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Homology of the Kac-Moody groups II

Dedicated to Professor Shôrô Araki on his 60th birthday

By

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§ 1. Introduction

Let G be a compact, connected, simply connected, simple Lie group and $\mathfrak{g}$ its Lie algebra. Let $X\langle n\rangle$ be the n-connected cover of the space X. Since $\pi_3(G) \cong \boldsymbol{Z}$ is the first non-trivial homotopy, there is an S^1-fibration

$$S^1 \longrightarrow \Omega G\langle 2\rangle \longrightarrow \Omega G.$$

(Notice that sometimes one likes to write $\Omega G\langle 3\rangle = \Omega(G\langle 3\rangle)$ instead of our $\Omega G\langle 2\rangle = (\Omega G)\langle 2\rangle$.) The homotopy type of the Kac-Moody group $\mathfrak{K}(\mathfrak{g}^{(1)})$ is $\Omega G\langle 2\rangle \times G$. (See [10] and [11].) Since the homology of G is known and $H_*(\Omega G\langle 2\rangle; \boldsymbol{Z})$ is finitely generated, we have only to determine $H_*(\Omega G\langle 2\rangle; \boldsymbol{Z}_{(p)})$ for all prime p to determine $H_*(\mathfrak{K}(\mathfrak{g}^{(1)}); \boldsymbol{Z})$.

The homology of G has non trivial p-torsions if and only if (G, p) is one of the following:

$$(Spin(n), 2)\ n \geq 7,\ (E_6, 2),\ (E_6, 3)$$
$$(E_7, 2),\ (E_7, 3),\ (E_8, 2),\ (E_8, 3),\ (E_8, 5),$$
$$(F_4, 2),\ (F_4, 3) \text{ and } (G_2, 2).$$

In [14], we computed $H_*(\Omega G\langle 2\rangle; \boldsymbol{Z}_{(p)})$ for such (G, p) except $(Spin(n), 2)$ and $(E_6, 2)$.

The purpose of this paper is to determine it for the groups whose homology has no p-torsion. The major problem in the above case is that it is very difficult to compute the Gysin sequence of $\boldsymbol{Z}_{(p)}$-coefficients directly. To avoid this problem, we consider the Bockstein spectral sequence of the Gysin sequence. By using the Serre spectral sequence associated with $\Omega G\langle 2\rangle \to \Omega G \to CP^\infty$, we can prove that the first non trivial p-torsion of $H_*(\Omega G\langle 2\rangle; \boldsymbol{Z}_{(p)})$ is order p for all G. (See Theorem 3.1.) This fact becomes the "seed" of our computation of the above Bockstein spectral sequence and also gives the result for $(E_6, 2)$.

We define $\boldsymbol{Z}_{(p)}$-modules $C(d, p)$ and $L(G, p)$ in § 3. Then the main result is

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Theorem 3.2. *If $H_*(G; \boldsymbol{Z}_{(p)})$ has no p-torsion, then*

$$H_*(\Omega G\langle 2\rangle; \boldsymbol{Z}_{(p)}) \cong C(d(G, p) - 1, p) \otimes_{\boldsymbol{Z}_{(p)}} L(G, p)$$

as a $\boldsymbol{Z}_{(p)}$-module.

§ 2. Bockstein spectral sequence

Let $(\boldsymbol{C} = \bigotimes_{j\geq 0} \boldsymbol{C}_j, \partial)$ be a differential graded commutative algebra over $\boldsymbol{Z}_{(p)}$ (or $\boldsymbol{Z}$) where $\boldsymbol{C}_j$ is free, $\partial \boldsymbol{C}_j \subset \boldsymbol{C}_{j-1}$ and ∂ is derivative: $\partial(x \cdot y) = (\partial x) \cdot y + (-1)^{|x|} x \cdot \partial y$. Let put $\boldsymbol{D} = H(\boldsymbol{C})$ and $\boldsymbol{E} = H(\boldsymbol{C} \otimes \boldsymbol{Z}/p)$. Then we have an exact couple

$$\begin{tikzcd} \boldsymbol{D} \arrow[rr, "i"] & & \boldsymbol{D} \arrow[dl, "j"] \\ & \boldsymbol{E} \arrow[ul, "k"] & \end{tikzcd}$$

where $i = \times p$, j is the mod p reduction ρ and k is the connected homomorphism d associated with the short exact sequence

$$0 \longrightarrow \boldsymbol{Z}_{(p)} \xrightarrow{\times p} \boldsymbol{Z}_{(p)} \xrightarrow{\rho} \boldsymbol{Z}/p \longrightarrow 0.$$

The derived couple of this exact couple is

$$\begin{tikzcd} \boldsymbol{D}_r \arrow[rr, "i_r"] & & \boldsymbol{D}_r \arrow[dl, "j_r"] \\ & \boldsymbol{E}_r \arrow[ul, "k_r"] & \end{tikzcd}$$

where $\boldsymbol{E}_r = k^{-1}(\operatorname{Im} i^r)/j(\operatorname{Ker} i^r)$, $\boldsymbol{D}_r = \operatorname{Im} i^r$, $i_r = i | \boldsymbol{D}_r$, k_r is the map induced from k naturally and $j_r = j \circ (i^r)^{-1}$. Let us introduce some notation. For $\alpha \in \operatorname{Ker} \partial$, we denote its class in $H(\boldsymbol{C})$ as $[\alpha]$. If $\partial\alpha \in p \cdot \boldsymbol{C}$, we write $\langle\alpha\rangle$ for the class of α in $H(\boldsymbol{C} \otimes \boldsymbol{Z}/p)$. For $a \in k^{-1}(\operatorname{Im} i^r) \subset H(\boldsymbol{C} \otimes \boldsymbol{Z}/p)$, we denote the corresponding element in $\boldsymbol{E}_r$-term as $\{a\}_r$.

Lemma 2.1. *Let $a \in k^{-1}(\operatorname{Im} i^r)$. Then there exists $b \in H(\boldsymbol{C} \otimes \boldsymbol{Z}/p)$ satisfying $d_r(\{a\}_r) = \{b\}$ and $d_{r+1}(\{a^p\}_{r+1}) = \{b \cdot a^{p-1}\}_{r+1}$*

Proof. From the assumption, we can take an element $\tilde{b} \in H_{2n-1}(\boldsymbol{C})$ so as to satisfy $da = i^r \tilde{b}$. Let $x \in \boldsymbol{C}_{2n}$ (respectively $y \in \boldsymbol{C}_{2n-1}$) be a representaive element of a (respectively $\tilde{b}$). Since $da = \left[\dfrac{\partial x}{p}\right]$, there is $u \in \boldsymbol{C}_{2n}$ satisfying

$$\frac{\partial x}{p} = p^r \cdot y + \partial u$$

in $\boldsymbol{C}_{2n-1}$. Using the fact that $\boldsymbol{C}_{2n-1}$ and $\boldsymbol{C}_{2n}$ are free, we obtain an equation

$$\frac{1}{p}\cdot\partial(x-p\cdot u)=p^r\cdot y.$$

So $\partial(x-p\cdot u)\in p\cdot \boldsymbol{C}_{2n-1}$ and $\langle x-p\cdot u\rangle=\langle x\rangle=a$. We put $b=\langle y\rangle$. Then we have

$$d_r(\{a\}_r)=j_r\circ k_r(\{a\}_r)=j_r(p^r\cdot y)=\{b\}_r.$$

Since d_r is derivative, $d_r(a^p)=0$ and

$$k_{r+1}(a^p)=d(a^p)=\left[\frac{1}{p}\cdot\partial(x-p\cdot u)^p\right]=[\partial(x-p\cdot u)\cdot(x-p\cdot u)^{p-1}]=$$
$$[p^{r+1}\cdot y\cdot(x-p\cdot u)^{p-1}].$$

Since $j_{r+1}[p^{r+1}\cdot y\cdot(x-p\cdot u)^{p-1}]=\{\langle y\cdot(x-p\cdot u)^{p-1}\rangle\}_{r+1}=\{b\cdot a^{p-1}\}_{r+1}$, we obtain $d_{r+1}(\{a^p\}_{r+1})=\{b\cdot a^{p-1}\}_{r+1}$.

§3. Proof of Theorem

Let G be a compact Lie group as in §1 and $G\langle 3\rangle$ be the 3-connected cover of G. For a graded module $A=\oplus A_i$ of finite type over $\boldsymbol{F}_p$, we define $P(A,q)=\sum(\dim A_i)q_i$. Let $m(1)=1<m(2)\le\cdots\le m(l)$ be the exponent of the Weyl group of G. Let $t\in H^2(\Omega G;\boldsymbol{F}_p)$ be a generator. Since G is compact and $H^*(\Omega G;\boldsymbol{F}_p)$ is a Hopf algebra, there exists an integer $d(G,p)$ satisfying

$$t^{p^{d(G,p)-1}}\neq 0 \text{ and } t^{p^{d(G,p)}}=0.$$

Now let us recall the result of Kono [13].

Theorem A (*Kono* [13], *Theorem* 2.). *If $H_*(G;\boldsymbol{Z}_{(p)})$ has no p-torsion, then $d(G,p)$ is given by the following*
(1) *For the classical groups,*

$$d(G,p)=\begin{cases} r(n,p) & \text{if } G=SU(n),\\ r(2n,p) & \text{if } G=Spin(2n+1),\ Spin(2n) \text{ or } Sp(n)\\ & \text{and } p \text{ is an odd prime},\\ 1 & \text{if } G=Sp(n) \text{ and } p=2,\end{cases}$$

where $p^{r(n,p)-1}<n\le p^{r(n,p)}$.

(2) *For the exceptional groups, $d(G,p)$ is given by the following table*:

G	G_2		F_4, E_6		E_7		E_8	
p	5	$\neq 5$	≤ 11	>11	$5\le p\le 17$	>17	$7\le p\le 29$	>29
$d(G,p)$	2	1	2	1	2	1	2	1

Theorem B (Kono [13], Theorem 1.)

$$P(H^*(\Omega G\langle 2\rangle; \boldsymbol{F}_p), q) = P(A(G, p), q)\cdot(1 + q^{2a(G,p)-1})$$

where $A(G, p)$ *is a graded algebra satisfying*

$$P(A(G, p), q)^{-1} = (\prod_{j=2}^{l}(1 - q^{2m(j)}))\cdot(1 - q^{2a(G,p)})$$

and $a(G, p) = p^{d(G,p)}$.

Since the fibration

$$\Omega G\langle 2\rangle \longrightarrow \Omega G \longrightarrow CP^\infty$$

is a Hopf fibration, the $\boldsymbol{Z}_{(p)}$-homology Serre spectral sequence of this fibration is the Hopf algebra spectral sequence. The homology of CP^∞ is the devided polynomial algebra, so

$$H_*(CP^\infty; \boldsymbol{Z}_{(p)}) \cong \boldsymbol{Z}_{(p)}[e_1, e_2, \dots, e_j, \cdots]/(e_j^p = p\cdot e_{j+1}, j > 0)$$

where $\deg e_j = 2p^{j-1}$. Since $H_*(\Omega G; \boldsymbol{Z}_{(p)})$ is zero at the odd dimensions and $H_{2n-1}(\Omega G\langle 2\rangle; \boldsymbol{Z}_{(p)})$ is zero for $n < a(G, p)$, $e_j \otimes 1$ is a permanent cycle for $j \le d(G, p)$ and $e_{d(G,p)+1}$ is transgressive to the generator b of

$$H_{2a(G,p)-1}(\Omega G\langle 2\rangle; \boldsymbol{Z}_{(p)})$$

which is the cyclic group $\boldsymbol{Z}/p^\alpha$ where $\alpha > 0$ by Theorem B. Since $e^p_{d(G,p)}$ is clearly transgressive to zero, $p\cdot b$ which is the transgressive image of $e_{d(G,p)+1} = e^p_{d(G,p)}$ is zero. Thus we have

Theorem 3.1. $H_{2n-1}(\Omega G\langle 2\rangle; \boldsymbol{Z}_{(p)})$ *is zero for* $n < a(G, p)$ *and*

$$H_{2a(G,p)-1}(\Omega G\langle 2\rangle; \boldsymbol{Z}_{(p)}) \cong \boldsymbol{Z}/p.$$

The homology Gysin sequence associated with an S^1-fibration $S^1 \to \Omega G\langle 2\rangle \xrightarrow{\Omega i} \Omega G$ is split to the following exact sequence:

$$0 \longrightarrow H_{2n}(\Omega G\langle 2\rangle; \boldsymbol{Z}_{(p)}) \xrightarrow{\Omega i_*} H_{2n}(\Omega G; \boldsymbol{Z}_{(p)}) \xrightarrow{\chi}$$
$$H_{2n-2}(\Omega G; \boldsymbol{Z}_{(p)}) \longrightarrow H_{2n-1}(\Omega G\langle 2\rangle; \boldsymbol{Z}_{(p)}) \longrightarrow 0.$$

Let $\boldsymbol{C}_{2n} = H_{2n}(\Omega G; \boldsymbol{Z}_{(p)})$, $\boldsymbol{C}_{2n-1} = H_{2n-2}(\Omega G; \boldsymbol{Z}_{(p)}) \otimes s$ and $\partial(\alpha \otimes 1) = \chi(\alpha) \otimes s$, $\partial(\alpha \otimes s) = 0$. Then $H_i(\Omega G\langle 2\rangle; \boldsymbol{Z}_{(p)}) = H_i(\boldsymbol{C}, \partial)$ for all i. By Theorem 3.1, there is a generator $b \in H_{2a(G,p)-1}(\Omega G\langle 2\rangle; \boldsymbol{F}_p)$ and there exists $a \in H_{2a(G,p)}(\Omega G\langle 2\rangle; \boldsymbol{F}_p)$ satisfying $d_1(\{a\}_1) = \{b\}_1$. First recall that P_1 = the image of

$$\Omega i_*: H_*(\Omega G\langle 2\rangle; \boldsymbol{F}_p) \longrightarrow H_*(\Omega G; \boldsymbol{F}_p)$$

is a polynomial algebra, since it is a Hopf subalgebra of a polynomial algebra. $H_*(\Omega G\langle 2\rangle; \boldsymbol{F}_p)$ is isomorphic to the tensor product of P_1 and $\Lambda(b)$. Hence for all $(G, p) \neq (Sp(n), 2)$, $(G_2, 5)$, the E_1-term of the Bockstein spectral sequence is generated by elements of degree not greater than $2a(G, p)$ by dimensional reasons. Consider the fibration

$$\Omega Sp(n-1)\langle 2\rangle \longrightarrow \Omega Sp(n)\langle 2\rangle \longrightarrow \Omega S^{4n-1}.$$

The spectral sequence of the above fibration collapses since the Poincaré series of $H_*(\Omega Sp(n)\langle 2\rangle; \boldsymbol{F}_2)$ is equal to the product of those of $H_*(\Omega S^{4n-1}; \boldsymbol{F}_2)$ and $H_*(\Omega Sp(n-1)\langle 2\rangle; \boldsymbol{F}_2)$. Therefore we can assume that $d_1(a) = b$ and d_r(other generators of deg $< 4n-2) = 0$ for $r \geq 1$. If d_r is non zero on the generator of deg $= 4n-2$, the rank of the rational homology of $\Omega Sp(n)\langle 2\rangle$ fails to match with the rank of $\boldsymbol{Q}[s_6, s_{10}, \ldots, s_{4n-2}]$ ($deg\ s_i = i$) at deg $= 4n-2$. Thus the generators except a are permanent cycles in the Bockstein spectral sequence. (The case $(G, p) = (G_2, 5)$ is clear by dimensional reasons.) Therefore by using (2.1) inductively, we obtain the equation

$$d_r(\{a^{p^{r-1}}\}_r) = \{b \cdot a^{p^{r-1}-1}\}_r$$

for all $r \geq 1$. Define a graded $\boldsymbol{Z}_{(p)}$-module $C(d, p)$ by

$$C(d, p)_j = \begin{cases} \boldsymbol{Z}_{(p)} & \text{if } j = 0 \\ \boldsymbol{Z}/p^{r-d} & \text{if } j+1 = 2p^r \cdot k,\ (k, p) = 1 \text{ and } r \geq d, \\ 0 & \text{otherwize.} \end{cases}$$

We also define a graded free $\boldsymbol{Z}_{(p)}$-module $L(G, p)$ so as to satisfy

$$P(L(G, p), q)^{-1} = \prod_{j=2}^{l} (1 - q^{2m(j)})$$

Now the Bockstein spectral sequence and Theorem A, B give the proof of the followng theorem.

Theorem 3.2. *If $H_*(G; \boldsymbol{Z}_{(p)})$ has no p-torsion, then*

$$H_*(\Omega G\langle 2\rangle; \boldsymbol{Z}_{(p)}) \cong C(d(G, p) - 1, p) \otimes_{\boldsymbol{Z}_{(p)}} L(G, p)$$

as a $\boldsymbol{Z}_{(p)}$-module.

Example. (1) For $G = SU(2)$, $L(G, p) = \boldsymbol{Z}_{(p)}$ and $d(G, p) = 1$ for all p. Then $H_*(\Omega SU(2)\langle 2\rangle; \boldsymbol{Z}_{(p)}) \cong C(0, p)$ which is the result of Serre [18].
(2) For $G = SU(3)$, $m(2) = 2$ and $L(G, p) = \boldsymbol{Z}_{(p)}[s]$ where deg $s = 4$. In the case $p = 2$, $d(G, 2) = 2$. Therefore

$$H_*(\Omega SU(3)\langle 2\rangle; \boldsymbol{Z}_{(2)}) \cong C(1, 2) \otimes_{\boldsymbol{Z}_{(2)}} \boldsymbol{Z}_{(2)}[s].$$

If p is an odd prime, then $d(G, p) = 1$ and

$$H_*(\Omega SU(3)\langle 2\rangle; \boldsymbol{Z}_{(p)}) \cong C(0, p) \otimes_{\boldsymbol{Z}_{(p)}} \boldsymbol{Z}_{(p)}[s].$$

Remark. As an application of (3.1), we can determine $H_*(\Omega E_6\langle 2\rangle; \boldsymbol{Z}_{(2)})$.

Since $H_*(\Omega E_6\langle 2\rangle; \boldsymbol{Z}/2) \cong \Lambda(y_{31}) \otimes_{\boldsymbol{Z}/2} \boldsymbol{Z}/2[h_8, h_{10}, h_{14}, h_{16}, h_{22}, y_{32}]$ where $\Lambda(\ )$ is the exterior algebra over $\boldsymbol{Z}/2$ and all subscripts designate the degrees of the elements (See [14].), one can deduce that y_{31} is Sq^1_* image by (3.1). Then the argument in [14] works well and we have

$$H_*(\Omega E_6\langle 2\rangle; \boldsymbol{Z}_{(2)}) \cong C(3, 2) \otimes_{\boldsymbol{Z}_{(2)}} \boldsymbol{Z}_{(2)}[s_8, s_{10}, s_{14}, s_{16}, s_{22}]$$

as the $\boldsymbol{Z}_{(2)}$-module where $\deg s_i = i$.

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