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Partially conformal qc mappings and the universal Teichmüller space

Dedicated to Professor Kôtaro Oikawa on his 60th birthday

By

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Introduction

Let $M(\Delta)$ be the set of all Beltrami coefficients on the unit disk Δ, that is, it is the set of all bounded measurable functions μ defined on Δ with $\|\mu\|_\infty = \mathrm{esssup}_\Delta|\mu(z)| < 1$. We denote by w_μ the unique quasiconformal (qc) self-mapping of Δ satisfying the Beltrami equation $w_{\bar z} = \mu w_z$ and leaving ± 1, i fixed. Two elements μ and ν in $M(\Delta)$ are called equivalent if $w_\mu = w_\nu$ on $\partial\Delta$. The universal Teichmüller space T is defined as the quotient space of $M(\Delta)$ with respect to this equivalence relation. This space T carries a natural metric, called the Teichmüller metric (cf. Lehto [3]), with respect to which the canonical projection $\Phi\colon M(\Delta) \to T$ is open as well as continuous.

Let V be a measurable subset of Δ, and set

$$M(V) = \{\mu \in M(\Delta);\ \mu|_{(\Delta - V)} = 0\}.$$

We denote the Banach space of all integrable holomorphic functions on Δ by A, and the characteristic function of a set Y by $\chi(Y)$. Our first result is a necessary condition for V to insure that the points which can be represented by quasiconformal mappings whose Beltrami coefficients are in $M(V)$ contain a non-empty open set in T.

Theorem 1. *Let V be a measurable subset of Δ with positive measure. If the interior of $\Phi(M(V))$ is not empty, then*

$$\inf\{\|\chi(V)\phi\|_1;\ \phi \in A,\ \|\phi\|_1 = 1\} > 0. \tag{1}$$

We denote the hyperbolic disk with center at $\zeta \in \Delta$ and hyperbolic radius ρ by $D(\zeta;\rho)$, and the hyperbolic area of $Y \subset \Delta$ by $\sigma(Y)$.

Definition 1. *A* measurable subset Y of Δ is *uniformly distributed in mean* if

$$\inf\left\{\frac{\sigma(Y \cap D(\zeta;\rho))}{\sigma(D(\zeta;\rho))};\ \zeta \in \Delta\right\} > 0 \qquad \text{for some } \rho > 0.$$

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Definition 2. *A denumerable subset S of Δ is ρ-scattered* ($\rho > 0$) if the collection of hyperbolic disks $\{D(\zeta; \rho); \zeta \in S\}$ covers Δ and the function $\sum_{\zeta\in S} \chi(D(\zeta; 2\rho))$ is bounded on Δ.

Our second result asserts that the subsets V satisfying an analytic condition (1) can be characterized in terms of the hyperbolic geometry;

Theorem 2. *For a measurable subset V of Δ the following three conditions are mutually equivalent.*

(a): *V satisfies the condition* (1).

(b): *V is uniformly distributed in mean.*

(c): *There is a ρ-scattered subset S of Δ for some positive ρ for which*

$$\inf\{\sigma(V \cap D(\zeta; \rho)); \zeta \in S\} > 0.$$

Our third result shows that the properties in Theorem 2 are quasiconformally invariant;

Theorem 3. Let V be a measurable subset of Δ and f be a quasiconformal self-mapping of Δ. Then V satisfies one of three conditions in Theorem 2 if and only if so does $f(V)$.

The above are improvement of results given in [5] for the case of the universal Teichmüller space.

§1. Proof of Theorem 2

We begin by providing fundamental facts which play important roles in the proof of Theorem 2. Though the first is geometrically almost clear, we include it for the sake of completeness.

Proposition 1. *A ρ-scattered set always exists for each positive ρ.*

Proof. Take a Fuchsian group G acting on Δ with Δ/G compact, and let Ω be a relatively compact fundamental domain for G. Cover $\bar{\Omega}$ with finitely many hyperbolic disks $D(\zeta_1; \rho), \dots, D(\zeta_k; \rho)$, where $\zeta_1, \dots, \zeta_k$ are in Ω. Then $S = \{g(\zeta_j); g \in G, 1 \le j \le k\}$ is ρ-scattered. In fact, firstly

$$\bigcup_{\zeta\in S} D(\zeta; \rho) \supset \bigcup_{g\in G} g(\bar{\Omega}) = \Delta.$$

Secondly, fix $z_0 \in \Omega$ and $\rho' > 0$ for which $\bar{\Omega} \subset D(z_0; \rho')$, and take finite subset G' of G such that $D(z_0; 2\rho + \rho') \subset \bigcup_{g\in G'} g(\bar{\Omega})$. Let $z \in \Delta$ and $z' \in \bar{\Omega} \cap \{g(z); g \in G\}$. Then

$$\sum_{\zeta\in S} \chi(D(\zeta; 2\rho))(z) = \#(S \cap D(z; 2\rho)) = \#(S \cap D(z'; 2\rho))$$

$$\le \#(S \cap D(z_0; 2\rho + \rho')) \le \#(S \cap \bigcup_{g\in G'} g(\bar{\Omega})) = k\#(G').$$

Lemma 1. *Let* $\{a_n\}_{n=1}^{\infty}$ *and* $\{A_n\}_{n=1}^{\infty}$ *be sequences of positive numbers such that* $1 \le \sum_1^\infty a_n$ *and* $\sum_1^\infty A_n \le m$. *Then there exists a (non-empty, not necessarily infinite) subset* N *of* $\mathbf{N}$ *such that* $A_n \le 2ma_n$ *for all* $n \in N$ *and* $\sum_{n\in N} a_n \ge 1/2$.

Proof. Set $N = \{n \in \mathbf{N};\ A_n \le 2ma_n\}$. Since

$$\sum_{n\notin N} a_n \le \frac{1}{2m} \sum_{n\notin N} A_n \le \frac{1}{2},$$

we have

$$\sum_{n\in N} a_n \ge 1 - \sum_{n\notin N} a_n \ge \frac{1}{2}.$$

For $0 < r < 1$, $0 < a < 1$ and $0 < b \le \sigma(\Delta_r)$, where $\Delta_r = \{z \in \mathbf{C};\ |z| < r\}$, we define

$$A(r, a) = \{\phi \in A;\ \|\phi\|_1 \le 1 \text{ and } \|\chi(\Delta_r)\phi\|_1 \ge a\},$$

and when $A(r, a) \ne \emptyset$, we set

$$\alpha(r, a, b) = \inf \|\chi(Y)\phi\|_1,$$

where infimum being taken over all $\phi \in A(r, a)$ and all measurable $Y \subset \Delta_r$ with $\sigma(Y) \ge b$.

Lemma 2. *If* $A(r, a) \ne \emptyset$, *then* $\alpha(r, a, b) > 0$.

Proof. Take sequences $\{\phi\}_{n=1}^{\infty}$ in $A(r, a)$ and $\{Y_n\}_{n=1}^{\infty}$ of measurable subsets of Δ_r with $\sigma(Y_n) \ge b$ for which $\lim \|\chi(Y_n)\phi_n\|_1 = \alpha(r, a, b)$. Since $A(r, a)$ is sequentially compact with respect to the topology induced by the locally uniform convergence, we may assume that ϕ_n converges to some $\phi \in A(r, a)$ locally uniformly, in particular, we have $\phi \ne 0$. Set $X = \{z \in \Delta_r;\ |\phi(z)| < 2\delta\}$, where a positive δ is chosen so small that the Euclidean area $m(X)$ of X is less than $c = (1 - r^2)^2 b/2$. Then

$$\begin{aligned} m(Y_n - X) &\ge m(Y_n) - m(X) \\ &> (1 - r^2)^2 \sigma(Y_n) - c \ge c. \end{aligned}$$

Since $|\phi_n| > \delta$ on $\Delta_r - X$ for sufficiently large n, we have

$$\|\chi(Y_n)\phi_n\|_1 > \delta m(Y_n - X) > c\delta \text{ for large } n,$$

hence we conclude that

$$\alpha(r, a, b) = \lim_{n\to\infty} \|\chi(Y_n)\phi_n\|_1 \ge c\delta > 0.$$

Proof of Theorem 2. (a) $\Rightarrow$ (b): Suppose that (b) does not hold. Then we can find a sequence $\{\zeta_n\}_{n=1}^{\infty}$ in Δ such that $\sigma(V \cap D(\zeta_n;\ n)) < 1/n$. Set $\gamma_n(z) = (z - \zeta_n)/(1 - \overline{\zeta_n} z)$ and $\phi_n = (\gamma_n')^2$, then we have $\phi_n \in A$ and $\|\phi_n\|_1 = \pi$. On the other hand, we have

$$\| \chi(V)\phi_n \|_1 = m(\gamma_n V) \leq m(\Delta - \Delta_{\tanh n}) + m((\gamma_n V) \cap \Delta_{\tanh n})$$
$$\leq \pi \operatorname{sech}^2 n + \frac{1}{n},$$

because

$$m((\gamma_n V) \cap \Delta_{\tanh n}) = m((\gamma_n V) \cap D(0; n))$$
$$\leq \sigma((\gamma_n V) \cap D(0; n)) = \sigma(V \cap D(\zeta_n; n)) < \frac{1}{n}.$$

Hence (a) does not hold.

(b) $\Rightarrow$ (c): This is obvious by Proposition 1.

(c) $\Rightarrow$ (a): Let $\{\zeta_n\}_{n=1}^{\infty}$ be an enumeration of a ρ-scattered set S. For ϕ in A with norm one, we put

$$a_n = \| \chi(D(\zeta_n; \rho))\phi \|_1, \ A_n = \| \chi(D(\zeta_n; 2\rho))\phi \|_1,$$
$$\text{and } m = \sup\{ \sum_{n=1}^{\infty} \chi(D(\zeta_n; 2\rho))(z); z \in \Delta \},$$

then we have

$$a_n < A_n \quad \text{for all } n,$$
$$\sum_{n=1}^{\infty} a_n \geq 1 \text{ and } \sum_{n=1}^{\infty} A_n \leq m.$$

Let N be a subset of $\mathbf{N}$ with the property in Lemma 1. For $n \in N$, we set

$$\phi_n = \frac{1}{A_n}(\phi \circ \eta_n \circ R)((\eta_n \circ R)')^2,$$

where $\eta_n(z) = (z + \zeta_n)/(1 + \overline{\zeta_n} z)$ and $R(z) = z\tanh(2\rho)$. Then $\phi_n \in A$ saitsfies

$$\| \phi_n \|_1 = \frac{1}{A_n} \| \chi(\eta_n \circ R(\Delta))\phi \|_1$$
$$= \frac{1}{A_n} \| \chi(D(\zeta_n; 2\rho))\phi \|_1 = 1$$

and similarly

$$\| \chi(\Delta_r)\phi_n \|_1 = \frac{a_n}{A_n} \geq \frac{1}{2m},$$

where $r = (\tanh \rho)/\tanh(2\rho)$, in other words $\phi_n \in A(r, 1/(2m))$. Next, we put

$$b = \inf\{ \sigma(V \cap D(\zeta_n; \rho)); n \in N \},$$

then subsets $V_n = R^{-1} \circ \eta_n^{-1}(V \cap D(\zeta_n; \rho))$ of Δ_r satisfy

$$\sigma(V_n) > \sigma(\eta_n^{-1}(V \cap D(\zeta_n; \rho))) \geq b.$$

We thus have by Lemma 2

$$\|\chi(V\cap D(\zeta_n;\,\rho))\phi\|_1 = A_n\|\chi(V_n)\phi_n\|_1$$
$$\geq \alpha A_n > \alpha a_n,$$

where $\alpha = \alpha(r,\ 1/(2\mathrm{m}),\ b)$, and consequently

$$\|\chi(V)\phi\|_1 \geq \frac{1}{m}\sum_{n\in N}\|\chi(V\cap D(\zeta_n;\,\rho))\phi\|_1$$
$$> \frac{\alpha}{m}\sum_{n\in N} a_n \geq \frac{\alpha}{2m}$$

by the definition of m and Lemma 1. Since the constants α and m are independent of ϕ, the condition (a) holds.

§2. Proof of Theorem 3

We prove that, for a quasiconformal self-mapping f of Δ and a measurable subset V of Δ, if $f(V)$ is uniformly distributed in mean then so is V. By performing preliminary Möbius transformations, it suffices to show that, for $\rho > 0$ and a K-quasiconformal self-mapping f of Δ with $f(0) = 0$, we have

$$\sigma(f(V)\cap D(0;\,\rho)) \leq b\sigma(V\cap D(0;\,\rho'))^a,$$

where a, b and ρ' are positive constants which depends only on K and ρ. Henceforth we denote by $c_j(j = 1, 2, \ldots)$ positive constants depending only on K and ρ.

By a distortion theorem of hyperbolic distances in Δ (for example, see Lehto-Virtanen [4, p.65]), we can find c_1 such that $f^{-1}(\overline{D(0;\,\rho)}) \subset D(0;\,c_1)$, and moreover if we take c_2 sufficiently small then for every z in $D(0;\,\rho)$, for the smallest hyperbolic disk $D'(z)$ with center at $f^{-1}(z)$ containing $f^{-1}(D(z))$, $D(z) = D(z;\,c_2)$, and for the smallest square Q that contains $D'(z)$ and whose sides are parallel to the axes, we have $D'(z) \subset D(0;\,c_1)$ and the Euclidean diameter of $f(Q)$ is less than the Euclidean distance between $f(Q)$ and $\partial\Delta$. Then, by the same argument with the proof of Theorem 2 in Gehring-Kelly [2], we see that

$$\frac{m(f(V)\cap D(z))}{m(D(z))} \leq c_3\left(\frac{m(V\cap D'(z))}{m(D'(z))}\right)^{c_4} \qquad (c_4 < 1)$$

for every z in $D(0;\,\rho)$. We also have

$$\sigma(f(V)\cap D(z)) \leq c_5 m(f(V)\cap D(z)),$$
$$m(D(z)) \leq \sigma(D(z)) = c_6,$$
$$m(V\cap D'(z)) \leq \sigma(V\cap D'(z)),$$
$$m(D'(z)) \geq c_7,$$

hence we get

$$\sigma(f(V)\cap D(z)) \leq c_8(\sigma(V\cap D'(z)))^{c_4}.$$

Choose a finite number of disks $D(z_1), \dots, D(z_n)$ so that they cover $D(0; \rho)$, thereby we have

$$\begin{aligned}\sigma(f(V)\cap D(0;\rho)) &\leq \sum_{k=1}^{n} \sigma(f(V)\cap D(z_k)) \\ &\leq c_8 \sum_{k=1}^{n} (\sigma(V\cap D'(z_k)))^{c_4} \\ &\leq c_9 \sigma(V\cap D(0; c_1))^{c_4}.\end{aligned}$$

This completes the proof.

§3. Proof of Theorem 1

We use the following inequality due to Reich and Strebel [6].

Theorem A. *For μ, ν in $M(\Delta)$ such that $w_\nu \circ w_\mu = \mathrm{id}$ on $\partial\Delta$, and for ψ in A with norm one, we have*

$$1 \leq \iint_\Delta |\psi| \frac{|1-\mu\psi/|\psi||^2}{1-|\mu|^2} \cdot \frac{1+|\nu\circ w_\mu|}{1-|\nu\circ w_\mu|}\, dxdy.$$

A Beltrami coefficient μ on Δ is said to be *extremal* if

$$\|\mu\|_\infty = \inf\{\|\nu\|_\infty;\ \nu\in M(\Delta),\ w_\nu = w_\mu \text{ on } \partial\Delta\}.$$

The extremality is characterized by Hamilton, Reich and Strebel as follows (for a proof, see [6] for example):

Theorem B. *A necessary and sufficient condition for μ in $M(\Delta)$ to be extremal is*

$$\|\mu\|_\infty = \sup\left\{\left|\iint_\Delta \mu\phi\, dxdy\right|;\ \phi\in A,\ \|\phi\|_1 = 1\right\}.$$

Proof of Theorem 1. Let τ be an arbitrary element in $M(V)$. We define a mapping $\tau_*: M(\Delta)\to M(\Delta)$ as $\tau_*(\mu)$ is the unique element of $M(\Delta)$ that satisfies $w_{\tau_*(\mu)} = w_\mu \circ (w_\tau)^{-1}$ for each $\mu\in M(\Delta)$. It is obvious that τ_* is a self-homeomorphism of $M(\Delta)$ with $\tau_*(\tau) = 0$, and that $\Phi(\mu) = \Phi(\nu)$ if and only if $\Phi(\tau_*(\mu)) = \Phi(\tau_*(\nu))$ for μ and ν in $M(\Delta)$. Hence there is a self-homeomorphism $\omega: T\to T$ such that

$$\omega\circ\Phi = \Phi\circ\tau_*, \tag{2}$$

in particular,

$$\omega(\Phi(\tau)) = \Phi(0). \tag{3}$$

We note that

(4) $$\tau_*(M(V)) = M(w_\tau(V)).$$

Suppose that V does not satisfy (1). Theorems 2 and 3 imply that neither does $V' = w_\tau(V)$. Then there exists a sequence $\{\phi_n\}_{n=1}^\infty$ in A such that

$$\|\phi_n\|_1 = 1 \quad \text{for all } n \quad \text{but} \lim_{n\to\infty} \|\chi(V')\phi_n\|_1 = 0.$$

By taking a subsequence, if necessary, we may assume that ϕ_n converges locally uniformly to some $\phi \in A$. Since $\|\chi(V')\phi\|_1 \leq \lim \|\chi(V')\phi_n\|_1 = 0$ and V' is not null, we have $\phi = 0$, that is, ϕ_n converges locally uniformly to zero. We can now choose $\{\phi_{n(j)}\}_{j=1}^\infty$, a subsequence of $\{\phi_n\}_{n=1}^\infty$, and $\{r_j\}_{j=1}^\infty$, an increasing sequence of numbers, so that

$$\lim_{j\to\infty} r_j = 1 \quad \text{and} \quad \|\chi(A_j)\phi_{n(j)}\|_1 > 1 - \frac{1}{j} \quad \text{for all} \quad j,$$

where $A_j = \{r_j \leq |z| < r_{j+1}\}$. For example, put $r_1 = 0$, $\phi_{n(1)} = \phi_1$ and $r_2 = 1/2$. After choosing $\{r_j\}_1^k$ and $\{\phi_{n(j)}\}_1^{k-1}$, take $n(k)$ ($> n(k-1)$) so large that $\|\chi(\bigcup_{j=1}^{k-1} A_j)\phi_{n(k)}\|_1 < 1/k$, next take $r_{k+1}(> r_k)$ so close to one that $\|\chi(A_k)\phi_{n(k)}\|_1 > 1 - 1/k$.

Set

$$\mu_0 = \sum_{j=1}^\infty \chi(A_j - V')|\phi_{n(j)}|/\phi_{n(j)}.$$

Then

$$\begin{aligned}\left|1 - \iint_\Delta \mu_0\phi_{n(j)}\,dxdy\right| &= \left|\iint_\Delta |\phi_{n(j)}| - \mu_0\phi_{n(j)}\,dxdy\right| \\ &\leq 2\iint_{\Delta-(A_j-V')} |\phi_{n(j)}|\,dxdy \\ &\leq 2(\|\chi(\Delta - A_j)\phi_{n(j)}\|_1 + \|\chi(V')\phi_{n(j)}\|_1) \\ &= o(1) \quad \text{as} \quad j\to\infty\end{aligned}$$

yields

$$\sup\left\{\left|\iint_\Delta \zeta\mu_0\phi\,dxdy\right|;\ \phi\in A,\ \|\phi\|_1 = 1\right\} \geq |\zeta| = \|\zeta\mu_0\|_\infty$$

for each ζ in Δ. Since the opposite inequality is trivial, we see by Theorem B that $\zeta\mu_0$ is extremal. Let μ be the Beltrami coefficient of $(w_{\zeta\mu_0})^{-1}$. Obviously, μ is extremal, hence Theorem B implies that there is a sequence $\{\psi_n\}_{n=1}^\infty$ in A such that

$$\|\psi_n\|_1 = 1 \text{ for all } n \quad \text{and} \lim_{n\to\infty} \iint_\Delta \mu\psi_n\,dxdy = \|\mu\|_\infty.$$

Notice that

$$|\mu| = |\zeta\mu_0 \circ (w_{\zeta\mu_0})^{-1}| = |\zeta|\chi(\Delta - V'')$$

where $V'' = w_{\zeta\mu_0}(V')$, and so

$$\begin{aligned}\|\chi(V'')\psi_n\|_1 &= 1 - \iint_{\Delta - V''} |\psi_n| dxdy \\ &\leq 1 - \frac{1}{|\zeta|}\left|\iint_{\Delta} \mu\psi_n dxdy\right| \\ &= o(1) \quad \text{as} \quad n \to \infty.\end{aligned}$$

Suppose that $\zeta\mu_0(\zeta \in \Delta - \{0\})$ were equivalent to some ν in $M(V')$. Then, applying Reich-Strebel's inequality to μ, ν and ψ_n, where μ and ψ_n are obtained from $\zeta\mu_0$ as above, we have

$$1 \leq \frac{1}{1 - |\zeta|^2} \iint_{\Delta - V''} |\psi_n| \left|1 - \frac{\mu\psi_n}{|\psi_n|}\right|^2 dxdy + \frac{1 + \|\nu\|_\infty}{1 - \|\nu\|_\infty} \|\chi(V'')\psi_n\|_1,$$

thus

$$\begin{aligned}1 - |\zeta|^2 &\leq \iint_{\Delta} |\psi_n||1 - \mu\psi_n/|\psi_n||^2\, dxdy + o(1) \\ &\leq 1 - 2\,\mathrm{Re} \iint_{\Delta} \mu\psi_n\, dxdy + |\zeta|^2 + o(1) \\ &= (1 - |\zeta|)^2 + o(1) \quad \text{as} \quad n \to \infty.\end{aligned}$$

This is a contradiction to $\zeta \neq 0$. Therefore

$$\Phi(\{\zeta\mu_0 ; \zeta \in \Delta - \{0\}\}) \cap \Phi(M(V')) = \emptyset,$$

which implies that $\Phi(\zeta\mu_0)$ approaches $\Phi(0)$ through the complement of $\Phi(M(V'))$ as ζ tends to zero, that is, $\Phi(0) \notin \mathrm{int}\ \Phi(M(V'))$. But (2), (3) and (4), we have

$$\Phi(\tau) \notin \mathrm{int}\ \Phi(M(V)),$$

and consequently we conclude that $\Phi(M(V))$ has no interior points. This completes the proof.

§4. Remark

In this section we remark that a part of Theorem 1 can be extended to general cases as we argumented in [5].

For a Fuchsian group Γ acting on Δ, which may or may not be elementary, and for a Γ-invariant measurable subset V of Δ with positive measure, we set

$$M(\Delta, \Gamma) = \{\mu \in M(\Delta) ; (\mu \circ \gamma)\bar{\gamma}'/\gamma' = \mu \text{ for all } \gamma \in \Gamma\},$$

$$M(V, \Gamma) = M(\Delta, \Gamma) \cap M(V).$$

Let C be a Γ-invariant closed subset of $\partial\Delta$ such that $\{\pm 1, i\} \subset C$. We say that μ and $\nu \in M(\Delta, \Gamma)$ are equivalent if $w_\mu = w_\nu$ on C. With this equivalence relation we define the Teichmüller space $T(\Gamma, C)$ as the quotient space of $M(\Delta, \Gamma)$, and denote the natural projection: $M(\Delta, \Gamma) \to T(\Gamma, C)$ by Φ.

We denote by $A(\Gamma, C)$ the Banach space of all holomorphic functions ϕ on Δ such that

$$(\phi \circ \gamma)(\gamma')^2 = \phi \quad \text{for all } \gamma \in \Gamma,$$

$$\|\phi\|_1 = \iint_{\Delta/\Gamma} |\phi|\, dxdy < \infty,$$

and furthermore ϕ can be continuously extended to $\partial\Delta - C$ so that $z^2\phi(z)$ is real there, proveded that $\partial\Delta \neq C$. Almost the same argument in the proof of Theorem 1 leads

Theorem 1′. *If $\Phi(0)$ is an interior point of $\Phi(M(V, \Gamma))$, then*

$$\inf\{\|\chi(V)\phi\|_1;\ \phi \in A(\Gamma, C),\ \|\phi\|_1 = 1\} > 0.$$

Sketch of Proof. Suppose that there exists a sequence $\{\phi_n\}$ in $A(\Gamma, C)$ such that $\|\phi_n\|_1 = 1$ and $\lim \|\chi(V)\phi_n\|_1 = 0$. We may assume that ϕ_n converges to zero locally uniformly on the borderded Riemann surface $R = (\bar{\Delta} - C)/\Gamma$. Let $\{R_m\}$ be an exhaustion of R. Then we can choose subsequences $\{R_{m(j)}\}$ and $\{\phi_{n(j)}\}$ so that

$$\|\chi(R_{m(j+1)} - R_{m(j)})\phi_{n(j)}\|_1 = 1 + o(1)$$

as we chose $\{r_j\}$ and $\{\phi_{n(j)}\}$ in the proof of Theorem 1. Set

$$\mu_0 = \sum_{j=1}^{\infty} \chi(R_{m(j+1)} - R_{m(j)} - V)|\phi_{n(j)}|/\phi_{n(j)}.$$

Then by using Theorems A and B for arbitrary Γ and C (see, for example, Gardiner [1, Ch. 6]), we can obtain as in the proof of Theorem 1

$$\Phi(\{t\mu_0;\ 0 < t < 1\}) \cap \Phi(M(V, \Gamma)) = \emptyset,$$

from which it follows that

$$\Phi(0) \notin \operatorname{int} \Phi(M(V, \Gamma)).$$

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