J. Math. Kyoto Univ. (JMKYAZ)
31-1 (1991) 121–132

On the homotopy group $\pi_{8n+4}(Sp(n))$ and the Hopf Invariant

By

Kaoru MORISUGI

The purpose of this note is to study the homotopy groups of $U(n)$ and $Sp(n)$ of the first dimension over the meta-stable range. Our method is to use the classical EHP-sequence.

§0. Notations

$\boldsymbol{F}=\boldsymbol{C}$ (the complex number) or $\boldsymbol{H}$(the quarternion).
n: a fixed positive integer.

$$QF=\begin{cases}\Sigma CP_n^\infty & \text{(The stunted complex projective spaces) if } \boldsymbol{F}=\boldsymbol{C},\\ Q_{n+1}^\infty & \text{(The stunted quaternionic quasi-projective space) if } \boldsymbol{F}=\boldsymbol{H}.\end{cases}$$

$d=\dim_{\boldsymbol{R}}\boldsymbol{F}$.

$$GF(n)=\begin{cases}U(n) & \text{if } \boldsymbol{F}=\boldsymbol{C},\\ Sp(n) & \text{if } \boldsymbol{F}=\boldsymbol{H}.\end{cases}$$

This paper is organized as follows: In §1 we prepare the required preliminaries. In §2 we state our results. In §3 we collect necessary lemmas for the proofs of the results in §2. §4 is devoted to the proofs of lemmas in §3. In §5 we give the proofs of our results. §6 consists of the corrections of my previous paper [M1] about the meta-stable homotopy groups of $Sp(n)$. In §7 (Appendix) we give a necessary condition for the existence of the Hopf invariant one map.

§1. Preliminaries

Since QF is $d(n+1)-2$ connected, by the suspension theorem, the suspension $E\colon \pi_k(QF)\to\pi_k^s(QF)$ is isomorphic for $k\leqq 2d(n+1)-4$ and onto for $k=2d(n+1)-3$. For $\pi_k(GF(n))$, the range $dn+d-2\leqq k\leqq 2d(n+1)-5$ is called the meta-stable range. In this note we shall investigate the homotophy group $\pi_{2d(n+1)-4}(GF(n))$. As is known, QF is a subcomplex of $GF(\infty)/GF(n)$ and the pair $(GF(\infty)/GF(n), QF)$ is $2dn+3d-3$ connected [J]. Recall that

Communicated by Prof. H. Toda, May 6, 1989

$$H^*(QF; Z) \cong Z\{\alpha_{n+1}, \alpha_{n+2}, \alpha_{n+3}, \cdots\}, \text{ where } \dim \alpha_j = dj - 1.$$

Let $i: S^{d(n+1)-1} \to QF$ be a generator of $\pi_{d(n+1)-1}(QF) \cong Z$. $H_{(2)}$ stands for the mod 2 reduction of the Hopf invariant,

$$H: \pi_{2d(n+1)-1}(\Sigma QF) \longrightarrow \pi_{2d(n+1)-1}(\Sigma QF \wedge QF) \cong Z.$$

The following theorem is the starting point of our investigation.

Theorem 1.1 (Ōshima [O1]). *If* $n + 1 \neq 2^t$, *then* $H_{(2)}(\pi_{2d(n+1)-1}(\Sigma QF)) = 0$. *Therefore, using EHP-sequence, there exists a short exact sequence;*

$$0 \longrightarrow Z/2 \xrightarrow{\Delta} \pi_{2d(n+1)-3}(QF) \xrightarrow{E} \pi^s_{2d(n+1)-3}(QF) \longrightarrow 0, \tag{1.1}$$

where $\Delta(1) = [i, i]$, *the Whitehead product, and* E *is the suspension.*

The following proposition is well known.

Proposition 1.2.

(1) There exists a short exact sequence;

$$0 \longrightarrow Z \longrightarrow \pi_{4n+1}(U/U(n)) \xrightarrow{\partial} \pi_{4n}(U(n)) \longrightarrow 0.$$

(2) *There is an isomorphism* [M 2];

$$\pi_{8n+4}(Sp(n)) \cong Z/2 \oplus \pi_{8n+5}(Sp/Sp(n)).$$

In the above proposition, from (1), in order to determine the group $\pi_{4n}(U(n))$, we have to solve the unstable James number problem and a group extension problem (See, for example [CK]). However fortunately in the quaternionic case, $\pi_{8n+4}(Sp(n))$ has no relation with James numbers.

§2. Statement of results

Theorem 2.1. *If* $n + 1 \neq 2^t$, *then the sequence* (1.1) *splits, that is,*

$$\pi_{2d(n+1)-3}(QF) \cong Z/2 \oplus \pi^s_{2d(n+1)-3}(QF).$$

Now recall that $\pi_{d(n+1)-2}(GF(n))$ is a finite cyclic group. We denote its generator by σ_n. Let $\langle \sigma_n, \sigma_n \rangle$ be the Samelson product of σ_n and itself. Then we have the following theorem;

Theorem 2.2. *Let* $n \geqq 1$ *and* $n + 1 \neq 2^t$. *Then the element* $\langle \sigma_n, \sigma_n \rangle$ *in* $\pi_{2d(n+1)-4}(GF(n))$ *gives a direct summand of order* 2.

From the above theorem we easily get the following corollary.

Corollary 2.3. *Let* $n \geqq 1$ *and* $n + 1 \neq 2^t$. *Then,*

$$\pi_{8n+4}(Sp(n)) \cong Z/2 \oplus Z/2 \oplus \pi^s_{8n+5}(Q^\infty_{n+1}).$$

Here the first $Z/2$ *summand is generated by* μ_{8n+1} *on* $S^3 = Sp(1)$ *and the second is* *generated* (*cf.* [O1]) *by the Samelson product* $\langle\sigma_n, \sigma_n\rangle$ *of the generator* $\sigma_n \in \pi_{4n+2}(Sp(n))$ *whcih is a cyclic group of order* $a(n)\cdot(2n+1)!$, *where* $a(k) = 1$ *or* 2 *according as* k *is even or odd.*

As an application of Corollary 2.3, we have Mimura-Toda's result[MT], ${}_2\pi_{20}(Sp(2)) \cong Z/2 \oplus Z/2 \oplus Z/2$, because ${}_2\pi_{21}^s(Q_3^\infty) \cong Z/2$ [M3], where ${}_2\pi_*(-)$ or ${}_2\pi_*^s(-)$ stands for its 2-primary component.

If there is an element of Hopf invariant one in $\pi_{2d(n+1)-1}(\Sigma QF)$, then from the EHP-sequence it is obvious that $\pi_{2d(n+1)-3}(QF) \cong \pi_{2d(n+1)-3}^s(QF)$. The following theorem gives a non-trivial example, which does not seem to follow from the well-known solution of Hopf invariant problem on spheres.

Theorem 2.4. *There exists an elemet* $f \in \pi_{31}(\Sigma Q_4^\infty)$ *such that* $H(f) = 1$. *Since* ${}_2\pi_{29}^s(Q_4^\infty) \cong Z/2$, *therefore,*

$$ {}_2\pi_{28}(Sp(3)) \cong Z/2 \oplus Z/2. $$

From Theorem 2.4 we get the result of Hubbuck-Ōshima [HO] which states that there exists a map $f \in \pi_{31}(\Sigma CP_7^\infty)$ with $H(f) = 1$.

§3. Lemmas for the proof of Theorem 2.1

Let M^k be the mod 2 Moore space, that is, $M^k = S^{k-1} \cup_2 e^k$. The folloiwng Lemma is a slight generalization of F. Cohen's observation (Proposition 11.4 in [C].)

Lemma 3.1. *Let* X *be a* $2m$*-connected complex. Assume that* $\pi_{2m+1}(X) \cong Z/2$ *or* Z. *Let* $i\colon S^{2m+1} \to X$ *be a generator of* $\pi_{2m+1}(X)$. *Suppose that* $H_{(2)}$ *is trivial on* $\pi_{4m+3}(\Sigma X)$. *Under these assumptions,* $[i, i] = 2x$ *for some* $x \in \pi_{4m+1}(X)$ *if and only if there exists a map* $f\colon M^{4m+3} \to \Sigma X$ *such that* $H(f) \neq 0$.

The followng lemma is easily verified.

Lemma 3.2. *In* $H^*(QF; Z/2) \cong Z/2\{\alpha_{n+1}, \alpha_{n+2}, \cdots\}$,

$$ Sq^k \alpha_{m+1} = \binom{m}{k/d} \alpha_{m+k/d+1} \text{ or } 0, $$

according as $k \equiv 0$ *mod* d *or not, where* Sq^k *is the Steenrod squaring operation.*

Let I be the following two-sided ideal of the mod 2 Steenrod algebra A;

$$ I_q = \{\textstyle\sum A(k) Sq^l A(q-k-l) \mid k \geqq d,\ l \not\equiv 0 \bmod d\}, $$

where $A(i)$ stands for an element of A of homogeneous dimension i.

Lemma 3.3. *Let* $n+1 = 2^{s+1}m + 2^s$ ($m \geqq 1$ *and* $s \geqq 0$). *Then,*

$$ Sq^{(n+1)d} = Sq^{2^s d} Sq^{2^{s+1}md} + \sum_{i=0}^{s-1} Sq^{2^i d} A(d(n+1-2^i)) \mod I_{d(n+1)}. $$

Lemma 3.4. *Let* $n+1 = 2^{s+1}m + 2^s$ ($m \geqq 1$ *and* $s \geqq 0$). *Then for any*

$f \in [M^{2d(n+1)-1}, \Sigma QF]$,

$$Sq^{2^i d}(\beta_{2n+2-2^i}) = 0 \text{ for } 0 \leqq i \leqq s, \text{ in } H^*(C_f; \boldsymbol{Z}/2),$$

where C_f is the mapping cone of f and $\beta_{n+j} \in H^{d(n+j)}(C_f; \boldsymbol{Z}/2)$ is the corresponding element to $\Sigma\alpha_{n+j} \in H^{d(n+j)}(\Sigma QF; \boldsymbol{Z}/2)$ under the obvious isomorphism for $1 \leqq j \leqq n + 1$.

Lemma 3.5. *If $n + 1 \neq 2^t$, then for any $f \in [M^{2d(n+1)-1}, \Sigma QF]$,*

$$Sq^{(n+1)d}(\beta_{n+1}) = 0 \text{ in } H^*(C_f; \boldsymbol{Z}/2).$$

§4. Proof of lemmas

Proof of Lemma 3.1. First, note that $(\Sigma i \wedge i)_*: \pi_{4m+3}(S^{4m+3}) \to \pi_{4m+3}(\Sigma X \wedge X)$ is epimorphic. So from the naturality of the EHP-sequence and from the assumption, it follows that the image of $P: \pi_{4m+3}(\Sigma X \wedge X) \to \pi_{4m+1}(X)$ is isomorphic to $\boldsymbol{Z}/2$ generated by $[i, i]$. Let

$$S^{4m+1} \xrightarrow{2} S^{4m+1} \xrightarrow{j} M^{4m+2} \xrightarrow{q} S^{4m+2}$$

be the usual cofiber sequence. If $[i, i] = 2x$ for some $x \in \pi_{4m+1}(X)$, then there exists a map $f: M^{4m+2} \to \Omega\Sigma X$ such that $E(x) = f \circ j$. We assert that $H(f) \neq 0$. Suppose that $H(f) = 0$. Then there exists a map $g: M^{4m+2} \to X$ such that $E(g) = f$. Since $E(x - g \circ j) = f \circ j - f \circ j = 0$, from the EHP-sequence, we see that $x - g \circ j$ belongs to the image of P. Therefore, $x - g \circ j = k[i, i]$ for some integer k. Since $[i, i]$ is of order 2, it follows that $[i, i] = 2x = 2x - 2(g \circ j) = 2k[i, i] = 0$. This is clearly a contradiction. Now suppose that there exists a map $f: M^{4m+2} \to \Omega\Sigma X$ such that $H(f) \neq 0$. By the suspension theorem, there exists a map $g: S^{4m+1} \to X$ such that $E(g) = f \circ j$. Since $E(2g) = 0$, we see that $2g = k[i, i]$ for some integer k. We assert that $2g \neq 0$. Suppose $2g = 0$. Then there exists a map $h: M^{4m+2} \to X$ such that $g = h \circ j$. Since $(f - E(h)) \circ j = 0$, there is a map $h': S^{4m+2} \to \Omega\Sigma X$ such that $h' \circ q = f - E(h)$. Then $H_{(2)}(h') = q^* H(h') = H(f - E(h)) = H(f) \neq 0$. This contradicts the assumption that $H_{(2)}: \pi_{4m+3}(\Sigma X) \to \boldsymbol{Z}/2$ is trivial. This completes the proof of Lemma 3.1.

Proof of Lemma 3.3. Induction on $s = \nu_2(n + 1) \geqq 0$. When $s = 0$, Consider the following Adem relation;

$$Sq^d Sq^{dn} = \sum_{j=0}^{d/2} \binom{dn - 1 - j}{d - 2j} Sq^{dn+d-j} Sq^j.$$

Since $s = 0$, thus since n is even, we have

$$Sq^{d(n+1)} = Sq^d Sq^{dn} \bmod I_{d(n+1)}.$$

Assume that Lemma 3.3 is true for n such that $\nu_2(n + 1) < s$. Consider the Adem relation, then we have

$$Sq^{2^s d} Sq^{2^{s+1}md} = \sum_{i=0}^{2^{s-1}} \binom{2^{s+1}md - id - 1}{(2^s - 2i)d} Sq^{(2^{s+1}m + 2^s - i)d} Sq^{id} \mod I_{d(n+1)}$$

$$= \binom{(2^{s+1}m - 1)d + d - 1}{2^s d} Sq^{(n+1)d} + \sum_{i=1}^{2^{s-1}} Sq^{(2^{s+1}m + 2^s - i)d} A(id) \mod I_{d(n+1)}$$

$$= Sq^{(n+1)d} + \sum_{i=1}^{2^{s-1}} Sq^{(2^{s+1}m + 2^s - i)d} A(id) \mod I_{d(n+1)}$$

Since $k_i = v_2(2^{s+1}m + 2^s - i) \leqq s - 1$ for $1 \leqq i \leqq 2^{s-1}$, by inductive assumption, we have

$$Sq^{(n+1)d} = Sq^{2^s d} Sq^{2^{s+1}md}$$

$$+ \sum_{i=1}^{2^{s-1}} \Big(\sum_{r=0}^{k_i} Sq^{2^r d} A((2^{s+1}m + 2^s - i - 2^r)d)\Big) A(id) \mod I_{d(n+1)}$$

Therefore, $Sq^{(n+1)d} = Sq^{2^s d} Sq^{2^{s+1}md} + \sum_{i=0}^{s-1} Sq^{2^i d} A(d(n+1-2^i)) \mod I_{d(n+1)}$.
This completes the proof of Lemma 3.3.

Proof of Lemma 3.4. First, note $I_{d(n+1)}(\beta_{n+1}) = 0$. Because $H^{d(n+j)}(C_f; \boldsymbol{Z}/2) \cong H^{d(n+j)}(\Sigma QF; \boldsymbol{Z}/2)$ for $1 \leqq j \leqq n+1$. We shall prove by induction on $i (0 \leqq i \leqq s)$.
For $i = 0$, $Sq^d(\beta_{2n+1}) = Sq^d Sq^d(\beta_{2n})$ by Lemma 3.2

$$= \begin{cases} (Sq^7 Sq^1 + Sq^6 Sq^2)(\beta_{2n}) & \text{if } \boldsymbol{F} = \boldsymbol{H} \\ Sq^3 Sq^1(\beta_{2n}) & \text{if } \boldsymbol{F} = \boldsymbol{C} \end{cases} \quad \text{by Adem relation}$$

$$= 0 \quad \text{by dimensional reason.}$$

Now $Sq^{2^i d}(\beta_{2n+2-2^i})$

$$= Sq^{2^i d}(Sq^{2^i d}(\beta_{2n+2-2^{i+1}})) \qquad \text{from Lemma 3.2 and } \binom{2n+1-2^{i+1}}{2^i} = 1$$

$$= \sum_{j=0}^{i-1} Sq^{(2^{i+1} - 2^j)d} Sq^{2^j d}(\beta_{2n+2-2^{i+1}}) \qquad \text{by Adem relation}$$

$$= \sum_{j=0}^{i-1} \Big(\sum_{r=0}^{j} Sq^{2^r d} A((2^{i+1} - 2^j - 2^r)d) Sq^{2^j d}(\beta_{2n+2-2^{i+1}})\Big) \qquad \text{from Lemma 3.3}$$

$$= \sum_{j=0}^{i-1} \sum_{r=0}^{j} Sq^{2^r d}(*\beta_{2n+2-2^r})$$

$$= 0 \qquad \text{by inductive assumption.}$$

This completes the proof of Lemma 3.4.

Proof of Lemma 3.5.

$$\begin{aligned} Sq^{d(n+1)}(\beta_{n+1}) &= \sum_{i=0}^{s} Sq^{2^i d} A((n+1-2^i)d)(\beta_{n+1}) && \text{by Lemma 3.3} \\ &= \sum_{i=0}^{s} Sq^{2^i d}(*\beta_{2n+2-2^i}) \\ &= 0 && \text{by Lemma 3.4.} \end{aligned}$$

This completes the proof of Lemma 3.5.

§5. Proof of theorems

Proof of Theorem 2.1. We shall applly Lemma 3.1 for this proof. Note that the necessary conditions are satisfied by Theorem 1.1. Let $f \in [M^{2d(n+1)-1}, \Sigma QF]$. Consider the cofiber sequence;

$$M^{2d(n+1)-1} \xrightarrow{f} \Sigma QF \xrightarrow{i} C_f \xrightarrow{p} M^{2d(n+1)}.$$

According to Boardman-Steer[BS], the following diagram commutes;

$$\begin{tikzcd} C_f \arrow[r, "\bar{\Delta}"] \arrow[d, "p"] & C_f \wedge C_f \\ M^{2d(n+1)} \arrow[r, "\Sigma H(f)"] & \Sigma QF \wedge \Sigma QF, \arrow[u, "i \wedge i"'] \end{tikzcd}$$

where $\bar{\Delta}$ stands for the reduced diagonal. Thus, since in our case $H(f)$ can be detected by the induced homomorphism of $H(f)$ on $H^{2d(n+1)}(\Sigma QF \wedge \Sigma QF; Z/2) \cong Z/2$, $H(f) = 0$ if and only if $\beta_{n+1}^2 = 0$ in $H^{2d(n+1)}(C_f; Z/2)$. From Lemma 3.5, $Sq^{d(n+1)}\beta_{n+1} = \beta_{n+1}^2 = 0$ for all $f \in [M^{2d(n+1)-1}, \Sigma QF]$. Therefore the proof of Theorem 2.1 follows from Lemma 3.1.

Remark. If $n \equiv 0 \bmod d$, then Theorem 2.1 can be proved easily as follows; Let $J = A\{Sq^l | l \not\equiv 0 \bmod d\}$ be a left ideal of the Steenrod algebra A. If $n \equiv 0 \bmod d$, then there is a following relation;

$$Sq^{d(n+1)} = Sq^{2d} Sq^{d(n-1)} + Sq^{dn} Sq^d \mod J.$$

Associated with the above relation, we can define the (unstable) secondary operation, say Φ. It is not difficult to see that Φ can be defined on $\alpha_{n+1} \in H^{(n+1)d}(QF \cup_{[i,i]} e^{2d(n+1)-2}; Z/2) \cong Z/2$ and that Φ detects the Whitehead product $[i, i]$. Thus it follows that $[i, i]$ cannot be divisible by 2 in $\pi_{2d(n+1)-1}(QF)$. (see Brown-Peterson [BP]).

Before we give the proof of Theorem 2.2, we need some notation and a lemma. Let $i_0: S^{d(n+1)-1} \to GF(\infty)/GF(n)$ be the bottom inclusion. Note that $i \in \pi_{d(n+1)-1}(QF)$ corresponds to $i_0 \in \pi_{d(n+1)-1}(GF(\infty)/GF(n))$ under the natural isomorphism. It is well known that

$$\partial[i_0, i_0] = \pm \langle \sigma_n, \sigma_n \rangle,$$

where $\partial \colon \pi_{2d(n+1)-3}(GF(\infty)/GF(n)) \to \pi_{2d(n+1)-4}(GF(n))$ is the connecting homomorphism associated with the fiber bundle;

$$GF(n) \longrightarrow GF(\infty) \longrightarrow GF(\infty)/GF(n).$$

So when $\boldsymbol{F} = \boldsymbol{H}$, the proof of Theorem 2.2 is obvious. Now let $\boldsymbol{F} = \boldsymbol{C}$. Let $\lambda_k \in \pi_{2k-1}(U) \cong \boldsymbol{Z}$ be a generator. We denote the usual bundle projection $U \to U/U(n)$ by j^n. We need the following lemma.

Lemma 5.1. *If $n \geqq 3$, then $j^n_*(\lambda_{2n+1})$ is divisible by 2 in $\pi_{4n+1}(U/U(n))$. However $j^2_*(\lambda_5)$ can not be divisible by 2.*

Proof. The idea of the proof is originally due to Crabb and Knapp. First observe that $j^2_*(\lambda_3)$ is divisible by 2 in $\pi_5(U/U(2))$. Since there exists an unstable Adams map $A \colon S^{12} \cup_2 e^{13} \to S^4 \cup_2 e^5$[O2], using the periodicity (See [CK] or Theorem 1.9 in [M2]), we see that $j^2_*(\lambda_{4k+3})$ is divisible by 2 in $\pi_{8k+5}(U/U(2))$. Therefore $j^n_*(\lambda_{4k+3})$ is divisble by 2 in $\pi_{8k+5}(U/U(n))$ for $k \geqq 0$. Next observe that $j^3_*(\lambda_5)$ is divisible by 2 in $\pi_9(U/U(3))$. Then by the same argument as above, we see that $j^3_*(\lambda_{4k+5})$ is divisble by 2 in $\pi_{8k+9}(U/U(3))$. Therefore $j^n_*(\lambda_{4k+5})$ is divisible by 2 in $\pi_{8k+9}(U/U(n))$ for $k \geqq 0$. The assertion in case that $n = 2$ follows from Theorem 2.1 and the exact sequence of Proposition 1.2. This completes the proof of Lemma 5.1.

Now we shall return to the proof of Theorem 2.2 in case of $\boldsymbol{F} = \boldsymbol{C}$. When $n = 2$, since $\pi_8(U(2)) \cong \boldsymbol{Z}/2$, the proof follows from Theorem 2.1 and the exact sequence (1) in Proposition 1.2. Let $n \geqq 3$. Suppose that $\langle \sigma_n, \sigma_n \rangle = 2x$ for some $x \in \pi_{4n}(U(n))$. Take an element $y \in \pi_{4n+1}(U/U(n))$ such that $\partial y = x$. Then from the exact sequence (1) in Proposition 1.2 we see that $[i_0, i_0] - 2y$ belongs to the image of j^n_*. From the above lemma, it follows that $[i_0, i_0]$ is divisible by 2 in $\pi_{4n+1}(U/U(n)) \cong \pi_{4n+1}(\Sigma CP^\infty_n)$. This contradicts Theorem 2.1.

Proof of Theorem 2.4. The proof comes from the following observations.

1) $\pi_{28}(Sp(3)) \cong \boldsymbol{Z}/2 \oplus \pi_{29}(Sp/Sp(3))$. ((2) in Proposition 1.2.)
2) $\pi_{29}(Sp/Sp(3)) \cong \pi_{29}(Q^\infty_4)$ (unstable). (By the connectivity of the pair $(Sp/Sp(n), Q^\infty_{n+1})$.)
3) Computation of the spectral sequence;

$$H_*(Q^\infty_4;\, {}_2\pi^s_*(S^0)) \Longrightarrow {}_2\pi^s_*(Q^\infty_4),$$

shows that ${}_2\pi^s_{29}(Q^\infty_4) \cong \boldsymbol{Z}/2$ generated by the class $\gamma_5 \otimes \eta\mu$, where γ_5 is a generator of $H_{19}(Q^\infty_4; \boldsymbol{Z}) \cong \boldsymbol{Z}$. This follows from [M3]. Note that $\pi_{30}(\Sigma Q^\infty_4) \cong \pi^s_{29}(Q^\infty_4)$.

4) Consider the following commutative diagram of the *EHP* sequences:

$$\begin{array}{ccccccc}
\pi_{31}(\Sigma Q_4^\infty \wedge Q_4^\infty) & \xrightarrow{P} & \pi_{29}(Q_4^\infty) & \xrightarrow{E} & \pi_{30}(\Sigma Q_4^\infty) & \longrightarrow & 0. \\
\uparrow{\scriptstyle (i\wedge i)_* \cong} & & \uparrow{\scriptstyle i_*} & & \uparrow{\scriptstyle i_*} & & \\
\pi_{31}(\Sigma S^{15} \wedge S^{15}) & \xrightarrow{P} & \pi_{29}(S^{15}) & \xrightarrow{E} & \pi_{30}(\Sigma S^{15}) & \longrightarrow & 0.
\end{array}$$

5) According to Toda's book [T], $P(\iota_{31}) = 2\sigma_{15}^2$. Since $i_*\Sigma\sigma_{15}^2 = 0$, (This follows from the above 3) or [M3]), from the *EHP*-sequence it follows that $i_*\sigma_{15}^2 \in$ Image of P. The image of P of this dimension is generated by $i_*[\iota_{15}, \iota_{15}] = 2i_*\sigma_{15}^2$. Therefore it follows that $i_*\sigma_{15}^2 = 0$. Thus we get ${}_2\pi_{29}(Q_4^\infty) \cong {}_2\pi_{29}^s(Q_4^\infty)$. Thus it follows from the *EHP*-sequence that there exists a map $f \in \pi_{31}(\Sigma Q_4^\infty)$ of Hopf invariant 1.

Remark. We don't know how to construct directly such a map f.

§6. Corrections of my paper Metastable homotopy groups of *Sp*(*n*)

There were many careless mistakes and transcription errors. Some of errors were pointed by D. Davis and A. T. Lundell. The author thanks them for their interests. The statements in Main Theorem of [M1] should be changed as follows. The statements corrected are underlined.

3) *If $\underline{n \geqq 3}$ (the case $n = 2$ is excluded, since $n = 2$ is not contained in the metastable range.), then*

$$\pi_{4n+12}(Sp(n)) \cong \underline{\boldsymbol{Z}/2 \oplus \boldsymbol{Z}/2}$$

4) *If $n \geqq 3$, then*

$$\pi_{4n+13}(Sp(n)) \cong \begin{cases} \underline{\boldsymbol{Z}/8} & \underline{\text{if } n \equiv 1 \bmod 4,} \\ \underline{\boldsymbol{Z}/(128,\ 4(n-3))} & \underline{\text{if } n \equiv 3 \bmod 4,} \\ \underline{\boldsymbol{Z}/2 + \boldsymbol{Z}/4 + \boldsymbol{Z}/8} & \text{if } n \equiv 6 \bmod 8, \\ \boldsymbol{Z}/2 + \boldsymbol{Z}/32 & \text{if } n \equiv 2 \bmod 8, \\ \boldsymbol{Z}/2 + \boldsymbol{Z}/2 + \boldsymbol{Z}/8 & \text{if } n \equiv 0 \bmod 8, \\ \boldsymbol{Z}/2 + \boldsymbol{Z}/2 + \boldsymbol{Z}/4 + \underline{\boldsymbol{Z}/(64,\ (n+4)/2)} & \underline{\text{if } n \equiv 4 \bmod 8,} \end{cases}$$

Here (a, b) means the greatest common divisor of a and *b.*

§7. The Hopf invariant and the e-invariant (Appendix)

Throughout this complex Y is assumed to satisfy the following conditions.

1) *Y is $2n$-connected.*
2) $\dim Y \leqq 4n+1$
3) $H_{2n+1}(Y; \boldsymbol{Z}) \cong \boldsymbol{Z}$.
4) *$H^*(\Sigma Y; \boldsymbol{Z})$ is free and is generated by even dimensional elements. Choose a basis $\{u_1, u_2, \dots, u_s\}$ of $H^*(\Sigma Y; \boldsymbol{Z})$ with* $\dim u_i = 2m_i$ *and* $m_i \leqq m_{i+1}$ $(m_1 = n+1)$.

Under these assumption, The reduced K-theory of ΣY, $K(\Sigma Y)$, is also free and we can choose a basis $\{x_1, x_2, \dots, x_s\}$ of $K(\Sigma Y)$ so that there exist rational numbers $c_{i,j}$ for $1 \leqq i, j \leqq s$ such that

$$ch(x_j) = \sum_{i=1}^{s} c_{i,j} u_i, \text{ with } c_{i,i} = 1 \text{ and } c_{i,j} = 0 \text{ if } i < j,$$

where ch is the Chern character. We denote the matrix $(c_{i,j})$ by C. For an integer k, let $A(k)$ be the diagonal matrix with diagonal entries, $\{k^{m_1}, k^{m_2}, \dots, k^{m_s}\}$.

Definition and Proposition A.1. *For any $k \in \boldsymbol{Z}$, all entries of the matrix $C^{-1}A(k)C$ are integral. We denote the j-th column vector of the matrix $k^{2n+2}E - C^{-1}A(k)C$ by $\boldsymbol{a}^j(k)$, where E is the unit matrix. Especially we denote $\boldsymbol{a}^1(2)$ by $\boldsymbol{h}$ which we call the Hopf vector.*

Proof. The existence of the Adams operation φ^k on $K(\Sigma Y)$ and the commutativity of the following diagram imply the assertion.

$$\begin{array}{ccc} K(\Sigma Y) & \xrightarrow{ch} & \prod H^{2l}(\Sigma Y; \boldsymbol{Q}) \\ \downarrow{\scriptstyle \varphi^k} & & \downarrow{\scriptstyle \Pi k^l} \\ K(\Sigma Y) & \xrightarrow{ch} & \prod H^{2l}(\Sigma Y; \boldsymbol{Q}). \end{array}$$

q.e.d.

Let $f \in \pi_{4n+3}(\Sigma Y)$. The *e-invariant vector* of f, $\boldsymbol{e}(f)$, is defined by

$$\boldsymbol{e}(f) = \begin{pmatrix} e_{\boldsymbol{C}}(f)(x_1) \\ e_{\boldsymbol{C}}(f)(x_2) \\ \vdots \\ e_{\boldsymbol{C}}(f)(x_s) \end{pmatrix},$$

where $e_{\boldsymbol{C}}$ is the Adams-Toda $e_{\boldsymbol{C}}$-invariant, that is,

$$e_{\boldsymbol{C}}(f)\colon \pi_{4n+3}(\Sigma Y) \longrightarrow \mathrm{Hom}(K(\Sigma Y), \boldsymbol{Q}/\boldsymbol{Z}).$$

The following theorem gives a relation between the Hopf invariant and the e-invariant. It is a slight generalization of Adams or Toda's observation in case $Y = S^{2n+1}$.

Theorem A.2. *Under the same assumption, let* $f \in \pi_{4n+3}(\Sigma Y)$. *Then for any* $k \in \boldsymbol{Z}$, *the inner product of the vector* $\boldsymbol{a}^j(k)$ *and the e-invariant vector*, $(\boldsymbol{e}(f), \boldsymbol{a}^j(k))$ *is always integral. And the* mod 2 *Hopf invariant of* f, $H_{(2)}(f)$, *is equal to the* mod 2 *reduction of the integer* $(\boldsymbol{e}(f), \boldsymbol{h})$.

Proof. Consider the cofiber, say C_f of $f\colon S^{4n+3} \to \Sigma Y$. The same argument as in the proof of Proposition A.1 implies that the matrix

$$\begin{pmatrix} C & 0 \\ {}^t\boldsymbol{e}(f) & 1 \end{pmatrix}^{-1} \begin{pmatrix} A(k) & 0 \\ 0 & k^{2n+2} \end{pmatrix} \begin{pmatrix} C & 0 \\ {}^t\boldsymbol{e}(f) & 1 \end{pmatrix}$$

is an integral matrix. Note the above matrix product is

$$\begin{pmatrix} C^{-1}A(k)C & 0 \\ {}^t\boldsymbol{e}(f)(k^{2n+2}E - C^{-1}A(k)C) & k^{2n+2} \end{pmatrix}.$$

Especially ${}^t\boldsymbol{e}(f)(k^{2n+2}E - C^{-1}A(k)C)$ is an integral row vector. Therefore, for any $k \in \boldsymbol{Z}$ and $1 \leqq j \leqq s$, $(\boldsymbol{e}(f), \boldsymbol{a}^j(k))$ is always integral.

Now recall that $\varphi^2(\xi) = \xi^2 \mod 2K(C_f)$ for $\xi \in K(C_f)$. This means that in $H^{**}(C_f; \boldsymbol{Q})$,

$$(\Pi 2^l)(ch(x_1)) = (ch(x_1))^2 \quad \mod 2ch(K(C_f)).$$

On the other hand, $(ch(x_1))^2 = u_1^2$ by our hypothesis 2). Let $v \in H^{4n+4}(C_f; \boldsymbol{Z}) \cong \boldsymbol{Z}$ be the generator. Then under our conditions of Y, according to Boardman-Steer [BS],

$$u_1^2 = H(f)v,$$

where $H(f)$ is the Hopf invariant of f.
Therefore there exists integers l_j for $1 \leqq j \leqq s+1$ such that

$$(\Pi 2^l)(ch(x_1)) = H(f)v + 2\sum_{j=1}^{s+1} l_j ch(x_j),$$

where x_{s+1} is an element of $K(C_f)$ such that $ch(x_{s+1}) = v$. This implies that the first column of the matrix $\begin{pmatrix} A(k) & 0 \\ 0 & k^{2n+2} \end{pmatrix} \begin{pmatrix} C & 0 \\ {}^t\boldsymbol{e}(f) & 1 \end{pmatrix}$ is equal to $2\begin{pmatrix} C & 0 \\ {}^t\boldsymbol{e}(f) & 1 \end{pmatrix} {}^t(l_1, \dots, l_{s+1}) + {}^t(0, \dots, 0, H(f))$. Thus, multiplying from the left $\begin{pmatrix} C & 0 \\ {}^t\boldsymbol{e}(f) & 1 \end{pmatrix}^{-1}$, we have the desired result.

Note that all entries of the Hopf vector $\boldsymbol{h}$ are always even, since $\varphi^2(x_1) \equiv x_1^2 \mod 2K(\Sigma Y)$ and $x_1^2 = 0$ in $K(\Sigma Y)$. Therefore the mod 2 reduction of the inner product $(\boldsymbol{h}, \boldsymbol{e}(f))$ is independent of choice of lifts of $x_i \in K(\Sigma Y)$ to $K(C_f)$.
The following theorem gives a necessary condition for existence of a map with

Hopf invariant one.

Theorem A.3. *Under the same assumption, if* $H_{(2)}: \pi_{4n+3}(\Sigma Y) \to \boldsymbol{Z}/2$ *is onto, then, there exists a non-zero vector* $\boldsymbol{e} \in (\boldsymbol{Q}/\boldsymbol{Z})^s$ *which satisfies the following*:
(1) *for any* $k \in \boldsymbol{Z}$ *and for any* $1 \leqq j \leqq s$, $(\boldsymbol{a}^j(k), \boldsymbol{e}) \in \boldsymbol{Z}$,
(2) $(\boldsymbol{h}, \boldsymbol{e})$ *is odd.*

Corollary A.4. *Under the same assumption, if* $H_{(2)}: \pi_{4n+3}(\Sigma Y) \to \boldsymbol{Z}/2$ *is onto, then, for any* $k \in \boldsymbol{Z}$, *there exist* (2)*-localized integers* $l_1(k), l_2(k), \dots, l_s(k)$ *such that*

$$\sum_{i=1}^{s} l_i(k) \cdot \det|\boldsymbol{a}^1(k), \boldsymbol{a}^2(k), \dots, \overset{\vee}{\boldsymbol{h}}, \dots, \boldsymbol{a}^s(k)| = \prod_{i=1}^{s} (k^{2n+2} - k^{m_i}),$$

where the symbol det | | *means the determinant of matrices.*

Proof. From the assumption, there exists a map $f: S^{4n+3} \to \Sigma Y$ such that $H_{(2)}(f) = 1$. By Theorem A.2, since $(\boldsymbol{e}(f), \boldsymbol{a}^i(k)) \in \boldsymbol{Z}$ for $1 \leqq i \leqq s$ and $H(f) \equiv (\boldsymbol{e}(f), \boldsymbol{h}) \equiv 1 \bmod 2$, there exist 2-localized integers l_i for $1 \leqq i \leqq s$ such that $(\boldsymbol{e}(f), \boldsymbol{a}^i(k)) = l_i(\boldsymbol{e}(f), \boldsymbol{h})$. Since $\boldsymbol{e}(f) \neq 0$, this implies that

$$\det|\boldsymbol{a}^1(k) - l_1\boldsymbol{h}, \dots, \boldsymbol{a}^s(k) - l_s\boldsymbol{h}| = 0.$$

The rest of the proof easily follows by linear algebra.

I believe that in case $Y = \Sigma CP_n^{2n}$ or Q_{n+1}^{2n+1}, the severe restriction on n comes from the Theorem A.3 and Corollary A.4 if one can compute effectively.

DEPARTMENT OF MATHEMATICS
WAKAYAMA UNIVERSITY

References

[BP] E. H. Brown and F. P. Peterson, Whitehead products and cohomology operations, Quart. J. Math. Oxford (2), **15** (1964), 116–120.
[BS] J. M. Boardman and B. Steer, On Hopf invariants, Comment. Math. Helv., **42** (1967), 180–221.
[C] F. R. Cohen, A course in some aspects of classical homotopy theory, Lecture Note in Math., Springer 1286 (1985).
[CK] M. C. Crabb and K. Knapp, James numbers, Math. Ann., **282** (1988), 395–422.
[HO] J. Hobbuck and H. Ōshima, Whitehead products in Stiefel manifolds and K-theory, Topology, **24**–4(1985), 487–494.
[J] I. M. James, The topology of Stiefel manifolds, London Math. Soc. Lecture Note 24 (1976).
[MT] M. Mimura and H. Toda, Homotopy groups of *SU*(3) *SU*(4) and *Sp*(2), J. Math. Kyoto Univ., **3** (1964), 217–250.
[M1] K. Morisugi, Metastable homotopy groups of *Sp*(*n*), J. Math. Kyoto Univ., **27**–2(1987), 367–380.
[M2] K. Morisugi, Homotopy groups of symplectic groups and the quaternionic James numbers, Osaka J. Math., **23** (1986), 867-880.

[M3] K. Morisugi, On the homotopy group $\pi_{4n+16}(Sp(n))$ for $n \geqq 4$, Bull. Fac. Edu. Wakayama Univ., **37** (1988), 19–23.

[O1] H. Ōshima, Whitehead products in Stiefel manifolds and Samelson products in classical groups, Advanced Studies in Pure Math., **9** (1986), Homotopy theory and related topics, 237–258.

[O2] S. Oka, Existence of the unstable Adams map, Mem. Fac. Sci., Kyushu Univ. Ser. A, **42** (1988), 95–108.

[T] H. Toda, Composition methods in homotopy groups of spheres, Ann. Math. Studies, 49 (1962).