum_{\substack{1\le i<j\le k_n\\|i-j|<[n^a]}}\exp\left(-\frac{u_n^2}{1+\delta}\right)$$

$$\leqq\delta k_n n^a\left\{\exp\left(-\frac{u_n^2}{2}\right)\right\}^{2/(1+\delta)}$$

$$\leqq\delta n^{1+a}(n^{-1+(\eta/2)})^{2/(1+\delta)}$$

$$\leqq\delta n^{-A_1},$$

where $A_1=-\left(1+a-\frac{2}{1+\delta}+\frac{\delta'-\delta}{1+\delta}\right)>0$.

In order to estimate $\Sigma_n^{(2)}$, we define for each $k=1, 2, \cdots, k_n-1$,

$$\delta_k=\sup\{\rho_m;\ k\leqq m\leqq k_n-1\}. \tag{16}$$

Then

$$\delta_k=k^{-\nu}\quad \text{for}\quad k=1, 2, \cdots, k_n-1.$$

Set $p=[n^a]$. If $p\leqq k_n-1$, then we have

$$\delta_p\leqq p^{-\nu}. \tag{17}$$

From the assumption of the lemma, (16) and (17), we have

$$|r_{ij}|\leqq\rho_{|i-j|}\leqq\delta_p\leqq p^{-\nu} \tag{18}$$

for i, j such that $1\leqq i<j\leqq k_n$ and $|i-j|\geqq p$. Choose $2>\eta_0>0$ so small that $\eta_0\leqq\delta'-\delta$ and $A_2=a\nu-\eta_0>0$. Then we have from (18)

$$\Sigma_n^{(2)}=\sum_{\substack{1\le i<j\le k_n\\|i-j|\ge[n^a]}}|r_{ij}|\exp\left(-\frac{u_n^2}{1+|r_{ij}|}\right)$$

$$\leqq[n^a]^{-\nu}\sum_{\substack{1\le i<j\le k_n\\|i-j|\ge[n^a]}}\exp(-u_n^2)\exp\left(\frac{u_n^2|r_{ij}|}{1+|r_{ij}|}\right)$$

$$\leqq[n^a]^{-\nu}n^2\exp(-u_n^2)\exp(u_n^2[n^a]^{-\nu})$$

$$=[n^a]^{-\nu}n^{\eta}\exp((2-\eta)(\log n)[n^a]^{-\nu})$$

$$\leqq K[n^a]^{-\nu}n^{\eta}$$

$$\leqq Kn^{-A_2}$$

for some constant K. This completes the proof.

From Lemmas 3.2 and 3.3 we can obtain the following

Lemma 3.4. *Let* $\{\xi_j;\ j=1, 2, \cdots, n\}$ *be* $N(0, 1)$*-random variables with* $\operatorname{cov}(\xi_i, \xi_j)=\Lambda_{ij}$ *such that* $\delta=\max|\Lambda_{ij}|<1$. *Let* $1\leqq l_1<l_2<\cdots<l_{k_n}<n$ *and* $k_n<n$ *be arbitrary positive*

integers. For a subsequence $\{\xi_{l_j};\ j=1, \cdots, k_n\}$ *of the sequence* $\{\xi_j;\ j=1, 2, \cdots, n\}$, *let* $r_{ij}=\Lambda_{l_i l_j}(i\neq j)$ *be such that* $|r_{ij}|\leqq\rho_{|i-j|}<1$ *and for some* $\nu>0$

$$\rho_m<m^{-\nu}, \qquad m=1, 2, \cdots, k_n-1. \tag{19}$$

Then there exist constants $\delta_0>0$, $2>\eta_0>0$ *and* K *depending only on* δ *and* ν *such that for all* $0<\eta<\eta_0$,

$$P\{\max_{1\leq j\leq k_n}\xi_{l_j}\leqq u_n\}\leqq\Phi(u_n)^{k_n}+Kn^{-\delta_0}. \tag{20}$$

where $u_n=\sqrt{(2-\eta)\log n}$, $0<2<\eta$.

In proving Theorem 3.3 we shall make use of Lemma 3.4.

Proof of Theorem 3.3. Let $\beta>0$ be a constant such that $\beta>2/\nu$ for $\nu>0$ given in the Theorem. Then for given $0<c<\infty$ and $n\in N$, we define the positive integer k_n of Lemma 3.4 by

$$k_n=\left[\frac{n}{c\log n+(\log n)^\beta}\right].$$

For this k_n and $i=1, \cdots, k_n$, we also define the partial sum

$$Y_{n,i}=\xi_{(i-1)([c\log n]+[(\log n)^\beta])+1}+\cdots+\xi_{i[c\log n]+(i-1)[(\log n)^\beta]}.$$

Then by the assumption of the theorem,

$$\mathrm{Var}(Y_{n,i})=\sigma^2_{[c\log n]} \tag{21}$$

and

$$\frac{Y_{n,i}}{\sigma_{[c\log n]}}\overset{d}{\sim}N(0, 1), \quad i=1, \cdots, k_n. \tag{22}$$

Define $r_n(i, j)=\text{correlation}(Y_{n,i}, Y_{n,j})$, $i\neq j$, and let $m=|i-j|$; then be the assumptions of the theorem we obtain for large n

$$\begin{aligned}
|r_n(i, j)|&=\left|\frac{E(Y_{n,i}Y_{n,j})}{\sigma^2_{[c\log n]}}\right|\\
&\leqq K|E\{(\xi_{(i-1)([c\log n]+[(\log n)^\beta])+1}+\cdots\\
&\quad+\xi_{i[c\log n]+(i-1)[(\log n)^\beta]})(\xi_{(j-1)([c\log n]+[(\log n)^\beta])+1}+\cdots\\
&\quad+\xi_{j[c\log n]+(j-1)[(\log n)^\beta]})\}|\\
&\leqq K[\log n]^2|r_{(j-i-1)[c\log n]+(j-i)(\log n)^\beta]}|\\
&\leqq K(\log n)^2\frac{1}{\{(m-1)[c\log n]+m[(\log n)^\beta]\}^\nu}\\
&\leqq\frac{K}{m^\nu(\log n)^{\beta\nu-2}}<m^{-\nu},
\end{aligned} \tag{23}$$

where K is a constant. In Lemma 3.4 if we set $\eta=2-(\sqrt{2}-\varepsilon\sqrt{c})^2$ for $0<\varepsilon<\sqrt{2/c}$, then $u_n=(\sqrt{2}-\varepsilon\sqrt{c})\sqrt{\log n}$. We now apply Lemma 3.4 for

$$\xi_{l_j}=\frac{Y_{n,j}}{\sigma_{[c\log n]}},\quad j=1,\cdots,k_n.$$

Thus from (14), (20) and (22), we obtain for some $\delta', \delta_0>0$ and all large n

$$\begin{aligned}
&\boldsymbol{P}\{\boldsymbol{C}(n,[c\log n])<\sqrt{2/c}-\varepsilon\}\\
&=\boldsymbol{P}\left\{\max_{0\le j\le n-[c\log n]}\frac{S_{j+[c\log n]}-S_j}{\sqrt{[c\log n]}\sigma_{[c\log n]}}<\sqrt{2/c}-\varepsilon\right\}\\
&\leqq\boldsymbol{P}\left\{\max_{1\le j\le k_n}\frac{Y_{n,j}}{\sigma_{[c\log n]}}<(\sqrt{2}-\varepsilon\sqrt{c})\sqrt{\log n}\right\}\\
&\leqq\Phi(u_n)^{k_n}+Kn^{-\delta_0}\\
&\leqq\exp(-n^{\delta'})+Kn^{-\delta_0},
\end{aligned}\tag{24}$$

where K is a constant. For given $k\in\boldsymbol{N}$, set $n'_k=\min\{n\in\boldsymbol{N};\ k=[c\log n]\}$ so that $k=[c\log n]$ if and only if $n'_k\leqq n<n'_{k+1}$. From (24) we get for k large and some $\delta''>0$,

$$\begin{aligned}
&\boldsymbol{P}\{\boldsymbol{C}(n'_k,[c\log n'_k])<\sqrt{2/c}-\varepsilon\}\\
&\quad\leqq\exp\{-(n'_k)^{\delta'}\}+K(n'_k)^{-\delta_0}\\
&\quad\leqq\exp\{-[\exp(k/c)]^{\delta'}\}+K[\exp(k/c)]^{-\delta_0}\\
&\quad\leqq K\exp(-\delta''k).
\end{aligned}$$

Thus

$$\sum_k\boldsymbol{P}\{\boldsymbol{C}(n'_k,[c\log n'_k])<\sqrt{2/c}-\varepsilon\}<\infty.$$

By the Borel-Cantelli lemma

$$\liminf_{k\to\infty}\boldsymbol{C}(n'_k,[c\log n'_k])\geqq\sqrt{2/c}-\varepsilon,\quad\text{a. s.}.$$

Since ε is arbitrary, we have

$$\liminf_{n\to\infty}\boldsymbol{C}(n,[c\log n])\geqq\sqrt{2/c},\quad\text{a. s.}.$$

Combining Theorems 2.1 and Theorem 3.3, we have

Theorem 3.4. *Under the assumptions of Theorem* 3.3, *we have for* $0<c<\infty$,

$$\lim_{n\to\infty}\boldsymbol{C}(n,[c\log n])=\sqrt{2/c},\quad\text{a. s.}.$$

§ 4. An extension

In this section we shall also assume that $\{\xi_j;\ j=1,2,\cdots\}$ is a stationary Gaussian sequence with $E\xi_1=0$ and $E\xi_1^2=1$.

4.1. Upper bound. To obtain an extension of Theorem 2.1, we shall make use of the similar techniques as in Chan [4] and Steinebach [20].

Theorem 4.1.1. *Let* $ES_n^2=\sigma_n^2$, *and the sequence* $\{a_n;\ n=1,2,\cdots\}$ *satisfy the conditions* (5) *and* (6). *Then we have*

$$\limsup_{n\to\infty} C^*(n, a_n) \leqq 1, \quad a.s..$$

Proof. Let $\varepsilon>0$ be given, and $\zeta>0$ be such that $0<\zeta<2\varepsilon+\varepsilon^2$. Let k be any integer such that $1\leqq k\leqq n^\zeta$. Then for large n

$$P\{\max_{1\le k\le n^\zeta} C^*(n, k)>1+\varepsilon\}$$

$$=P\left\{\max_{1\le k\le n^\zeta}\max_{0\le j\le n-k}\frac{S_{j+k}-S_j}{\sqrt{2\log n}\,\sigma_k}>1+\varepsilon\right\}$$

$$=\sum_{1\le k\le n^\zeta}\sum_{0\le j\le n-k}P\left\{\frac{S_k}{\sigma_k}>(1+\varepsilon)\sqrt{2\log n}\right\}$$

$$\leqq Kn^{1+\zeta-(1+\varepsilon)^2}=Kn^{-r}, \tag{25}$$

where K is a constant and $r=2\varepsilon+\varepsilon^2-\zeta>0$. Take $\theta>1$ and consider the following sequence of integers $\{[\theta^i]; i=1, 2, \cdots\}$. Then for large i, (25) yields

$$P\{\max_{1\le k\le[\theta^i]^\zeta} C^*([\theta^i], k)>1+\varepsilon\}\leqq K[\theta^i]^{-r}.$$

Thus

$$\sum_i P\{\max_{1\le k\le[\theta^i]^\zeta} C^*([\theta^i], k)>1+\varepsilon\}<\infty.$$

By the Borel-Cantelli lemma,

$$\limsup_{i\to\infty}\{\max_{1\le k\le[\theta^i]^\zeta} C^*([\theta^i], k)\}\leqq 1+\varepsilon, \quad \text{a.s..} \tag{26}$$

For any $\theta>1$ and given i, suppose $[\theta^{i-1}]\leqq n\leqq[\theta^i]$. Then it is obvious that

$$\max_{1\le k\le n^\zeta} C^*(n, k)\leqq \max_{1\le k\le[\theta^i]^\zeta} C^*([\theta^i], k)\left(\frac{\log[\theta^i]}{\log[\theta^{i-1}]}\right)^{1/2}. \tag{27}$$

Since $\dfrac{\log[\theta^i]}{[\theta^i]}$ is decreasing,

$$1\leqq\frac{\log[\theta^i]}{\log[\theta^{i-1}]}\leqq\frac{[\theta^i]}{[\theta^{i-1}]}\longrightarrow 1 \tag{28}$$

as $i\to\infty$ and $\theta\to 1$. From (26), (27) and (28), we get

$$\limsup_{n\to\infty}\{\max_{1\le k\le n^\zeta} C^*(n, k)\}\leqq 1+\varepsilon, \quad \text{a.s..} \tag{29}$$

We note that whenever $a_n<n^\zeta$ for large n,

$$C^*(n, a_n)\leqq \max_{1\le k\le n^\zeta} C^*(n, k). \tag{30}$$

Thus from (29) and (30) the proof is complete.

4.2. Lower bound.

Theorem 4.2.1. *Let* $ES_n^2=\sigma_n^2$ *and* $r_n=E\xi_1\xi_{1+n}\leqq 0$, $n\geqq 1$. *Let the sequence* $\{a_n; n=1, 2, \cdots\}$ *satisfy the conditions* (5) *and* (6). *Then*

$$\liminf_{n\to\infty} C^*(n, a_n)\geqq 1, \quad a.s..$$

Proof. The proof is very similar to that of Theorem 3.1. But we should remark that for given a_n, if we define the positive integer h_n by

$$h_n=\left[\frac{n}{a_n}\right].$$

then there exists $\delta''>0$ such that

$$h_n \geqq n^{1-(\delta''/2)}.$$

since $a_n<n^{\zeta}$ for large n.

From Theorems 4.1.1 and 4.2.1 we can conclude

Theorem 4.2.2. *Under the asumptions of Theorem* 4.2.1, *we have*

$$\lim_{n\to\infty} C^*(n, a_n)=1, \quad a.s..$$

Remark 1. For each $0<c<\infty$, set $a_n=[c\log n]$ in Theorem 4.2.2. Then Theorem 3.2 follows immediately from the fact that with probability 1

$$\begin{aligned}1&=\lim_{n\to\infty} C^*(n, [c\log n])\\ &=\lim_{n\to\infty}\max_{0\le j\le n-[c\log n]}\frac{S_{j+[c\log n]}-S_j}{\sqrt{[c\log n]}\sigma_{[c\log n]}}\left(\frac{[c\log n]}{2\log n}\right)^{1/2}\\ &=\lim_{n\to\infty} C(n, [c\log n])\sqrt{c/2}.\end{aligned}$$

Theorem 4.2.3. *Let* $ES_n^2=\sigma_n^2$ *and* $r_n=E\xi_1\xi_{1+n}$ *such that*

(i) σ_n *is a regulary varying function,* (31)

(ii) $\lim_{n\to\infty} n^{\nu}r_n=0$ *for some* $\nu>0$. (32)

Assume that the sequence $\{a_n;\ n=1, 2, \cdots\}$ *satisfies the conditions* (5), (6) *and* (7). *Then we have*

$$\liminf_{n\to\infty} C^*(n, a_n)\geqq 1, \quad a.s..$$

Before proving the theorem, we need some lemmas.

Lemma 4.1. *Let the sequence* $\{a_n;\ n=1, 2, \cdots\}$ *satisfy the conditions* (5), (6) *and* (7). *Let* $ES_n^2=\sigma_n^2$ *and further, for some* $\alpha>0$, $\sigma_n=n^{\alpha}h(n)$ *where* $h(n)$ *is a slowly vrrying function. For* $\theta>1$ *and a positive integer* i, *let* $[\theta^i]\leqq n\leqq[\theta^{i+1}]$ *and*

$$N_i=\{n;\ n-a_n\leqq[\theta^i]-a_{[\theta^i]}\}.$$

Then we have

$$\text{(i)}\quad \limsup_{i\to\infty}\max_{[\theta^i]\le n\le[\theta^{i+1}]}\max_{0\le j\le[\theta^i]-a_{[\theta^i]}}\frac{|S_{j+a_n}-S_{j+a_{[\theta^i]}}|}{\sqrt{2\log n}\,\sigma_{a_n}}=0, \quad a.s.. \tag{33}$$

and

$$\text{(ii)}\quad \limsup_{i\to\infty}\max_{n\in N_i}\max_{n-a_n\le j\le[\theta^i]-a_{[\theta^i]}}\frac{|S_{j+a_n}-S_j|}{\sqrt{2\log n}\,\sigma_{a_n}}=0, \quad a.s.. \tag{34}$$

Proof. First let us prove (i). From (7) we see that

$$\frac{a_n}{a_{[\theta^i]}} \leqq \left(\frac{n}{[\theta^i]}\right)^{\zeta_0} \leqq \theta^{\zeta_0}$$

and

$$\frac{a_n}{a_n - a_{[\theta^i]}} \geqq \frac{\theta^{\zeta_0}}{\theta^{\zeta_0}-1} > 0.$$

Thus by (5) and (31), for large i and any $\varepsilon>0$ we can take $M>0$ such that

$$\frac{\sigma_{a_n}}{\sigma_{a_n - a_{[\theta^i]}}} \geqq M \tag{35}$$

and $M\varepsilon > \sqrt{2}$.
From (35) we get for large i and any $\varepsilon>0$

$$\begin{aligned} &P\left\{\max_{0\leqq j\leqq [\theta^i]-a_{[\theta^i]}} \frac{|S_{j+a_n} - S_{j+a_{[\theta^i]}}|}{\sqrt{2\log n}\,\sigma_{a_n}} \geqq \varepsilon\right\} \\ &\leqq [\theta^i] P\left\{\frac{|S_{a_n} - S_{a_{[\theta^i]}}|}{\sigma_{a_n - a_{[\theta^i]}}} \geqq \varepsilon\sqrt{2\log n}\,\frac{\sigma_{a_n}}{\sigma_{a_n - a_{[\theta^i]}}}\right\} \\ &\leqq K\theta^{-(\varepsilon^2 M^2 - 1)i} \end{aligned} \tag{36}$$

where K is a constant. Thus we have for large i

$$\begin{aligned} &P\left\{\max_{[\theta^i]\leqq n\leqq [\theta^{i+1}]} \max_{0\leqq j\leqq [\theta^i]-a_{[\theta^i]}} \frac{|S_{j+a_n} - S_{j+a_{[\theta^i]}}|}{\sqrt{2\log n}\,\sigma_{a_n}} \geqq \varepsilon\right\} \\ &\leqq K\theta^{-(\varepsilon^2 M^2 - 2)i}. \end{aligned}$$

Therefore, we get

$$\sum_i P\left\{\max_{[\theta^i]\leqq n\leqq [\theta^{i+1}]} \max_{0\leqq j\leqq [\theta^i]-a_{[\theta^i]}} \frac{|S_{j+a_n} - S_{j+a_{[\theta^i]}}|}{\sqrt{2\log n}\,\sigma_{a_n}} \geqq \varepsilon\right\} < \infty.$$

By the Borel-Cantelli lemma, (i) follows. Similarly, we can obtain (ii) when $n \in N_i$.

Lemma 4.2. *Under the assumptions of Lemma* 4.1, *we have*

$$\liminf_{n\to\infty} C^*(n, a_n) \geqq \lim_{\theta\downarrow 1} \liminf_{i\to\infty} C^*([\theta^i], a_{[\theta^i]}), \quad a.s.. \tag{37}$$

Proof. First consider the case $n \notin N_i$ in $[\theta^i] \leqq n \leqq [\theta^{i+1}]$. Then

$$\begin{aligned} C^*(n, a_n) &\geqq \max_{0\leqq j\leqq [\theta^i]-a_{[\theta^i]}} \frac{S_{j+a_n} - S_j}{\sqrt{2\log n}\,\sigma_{a_n}} \\ &\geqq \max_{0\leqq j\leqq [\theta^i]-a_{[\theta^i]}} \frac{S_{j+a_{[\theta^i]}} - S_j}{\sqrt{2\log n}\,\sigma_{a_n}} \\ &\quad - \max_{n\notin N_i} \max_{0\leqq j\leqq [\theta^i]-a_{[\theta^i]}} \frac{|S_{j+a_n} - S_{j+a_{[\theta^i]}}|}{\sqrt{2\log n}\,\sigma_{a_n}}. \end{aligned} \tag{38}$$

From (33) and (38) we get

$$\liminf_{n\to\infty} C^*(n,\ a_n) \geqq \liminf_{i\to\infty} \max_{0\leq j\leq[\theta^i]-a_{[\theta^i]}} \frac{S_{j+a_{[\theta^i]}}-S_j}{\sqrt{2\log n}\,\sigma_{a_n}}$$

$$\geqq \liminf_{i\to\infty} C^*([\theta^i],\ a_{[\theta^i]}) \max_{[\theta^i]\leq n\leq[\theta^{i+1}]} \left(\frac{\sigma_{a_{[\theta^i]}}}{\sigma_{a_n}}\right)\left(\frac{\log[\theta^i]}{\log[\theta^{i+1}]}\right)^{1/2}, \quad \text{a. s.}. \tag{39}$$

Since $\frac{\log n}{n}$ is decreasing and

$$\min_{[\theta^i]\leq n\leq[\theta^{i+1}]} \frac{\sigma_{a_{[\theta^i]}}}{\sigma_{a_n}} \longrightarrow 1, \quad \text{as } i\to\infty \quad \text{and} \quad \theta\to 1. \tag{40}$$

(37) follows from (39). Next consider the case $n\in N_i$. Then

$$\begin{aligned} C^*(n,\ a_n) \geqq & \max_{0\leq j\leq[\theta^i]-a_{[\theta^i]}} \frac{S_{j+a_{[\theta^i]}}-S_j}{\sqrt{2\log n}\,\sigma_{a_n}} \\ & - \max_{n\in N_i} \max_{0\leq j\leq[\theta^i]-a_{[\theta^i]}} \frac{|S_{j+a_n}-S_{j+a_{[\theta^i]}}|}{\sqrt{2\log n}\,\sigma_{a_n}} \\ & - \max_{n\in N_i} \max_{n-a_n\leq j\leq[\theta^i]-a_{[\theta^i]}} \frac{|S_{j+a_n}-S_j|}{\sqrt{2\log n}\,\sigma_{a_n}}. \end{aligned} \tag{41}$$

From (33), (34) and (41), we obtain (39) again. Then in the same way, (37) can be concluded and the proof is complete.

Now we are ready to prove Theorem 4.2.3.

Proof of Theorem 4.2.3. Proceeding to the same line as in the proof of Theorem 3.3, we can deduce that for all large n and $0<\varepsilon<1$

$$P\{C^*(n,\ a_n)<1-\varepsilon\} \leqq \exp\left(-\frac{1}{2}n^{\delta'}\right)+Kn^{-\delta_0}, \tag{42}$$

where K, δ' and δ_0 are positive constants. Take $\theta>1$ and consider the following sequence of integers $\{[\theta^i];\ i=1, 2, \cdots\}$. Then the inequality (42) yields for large number i

$$\begin{aligned} & P\{C^*([\theta^i],\ a_{[\theta^i]})<1-\varepsilon\} \\ & \leqq \exp\left(-\frac{1}{2}[\theta^i]^{\delta'}\right)+K[\theta^i]^{-\delta_0}. \end{aligned}$$

Thus the series

$$\sum_i P\{C^*([\theta^i],\ a_{[\theta^i]})<1-\varepsilon\}$$

is convergent and

$$\liminf_{i\to\infty} C^*([\theta^i],\ a_{[\theta^i]}) \geqq 1-\varepsilon, \quad \text{a. s.}.$$

For given i, set $[\theta^i]\leqq n\leqq[\theta^{i+1}]$. Then from (37) of Lemma 4.2 we have the result.

Combinig Theorems 4.1.1 and 4.2.3, we obtain

Theorem 4.2.4. *Under the assumptions of Theorem* 4.2.3, *we have*

$$\lim_{n\to\infty} C^*(n,\ a_n)=1, \quad a.s..$$

Remark 2. As in Remark 1, take $a_n=[c\log n]$ in Theorem 4.2.4. Then Theorem 3.4 follows from the observation that

$$\lim_{n\to\infty}\frac{[c\log n]}{2\log n}=\frac{c}{2}.$$

Corollary. (Deo [10]) *Let* $\{\xi_j\,;\,j=1,2,\cdots\}$ *have the correlation function* $r_n=E\xi_1\xi_{1+n}$ *such that*

$$\lim_{n\to\infty} n^{1+\beta}r_n=0 \qquad \text{for some } \beta>0. \tag{43}$$

Set

$$0<\sigma^2=1+2\sum_{j=1}^{\infty}r_j.$$

Then for each $0<c<\infty$

$$\lim_{n\to\infty}\boldsymbol{D}(n,[c\log n])=\sigma\sqrt{2/c},\quad a.s..$$

Proof. As in Ibragimov [12], we have from the above assumption that

$$\begin{aligned}\sigma_n^2&=E(\xi_1+\cdots\xi_n)^2=n+2\sum_{1\le i<j\le n}E\xi_i\xi_j\\&=n+2\sum_{j=1}^{n-1}(n-j)E\xi_1\xi_{1+j}\\&=n\left(1+2\sum_{j=1}^{n-1}\left(1-\frac{j}{n}\right)r_j\right)\\&=n\sigma^2\{1+o(1)\}.\end{aligned}\tag{44}$$

Since (43) and (44) imply the conditions of Theorem 4.2.4, we have with probability 1

$$\begin{aligned}1&=\lim_{n\to\infty}\boldsymbol{C}^*(n,[c\log n])\\&=\lim_{n\to\infty}\max_{0\le j\le n-[c\log n]}\frac{S_{j+[c\log n]}-S_j}{\sqrt{2\log n}\,\sigma_{[c\log n]}}\\&=\lim_{n\to\infty}\max_{0\le j\le n-[c\log n]}\frac{S_{j+[c\log n]}-S_j}{[c\log n]}\left(\frac{[c\log n]}{2\log n}\right)^{1/2}\frac{1}{\sigma}\\&=\lim_{n\to\infty}\boldsymbol{D}(n,[c\log n])\sqrt{c/2}\,\frac{1}{\sigma}.\end{aligned}$$

§ 5. Examples

First we shall give an example of Theorem 3.4. Suppose $\{\xi_i\,;\,i=0,\pm1,\pm2,\cdots\}$ is a strictly stationary sequence with

$$E\xi_i=0 \quad\text{and}\quad 0<\mathrm{Var}(\xi_i)<\infty. \tag{45}$$

For all $-\infty\leqq i\leqq j\leqq\infty$, let F_i^j denote the Borel σ-field of events generated by random variables $\xi_k(i\leqq k\leqq j)$. For each $n=1,2,\cdots$, we define the dependence coefficient:

$$\rho(n)=\sup_{f,g}|\mathrm{correlation}(f,g)|,\; f\in L_2(F_{-\infty}^k),\; g\in L_2(F_{k+n}^{\infty}), \tag{46}$$

where $L_2(F^j_i)$ denotes the collection of all F^j_i-measurable random variables. Then the sequence $\{\xi_i;\ i=0, \pm 1, \pm 2, \cdots\}$ is called a *ρ-mixing* if $\rho(n)\to 0$ as $n\to\infty$. From Ibragimov [13]-[14] and Bradley [3] we can see that if

$$\sum_{n=1}^{\infty} \rho(2^n)<\infty \tag{47}$$

then

(i) the sequence $\{\xi_i;\ i=0, \pm 1, \pm 2, \cdots\}$ has a continuous spectral density $f(\lambda)$ and

(ii) when $f(0)\neq 0$, $\sigma_n^2 \sim 2\pi f(0)n$ as $n\to\infty$. (48)

From these facts we can deduce

Example 5.1. Let $\{\xi_j;\ j=1, 2, \cdots\}$ be a stationary Gaussian sequence with $E\xi_1=0$, $E\xi_1^2=1$ and

$$\rho(n)\leqq n^{-\nu} \quad \text{for some } \nu>0 \tag{49}$$

and further $f(0)\neq 0$. Then we obtain the result of Theorem 3.4. For, (49) implies (47) and (15). Furthermore, since $f(0)\neq 0$, (48) implies $\sigma_n^2\to\infty$ as $n\to\infty$. Thus the conditions of Theorem 3.4 are satisfied.

Example 5.2. Let $\{X(t);\ -\infty<t<\infty\}$ be a fractional Brownian motion with the covariance function

$$E\{X(t)X(s)\}=\frac{1}{2}\{|t|^{2\alpha}+|s|^{2\alpha}-|t-s|^{2\alpha}\}, \quad 0<\alpha<1.$$

Then

$$E\{\{X(t)-X(s)\}^2\}=|t-s|^{2\alpha}.$$

Define random variables

$$\xi_n=X(n)-X(n-1), \qquad n=1, 2, \cdots,$$

$$S_n=\sum_{i=1}^{n}\xi_i \quad \text{and} \quad X(0)=0.$$

Then

$$ES_n^2=EX(n)^2=n^{2\alpha}$$

and $\{\xi_n;\ n=1, 2, \cdots\}$ is a stationary Gaussian sequence with $E\xi_1=0$ and $E\xi_1^2=1$.

(i) $0<\alpha\leqq\frac{1}{2}$ iff $E\xi_n\xi_m\leqq 0$, $n\neq m$.

In this case we have from Theorem 3.2

$$\lim_{n\to\infty} C(n, [c\log n])=\sqrt{2/c}, \quad \text{a. s.}. \tag{50}$$

In particular, if $\alpha=1/2$, then $\{\xi_n;\ n=1, 2, \cdots\}$ is an i. i. d. Gaussian sequence with $E\xi_1=0$ and $E\xi_1^2=1$, and it is well known that (50) is the result of Erdös-Rényi law for i. i. d. $N(0, 1)$-sequence.

(ii) Let $\frac{1}{2}<\alpha<1$, then

$$r_n = O(n^{-(2-2\alpha)}), \quad \text{as} \quad n \longrightarrow \infty.$$

Thus there exists a number ν with $0<\nu<2-2\alpha$ such that

$$\lim_{n\to\infty} n^{\nu} r_n = 0.$$

From Theorem 3.4 we also obtain (50).

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