J. Math. Kyoto Univ. (JMKYAZ)
31-1 (1991) 71-82

On the structure of infinitesimal automorphisms of linear Poisson manifolds I

Dedicated to Professor Noboru Tanaka on his sixtieth birthday

By

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Introduction

Let M be a smooth manifold. A Poisson structure on M is defined as a Lie algebra structure $\{\cdot,\cdot\}$ on $C^\infty(M)$ satisfying Leibniz identity. Let $x_1, x_2, \ldots, x_n$ be local coordinates on M. Then as is usual [6], this is equal to giving an antisymmetric contravariant 2-tensor P on M which satisfies Jacobi identity. In the local coordinates expression, P satisfies:

$$P = \frac{1}{2} \sum_{1 \leqslant i,j \leqslant n} P_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \quad \text{with } P_{ij} = -P_{ji}, \tag{0.1}$$

$$\sum_{1 \leqslant \ell \leqslant n} \left(P_{i\ell} \frac{\partial P_{jk}}{\partial x_\ell} + P_{j\ell} \frac{\partial P_{ki}}{\partial x_\ell} + P_{k\ell} \frac{\partial P_{ij}}{\partial x_\ell} \right) = 0, \quad \text{for } 1 \leqq i, j, k \leqq n. \tag{0.2}$$

The corresponding Lie algebra structure on $C^\infty(M)$ is called a Poisson structure on M.

Next we shall define here a linear Poisson manifold, which is one of the most important examples of Poisson manifolds. Let G be a connected Lie group whose Lie algebra is $\mathfrak{g}$. Let $\mathfrak{g}^*$ be the dual space of $\mathfrak{g}$. If $x_1, x_2, \ldots, x_n$ is a basis of $\mathfrak{g}$ satisfying

$$[x_i, x_j] = \sum_{k=1}^{n} c_{ijk} x_k, \tag{0.3}$$

then from this bracket operation, we can define the Poisson bracket $\{\cdot,\cdot\}$ on $C^\infty(\mathfrak{g}^*)$ as follows:

$$\{f, g\} = \sum_{1 \leqslant i,j,k \leqslant n} c_{ijk} x_k \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_j}, \tag{0.4}$$

where $C^\infty(\mathfrak{g}^*)$ denotes an algebra of C^∞-function on $\mathfrak{g}^*$. Note that each x_k is considered as a linear function on $\mathfrak{g}^*$. By this Poisson bracket, $C^\infty(\mathfrak{g}^*)$ becomes a

Communicated by Prof. H. Toda Feb. 1, 1989
Revised May 31, 1990

Lie algebra because (c_{ijk}) are structure constants of a Lie algebra $\mathfrak{g}$. Thus in this case, a Poisson tensor P on $\mathfrak{g}^*$ is given by

$$P = \frac{1}{2} \sum_{1 \leq i,j,k \leq n} c_{ijk} x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}. \tag{0.5}$$

Using this tensor P, (0.4) is also written as

$$\{f, g\} = \langle df \wedge dg | P \rangle. \tag{0.6}$$

A linear Poisson manifold is a pair $(\mathfrak{g}^*, P)$. We may often simply write $\mathfrak{g}^*$ for $(\mathfrak{g}^*, P)$. By the rank of P at $\mu \in \mathfrak{g}^*$, we shall mean the rank of the skew-symmetric matrix $(P_{ij}(\mu))$. In an article [2], A. Lichnerowicz studied a Poisson manifold with the constant rank.

In the present paper, we shall treat the case $G = SL(2, R)$, and study infinitesimal automorphisms of $sl(2, R)^*$ with a natural Poisson structure. By the theorem of Kirillov-Kostant-Souriau (see Abraham and Marsden [1]), each coadjoint orbit has the canonical symplectic structure. In our case, each coadjoint orbit is noncompact, except for the origin, and therefore, we are able to obtain interesting results for infinitesimal automorphisms.
A part of this paper was announced in [5].

1. Casimir functions and infinitesimal automorphisms

From this point, we would like to confine ourselves to the case $G = SL(2, R)$, with all the elements considered here as C^∞. We will identify $\mathfrak{g}^*$ with R^3. Let x, y and z be a basis of $\mathfrak{g} = sl(2, R)$, satisfying the following relations:

$$[x, y] = -z, \ [y, z] = x, \ [z, x] = y. \tag{1.1}$$

(If we regard x, y, z as linear functions on $\mathfrak{g}^*$, we should write: $\{x, y\} = -z$, $\{y, z\} = x$, $\{z, x\} = y$.) Then the corresponding linear Poisson tensor P is given by

$$P = -z\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + x\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + y\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x}. \tag{1.2}$$

As is easily seen, the rank of P is two, except for the orgin.

Given $f \in C^\infty(\mathfrak{g}^*)$, $\{f, \cdot\}$ defines a derivation of $C^\infty(\mathfrak{g}^*)$. Hence there corresponds a vector field ξ_f, which we call the Hamiltonian vector field.

A Casimir function on $\mathfrak{g}^*$ is a function C such that $\{C, f\} = 0$ for all function f. In order words, C is an element of the center of the Lie algebra $C^\infty(\mathfrak{g}^*)$. We denote by $\mathscr{C}$ the set of Casimir functions. By simple consideration, we know that for each element C of $\mathscr{C}$ there exists a function ϕ of one variable such that $C(x, y, z) = \phi(x^2 + y^2 - z^2)$.

A coadjoint orbit $G \cdot \mu$ passing through $\mu \neq 0$ is also called a symplectic leaf. There are three kinds of symplectic leaves: circular conics, hyperboloids of 1-sheet and hyperboloids of 2-sheets. Each symplectic leaf is the common level manifold of Casimir functions.

By an infinitesimal automorphism of $\mathfrak{g}^*$, we mean a smooth vector field X on $\mathfrak{g}^*$ such that

$$L(X)P = 0, \tag{1.3}$$

where $L(X)$ denotes the Lie derivative along X.

Next we shall define some Lie algebras. Let $\mathscr{L}$ be the Lie algebra consisting of all infinitesimal automorphisms. Suppose that for a smooth vector field Y on $\mathfrak{g}^*$ there exists a Casimir function C_Y depending on Y, such that $L(Y)P = C_Y P$. Then we denote by $\mathscr{L}^c$ the Lie algebra consisting of such vector fields Y. Let $\mathscr{N}$ be the Lie algebra obtained as the normalizer of $\mathscr{L}$ in $\mathscr{L}^c$, that is, $\mathscr{N} = \{X \in \mathscr{L}^c \mid [X, \mathscr{L}] \subset \mathscr{L}\}$. Let $\mathscr{I}$ be a Lie subalgebra of $\mathscr{L}$ consisting of vector fields X such that each X is tangent to symplectic leaves. And we denote by $\mathscr{H}$ the Lie algebra of Hamiltonian vector fields. Then there is a canonical inclusion relation: $\mathscr{L}^c \supset \mathscr{N} \supset \mathscr{L} \supset \mathscr{I} \supset \mathscr{H}$. Direct calculation shows that both Lie subalgebras $\mathscr{I}$ and $\mathscr{H}$ are ideals of $\mathscr{L}$. Let $X = f\partial_x + g\partial_y + h\partial_z$ be a vector field of $\mathscr{L}$. Then three functions f, g and h must satisfy:

$$\begin{cases} f = xg_y - yg_x + zh_x + xh_z, \\ g = yf_x - xf_y + zh_y + yh_z, \\ h = zf_x + xf_z + zg_y + yg_z. \end{cases} \tag{1.4}$$

Put $\operatorname{div} X = f_x + g_y + h_z$. Then (1.4) is equivalent to the following:

$$\begin{cases} x \cdot \operatorname{div} X = (xf + yg - zh)_x, \\ y \cdot \operatorname{div} X = (xf + yg - zh)_y, \\ z \cdot \operatorname{div} X = -(xf + yg - zh)_z. \end{cases} \tag{1.5}$$

Next we shall investigate that under what conditions a vector field X is contained in $\mathscr{I}$. Put $X = f\partial_x + g\partial_y + h\partial_z$. Since X is tangent to each orbit, it satisfies: $X(x^2 + y^2 - z^2) = 0$. Hence we have

$$xf + yg - zh = 0. \tag{1.6}$$

Taking (1.6) into account, we have from (1.5), $x \cdot \operatorname{div} X = 0$, $y \cdot \operatorname{div} X = 0$ and $z \cdot \operatorname{div} X = 0$. This means $(x^2 + y^2 + z^2) \cdot \operatorname{div} X = 0$ except for the orgin. Since X is smooth on R^3, we have $\operatorname{div} X = 0$ on R^3. Conversely, if a vector field X satisfies (1.6) and $\operatorname{div} X = 0$, then it is clear that X is contained in $\mathscr{I}$. Thus we have proved:

Proposition 1.1. *For a smooth vector field $X = f\partial_x + g\partial_y + h\partial_z$ defined on R^3, X is contained in $\mathscr{I}$ if and only if*

$$\begin{cases} f_x + g_y + h_z = 0. \\ xf + yg - zh = 0. \end{cases} \tag{1.7}$$

Now we shall clarify the gap between $\mathscr{L}$ and $\mathscr{I}$.

Put $f_1 = xm(x^2 + y^2 - z^2)/(x^2 + y^2)$, $g_1 = ym(x^2 + y^2 - z^2)/(x^2 + y^2)$ and $h_1 = 0$,
where $m(u)$ is a smooth function of one variable which is defined by

$$(1.8) \qquad m(u) = \begin{cases} 0 & u \leqq 0, \\ \exp(-1/u^2) & u > 0. \end{cases}$$

Then f_1, g_1 and h_1 satisfy the relation (1.4), and hence $Y = f_1\partial_x + g_1\partial_y$ belongs to $\mathscr{L}$. In our case, three functions f_1, g_1 and h_1 do not satisfy the relations (1.7). In fact, it holds that

$$\begin{cases} xf_1 + yg_1 - zh_1 = m(x^2 + y^2 - z^2), \\ (f_1)_x + (g_1)_y + (h_1)_z = 2m'(x^2 + y^2 - z^2). \end{cases}$$

Hence $Y = f_1\partial_x + g_1\partial_y$ does not belong to $\mathscr{I}$.

Next we shall clarify the gap between $\mathscr{I}$ and $\mathscr{H}$. For this purpose, it is convenient to introduce the cylindrical coordinates (r, θ, z). Let $G\cdot\mu$ be an orbit satisfying $x^2 + y^2 - z^2 = c \neq 0$. We shall write the Poisson tensor P and the symplectic form ω on $G\cdot\mu$ in the cylindrical coordinates. Note that $\partial_x = \cos\theta\partial_r - \sin\theta/r\partial_\theta$, $\partial_y = \sin\theta\partial_r + \cos\theta/r\partial_\theta$ and $\partial_z = \partial_z$. Then we have $P = z/r\,\partial_\theta \wedge \partial_r + \partial_\theta \wedge \partial_z$.

Put $\omega = \alpha dr \wedge dz + \beta dr \wedge d\theta + \gamma d\theta \wedge dz$. Then we obtain

$$(1.9) \qquad \omega = (\gamma - (z/r)\beta) d\theta \wedge dz,$$

since $rdr = zdz$ on $G\cdot\mu$.

For any smooth function F on $G\cdot\mu$, we have

$$(1.10) \qquad X_F = (z/r)F_\theta\partial_r - ((z/r)F_r + F_z)\partial_\theta + F_\theta\partial_z.$$

In particular, $X_\theta = (z/r)\partial_r + \partial_z$ and $X_z = -\partial_\theta$. For these two vector fields X_θ and X_z, we have

$$\begin{aligned} \omega(X_\theta, X_z) &= \gamma - (z/r)\beta \\ &= \{\theta, z\} = 1. \end{aligned}$$

Hence (1.9) can be rewritten on $G\cdot\mu$ as

$$(1.11) \qquad \omega = d\theta \wedge dz.$$

Recall the function $m(u)$ defined in (1.8). For $r > 0$, we define a smooth vector field X using the smooth function $m(u)$ as follows:

$$(1.12) \qquad X = (z/r)m(r^2 - z^2)\partial_r + m(r^2 - z^2)\partial_z.$$

Let $G\cdot\mu$ be a symplectic leaf defined by $x^2 + y^2 - z^2 = c > 0$, that is, $G\cdot\mu$ is a hyperboloid of 1-sheet. Since $r > 0$ on this symplectic leaf $G\cdot\mu$, the smooth vector field X is well-defined. Then $r^2 - z^2 = x^2 + y^2 - z^2 = c$ on $G\cdot\mu$ so that we have $i(X)\omega = -m(r^2 - z^2)d\theta = \text{const.}\, d\theta$. Note that θ is not a globally defined function on $G\cdot\mu$. Hence X is not a Hamiltonian vector field on the symplectic manifold $G\cdot\mu$. Such a vector field exists because $G\cdot\mu$ is not simply connected. ($G\cdot\mu$ is homeomorphic to a cylinder.)

If we extend this vector field X to the whole space $\mathfrak{g}^*$, we have the following new vector field $\tilde{X}$:

$$(1.13)\quad \tilde{X} = xz\frac{m(x^2+y^2-z^2)}{x^2+y^2}\partial_x + yz\frac{m(x^2+y^2-z^2)}{x^2+y^2}\partial_y + m(x^2+y^2-z^2)\partial_z.$$

This vector field $\tilde{X}$ is smooth on $\mathfrak{g}^*$ and three coefficients of $\tilde{X}$ satisfy the relations (1.7). Hence $\tilde{X}$ is an element of $\mathscr{I}$. If $\tilde{X}$ is a Hamiltonian vector field on $\mathfrak{g}^*$, then $\tilde{X}|_{G\cdot\mu} = X$ must be also a Hamiltonian vector field on $G\cdot\mu$. But X is not Hamiltonian, as we have seen before. Hence $\tilde{X}$ is not contained in $\mathscr{H}$. Thus we have proved:

Theorem 1.2. *The ideal $\mathscr{I}$ is strictly contained in $\mathscr{L}$ and the ideal $\mathscr{H}$ is strictly contained in $\mathscr{I}$.*

Finally we shall characterize a Lie algebra $\mathscr{N}$. First we prove:

Lemma 1.3. *Let $X\in\mathscr{L}^c$ and put $L(X)P = C_XP$, $C_X\in\mathscr{C}$. Then X belongs to $\mathscr{N}$ if and only if $Y(C_X)=0$ for all $Y\in\mathscr{L}$.*

Proof. Let $X\in\mathscr{N}$. Since $[X, Y]\in\mathscr{L}$, we have

$$\begin{aligned} 0 = L([X, Y])P &= L(X)L(Y)P - L(Y)L(X)P \\ &= -L(Y)L(X)P = \{-Y(C_X)\}P. \end{aligned}$$

Thus $Y(C_X)=0$ on $M = R^3 - \{0\}$. By the smoothness of $Y(C_X)$, we have $Y(C_X) = 0$ on R^3. The converse is easily proved. q.e.d.

Recall the vector field $Y\in\mathscr{L}$ defined in the proof of Theorem 1.2. Here we define another vector field Z by

$$Z = \frac{ys(x^2+y^2-z^2)}{y^2-z^2}\partial_y + \frac{zs(x^2+y^2-z^2)}{y^2-z^2}\partial_z$$

where $s(u)$ is a smooth function of one variable defined by

$$s(u) = \begin{cases} 0 & u>0, \\ \exp(-1/u^2) & u\leqq 0. \end{cases}$$

Then the vector field Z is an element of $\mathscr{L}$. Using these two vector fields Y and Z, we shall prove

Proposition 1.4. *$\mathscr{N}/\mathscr{L}$ is isomorphic to R.*

Proof. Let X be any element of $\mathscr{N}$, and C_X be the corresponding Casimir function. Put $C_X(x, y, z) = \phi(x^2+y^2-z^2)$. Let Y and Z two vector fields of $\mathscr{L}$ stated above. Then by Lemma 1.3, we have

$$0 = Y(C_X) = Y(\phi) = \frac{xm(x^2 + y^2 - z^2)}{x^2 + y^2} \cdot \frac{\partial\phi}{\partial x} + \frac{ym(x^2 + y^2 - z^2)}{x^2 + y^2} \cdot \frac{\partial\phi}{\partial y}.$$

In the region $z = 0$ and $x^2 + y^2 > 0$, $m(u)$ is positive. Hence $d\phi/du = 0$ if $u > 0$.

On the other hand, we also have

$$0 = Z(C_X) = Z(\phi) = \frac{ys(x^2 + y^2 - z^2)}{y^2 - z^2} \frac{\partial\phi}{\partial y} + \frac{zs(x^2 + y^2 - z^2)}{y^2 - z^2} \frac{\partial\phi}{\partial z}.$$

In the region $x = 0$ and $y^2 - z^2 < 0$, $s(u)$ is positive. Hence $d\phi/du = 0$ if $u < 0$. By the smoothness of $\phi(u)$, we obtain that $\phi(u)$ is constant on R^3, and $C_X(x, y, z)$ is constant. Put $W = -(x\partial_x + y\partial_y + z\partial_z)$. Then W satisfies $L(W)P = P$ so that W is an element of $\mathscr{L}^c$. By Lemma 1.3, W is also an element of $\mathscr{N}$. Thus the linear mapping $T: \mathscr{N} \to R$ defined by $X \to C_X$ is surjective. Since $\mathrm{Ker}(T) = \mathscr{L}$, we get $\mathscr{N}/\mathscr{L} = R$. q.e.d.

2. Derivations — A formal version

Let $x = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$, $y = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$ and $z = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix}$ be a basis of $sl(2, R)$. Then this basis satisfies the relations (1.1). Let F_p be a space of homogeneous polynomials $f(x, y, z)$ with $\deg(f) = p + 1$, and put $F = \sum_{p \geqq 0} F_p$. We can also define a Poisson bracket $\{\cdot, \cdot\}$ on F, using a linear Poisson tensor P defined by (1.2). Since it is clear that $\{F_p, F_q\} \subset F_{p+q}$ $(p, q \geqq 0)$, F becomes a graded Lie algebra.

A space $h = \{x\}$ is a Cartan subalgebra of $sl(2, R)$ and a root decomposition of $sl(2, R)$ with respect to h is given by:

$$\begin{aligned} sl(2, R) &= g_{-1} + g_0 + g_1 \\ &= \{y + z\} + \{x\} + \{y - z\}. \end{aligned} \tag{2.1}$$

The first result is:

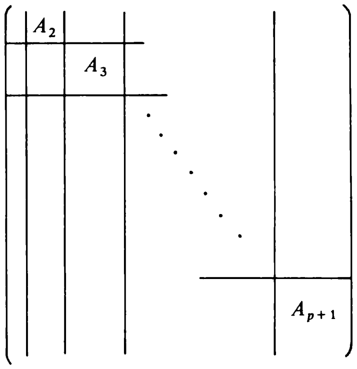
Proposition 2.1. *Each space F_p $(p \geqq 2)$ is generated by F_1. Namely it holds that $F_2 = \{F_1, F_1\}$, $F_3 = \{F_1, F_2\}, \ldots, F_p = \{F_1, F_{p-1}\}$.*

To prove the above proposition, we need the following lemma. This lemma is easily proved and we omit the proof.

Lemma 2.2. *Let A_k be a $k \times k$ matrix with $(A_k)_{i,b+1} = -2(k - i)$, $(A_k)_{i+1,i} = -2i$ and other elements $(A_k)_{a,b} = 0$. Then the* rank *$A_k = k$ if k is an even number, and the* rank *$A_k = k - 1$ if k is an odd number.*

Proof of Proposition 2.1. We prove $\{F_1, F_{p-1}\} = F_p$ $(p \geqq 2)$. First decompose F_p into two subspaces as follows: $F_p = V_1 + V_2$, where $V_1 = \{x^{p+1}, x^p y, \ldots, xz^p\}$ and $V_2 = \{y^{p+1}, y^p z, \ldots, z^{p+1}\}$. Since $\{x^2, F_{p-1}\} \subset V_1$, we can represent $\{x^2, F_{p-1}\}$ as a matrix with respect to the basis of V_1. Then we have the following matrix Q.

$Q =$



If p is even, by Lemma 2.2, rank $Q = \dim V_1 - (p/2 + 1)$. To make up for deficiencies of $\{x^2, F_{p-1}\}$, we need

$$p/2\left\{\begin{aligned}
&\{xy,\ x^{p-1}z\} = x^{p+1} - x^{p-1}y^2 + (p-1)x^{p-1}z^2,\\
&\{xy,\ x^{p-3}y^2z\} = x^{p-1}y^2 - x^{p-3}y^4 + (p-5)x^{p-3}y^2z^2,\\
&\qquad\vdots\\
&\{xy,\ xy^{p-2}z\} = x^3y^{p-2} - xy^p - (p-3)xy^{p-2}z^2,\\
&\{xz,\ x^{p-1}y\} = -x^{p+1} + (p-1)x^{p-1}y^2 - x^{p-1}z^2.
\end{aligned}\right.$$

It is easy to see that all these brackets span the subspace V_1. (If $p = 2$, we should use $\{y^2, yz\}$ instead of $\{xz, xy\}$.)

Next put $W = \{y^p, y^{p-1}z, \dots, z^p\}$. Then we can conclude that $\{xz, W\}$, $\{xy, y^{p-1}z\}$ and $\{xy, y^{p-2}z^2\}$ span the subspace V_2.

If p is odd, the result is obtained by a similar method as above. In fact, $\{x^2, F_{p-1}\}$, $\{xy, x^{p-1}z\}$, $\{xy, x^{p-3}y^2z\}, \dots$, $\{xy, x^2y^{p-3}z\}$ and $\{xz, x^{p-1}y\}$ span the subspace V_1. Moreover, $\{xz, W\}$ and $\{xy, y^{p-1}z\}$ span the subspace V_2. q.e.d.

A linear mapping $c\colon F \to F$ is called a derivation if it satisfies

$$c\{f, g\} = \{c(f), g\} + \{f, c(g)\} \quad \text{for any } f, g \in F. \tag{2.2}$$

We shall determine all derivations of F. We adopt the same method as that of T. Morimoto [4].

If a derivation $c\colon F \to F$ satisfies: $c(F_p) \subset F_{p+r}$ for any p, we say that the degree of c is r, and write as $\deg c = r$.

For any derivation c, we denote by $c_p^{(k)}$ the Hom (F_p, F_{p+k})-component of c. Define a new derivation $c^{(k)}$ by $c^{(k)}|F_p = c_p^{(k)}$. Then $c^{(k)}$ is a derivation of degree k, and c is written as $c = \sum_k c^{(k)}$.

For determining derivations of each degree, it is useful to obtain derect sum decomposition of F_p with respect to the action of ad(x) on F_p.

Proposition 2.3. *All eigen values of a linear mapping* $\mathrm{ad}(x)\colon F_p \to F_p$ *are* 0, ± 1, $\pm 2, \ldots, \pm(p+1)$. *Let* $F_p(k)$ *be an eigen space corresponding to an eigen value* k. *Then we obtain* $F_p = \sum_{k=-(p+1)}^{p+1} F_p(k)$, *where each* $F_p(k)$ *is given by*:

(i) *If* p *is even, say* $p = 2m$,

$$F_p(0) = \langle x^{2m+1},\ x^{2m-1}(y^2 - z^2), \ldots, x(y^2 - z^2)^m \rangle,$$
$$F_p(1) = \langle x^{2m}(y - z),\ x^{2(m-1)}(y - z)(y^2 - z^2), \ldots, (y - z)(y^2 - z^2)^m \rangle,$$
$$F_p(-1) = \langle x^{2m}(y + z),\ x^{2(m-1)}(y + z)(y^2 - z^2), \ldots, (y + z)(y^2 - z^2)^m \rangle,$$
$$F_p(2) = \langle x^{2m-1}(y - z)^2,\ x^{2m-3}(y - z)^2(y^2 - z^2), \ldots, x(y - z)^2(y^2 - z^2)^{m-1} \rangle,$$
$$F_p(-2) = \langle x^{2m-1}(y + z)^2,\ x^{2m-3}(y + z)^2(y^2 - z^2), \ldots, x(y + z)^2(y^2 - z^2)^{m-1} \rangle,$$
$$\vdots$$
$$F_p(2m+1) = \langle (y - z)^{2m+1} \rangle,$$
$$F_p(-(2m+1)) = \langle (y + z)^{2m+1} \rangle.$$

(ii) *If* p *is odd, say* $p = 2m - 1$,

$$F_p(0) = \langle x^{2m},\ x^{2(m-1)}(y^2 - z^2), \ldots, x^2(y^2 - z^2)^{m-1},\ (y^2 - z^2)^m \rangle,$$
$$F_p(1) = \langle x^{2m-1}(y - z),\ x^{2m-3}(y - z)(y^2 - z^2), \ldots, x(y - z)(y^2 - z^2)^{m-1} \rangle,$$
$$F_p(-1) = \langle x^{2m-1}(y + z),\ x^{2m-3}(y + z)(y^2 - z^2), \ldots, x(y + z)(y^2 - z^2)^{m-1} \rangle,$$
$$F_p(2) = \langle x^{2m-2}(y - z)^2,\ x^{2m-4}(y - z)^2(y^2 - z^2), \ldots, x(y - z)^2(y^2 - z^2)^{m-1} \rangle,$$
$$F_p(-2) = \langle x^{2m-2}(y + z)^2,\ x^{2m-4}(y + z)^2(y^2 - z^2), \ldots, x(y + z)^2(y^2 - z^2)^{m-1} \rangle,$$
$$\vdots$$
$$F_p(2m) = \langle (y - z)^{2m} \rangle,$$
$$F_p(-2m) = \langle (y + z)^{2m} \rangle.$$

The above proposition can be easily proved by direct calculations. (Taking an equation (2.1) into consideration, the proposition is almost obvious.)

In the rest of this section, we shall determine an explicit form of a derivation of each degree. First we prove:

Proposition 2.4. *For a derivation* c, *if* $\deg c \leqq -1$, *then* $c = 0$.

Proof. If $\deg c \leqq -2$, c satisfies $c(F_0) = c(F_1) = 0$. Combining this with Proposition 2.1, we easily have $c = 0$ on F. Next let $\deg c = -1$. Note that $c(F_0) = 0$. Since $c(F_1) \subset F_0$, we can write each element of $c(F_1)$ as follows: $c(x^2) = a_1x + b_1y + c_1z$, $c(xy) = a_2x + b_2y + c_2z$, $c(xz) = a_3x + b_3y + c_3z$, $c(y^2) = a_4x + b_4y + c_4z$, $c(yz) = a_5x + b_5y + c_5z$, $c(z^2) = a_6x + b_6y + c_6z$. Then by equations $0 = c\{x, x^2\} = \{x, c(x^2)\}$, $0 = c\{y, y^2\} = \{y, c(y^2)\}$, and $0 = c\{z, z^2\} = \{z, c(z^2)\}$, we easily have $b_1 = c_1 = a_4 = c_4 = a_6 = b_6 = 0$. On the other hand, by equations $c\{x, y^2\} = \{x, c(y^2)\} = -2c(yz)$, we have $a_5 = b_5 = 0$ and $2c_5$

$= b_4$. Similarly by equations $c\{y, z^2\} = \{y, c(z^2)\} = 2c(xz)$, we have $b_3 = c_3 = 0$ and $c_6 = 2a_3$. Equations $c\{x, z^2\} = \{x, c(z^2)\} = -2c(yz)$ mean $c_5 = c_6 = 0$ and hence $b_4 = a_3 = 0$. Equations $c\{x, xy\} = \{x, c(xy)\} = -c(xz) = 0$ mean $b_2 = c_2 = 0$. Equations $c(y, xy\} = \{y, c(xy)\} = c(yz) = 0$ mean $a_2 = 0$. Finally equations $c\{y, x^2\} = \{y, c(x^2)\} = 2c(xz) = 0$ mean $a_1 = 0$. Thus all coefficients of $c(F_1)$ vanish and hence by Proposition 2.1, we get $c = 0$ on F. q.e.d.

By the above proposition, we know that any derivation c can be written as $c = \sum_{k \geqq 0} c^{(k)}$. Next we shall determine a derivation c with $\deg c = p \geqq 0$. Consider the adjoint action of $F_0 = sl(2, R)$ over F_p. Since F_0 is a simple Lie algebra, it holds $H^1(F_0, F_p) = 0$. This means that for a derivation c, there exists an f of F_p such that $c|F_0 = \mathrm{ad}(f)$. Thus $(c - \mathrm{ad}(f))(F_0) = 0$. By this reason, hereafter, we always assume that a derivation c with non-negative degree satisfies $c(F_0) = 0$. (We are interested in only "outer" derivations.)

Proposition 2.5. *Let* $\deg c = p \geqq 0$. *Since* $c(F_1) \subset F_{p+1}$, *according to the direct sum decomposition of* F_{p+1}, *we can put*: $c(x^2) = \sum_{i=-p-2}^{p+2} a_i$, $c(y^2) = \sum_{i=-p-2}^{p+2} b_i$, $c(z^2) = \sum_{i=-p-2}^{p+2} c_i$, $c(yz) = \sum_{i=-p-2}^{p+2} r_i$. *Then* $c(x^2) = a_0$, $c(y^2) = r_{-2} + b_0 - r_2$, $c(z^2) = r_{-2} - b_0 - r_2$ *and* $c(yz) = r_{-2} + r_2$. *Moreover* $a_0 + 2b_0 \in \mathscr{C}$, *where* $\mathscr{C}$ *is a space of Casimir functions in* F.

Proof. Note that $c(F_0) = 0$. By the eqution $0 = c\{x, x^2\} = \{x, c(x^2)\} = \sum ia_i$, we have $a_i = 0$ if $i \neq 0$. Other equations $c\{x, y^2\} = \{x, c(y^2)\} = -2c(yz)$ imply that $\{x, \sum b_i\} = \sum ib_i = -2\sum r_i$. Thus we get $r_0 = 0$ and $b_i = -(2/i)r_i$ if $i \neq 0$. Similarly, the following equations $c\{x, z^2\} = \{x, c(z^2)\} = -2c(yz)$ mean that $c_i = -(2/i)r_i$ if $i \neq 0$. Using $c\{x, yz\} = \{x, c(yz)\} = -c(y^2) - c(z^2)$, we have

$$\begin{cases} -b_i - c_i = ir_i = (4/i)\mathrm{r}_i & (i \neq 0), \\ -b_0 - c_0 = 0. \end{cases}$$

Hence $r_i = 0$ is $i \neq \pm 2$, and we obtain: $c(y^2) = r_{-2} + b_0 - r_2$, $c(z^2) = r_{-2} - b_0 - r_2$ and $c(yz) = r_{-2} + r_2$. It generally holds that $c(\mathscr{C}) \subset \mathscr{C}$. Thus $c(x^2 + y^2 - z^2) \in \mathscr{C}$ and finally we have $a_0 + 2b_0 \in \mathscr{C}$. q.e.d.

The above proposition makes it easy to determine derivatins of positive degree. In fact, using Proposition 2.5, we prove

Proposition 2.6. (i) *If* $\deg c = 2m - 1$ $(m \geqq 1)$, *then c is an inner derivation.*
(ii) *If* $\deg c = 2m$ $(m \geqq 0)$, *then c is an outer derivation. More precisely, c is essentially defined as follows*:
For any $p \geqq 0$, $c(u_p) = pu_p(x^2 + y^2 - z^2)^m$ *for any* $u_p \in F_p$.

Proof. (i) Notes that $c(F_1) \subset F_{2m}$, and that there are no Casimir functions in F_{2m} except for 0. So by Proposition 2.5, it holds that $b_0 = -a_0/2$. According to the direct sum decomposition of F_{2m}, a_0, r_2 and r_{-2} are written as follows:

$$(2.3)\quad \begin{cases} a_0 = a_1x^{2m+1} + a_2x^{2m-1}(y^2 - z^2) + \cdots + a_{m+1}x(y^2 - z^2)^m, \\ r_2 = c_1x^{2m-1}(y - z)^2 + c_2x^{2m-3}(y - z)^2(y^2 - z^2) \\ \qquad\qquad + \cdots + c_mx(y - z)^2(y^2 - z^2)^{m-1}, \\ r_{-2} = d_1x^{2m-1}(y + z)^2 + d_2x^{2m-3}(y + z)^2(y^2 - z^2) \\ \qquad\qquad + \cdots + d_mx(y + z)^2(y^2 - z^2)^{m-1}. \end{cases}$$

Again recall that $c(F_0) = 0$. Substituting (2.3) into $\{y, c(y^2)\} = c\{y, y^2\} = 0$ and equating coefficients of z^k ($k = 1, 2, \dots, 2m + 1$) to zero, we have $a_i = c_j = d_j = 0$ ($1 \leqq i \leqq m + 1$, $1 \leqq j \leqq m$). Thus $c(x^2) = c(y^2) = c(z^2) = 0$. Since $c(F_0) = 0$, we also have $c(xy) = c(xz) = c(yz) = 0$. For example, $c(xy) = (1/2)c\,\{z, y^2\} = (1/2)\{z, c(y^2)\} = 0$. By Proposition 2.1, $c(F_p) = 0$ ($p \geqq 1$), and hence $c = 0$ on F. This means that c is an inner derivation of F by the previous remark.

(ii) In this case, $c(F_1) \subset F_{2m+1}$. And the space F_{2m+1} contains one dimensional subspace of Casimir functions whose basis is $(x^2 + y^2 - z^2)^{m+1}$. Hence we need put $b_0 = (K(x^2 + y^2 - z^2)^{m+1} - a_0)/2$. According to the direct sum decomposition of F_{2m+1}, a_0, r_2 and r_{-2} are written as follows:

$$(2.4)\quad \begin{cases} a_0 = a_1x^{2m+2} + a_2x^{2m}(y^2 - z^2) + \cdots + a_{m+2}(y^2 - z^2)^{m+1}, \\ r_2 = c_1x^{2m}(y - z)^2 + c_2x^{2m-2}(y - z)^2(y^2 - z^2) \\ \qquad\qquad + \cdots + c_{m-1}(y - z)^2(y^2 - z^2)^m, \\ r_{-2} = d_1x^{2m}(y + z)^2 + d_2x^{2m-2}(y + z)^2(y^2 - z^2) \\ \qquad\qquad + \cdots + d_{m+1}(y + z)^2(y^2 - z^2)^m. \end{cases}$$

Substituting (2.4) into $\{y, c(y^2)\} = 0$, and equating coefficients of z^k ($k = 1, 2, \dots, 2m + 2$) to zero, we have

$$(2.5)\quad \begin{cases} c(x^2) = \alpha x^2(x^2 + y^2 - z^2)^m - f, \\ c(y^2) = \alpha y^2(x^2 + y^2 - z^2)^m - g, \\ c(y^2) = \alpha(z^2) = \alpha z^2(x^2 + y^2 - z^2)^m + g, \end{cases}$$

where $\alpha = (m + 1)a_1 - a_2$, $f = (ma_1 - a_2)(x^2 + y^2 - z^2)^{m+1}$, $g = (1/2)(a_1 - k)(x^2 + y^2 - z^2)^{m+1}$.

Then taking account of $c(F_0) = 0$ and $f, g \in \mathscr{C}$, we also have

$$(2.6)\quad \begin{cases} c(xy) = \{c(y^2), z\}/2 = \alpha xy(x^2 + y^2 - z^2)^m, \\ c(xz) = \{y, c(z^2)\}/2 = \alpha xz(x^2 + y^2 - z^2)^m, \\ c(yz) = \{c(y^2), x\}/2 = \alpha yz(x^2 + y^2 - z^2)^m. \end{cases}$$

Hence a derivation c can be essentially written as $c(u_1) = u_1(x^2 + y^2 - z^2)^m$ for any $u_1 \in F_1$. Since F_1 generates F_p ($p \geqq 2$), we also obtain that $c(u_p) = pu_p(x^2 + y^2 - z^2)^m$ for any $u_p \in F_p$.

Now we shall prove that such a derivation c defined in this way is an outer derivation. Since $c(\mathscr{C}) \subset \mathscr{C}$, a derivation $c: F \to F$ induces a derivation $\bar{c}: L \to L$. More precisely, $\bar{c}$ is defined by $\bar{c}(X_h) = X_{c(h)}$, where $X_h \in L$ is a Hamiltonian vector field corresponding to $h \in F$. By an easy calculation, we know that $\bar{c}$ is given by

$$\bar{c} = \mathrm{ad}((x^2 + y^2 - z^2)^m (z\partial x + y\partial y + z\partial z)).$$

A vector field $(x^2 + y^2 - z^2)^m (z\partial x + y\partial y + z\partial z)$ is not an element of L. So $\bar{c}$ is an outer derivation of L and thus c is also an outer derivation of F. q.e.d.

We have completely determined the derivation algebra of F. We shall resume it in

Theorem 2.7. *Let $c: F \to F$ be any derivation. Then $c \equiv \sum_{m \geqq 0} \alpha_m c^{(2m)}$ (mod $\mathrm{ad}(F)$), where $c^{(2m)}$ is a derivation defined by*

$$c^{(2m)}(u_p) = p u_p (x^2 + y^2 - z^2)^m \quad \textit{for all } u_p \in F_p,$$

and α_m is some constant depending on c. In particular, all $c^{(2m)}$ are outer derivations, hence $H^1(F, F)$ is infinite dimensional.

3. Application

Let us consider the C^∞-version of the result obtained in the previous section. Recall the definition of Lie algebras $\mathscr{L}^c$, $\mathscr{L}$ and $\mathscr{N}$. Put $X = (x^2 + y^2 - z^2)^m (x\partial x + y\partial y + z\partial z)$, $(m \geqq 0)$. Then we have known that $\mathrm{ad}(X)$ is an outer derivation of L. If we regard X as a smooth vector field on R^3, X is an element of $\mathscr{L}^c$. (Note that $L(X)P = -(x^2 + y^2 - z^2)^m P$). For such a smooth vector field X, we shall prove

Proposition 3.1. $\mathrm{ad}(X)$ *is a derivation of $\mathscr{L}$ if and only if $m = 0$.*

Proof. By the definitin of $\mathscr{L}^c$, $\mathrm{ad}(X)$ is a derivation of $\mathscr{L}$ if and only if $X \in \mathscr{N}$. Hence by Lemma 1.3, X must satisfy $Y(C_X) = 0$ for all $Y \in \mathscr{L}$. In this case, $C_X = -(x^2 + y^2 - z^2)^m$, $(m \geqq 0)$. Let Y be a vector field appeared in the proof of Theorem 1.2. Then from the equation $Y(C_X) = 0$, we have $m = 0$. Conversely $\mathrm{ad}(x\partial x + y\partial y + z\partial z)$ is clearly a derivation of $\mathscr{L}$. q.e.d.

As the first step of studies of a linear Poisson manifold, we have studied infinitesimal automorphisms defined on $sl(2, R)^*$. If we take $sl(3, R)$ as $\mathfrak{g}$, we shall encounter some difficulties. For example the rank of the Poisson tensor takes the values 4 and 6, except for the origin. A linear Poisson manifold $sl(3, R)^*$ itself has interesting structures [3].

For a linear Poisson manifold with a compact Lie group, the circumstances are entirely different from the case of noncompact Lie groups. We shall treat these problems elsewhere.

The author would like to express his gratitude to Profesor P. Molino for his introduction of the geometry of Poisson manifolds.

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