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# Construction of irreducible unitary representations of the infinite symmetric group $\mathfrak{S}_\infty$

Dedicated to Professor Nobuhiko Tatsuuma on his sixtieth birthday

By

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**Introduction.**

For a set $I$, we denote by $\mathfrak{S}_I$ the group of all finite permutations on $I$. In this paper, we study irreducible unitary representations (=IURs) of the infinite symmetric group $\mathfrak{S}_N$, denoted also by $\mathfrak{S}_\infty$. We consider it as an infinite discrete group, of non type I, and apply our results in the previous paper [DG] (=[8]), getting a big family of completely new type of IURs.

Representations of the infinite symmetric group have been studied from many standpoints. All the indecomposable positive-definite class functions (or characters) have already been determined by Thoma [21]. They are also studied recently by Vershik and Kerov from different points of view ([9], [22], [23]). When we introduce a certain non-discrete topology in $\mathfrak{S}_\infty$, it becomes of type I and its IURs can be completely determined as shown by Lieberman ([11], [12]). Cf. also O'lshanskii [17] from this point of view. We have also other works ([3], [5], [7] etc.), rather operator algebra theoretic.

Very recently a new type of IURs has been constructed by Obata [16]. Discussions with him on his study and on Saito's [18] are one of our motivations of the present work, and discussions with Hashizume on his work [6] were also inspiring.

In our previous paper [DG], we studied a general theory of representations of infinite discrete groups, and applied it to wreath product groups $\mathfrak{S}_A(T)=D_A(T)\rtimes\mathfrak{S}_A$ of a group $T$ with the permutation group $\mathfrak{S}_A$, where $D_A(T)=\prod'_{\alpha\in A}T_\alpha$, $T_\alpha=T$ $(\alpha\in A)$, is the restricted direct product. We consider a family $\mathfrak{A}(\mathfrak{S}_A(T))$ of subgroups of the form $H=\prod'_{\gamma\in\Gamma}\mathfrak{S}_{A_\gamma}(T_\gamma)$, where $A=\coprod_{\gamma\in\Gamma}A_\gamma$ is a partition of $A$ and $T_\gamma$'s are subgroups of $T$. Further consider a family $\mathfrak{R}_H$ of IURs of $H$ coming naturally from characters $\chi_\gamma$ of $\mathfrak{S}_{A_\gamma}$, IURs $\rho^\gamma_{T_\gamma}$ of $T_\gamma$ and reference vectors to form tensor products, and put $\mathfrak{R}(\mathfrak{S}_A(T))=\bigcup_H\mathfrak{R}_H$ $(H\in\mathfrak{A}(\mathfrak{S}_A(T)))$. Then, in case $|T|<\infty$, the induced representations

$$\mathrm{Ind}_H^{\mathfrak{S}_A(T)}\pi,\quad H\in\mathfrak{A}(\mathfrak{S}_A(T)),\quad \pi\in\mathfrak{R}_H\subset\mathfrak{R}(\mathfrak{S}_A(T)),$$

give always IURs of $\mathfrak{S}_A(T)$ if $|\Gamma_f|\leqq 1$ with $\Gamma_f=\{\gamma\in\Gamma;\,|A_\gamma|<\infty\}$ and $\mathrm{Ind}_{T_\gamma}^T\rho^\gamma_{T_\gamma}$ is irreducible for $\gamma\in\Gamma_f$. Moreover the equivalence relations among these IURs are also completely determined.

For our study on the infinite symmetric group $G=\mathfrak{S}_N$ in the present paper, we

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apply fully these results. Start with a set $A_\gamma$ and a finite group $T_\gamma$, then the wreath product $\mathfrak{S}_{A_\gamma}(T_\gamma)$ is imbedded into $G$ as follows. First take a faithful permutation representation of $T_\gamma$ and imbed it into the symmetric group $\mathfrak{S}_{n(\gamma)}$ of degree $n(\gamma)$. Second, take a set $\mathfrak{J}_\gamma=\{J_{\gamma,\alpha}\,;\,\alpha\in A_\gamma\}$ of ordered $n(\gamma)$-sets $J_{\gamma,\alpha}$ parametrized by $\alpha\in A_\gamma$. Here an ordered $n(\gamma)$-set is an ordered set of $n(\gamma)$ different elements in $N$. Then, for $J_{\gamma,\alpha}=(p_1, p_2, \cdots, p_n)$, $n=n(\gamma)$, put $\bar{J}_{\gamma,\alpha}=\{p_1, p_2, \cdots, p_n\}$ the underlying subset and define a map $\iota_\alpha$ from $\bar{J}_{\gamma,\alpha}$ onto $\{1, 2, \cdots, n\}$ as $\iota_\alpha(p_j)=j$, $1\leqq j\leqq n$. Further define an isomorphism $\mathfrak{S}_n\to\mathfrak{S}_{\bar{J}_{\gamma,\alpha}}$ by $\sigma\mapsto\iota_\alpha^{-1}\circ\sigma\circ\iota_\alpha$. Assume that $\bar{J}_{\gamma,\alpha}$'s are mutually disjoint for $\mathfrak{J}_\gamma$, we get an imbedding $\psi_\gamma\,;\,\mathfrak{S}_{A_\gamma}(T_\gamma)\to\mathfrak{S}_N=G$. Such subgroups of $G$ are called *of wreath product type*. We consider a family $\mathfrak{A}$ of subgroups of $G$ obtained as images of "saturated" imbeddings $\psi_N\otimes(\otimes_{\gamma\in\Gamma}\psi_\gamma)$ of groups of type

$$\mathfrak{S}_N\times\prod'_{\gamma\in\Gamma}\mathfrak{S}_{A_\gamma}(T_\gamma)\qquad\text{with }\infty>N\geqq 0,\ |A_\gamma|=\infty\ (\gamma\in\Gamma),$$

where $\psi_N:\mathfrak{S}_N\to\mathfrak{S}_N$ and $\psi_\gamma$'s should satisfy certain conditions (cf. (B1)-(B3) in § 2.4).

For every $H\in\mathfrak{A}$, we consider a family $\mathfrak{R}_H$ of its IURs obtained similarly as above, and put $\mathfrak{R}=\bigcup_{H\in\mathfrak{A}}\mathfrak{R}_H$. Then the induced representations

$$\mathrm{Ind}_H^G\pi,\qquad H\in\mathfrak{A},\qquad \pi\in\mathfrak{R}_H\subset\mathfrak{R},$$

are the principal object of our study. We prove that they are all irreducible (Theorem 7.1) and also determine completely the equivalence relation among them (Theorem 8.9). As in the case of the wreath product $\mathfrak{S}_A(T)$, we have an interesting equivalence relation (Equivalence 2 in § 8.11) other than the usual one (Equivalence 1 loc. cit.) coming from inner automorphisms of $G$. We note that our method here is quite different from other ones and gives us a completely new big family of IURs which we call *standard* together with subgroups in $\mathfrak{A}$. Furthermore the approaches through *AF*-algebras as in [1] and [19] can not give us such results eventhough they are powerful to study factor representations.

As an important step to arrive at our final results, we prove in § 3 that the conditions (GRP1)-(GRP2) (cf. § 1.1.4) hold for the family $\mathfrak{A}$ of standard subgroups. Further we study how far the condition (REP) (cf. § 1.1.4) holds for the family $\mathfrak{R}$ of IURs of subgroups in $\mathfrak{A}$. Let $\mathfrak{R}_f$ be the family of all finite-dimensional IURs of $H\in\mathfrak{A}$, then $\mathfrak{R}_f\subset\mathfrak{R}$ and (REP) holds for it. Hence we can prove by Theorem 1.2 that the induced representations $\mathrm{Ind}_H^G\pi$, $H\in\mathfrak{A}$, $\pi\in\mathfrak{R}_H\cap\mathfrak{R}_f$, are all irreducible and the equivalence relations among them are all *elementary*, i.e., coming from inner automorphisms of $G$ (Theorem 5.1). All the IURs of $G$ constructed so far were in this family.

This paper is organized as follows. In § 1, we summarize the results in [DG] necessary to this paper and also give a general lemma for applying the boundedness conditions ($\mathrm{B}_x$) and ($\mathrm{C}_x$) (for the definition, see § 1.3). In § 2, we study the properties of subgroups of wreath product type, and in § 3, using these results, we study standard subgroups of $G$ and prove that $\mathfrak{A}$ satisfies the conditions (GRP1)-(GRP2). In §§ 4-5, we treat completely the inducing up of finite-dimensional IURs. In § 6, the family $\mathfrak{R}$ of IURs of standard subgroups is introduced and studied. In § 7, the irreducibility of the induced representations $\mathrm{Ind}_H^G\pi$, $H\in\mathfrak{A}$, $\pi\in\mathfrak{R}_H\subset\mathfrak{R}$, is established. In § 8, the equivalence relations among them are completely analyzed. The main tool in these sections, §§ 7-8, is the boundedness conditions ($B_x$) and ($C_x$), and the results in [DG] are essentially applied in § 8. We add as Appendix a simple proof of Moore's criterion for the unitary equivalence between tensor products of IURs.

CONVENTION. To refer Theorem 1.1 or § 1.1 in [DG], we sometimes refer it as Theorem DG1.1 or § DG1.1 for brevity.

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## § 1. Fundamental tools and methods

We summarize here the results in the previous paper [DG] and give a general lemma, which are necessary in this paper. At the same time, we prepare some notations and tools for later use.

### 1.1. Induced representations for discrete groups

**1.1.1. Induced representations.** Let $G$ be a discrete group and $H$ its subgroup. Take a unitary representation $\pi$ of $H$ on a Hilbert space $V(\pi)$. We realize the induced representation $U_\pi=\mathrm{Ind}_H^G\pi$ of $G$ as follows. The Hilbert space $\mathcal{H}(U_\pi)$ for $U_\pi$ is the space of $V(\pi)$-valued functions on $G$ such that

$$(1.1)\qquad \begin{aligned} &f(hg)=\pi(h)f(g) \qquad (h\in H,\ g\in G)\,,\\ &\|f^2\|=\textstyle\sum_{g\in H\backslash G}\|f(g)\|^2<\infty\,, \end{aligned}$$

where the summation runs over a section of $H\backslash G$ in $G$. (The convenient notation $g\in H\backslash G$ will be used throughout this paper.) The operator $U_\pi(g_0)$ for $g_0\in G$ is given by

$$(1.2)\qquad U_\pi(g_0)f(g)=f(gg_0) \qquad (g\in G)\,.$$

We introduce the notion of induced vectors. For a vector $v\in V(\pi)$, define an $f\in \mathcal{H}(U_\pi)$ such that $f(e)=v$ and $f(g)=0$ outside of $H$. Here $e$ denotes the unit element of $G$. This $f$ is called the *induced vector* of $v$ and is denoted by $\mathrm{Ind}_H^G v$.

The set of all induced vectors in $\mathcal{H}(U_\pi)$ is cyclic in the sense that the $G$-invariant subspace containg it is everywhere dense.

**1.1.2. Intertwining operators for induced representations.** Let $H_1$, $H_2$ be two subgroups of $G$, and $\pi_i$ a unitary representation of $H_i$ for $i=1, 2$. Put $U_{\pi_i}=\mathrm{Ind}_{H_i}^G\pi_i$, and let $\mathrm{Hom}_G(U_{\pi_1}, U_{\pi_2})=\mathrm{Hom}(U_{\pi_1}, U_{\pi_2}; G)$ be the space of intertwining operators of $U_{\pi_1}$ with $U_{\pi_2}$. Then, every $T\in\mathrm{Hom}_G(U_{\pi_1}, U_{\pi_2})$ is given by a kernel $K(g_2, g_1)$, $g_2$, $g_1\in G$, with values in $\boldsymbol{B}(V(\pi_1), V(\pi_2))$, the space of bounded linear operators of $V(\pi_1)$ into $V(\pi_2)$, as

$$(1.3)\qquad (Tf)(g)=\sum_{g'\in H_1\backslash G} K(g,g')f(g') \qquad (f\in\mathcal{H}(U_{\pi_1})),$$

and the kernel satisfies the conditions

(A) $K(h_2g_2, h_1g_1)=\pi_2(h_2)K(g_2, g_1)\pi_1(h_1)^{-1} \qquad (h_i\in H_i,\ g_i\in G_i)$,

(B) $\sum_{g\in H_2\backslash G}\|K(g,\xi)v\|^2\leqq M\|v\|^2 \qquad (\xi\in G,\ v\in V(\pi_1))$,

(C) $\sum_{g\in H_1\backslash G}\|K(\xi,g)^*w\|^2\leqq M\|w\|^2 \qquad (\xi\in G,\ w\in V(\pi_2))$,

(D) $K(g_2g, g_1g)=K(g_2, g_1) \qquad (g_2, g_1, g\in G)$,

where $M$ is a positive constant. The conditions (B) and (C) guarantee the boundedness of the operator defined under (A) by the right hand side of (1.3).

Conversely we know by Mackey [13] that every kernel function $K$ satisfying (A)-(D) defines an intertwining operator $T\in\mathrm{Hom}_G(U_{\pi_1}, U_{\pi_2})$ by (1.3) if $\dim\mathrm{Hom}_G(U_{\pi_1}, U_{\pi_2})<\infty$, and that, in general, $\dim\mathrm{Hom}_G(U_{\pi_1}, U_{\pi_2})$ is equal to the dimension of the space of kernel functions satisfying (A)-(D).

**1.1.3. Boundedness conditions.** Let $K$ be the kernel of a $T\in\mathrm{Hom}_G(U_{\pi_1}, U_{\pi_2})$. Then it is determined by $k(g)=K(g, e)$. The functions $k$ satisfies

$$k(h_2gh_1)=\pi_2(h_2)k(g)\pi_1(h_1) \qquad (h_i\in H_i,\ g\in G).$$

For $x\in G$, we put ${}^xg=xgx^{-1}$ $(g\in G)$, $H_2{}^x=x^{-1}H_2x$ and

$$(1.4)\qquad (\pi_2{}^x)(h)=\pi_2({}^xh) \qquad (h\in H_2{}^x).$$

Then $L=k(x)$ belongs to $\mathrm{Hom}(\pi_1, \pi_2{}^x; H_1\cap H_2{}^x)$, that is,

$$(1.5)\qquad L\circ\pi_1(h)=(\pi_2{}^x)(h)\circ L \qquad (h\in H_1\cap H_2{}^x).$$

Further $L$ determines $K(g_2, g_1)$ for $g_2g_1{}^{-1}\in H_2xH_1$. Rewriting the conditions (B) and (C) for this part of $K$, we get the following two conditions for $L$: there exists a positive constant $M$ such that for $v\in V(\pi_1)$ and $w\in V(\pi_2)$,

($\mathrm{B}_x$) $\sum_{h_1}\|L\pi_1(h_1)v\|^2\leqq M\|v\|^2 \qquad (h_1\in(H_1\cap x^{-1}H_2x)\backslash H_1)$,

($\mathrm{C}_x$) $\sum_{h_2}\|L^*\pi_2(h_2)w\|^2\leqq M\|w\|^2 \qquad (h_2\in(H_2\cap xH_1x^{-1})\backslash H_2)$.

Conversely there holds the following

**Lemma 1.1** [13]. *For an $x\in G$, put*

$$(1.8)\qquad d_x=\dim\{L\in\mathrm{Hom}(\pi_1, \pi_2{}^x; H_1\cap H_2{}^x);\ L \textit{ satisfies } (\mathrm{B}_x) \textit{ and } (\mathrm{C}_x)\}.$$

*For a complete system $X$ of representatives of $H_2\backslash G/H_1$, we have*

$$(1.7)\qquad \dim\mathrm{Hom}_G(U_{\pi_1}, U_{\pi_2})=\textstyle\sum_{x\in X}d_x.$$

In the sequel, we call the conditions ($\mathrm{B}_x$) and ($\mathrm{C}_x$) the *boundedness conditions.*

**1.1.4. Irreducibility and equivalence relations.** Let $\mathfrak{A}$ be a family of subgroups of $G$. Consider the following two conditions on $\mathfrak{A}$.

(GRP1) Let $H\in\mathfrak{A}$ and $g\in G$. (i) If $[H: H\cap gHg^{-1}]<\infty$, then $H\subset gHg^{-1}$. (ii) If $H\subset gHg^{-1}$, then $g\in H$.

(GRP2) Let $H_1, H_2\in\mathfrak{A}$, and $g\in G$. If $[H_1: H_1\cap gH_2g^{-1}]<\infty$ and $[gH_2g^{-1}: H_1\cap gH_2g^{-1}]<\infty$, then $H_1=gH_2g^{-1}$.

Under the condition (ii) in (GRP1), the normalizer $N_G(H)$ of $H$ in $G$ is equal to $H$ itself. We note that (GRP1) is devided into (i) and (ii) for later convenience.

Let $\mathfrak{R}$ be a family of irreducible unitary representations (=IURs) of groups in $\mathfrak{A}$. For a pair $\{\pi_1, \pi_2\}$ of elements in $\mathfrak{R}$, we consider the following condition.

(REP) Let $\pi_i$ be an IUR of $H_i\in\mathfrak{A}$ for $i=1, 2$. Suppose, for an $x\in G$, $L\in \mathrm{Hom}(\pi_1, \pi_2{}^x; H_1\cap H_2{}^x)$ satisfies $(\mathrm{B}_x)$ and $(\mathrm{C}_x)$. Then $L=0$ unless

$$[H_1: H_1\cap x^{-1}H_2x]<\infty \quad \text{and} \quad [H_2: xH_1x^{-1}\cap H_2]<\infty\,. \tag{1.8}$$

We say that (REP) holds for $\mathfrak{R}$ if it holds for any pair $\pi_1, \pi_2\in\mathfrak{R}$.

**Theorem 1.2** [DG, Th. 1.10]. (i) *Assume that* $\mathfrak{A}$ *satisfies* (GRP1). *Let* $\pi\in\mathfrak{R}$ *be an IUR of* $H\in\mathfrak{A}$. *If the condition* (REP) *holds for* $\pi_1=\pi_2=\pi$, *then the induced representation* $U_\pi=\mathrm{Ind}_H^G\pi$ *is irreducible.*

(ii) *Assume that* $\mathfrak{A}$ *satisfies* (GRP1)-(GEP2). *Let* $\pi_i\in\mathfrak{R}$ *be an IUR of* $H_i\in\mathfrak{A}$ *for* $i=1, 2$. *If the condition* (REP) *holds for any pairing* $\{\pi_i, \pi_j\}$ $(i, j=1, 2)$, *then* $U_{\pi_i}=\mathrm{Ind}_{H_i}^G\pi_i$ *are irreducible, and they are mutually equivalent if and only if, for an* $x\in G$,

$$H_1=H_2{}^x, \quad \textit{and} \quad \pi_1\cong\pi_2{}^x \ \textit{ for } H_1=H_2{}^x\,. \tag{1.9}$$

(iii) *Assume that* (GRP1)-(GRP2) *hold for* $\mathfrak{A}$ *and* (REP) *holds for* $\mathfrak{R}$. *Then the induced representations of* $\pi\in\mathfrak{R}$ *are all irreducible, and the conclusion in* (ii) *holds for any pair* $\pi_1, \pi_2\in\mathfrak{R}$.

We call *elementary* the equivalence relation $U_{\pi_1}\cong U_{\pi_2}$ if it comes from the relation (1.9).

**Remark 1.3.** Let $X_f$ be the subset of $X$ consisting of $x$ for which (1.8) holds. Then, under the condition (REP), we have

$$\dim \mathrm{Hom}_G(U_{\pi_1}, U_{\pi_2})=\textstyle\sum_{x\in X_f}\dim \mathrm{Hom}(\pi_1, \pi_2{}^x; H_1\cap H_2{}^x)\,. \tag{1.10}$$

We know from [10] that for any pair of finite-dimensional IURs $\pi_i$ of any subgroups $H_i$ $(i=1, 2)$, there holds the condition (REP). Hence we get the following

**Corollary 1.4.** *Assume* (GRP1)-(GRP2) *hold for* $\mathfrak{A}$, *and* $\mathfrak{R}$ *consists of finite-dimensional IURs. Then the condition* (REP) *holds for* $\mathfrak{R}$. *Hence the induced representations* $U_\pi$, $\pi\in\mathfrak{R}$, *are all irreducible, and the equivalence relations among them are all elementary.*

**Remark 1.5.** For $G=\mathfrak{S}_\infty$, the infinite symmetric group, Obata's case in [16] and our simple case in § 4 can be controlled by this corollary (cf. Remark 1.8). However, in our general case, the situation is not so simple that it cannot be controlled only by the above criterions, and we have non-elementary equivalence relation essentially (cf. § 8.10 and also Theorem 1.9).

### 1.2. Wreath product groups and their representations

**1.2.1. Wreath products.** For a set $I$, we denote by $\mathfrak{S}_I$ the group of all finite permutations on $I$. A permutation $\sigma$ is called *finite* if it leaves invariant almost all elements in $I$, or $\sigma(i)=i$ except a finite number of $i\in I$. If $J$ is a subset of $I$, $\mathfrak{S}_J$ is canonically imbedded into $\mathfrak{S}_I$.

Let $G_\alpha$ $(\alpha\in A)$ be a family of discrete groups with an index set $A$. Then the restricted direct product $\prod'_{\alpha\in A}G_\alpha$ is defined as the subgroup of the direct product $\prod_{\alpha\in A}G_\alpha$ consisting of $g=(g_\alpha)_{\alpha\in A}$ with $g_\alpha=e_\alpha$ for almost all $\alpha\in A$, where $e_\alpha$ denotes the unit element of $G_\alpha$.

Let $T$ be a discrete group and define the wreath product $\mathfrak{S}_A(T)$ of $T$ with $\mathfrak{S}_A$ (cf. [20, Chap. 2, § 10]) as

$$\mathfrak{S}_A(T)=D_A(T)\rtimes\mathfrak{S}_A,\quad D_A(T)=\textstyle\prod'_{\alpha\in A}T_\alpha \quad \text{with } T_\alpha=T\ (\alpha\in A)\,, \tag{1.19}$$

for which the product is given by

$$\sigma\cdot(t_\alpha)_{\alpha\in A}\cdot\sigma^{-1}=(t'_\alpha)_{\alpha\in A} \quad \text{with } t'_\alpha=t_{\sigma^{-1}(\alpha)} \quad (\sigma\in\mathfrak{S}_A,\ t_\alpha\in T_\alpha)\,. \tag{1.12}$$

An element $\sigma\in\mathfrak{S}_A$ which is imbedded into $\mathfrak{S}_A(T)$, is denoted again by $\sigma$ or sometimes by $1\times\sigma$ to avoid the confusion.

IURs of a wreath product group $\mathfrak{S}_A(T)$ were studied in detail in [DG], and the results necessary in this paper are summarized below, except some detailed accounts on intertwining operators [DG, § 7].

**1.2.2. Representations of a restricted direct product group.** First consider representations of a restricted direct product $G_A=\prod'_{\alpha\in A}G_\alpha$ of discrete groups. A unitary representation (=UR) of $G_A$ is called *factorizable* if it is equivalent to a direct product $\otimes^a_{\alpha\in A}\pi_\alpha$ of URs $\pi_\alpha$ of $G_\alpha$ with respect to a reference vector $a=(a_\alpha)_{\alpha\in A}$. Here $\pi^a=\otimes^a_{\alpha\in A}\pi_\alpha$ is defined as follows. The representation space $V(\pi^a)$ is the tensor product $\otimes^a_{\alpha\in A}V_\alpha=\otimes_{\alpha\in A}\{V_\alpha,\ a_\alpha\}$ of $V_\alpha=V(\pi_\alpha)$ with respect to the reference vector $a=(a_\alpha)_{\alpha\in A}$, $a_\alpha\in V(\pi_\alpha)$, $\|a_\alpha\|=1$, and

$$\pi^a(g)=\otimes^a_{\alpha\in A}\pi_\alpha(g_\alpha) \quad \text{for } g=(g_\alpha)_{\alpha\in A}\in G_A\,. \tag{1.13}$$

(For infinite tensor products of Hilbert spaces, cf. [4], [15] or § DG2.) This representation is irreducible if and only if so is every $\pi_\alpha$ of $G_\alpha$.

Consider $v=(v_\alpha)_{\alpha\in A}$ with $v_\alpha\in V_\alpha$, $\|v_\alpha\|=1$, and a formal vector $\otimes_{\alpha\in A}v_\alpha$. Then it can be considered to belong to $\otimes^a_{\alpha\in A}V_\alpha$ if and only if

$$\textstyle\sum_{\alpha\in A}|1-\langle v_\alpha,\ a_\alpha\rangle|<\infty\,, \tag{1.14}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product on $V_\alpha$. We call this relation the *Neumann-*

*equivalence* and denote it as $v\overset{N}{\cong}a$. We consider another weaker relation

$$\sum_{\alpha\in A}(1-|\langle v_\alpha,\ a_\alpha\rangle|)<\infty\,, \tag{1.15}$$

and call it the *Moore-equivalence* and denote it as $v\cong a$.

We know from [14, Th. 5] the following equivalence criterion (see Appendix, for a proof).

**Lemma 1.6.** *Let* $\otimes^a_{\alpha\in A}\pi_\alpha$ *and* $\otimes^b_{\alpha\in A}\pi'_\alpha$ *be two factorizable representations of* $G_A=\prod'_{\alpha\in A}G_\alpha$, *where* $\pi_\alpha$, $\pi'_\alpha$ *are IURs of* $G_\alpha$ $(\alpha\in A)$, $a=(a_\alpha)_{\alpha\in A}$, $a_\alpha\in V(\pi_\alpha)$, $\|a_\alpha\|=1$ *and* $b=(b_\alpha)_{\alpha\in A}$, $b_\alpha\in V(\pi'_\alpha)$, $\|b_\alpha\|=1$. *Then they are mutually equivalent if and only if* $\pi_\alpha\cong\pi'_\alpha$ *for every* $\alpha\in A$ *and*

$$a\cong(K_\alpha b_\alpha)_{\alpha\in A}\,, \tag{1.16}$$

*where* $K_\alpha$ *is a unitary intertwining operator of* $\pi'_\alpha$ *with* $\pi_\alpha$.

We say for (1.16) that $a$ and $b$ are *Moore-equivalent in an extended sense*. But we should note that this time there essentially enter representations of groups, or $K_\alpha\in\mathrm{Hom}\,(\pi'_\alpha,\ \pi_\alpha\,;\ G_\alpha)$.

**1.2.3. IURs of a wreath product group $\mathfrak{S}_A(T)$.** Let $T$ be a finite group. We consider IURs of the wreath product $\mathfrak{S}_A(T)=D_A(T)\rtimes\mathfrak{S}_A$, which come from factorizable URs of $D_A(T)$ or its subgroups.

For a UR $\pi$ of $D_A(T)=\prod'_{\alpha\in A}T_\alpha$ with $T_\alpha=T$ $(\alpha\in A)$, we put

$$({}^\sigma\pi)(t)=\pi(\sigma^{-1}t\sigma) \qquad (t\in D_A(T)) \tag{1.17}$$

and $\mathfrak{S}_A(\pi)=\{\sigma\in\mathfrak{S}_A\,;\ {}^\sigma\pi\cong\pi\}$ the stationary subgroup of $\pi$. The reason why we restrict ourselves here to the case of factorizable $\pi$, is that, for infactorizable $\pi$, we know almost nothing about $\mathfrak{S}_A(\pi)$ and intertwining operators for $\sigma\in\mathfrak{S}_A(\pi)$.

First take a UR $\rho_T$ of $T$ and consider a factorizable UR $\pi^a=\otimes^a_{\alpha\in A}\pi_\alpha$ with $\pi_\alpha=\rho_T$ $(\alpha\in A)$ with respect to a reference vector $a=(a_\alpha)_{\alpha\in A}$, $a_\alpha\in V(\pi_\alpha)=V(\rho_T)$, $\|a_\alpha\|=1$. Then, for $\pi=\pi^a$, we have $\mathfrak{S}_A(\pi)=\mathfrak{S}_A$. More exactly an intertwining operator $I_\sigma$ of ${}^\sigma\pi$ with $\pi$ is given as follows. For a decomposable vector $v=\otimes_{\alpha\in A}v_\alpha$ in $V(\pi)=\otimes^a_{\alpha\in A}V_\alpha$, $v_\alpha\in V_\alpha=V(\rho_T)$, put

$$I_\sigma v=\otimes_{\alpha\in A}v'_\alpha \qquad \text{with } v'_\alpha=v_{\sigma^{-1}(\alpha)}\ (\alpha\in A)\,. \tag{1.18}$$

Then $I_{\sigma\sigma'}=I_\sigma I_{\sigma'}$ $(\sigma,\ \sigma'\in\mathfrak{S}_A)$ and

$$I_\sigma\circ({}^\sigma\pi)(t)=\pi(t)\circ I_\sigma \qquad (t\in D_A(T),\ \sigma\in\mathfrak{S}_A=\mathfrak{S}_A(\pi))\,. \tag{1.19}$$

Take now a character $\chi$ of $\mathfrak{S}_A$ ($\chi=\mathbf{1}$ or sgn) and put

$$\Pi(t\cdot\sigma)=(\pi(t)I_\sigma)\cdot\chi(\sigma) \qquad (t\in D_A(T),\ \sigma\in\mathfrak{S}_A)\,. \tag{1.20}$$

Then $\Pi=\Pi(\pi,\chi)$ gives a UR of $\mathfrak{S}_A(T)$, which is determined by the datum $Q=\{A,\ \rho_T,\ \chi,\ a=(a_\alpha)_{\alpha\in A}\}$ and is also denoted by $\Pi(Q)$. If $\rho_T$ is irreducible, then so is $\Pi(Q)$. This type of IURs are called *elementary*, and we call $\Pi(Q)$ a WP-induced representation of $\rho_T$.

Generalizing this process, we obtain IURs of $\mathfrak{S}_A(T)$ which will be called *standard.* A standard representation $\rho(Q)$ is determined by a datum

$$Q=\{(A_\gamma,\ \rho^\gamma_{T_\gamma},\ \chi_\gamma)_{\gamma\in\Gamma},\ (a(\gamma))_{\gamma\in\Gamma},\ (b_\gamma)_{\gamma\in\Gamma}\}, \tag{1.21}$$

where $(A_\gamma)_{\gamma\in\Gamma}$ is a partition of $A$, $T_\gamma$ a subgroup of $T$, $\rho^\gamma_{T_\gamma}$ an IUR of $T_\gamma$, $\chi_\gamma$ a character of $\mathfrak{S}_{A_\gamma}$, and

$$\begin{aligned} &a(\gamma)=(a_\alpha)_{\alpha\in A_\gamma},\ a_\alpha\in V_\alpha=V(\rho^\gamma_{T_\gamma}),\ \|a_\alpha\|=1 \qquad (\alpha\in A_\gamma), \\ &b_\gamma\in\otimes^{a(\gamma)}_{\alpha\in A_\gamma}V_\alpha,\ \|b_\gamma\|=1 \qquad (\gamma\in\Gamma). \end{aligned} \tag{1.22}$$

To give $\rho(Q)$, first consider elementary IURs $\Pi(Q_\gamma)$ of $\mathfrak{S}_{A_\gamma}(T_\gamma)$ with data $Q_\gamma=\{A_\gamma,\ \rho^\gamma_{T_\gamma},\ \chi_\gamma,\ a(\gamma)\}$. Then, consider a subgroup of $\mathfrak{S}_A(T)$ given as

$$H=H(Q)\equiv\prod'_{\gamma\in\Gamma}\mathfrak{S}_{A_\gamma}(T_\gamma) \tag{1.23}$$

and an IUR of $H$ through a tensor product

$$\pi(Q)=\otimes^b_{\gamma\in\Gamma}\Pi(Q_\gamma) \qquad \text{with respect to } b=(b_\gamma)_{\gamma\in\Gamma}, \tag{1.24}$$

and finally induce it up from $H$ to $\mathfrak{S}_A(T)$:

$$\rho(Q)=\mathrm{Ind}\,(\pi(Q)\,;\,H\uparrow\mathfrak{S}_A(T)). \tag{1.25}$$

Now we assume $A$ is countably infinite, and further restrict ourselves only to the case which will be necessary in this paper. Put

$$\Gamma_f=\{\gamma\in\Gamma\,;\,|A_\gamma|<\infty\}, \qquad \Gamma_\infty=\{\gamma\in\Gamma\,;\,|A_\gamma|=\infty\}, \tag{1.26}$$

where $|A_\gamma|$ denotes the cardinal number of $A_\gamma$. Consider the following condition on $Q$:

$$|\Gamma_f|=|\Gamma\setminus\Gamma_\infty|\leqq 1. \tag{Q1}$$

Then we have

**Theorem 1.7** (cf. Th. DG4.2). *Let $T$ be a finite group and $\mathfrak{S}_A(T)$ the wreath product group. Then the induced representation $\rho(Q)$ of $\mathfrak{S}_A(T)$ is irreducible for $Q$ in* (1.21) *satisfying the condition* (Q1), *if $\mathrm{Ind}^T_{T_\gamma}\rho^\gamma_{T_\gamma}$ is irreducible for $\gamma\in\Gamma_f$.*

**Remark 1.8.** In relation to Theorem 1.2, we remark here the following. Put $G=\mathfrak{S}_A(T)$ and let $\mathfrak{A}$ be the set of subgroups $H$ in (1.23) of $G$ for which the condition (Q1) holds and $T_\gamma=T$ for $\gamma\in\Gamma_f$. Then we can prove that the conditions (GRP1)-(GRP2) hold for $\mathfrak{A}$ (cf. Theorem DG3.2 for (GRP1)). Further, let $\mathfrak{R}_f$ be the set of finite-dimensional IURs of $H\in\mathfrak{A}$ (a special subclass of $\pi(Q)$'s). Then the condition (REP) holds for $\mathfrak{R}_f$.

**1.2.4. Commutativity of two kinds of inducing processes.** Let $T$ be a group and $S$ its subgroup. Consider wreath product groups $\mathfrak{S}_A(S)$ and $\mathfrak{S}_A(T)$. Then we have two kinds of inducing $S\uparrow T$ and $\mathfrak{S}_A(S)\uparrow\mathfrak{S}_A(T)$ of representations. We give here a certain commutativity of these inducing processes.

Let us start with a datum

$$R=\{A,\ \rho_S,\ \chi,\ a=(a_\alpha)_{\alpha\in A}\}$$

for an elementary representation of $\mathfrak{S}_A(S)$.

On the one hand, put $\tilde{\rho}_T=\mathrm{Ind}_S^T\rho_S$, and let $\tilde{a}_\alpha=\mathrm{Ind}_S^T a_\alpha\in V(\tilde{\rho}_T)$ be the induced vector of $a_\alpha\in V(\rho_S)$. Then $\tilde{a}=(\tilde{a}_\alpha)_{\alpha\in A}$ is a reference vector for $(\tilde{V}_\alpha)_{\alpha\in A}$ with $\tilde{V}_\alpha=V(\tilde{\rho}_T)$. We denote $\tilde{a}$ also by $\mathrm{Ind}_S^T a$ in abuse of notation. Thus we get a new datum for $\mathfrak{S}_A(T)$ as

$$\tilde{R}=\{A,\ \tilde{\rho}_T,\ \chi,\ \tilde{a}\}\qquad \text{with } \tilde{\rho}_T=\mathrm{Ind}_S^T\rho,\ \tilde{a}=\mathrm{Ind}_S^T a\ , \tag{1.27}$$

and correspondingly an elementary representation $\rho(\tilde{R})$ of $\mathfrak{S}_A(T)$. On the other hand, we have the induced representation $\mathrm{Ind}\,(\rho(R)\,;\,\mathfrak{S}_A(S)\uparrow\mathfrak{S}_A(T))$.

As the commutativity of inducing processes (WP-inducing and usual inducing), we mean the following

**Theorem 1.9** (cf. Th. DG3.13). *Let $R$ be a datum for an elementary representation of $\mathfrak{S}_A(S)$. Then the two representations $\rho(\tilde{R})$ and $\mathrm{Ind}\,(\rho(R)\,;\,\mathfrak{S}_A(S)\uparrow\mathfrak{S}_A(T))$ of $\mathfrak{S}_A(T)$ are canonically equivalent to each other.*

*A similar assertion holds for standard representations of $\mathfrak{S}_A(S)$ and $\mathfrak{S}_A(T)$.*

**1.2.5. Equivalence relations between standard representations.** Take two standard representation $\rho(Q_1)$, $\rho(Q_2)$ of $\mathfrak{S}_A(T)$, and let the corresponding data be

$$\begin{aligned}Q_1&=\{(A_\gamma,\ \rho^\gamma_{T_{1\gamma}},\ \chi_{1\gamma})_{\gamma\in\Gamma},\ (a_1(\gamma))_{\gamma\in\Gamma},\ (b_{1\gamma})_{\gamma\in\Gamma}\},\\ Q_2&=\{(B_\delta,\ \rho^\delta_{T_{2\delta}},\ \chi_{2\delta})_{\delta\in\Delta},\ (a_2(\delta))_{\delta\in\Delta},\ (b_{2\delta})_{\delta\in\Delta}\},\end{aligned}\tag{1.28}$$

where, in particular, $(A_\gamma)_{\gamma\in\Gamma}$ and $(B_\delta)_{\delta\in\Delta}$ are partitions of $A$, and $T_{1\gamma}$ and $T_{2\delta}$ are subgroups of $T$.

For an element $\zeta$ of $\mathfrak{S}_A$, we call an *adjustment* of $Q_2$ by $\zeta$ the following datum:

$$^{\zeta}Q_2=\{(\zeta(B_\delta),\ \rho^\delta_{T_{2\delta}},\ \chi_\delta)_{\delta\in\Delta},\ (a_2(\delta))_{\delta\in\Delta},\ (b_{2\delta})_{\delta\in\Delta}\}. \tag{1.29}$$

Then $\rho(Q_2)$ is equivalent to $\rho(^{\zeta}Q_2)$ in a trivial fashion.

**Theorem 1.10** (cf. Th. DG4.5). *Assume that two data $Q_1$ and $Q_2$ satisfy the condition* (Q1), *i.e., $|\Gamma_f|\leqq 1$, $|\Delta_f|\leqq 1$, and that both $\rho(Q_1)$ and $\rho(Q_2)$ are irreducible. Then they are mutually equivalent if and only if the following three conditions hold.*

(EQU1) *Replacing $Q_2$ by its adjustment by an element in $\mathfrak{S}_A$ if necessary, we have a $1-1$ correspondence $\kappa$ of $\Gamma$ onto $\Delta$ such that $A_\gamma=B_{\kappa(\gamma)}$ for $\gamma\in\Gamma$. Further $\chi_\gamma=\chi_{\kappa(\gamma)}$ for $\gamma\in\Gamma$, and*

$$\mathrm{Ind}^T_{T_{1\gamma}}\rho^\gamma_{T_{1\delta}}\cong\mathrm{Ind}^T_{T_{2\delta}}\rho^\delta_{T_{2\delta}}\qquad \textit{for } \gamma\in\Gamma_f \textit{ and } \delta=\kappa(\gamma)\,. \tag{1.30}$$

(EQU2) *For $\gamma\in\Gamma_\infty$, put $T_{2\gamma}=T_{2\kappa(\gamma)}$ and $T_{0\gamma}=T_{1\gamma}\cap T_{2\gamma}$. Then, for every $\gamma\in\Gamma_\infty$, there exist an IUR $\rho^\gamma_{T_{0\gamma}}$ of $T_{0\gamma}$ and a reference vector $a_0(\gamma)=(a_{0\alpha})_{\alpha\in A_\gamma}$, $a_{0\alpha}\in V(\rho^\gamma_{T_{0\gamma}})$, $\|a_{0\alpha}\|=1$, such that for $j=1,2$,*

$$\rho^\gamma_{T_{j\gamma}}\cong\mathrm{Ind}\,(\rho^\gamma_{T_{0\gamma}}\,;\,T_{0\gamma}\uparrow T_{j\gamma})\,, \tag{1.31}$$

*and $a_j(\gamma)$ is Moore-equivalent to $\mathrm{Ind}\,(a_0(\gamma)\,;\,T_{0\gamma}\uparrow T_{j\gamma})$ in the extended sense.*

(EQU3) *Replace* $\delta=\kappa(\gamma)$ *by* $\gamma$ *and put*

$$Q_{j\gamma}=\{A_\gamma,\ \rho^\gamma_{T_{j\gamma}},\ \chi_{j\gamma},\ a_j(\gamma)\},\qquad 0\leqq j\leqq 2,$$

*with* $\chi_{0\gamma}=\chi_{1\gamma}$ $(=\chi_{2\gamma})$ *and consider IURs* $\Pi(Q_{j\gamma})$ *of* $H_{j\gamma}=\mathfrak{S}_{A_\gamma}(T_{j\gamma})$. *Then there exists a unit vector* $b_{0\gamma}\in V(\Pi(Q_{0\gamma}))$ *for every* $\gamma\in\Gamma_\infty$ *such that* $(b_{j\gamma})_{\gamma\in\Gamma_\infty}$, $j=1, 2$, *are respectively Moore-equivalent in the extended sense to*

$$(\tilde{b}_{j\gamma})_{\gamma\in\Gamma_\infty}\qquad \text{with } \tilde{b}_{j\gamma}=\mathrm{Ind}\,(b_{0\gamma};\ H_{0\gamma}\uparrow H_{j\gamma})$$

*with respect to* $\Pi(Q_{j\gamma})$ *and* $\mathrm{Ind}\,(\Pi(Q_{0\gamma});\ H_{0\gamma}\uparrow H_{j\gamma})$.

Here note that under the condition (EQU2) the IUR $\Pi(Q_{j\gamma})$ is equivalent to the induced one $\mathrm{Ind}\,(\Pi(Q_{0\gamma});\ H_{0\gamma}\uparrow H_{j\gamma})$ for $j=1, 2$, by Theorem 1.9.

**Remark 1.11.** The intertwining operators for the equivalence $\rho(Q_1)\cong\rho(Q_2)$, unique up to scalar multiples, can be easily written down using the explicit form of non-zero $L\in\mathrm{Hom}\,(\pi(Q_1), \pi(Q_2);\ H_1\cap H_2)$ satisfying the boundedness conditions ($\mathrm{B}_e$) and ($\mathrm{C}_e$), given in Theorem DG7.8, where $H_j=H(Q_j)$, $j=1, 2$. This explicit form of intertwinig operators plays an important role in our later discussions on the unitary equivalence among the standard IURs of $\mathfrak{S}_\infty$ (cf. §8), however we do not reproduce it here since we need still some more notations for that.

### 1.3. A general lemma for the study of intertwining operators

We give a general lemma which will play an important role for studying the boundedness conditions ($\mathrm{B}_x$), ($\mathrm{C}_x$), and then in §8 the equivalence relations between standard IURs constructed in §7.

Let $T$ be a finite group, $S$ its subgroup, and $\rho$ an IUR of $T$. Put $V_1=V(\rho)$ and consider a unitary $S$-module $V_2$. Take a complex Hilbert space $W_1$, and consider trivially $V_1\otimes W_1$ as a $T$-module. Now take an $L\in\mathrm{Hom}_S(V_1\otimes W_1, V_2)$ and evaluate the following sum for $u_1\in V_1\otimes W_1$:

$$J(u_1)=\sum_{t\in S\backslash T}\|L\rho(t)u_1\|^2=\frac{1}{|S|}\sum_{t\in T}\|L\rho(t)u_1\|^2. \tag{1.32}$$

Detailed evaluations of this kind of sums were necessary to prove the results in [DG] on irreducibility and equivalence relations for standard IURs of a wreath product group $\mathfrak{S}_A(T)$ (cf. Theorem 1.10 just above). Here we restrict ourselves to give a simple result as follows.

**Lemma 1.12.** *Let* $U$ *be any isometric linear operator of* $V_1$ *into* $V_1$ *and extend it naturally to* $V_1\otimes W_1$ *as such an operator. Then* $J(Uu_1)=J(u_1)$ *for* $u_1\in V_1\otimes W_1$. *Further*

$$\sup_{\|u_1\|\leqq 1} J(u_1)\geqq\|L\|^2. \tag{1.33}$$

*Proof.* Take a CONS $\{v_j;\ 1\leqq j\leqq d(\rho)\equiv\dim\rho\}$ of $V_1$. Then a unit vector $u_1$ is expressed as $u_1=\sum_j v_j\otimes w_j$ with $w_j\in W_1$ such that $\sum_j\|w_j\|^2=1$. By a simple calculation, we get

$$\sum_{t\in T}\|L\rho(t)u_1\|^2=\frac{|T|}{d(\rho)}\sum_{i,j}\|L(v_i\otimes w_j)\|^2.$$

From this the first equality follows immediately.

For (1.33), it is enough to use certain generalizations of Lemma DG5.6 (ii) and Lemma DG5.7. Q.E.D.

## § 2. Certain subgroups of $G=\mathfrak{S}_\infty$

### 2.1. Permutation groups and wreath products

Let $I$ be a set. An element $\sigma\in\mathfrak{S}_I$ can be represented by a matrix $M(\sigma)$ with suffices $I\times I$ as follows: the components of $M(\sigma)$ are equal to 1 at $(\sigma(i), i)$, $i\in I$, and zero elsewhere. By definition, $(\sigma\sigma')(i)=\sigma(\sigma'(i))$, $\sigma, \sigma'\in\mathfrak{S}_I$, and so we have $M(\sigma\sigma')=M(\sigma)M(\sigma')$ in the usual multiplication rule of matrices. Put $G=\mathfrak{S}_{\boldsymbol{N}}$ with $\boldsymbol{N}=\{1, 2, 3, \cdots\}$, the set of all natural numbers. We denote $G$ also by $\mathfrak{S}_\infty$ frequently.

Let $T$ be a finite group and $\mathfrak{S}_A(T)$ the wreath product of $T$ with $\mathfrak{S}_A$ given in (1.11)-(1.12). We define certain types of subgroups of $G$ by imbeddig the wreath product groups and taking their restricted direct products.

The results on IURs of the wreath product groups in [DG] will be applied to construct a new type of IURs of the groups $\mathfrak{S}_\infty$ in later sections.

### 2.2. Imbedding of the wreath product groups into $G$

An ordered set $J=(p_1, p_2, \cdots, p_n)$ of different $n$ integers $p_j\in\boldsymbol{N}$ is called an *ordered n-set*. We denote by $\bar{J}$ the underlying set $\{p_j, p_2, \cdots, p_n\}$ of $J$. Now let $\mathfrak{J}_n$ be a family of infinite number of ordered $n$-sets $J_\alpha$ $(\alpha\in A)$ such that $\bar{J}_\alpha$'s are disjoint mutually. Take a subgroup $T_n$ of $\mathfrak{S}_n=\mathfrak{S}_{\boldsymbol{N}_n}$ with $\boldsymbol{N}_n=\{1, 2, \cdots, n\}$. Let $\iota_\alpha$ be the order-preserving correspondence between $J_\alpha$ and $I_n=(1, 2, \cdots, n)$ such that $\iota_\alpha(p_j)=j$ $(1\leqq j\leqq n)$. Using this $\iota_\alpha$, we imbed $T_n$ into $\mathfrak{S}_{\bar{J}_\alpha}$ as

$$\varphi_\alpha:\mathfrak{S}_n\ni t\longmapsto\varphi_\alpha(t)=\iota_\alpha^{-1}\circ t\circ\iota_\alpha\in\mathfrak{S}_{\bar{J}_\alpha}, \tag{2.1}$$

or $\varphi_\alpha(t)p_j=p_{t(j)}$ for $J_\alpha=(p_1, p_2, \cdots, p_n)$. Denote by $T_\alpha$ the image $\varphi_\alpha(T_n)\subset\mathfrak{S}_{\bar{J}_\alpha}\subset G$ of $T_n$.

For a given $(\mathfrak{J}_n, T_n)$, we define a subgroup $D(\mathfrak{J}_n, T_n)$ and $H(\mathfrak{J}_n, T_n)$ of $G$ as follows:

$$\begin{aligned} &D(\mathfrak{J}_n, T_n)=\textstyle\prod'_{\alpha\in A}T_\alpha \qquad \text{with } T_\alpha=\varphi_\alpha(T_n),\\ &H(\mathfrak{J}_n, T_n)=D(\mathfrak{J}_n, T_n)\rtimes\mathfrak{S}_A, \end{aligned}\tag{2.2}$$

where $D(\mathfrak{J}_n, T_n)\rtimes\mathfrak{S}_A$ denotes a semidirect product of $D(\mathfrak{J}_n, T_n)$ and $\mathfrak{S}_A$ such that, for $(t_\alpha)_{\alpha\in A}\in D(\mathfrak{J}_n, T_n)$ and $\sigma\in\mathfrak{S}_A$,

$$\sigma^{-1}\cdot(t_\alpha)\cdot\sigma=(t'_\alpha)\qquad\text{with } \varphi_\alpha^{-1}(t'_\alpha)=(\varphi_{\sigma(\alpha)})^{-1}(t_{\sigma(\alpha)})\in T_n. \tag{2.3}$$

We know that $H(\mathfrak{J}_n, T_n)$ is canonically isomorphic to the wreath product $\mathfrak{S}_A(T_n)=D_A(T_n)\rtimes\mathfrak{S}_A$. In other words, the datum $\mathfrak{J}_n=\{J_\alpha; \alpha\in A\}$ gives an imbedding of the wreath product $\mathfrak{S}_A(T_n)$ into $G$. We call this type of subgroups of $G$ *of wreath product type*.

Every element $(s_\alpha)\cdot\sigma=(s_\alpha)\times\sigma$ in $H(\mathfrak{J}_n, \mathfrak{S}_n)\supset H(\mathfrak{J}_n, T_n)$ can be represented by an $A\times A$-matrix $M_n((s_\alpha)\cdot\sigma)$ whose entry at $(\sigma(\alpha), \alpha)$ for $\alpha\in A$ is $\tilde{s}_\alpha=\varphi_\alpha{}^{-1}(s_\alpha)\in T_n\subset\mathfrak{S}_n$, identified with its matrix $M(s_\alpha)$, and is zero elsewhere. Then, as is easily seen,

$$M_n(h_1h_2)=M_n(h_1)M_n(h_2) \qquad (h_1, h_2\in H(\mathfrak{J}_n, T_n)). \tag{2.4}$$

Note that, to arrive at subgroups of type $H(\mathfrak{J}_n, T_n)$, we can start with a wreath product group $\mathfrak{S}_A(T)$ with an arbitrary finite group $T$ and a countably infinite index set $A$. Take any faithful permutation representation of $T$ into $\mathfrak{S}_n$. Let $T_n$ be the image of $T$ in $\mathfrak{S}_n$ and take $\mathfrak{J}_n$ in addition, then we get a subgroup $H(\mathfrak{J}_n, T_n)\cong\mathfrak{S}_A(T)$.

### 2.3. Properties of subgroups of $G$ of wreath product type

We remark certain properties of this type of subgroups.

**Lemma 2.1.** *Take a* $\xi\in\mathfrak{S}_n$, *and put*

$$\xi\circ\mathfrak{J}_n=\{\xi\circ J_\alpha\,;\,\alpha\in A\}, \qquad \xi\circ T_n=\xi\cdot T_n\cdot\xi^{-1}, \tag{2.5}$$

*where, for* $J_\alpha=(p_1, p_2, \cdots, p_n)$,

$$\xi\circ J_\alpha=(p_{\xi^{-1}(1)}, p_{\xi^{-1}(2)}, \cdots, p_{\xi^{-1}(n)}). \tag{2.6}$$

*Then* $H(\mathfrak{J}_n, T_n)=H(\xi\circ\mathfrak{J}_n, \xi\circ T_n)$.

**Note 2.2.** When $T_n=\mathfrak{S}_n$, the orders in $J_\alpha$'s are superfluous in the sense that the imbedded subgroup $H(\mathfrak{J}_n, \mathfrak{S}_n)$ is determined only by the family of *non-ordered* sets $\bar{\mathfrak{J}}_n=\{\bar{J}_\alpha\,;\,\alpha\in A\}$. However the orders are not superfluous in the sense that the imbeddings of $\mathfrak{S}_A(T_n)$ into $G$, determined by various $\mathfrak{J}_n$'s with the same underlying $\bar{\mathfrak{J}}_n$, are in general not conjugate under $\mathrm{Int}(G)$. This fact is essential when we induce an IUR of $H(\mathfrak{J}_n, T_n)$ from that of $\mathfrak{S}_A(T_n)$ through the imbedding $\mathfrak{S}_A(T_n)\to H(\mathfrak{J}_n, T_n)\subset G$, and then induce it up from $H(\mathfrak{J}_n, T_n)$ to $G$ (see latter sections). Thus we should keep not only non-ordered $\bar{\mathfrak{J}}_n$ but also ordered $\mathfrak{J}_n$ even when $T_n=\mathfrak{S}_n$.

**Note 2.3.** From Lemma 2.1, we see that for some fixed $\alpha_0\in A$, we may fix without loss of generality the order in $J_{\alpha_0}=(p_1, p_2, \cdots, p_n)$ once for all, for instance, as $p_1<p_2<\cdots<p_n$. However we do not do so for the sake of convenience (see, for example, Lemma 3.1). We note further that if $n=1$, then necessarily $T_n=\{1\}$, and so $H(\mathfrak{J}_n, T_n)=\mathfrak{S}_C$ with $C=\mathrm{supp}(\mathfrak{J}_n)\subset \boldsymbol{N}$, the union of $\bar{J}_\alpha$ $(\alpha\in A)$.

Let $\mathfrak{J}'_m=\{J'_\beta\,;\,\beta\in B)$ be another family of ordered $m$-sets. Then we say that $\mathfrak{J}_n$ is a *refinement* of $\mathfrak{J}'_m$ if $m=Nn$ with an integer $N$, and every $\bar{J}'_\beta$ is a union of $N$ number of $\bar{J}_\alpha$, $\alpha\in A$. Further we say that $\mathfrak{J}_n$ is *equivalent* to $\mathfrak{J}'_m$ under $T_n\subset\mathfrak{S}_n$ if (1) $\bar{\mathfrak{J}}_n=\{\bar{J}_\alpha\,;\,\alpha\in A\}$ coincides with $\bar{\mathfrak{J}}'_m$ (in particular, $m=n$), and (2) for every $\alpha$, there exists $s_\alpha\in T_n$ such that

$$s_\alpha\circ J_\alpha=J'_\beta,$$

where $\beta\in B$ is determined by $\alpha$ as $\bar{J}_\alpha=\bar{J}'_\beta$.

The meaning of these definitions can be seen from the following lemmas.

**Lemma 2.4.** *Let $H=H(\mathfrak{J}_n, T_n)$, $H'=H(\mathfrak{J}'_m, T'_m)$ with $\mathfrak{J}_n=\{J_\alpha; \alpha\in A\}$, $\mathfrak{J}'_m=\{J'_\beta; \beta\in B\}$, and $T_n$, $T'_m$ subgroups of $\mathfrak{S}_n$, $\mathfrak{S}_m$ respectively.*

(i) *Assume that $H'\subset H$. Then $\mathfrak{J}_n$ contains a refinement of $\mathfrak{J}'_m$, and so $m=Nn$ for some integer $N$, and* $\mathrm{supp}(\mathfrak{J}_n)\supset\mathrm{supp}(\mathfrak{J}'_m)$.

(ii) *Let $m=n$ and suppose* $\mathrm{supp}(\mathfrak{J}_n)=\mathrm{supp}(\mathfrak{J}'_m)$. *Then $H'\subset H$ if and only if there exists a $\xi\in\mathfrak{S}_n$ such that $\xi\circ T'_n\subset T_n$, and $\xi\circ\mathfrak{J}'_n$ is equivalent to $\mathfrak{J}_n$ under $T_n$.*

*Proof.* (i) Assume that $H'\subset H$. Then, for $\alpha\in A$ and $\beta\in B$, $\bar{J}'_\beta$ contains $\bar{J}_\alpha$ in total or has no intersection with it: $\bar{J}'_\beta\supset\bar{J}_\alpha$ or $\bar{J}'_\beta\cap\bar{J}_\alpha=\emptyset$. Therefore every $\bar{J}'_\beta$ is a union of some number of $\bar{J}_\alpha$, $\alpha\in A$. Let this number be $N$, then $m=Nn$.

(ii) Assume $m=n$ and $\mathrm{supp}(\mathfrak{J}_n)=\mathrm{supp}(\mathfrak{J}'_m)$. Then we can use the same set of indices $A$ for $\mathfrak{J}_n$ and $\mathfrak{J}'_n$, and we have, for every $\alpha\in A$, a unique $\eta_\alpha\in\mathfrak{S}_n$ such that $J'_\alpha=\eta_\alpha\circ J_\alpha$. Let us express an element $h'$ in $H'=H(\mathfrak{J}'_n, T'_n)$ according to the decomposition analogous to (2.2) but for $\mathfrak{J}'_n$. Let $\varphi'_\alpha: \mathfrak{S}_n\to\mathfrak{S}_{\bar{J}'_\alpha}$ be analogous to $\varphi_\alpha$. Then, for $\mathfrak{J}'_n$,

$$h'=(t'_\alpha)\cdot\sigma' \qquad \text{with } t'_\alpha=\varphi'_\alpha(\tau'_\alpha),\ \tau'_\alpha\in T'_n,\ \sigma'\in\mathfrak{S}_A\,.$$

Since $H(\mathfrak{J}'_n, T'_n)\subset H(\mathfrak{J}'_n, \mathfrak{S}_n)=H(\mathfrak{J}_n, \mathfrak{S}_n)$, the element $h'$ has also an expression for $\mathfrak{J}_n$ according to (2.2) as an element of $H(\mathfrak{J}_n, \mathfrak{S}_n)$. Let it be as, for $\mathfrak{J}_n$,

$$h'\longleftrightarrow(t_\alpha)\cdot\sigma \qquad \text{with } t_\alpha=\varphi_\alpha(\tau_\alpha),\ \tau_\alpha\in\mathfrak{S}_n,\ \sigma\in\mathfrak{S}_A\,.$$

Then we have $\sigma=\sigma'$ and

$$\tau_\alpha=\eta_\alpha\tau'_\alpha\eta_{\sigma(\alpha)}{}^{-1} \qquad (\alpha\in A)\,. \tag{2.7}$$

Therefore, to have $H'\subset H$, it is necessary and sufficient that $\eta_\alpha T'_n\eta_{\sigma(\alpha)}{}^{-1}\subset T_n$ for $\alpha\in A$, $\sigma\in\mathfrak{S}_A$. Put $\xi=\eta_{\alpha_0}$ for a fixed $\alpha_0\in A$, then $\xi_\alpha=\eta_\alpha\eta_{\alpha_0}{}^{-1}=\eta_\alpha\xi^{-1}$ is in $T_n$ for any $\alpha\in A$. Hence $\eta_\alpha=\xi_\alpha\xi$ with $\xi_\alpha\in T_n$ for a fixed $\xi\in\mathfrak{S}_n$. This means that $\xi\circ\mathfrak{J}'_n$ is equivalent to $\mathfrak{J}_n$ under $T_n$.

Moreover we have

$$T_n\supset\eta_\alpha T'_n\eta_{\sigma(\alpha)}{}^{-1}=\xi_\alpha\xi T'_n\xi^{-1}\xi_{\sigma(\alpha)}{}^{-1}=\xi_\alpha(\xi\circ T'_n)\xi_{\sigma(\alpha)}{}^{-1}\,.$$

Hence $T_n\supset\xi\circ T'_n$. This completes the proof of the lemma. Q.E.D.

In general, when $N\geqq 2$, a necessary and sufficient condition for $H'\subset H$, is a little more complicated than above as seen below.

**Lemma 2.5.** *Let $H=H(\mathfrak{J}_n, T_n)$, $H'=H(\mathfrak{J}'_m, T'_m)$ be as in Lemma 2.4. Suppose* $\mathrm{supp}(\mathfrak{J}_n)=\mathrm{supp}(\mathfrak{J}'_m)$. *Then, $H'\subset H$ if and only if the following two conditions* (J1) *and* (J2) *hold.*

(J1) *$\mathfrak{J}_n$ is a refinement of $\mathfrak{J}'_m$, in particular, $m=Nn$ with an integer $N$.*

(J2) *For every $\beta\in B$, let $A(\beta)=\{\alpha\in A; \bar{J}_\alpha\subset\bar{J}'_\beta\}$. Fix a numbering $\alpha_1, \alpha_2, \cdots, \alpha_N$ of elements in $A(\beta)$ and define an ordered $m$-set $J^0_\beta$ as*

$$J^0_\beta=(J_{\alpha_1}, J_{\alpha_2}, \cdots, J_{\alpha_N})\,, \tag{$*$}$$

*and put $\mathfrak{J}^0_m=\{J^0_\beta; \beta\in B\}$. Then there exists a $\xi\in\mathfrak{S}_m$ satisfying*

(J2i) *$\xi\circ\mathfrak{J}'_m$ is equivalent to $\mathfrak{J}^0_m$ under $T(m)=T_n\rtimes\mathfrak{S}_N$, where $T(m)$ is canonically imbedded into $\mathfrak{S}_m$ according to (*), and $\xi\circ\mathfrak{J}'_m$ is as in Lemma 2.1, and*

(J2ii) *$\xi\circ T'_m\subset T(m)$ with $\xi\circ T'_m=\xi\cdot T'_m\cdot\xi^{-1}$.*

*Proof.* From Lemma 2.4(i), we may assume from the beginning that (J1) holds. Now let $D^0$ (resp. $H^0$) be the subgroup of $H$ consisting of all elements which leave every subset $\bar{J}^0_\beta\subset \boldsymbol{N}$ stable for $\beta\in B$ (resp. which permute subsets $\bar{J}^0_\beta$, $\beta\in B$). It is not difficult to see that $D^0=D(\mathfrak{J}^0_m, T(m))$, and then we get $H^0=D^0\rtimes\mathfrak{S}_B=H(\mathfrak{J}^0_m, T(m))$.

Therefore, $H'\subset H$ is equivalent to $H'\subset H^0$. Now apply Lemma 2.4(ii) to $H'=H(\mathfrak{J}'_m, T'_m)$ and $H^0=H(\mathfrak{J}^0_m, T(m))$, then we get the desired result. Q.E.D.

Moreover we have the following

**Lemma 2.6.** *Let $H=H(\mathfrak{J}_n, T_n)$ with $\mathfrak{J}_n=\{J_\alpha\,;\,\alpha\in A\}$ be as in Lemma 2.4. Then for $g\in G$,*

$$gHg^{-1}=H({}^g\mathfrak{J}_n, T_n)\qquad \text{with } {}^g\mathfrak{J}_n=\{{}^gJ_\alpha\,;\,\alpha\in A\},$$

*where*

$${}^gJ_\alpha=(g(p_1), g(p_2), \cdots, g(p_n))\qquad \text{for } J_\alpha=(p_1, p_2, \cdots, p_n).$$

*Proof.* Take $\alpha\in A$, and let $J_\alpha=(p_1, p_2, \cdots, p_n)$, ${}^gJ_\alpha=(q_1, q_2, \cdots, q_n)$. Put $\iota_\alpha(p_j)=j$, $\iota'_\alpha(q_j)=j$ $(1\leqq j\leqq n)$. For any $t\in T_n\subset\mathfrak{S}_n$, $\varphi_\alpha(t)=\iota_\alpha^{-1}\circ t\circ\iota_\alpha$ acts on $p_j$ as $\varphi_\alpha(t)(p_j)=p_{t(j)}$. On the other hand, $g\cdot\varphi_\alpha(t)\cdot g^{-1}$ acts on $q_j$ as

$$(g\cdot\varphi_\alpha(t)\cdot g^{-1})(q_j)=(g\cdot\varphi_\alpha(t))(p_j)=g(p_{t(j)})=q_{t(j)}\,.$$

This proves that, through $\iota'_\alpha$, we have also the group $T_n$ as the canonical subgroup acting on ${}^gJ_\alpha$. This, in turn, proves our assertion. Q.E.D.

It follows from Lemmas 2.4 and 2.6 the following important property of these subgroups.

**Proposition 2.7.** *Let $H=H(\mathfrak{J}_n, T_n)$ be as in Lemma 2.4. If $g\in G$ satisfies $gHg^{-1}\subset H$, then $gHg^{-1}=H$, that is, $g$ belongs to the normalizer $N_G(H)$ of $H$ in $G$. Moreover, if $\bar{\mathfrak{J}}_n=\{\bar{J}_\alpha\,;\,\alpha\in A\}$ is a partition of $\boldsymbol{N}$, then $N_G(H)=H$.*

*Proof.* Since $gHg^{-1}=H({}^g\mathfrak{J}_n, T_n)$ is contained in $H=H(\mathfrak{J}_n, T_n)$, and since ${}^gJ_\alpha=J_\alpha$ for almost all $\alpha\in A$, we get $\mathrm{supp}\,(\mathfrak{J}_n)=\mathrm{supp}\,(\mathfrak{J}'_m)$, and so we see from Lemma 2.4 that ${}^g\mathfrak{J}_n$ is equivalent to $\mathfrak{J}_n$ under $T_n$. Hence we get $gHg^{-1}=H$.

Now assume that $\mathfrak{J}_n$ is a partition of $\boldsymbol{N}$ and take a $g\in N_G(H)$. Then, since ${}^g\bar{\mathfrak{J}}_n\equiv({}^g\mathfrak{J}_n)^-=\bar{\mathfrak{J}}_n$, we see that $g\in H(\mathfrak{J}_n, \mathfrak{S}_n)$. Define $\sigma\in\mathfrak{S}_A$ by ${}^g\bar{J}_\alpha=\bar{J}_{\sigma(\alpha)}$, and $\tilde{s}_\alpha\in\mathfrak{S}_n$ by $\tilde{s}_\alpha=\iota_{\sigma(\alpha)}\circ(g\,|\,\bar{J}_\alpha)\circ\iota_\alpha^{-1}$. Then the matrix $M_n(g)$ for $g$ is given by these data, that is, components are $\tilde{s}_\alpha$ at $(\sigma(\alpha), \alpha)$ for $\alpha\in A$, and zero elsewhere. Since $\sigma\in\mathfrak{S}_A\hookrightarrow H$, we get $g'=g\sigma^{-1}\in N_G(H)$.

On the other hand, $M_n(g')$ is a blockwise diagonal matrix with diagonal components $\tilde{s}_\alpha$ at $(\alpha, \alpha)$. Take $\tau\in\mathfrak{S}_A\hookrightarrow H$ and consider $g'\tau g'^{-1}\in H$. Then its matrix $M_n(g'\tau g'^{-1})$ has components $\tilde{s}_{\tau(\alpha)}\tilde{s}_\alpha^{-1}\in T_n$ at $(\tau(\alpha), \alpha)$ and zero elsewhere. Note that there exists a finite subset $A_f$ of $A$ such that $\tilde{s}_\alpha=1$ if $\alpha\notin A_f$. Take $\tau\in\mathfrak{S}_A$ such that $\tau(\alpha)\notin A_f$ for any $\alpha\in A_f$. Then for $\alpha\in A_f$, $\tilde{s}_{\tau(\alpha)}\tilde{s}_\alpha{}^{-1}=\tilde{s}_\alpha{}^{-1}\in T_n$, that is, $\tilde{s}_\alpha\in T_n$. This proves that $g'\in H=H(\mathfrak{J}_n, T_n)$ and so is $g=g'\sigma$. Q.E.D.

**Note 2.8.** By Proposition 2.7, we see that $H$ satisfies the condition (ii) in (GRP1) if $\mathfrak{J}_n$ is a partition of $\boldsymbol{N}$.

### 2.4. Standard subgroups of $G$

Now take $\mathfrak{b}=\{F, (\mathfrak{J}_\gamma, T_\gamma); \gamma\in\Gamma\}$ with an index set $\Gamma$, finite or infinite, such that

(B1) $F$ is a finite subset of $\boldsymbol{N}$;

(B2) for every $\gamma\in\Gamma$, there corresponds $\infty>n(\gamma)\geqq 1$, and $\mathfrak{J}_\gamma=\{J_{\gamma,\alpha}; \alpha\in A_\gamma\}$ an infinite set of ordered $n(\gamma)$-sets $J_{\gamma,\alpha}$, and $T_\gamma$ a subgroup of $\mathfrak{S}_{n(\gamma)}$;

(B3) $\{F, \bar{J}_{\gamma,\alpha}; \gamma\in\Gamma, \alpha\in A_\gamma\}$ is a partition of $\boldsymbol{N}$.

For this set $\mathfrak{b}$, we define a subgroup $H=H^{\mathfrak{b}}$ of $G$ as follows:

$$H=H^{\mathfrak{b}}=H_f H_\infty, \tag{2.8}$$

where

$$H_f=\mathfrak{S}_F, \quad H_\infty={\prod}'_{\gamma\in\Gamma}H_\gamma \qquad \text{with } H_\gamma=H(\mathfrak{J}_\gamma, T_\gamma). \tag{2.9}$$

Here $H_\gamma=H(\mathfrak{J}_\gamma, T_\gamma)\cong\mathfrak{S}_{A_\gamma}(T_\gamma)$ is given as in § 2.2 for $(\mathfrak{J}_n, T_n)=(\mathfrak{J}_\gamma, T_\gamma)$ with $n=n(\gamma)$. We call $H_f$ and $H_\infty$ the finite and the infinite part of $H$ respectively.

Furthermore we call *standard* this type of subgroups of $G$, and denote by $\mathfrak{B}$ the family of all such $\mathfrak{b}$ that satisfy (B1), (B2) and (B3) above, and by $\mathfrak{A}$ the set of all standard subgroups: $\mathfrak{A}=\{H^{\mathfrak{b}}; \mathfrak{b}\in\mathfrak{B}\}$.

In the following, we study the set of representations of $G$ obtained by inducing up IURs of $H$ in $\mathfrak{A}$, thus getting a big family of new IURs of $G$. For this purpose, we give in the next section some important properties of these subgroups, which are similar as those for $H(\mathfrak{J}_n, T_n)$ given in § 2.3. From this study, we can see that the conditions (GRP1) and (GRP2) hold for our set of standard subgroups $\mathfrak{A}$.

**Notation 2.9.** We have introduced the following notations which will be utilized frequently in the sequel. Let $J=(p_1, p_2, \cdots, p_n)$ be an ordered set of natural numbers and $T$ a subgroup of $\mathfrak{S}_n$. Then, for a $\xi\in\mathfrak{S}_n$,

$$\xi\circ J=(p_{\xi^{-1}(1)}, p_{\xi^{-1}(2)}, \cdots, p_{\xi^{-1}(n)}), \qquad \xi\circ T=\xi\cdot T\cdot\xi^{-1},$$

and for $g\in\mathfrak{S}_\infty$,

$$^{g}J=gJ=(g(p_1), g(p_2), \cdots, g(p_n)).$$

Under this notation, for $J_\alpha$ in $\mathfrak{J}_n$ and $\xi\in\mathfrak{S}_n$, we have $\xi\circ J_\alpha=\varphi_\alpha(\xi^{-1})J_\alpha$.

## § 3. Properties of standard subgroups of $G$

In this section, we investigate some properties of subgroups $H$ in $\mathfrak{A}$, and establish the conditions (GRP1) and (GRP2) for $\mathfrak{A}$, which will be necessary to study the irreducibility and the equivalence relation of the representations of $G$ induced from those of $H\in\mathfrak{A}$.

### 3.1. The conjugate of $H$ under $g\in G$

To begin with, we remark the following elementary fact, a straightforward generalization of Lemma 2.6. Recall that a datum $\mathfrak{b}\in\mathfrak{B}$ is of the form

$$\mathfrak{b}=\{F, (\mathfrak{J}_\gamma, T_\gamma);\ \gamma\in\Gamma\}, \qquad \mathfrak{J}_\gamma=\{J_{\gamma,\alpha};\ \alpha\in A_\gamma\}, \tag{3.1}$$

where $T_\gamma$ is a finite group faithfully represented in $\mathfrak{S}_{n(\gamma)}$, each $J_{\gamma,\alpha}$ is an ordered $n(\gamma)$-set and so $|J_{\gamma,\alpha}|=n(\gamma)$, and $|A_\gamma|=\infty$.

**Lemma 3.1.** *Let $H=H^{\mathfrak{b}}\in\mathfrak{A}$ with $\mathfrak{b}\in\mathfrak{B}$ in* (3.1). *For any $g\in G$, the group $gHg^{-1}$ belongs again to $\mathfrak{A}$ and corresponds to a datum ${}^g\mathfrak{b}\in\mathfrak{B}$, i.e., $gHg^{-1}=H^{{}^g\mathfrak{b}}$. Here*

$$ {}^g\mathfrak{b}=\{{}^gF, ({}^g\mathfrak{J}_\gamma, T_\gamma);\ \gamma\in\Gamma\} \tag{3.2}$$

$$\text{with } {}^gF=\{g(i);\ i\in F\}, \quad {}^g\mathfrak{J}_\gamma=\{{}^gJ_{\gamma,\alpha};\ \alpha\in A_\gamma\}.$$

Let $\tilde{\mathfrak{S}}_N$ denote the group of all permutations on $N$. Then the above lemma holds also for any $g\in\tilde{\mathfrak{S}}_N$. Further we know that $\mathrm{Aut}(\mathfrak{S}_N)\cong\tilde{\mathfrak{S}}_N$.

### 3.2. Conditions for the inclusion $H'\subset H$

Now consider the relations between two subgroups $H=H^{\mathfrak{b}}$, $H'=H^{\mathfrak{b}'}$ in $\mathfrak{A}$ with $\mathfrak{b}$, $\mathfrak{b}'\in\mathfrak{B}$ in (3.1) and

$$\mathfrak{b}'=\{F', (\mathfrak{J}'_\delta, T'_\delta);\ \delta\in\Delta\}, \quad \mathfrak{J}'_\delta=\{J'_{\delta,\beta};\ \beta\in B_\delta\}. \tag{3.3}$$

First we give a necessary and sufficient condition for the inclusion $H'\subset H$. To do so, we introduce some definitions. Put

$$\bar{\mathfrak{b}}=\{F, \bar{\mathfrak{J}}_\gamma;\ \gamma\in\Gamma\} \qquad \text{with } \bar{\mathfrak{J}}_\gamma=\{\bar{J}_{\gamma,\alpha};\ \alpha\in A_\gamma\}, \tag{3.4}$$

where $\bar{J}_{\gamma,\alpha}$ denotes the underlying set of $J_{\gamma,\alpha}$, and call $\bar{\mathfrak{b}}$ the underlying two-step partition of $N$ for $\mathfrak{b}$. We say that $\bar{\mathfrak{b}}$ is a *refinement* of $\bar{\mathfrak{b}}'=\{F', \bar{\mathfrak{J}}'_\delta;\ \delta\in\Delta\}$ if there hold the following two conditions (R1), (R2).

(R1) (i) $F=F'$; or (ii) $F=\emptyset$ and $F'=\bigcup_{1\le i\le p}\bar{J}_{\gamma,\alpha_i}$ for some $\gamma\in\Gamma$ and $\alpha_1, \alpha_2, \cdots, \alpha_p\in A_\gamma$, and in addition $p=1$ if $n(\gamma)>1$.

(R2) If $\mathrm{supp}(\mathfrak{J}_\gamma)\supset F'$, put $A_\gamma^0=A_\gamma\setminus\{\alpha_1, \alpha_2, \cdots, \alpha_p\}$ with $\alpha_1, \alpha_2, \cdots, \alpha_p\in A_\gamma$ such that $F'=\bigcup_{1\le i\le p}\bar{J}_{\gamma,\alpha_i}$, and otherwise put $A_\gamma^0=A_\gamma$. (Note that $A_\gamma^0\ne A_\gamma$ for at most one $\gamma\in\Gamma$.) Then there exist (1) a partition of each $A_\gamma^0$ as $A_\gamma^0=\bigcup_{\delta\in\Delta}A_\gamma^\delta$ and (2) a partition of each $A_\gamma^\delta$ as

$$A_\gamma^\delta=\bigcup_\beta A_\gamma^\delta(\beta) \qquad (\beta\in B_\delta),$$

such that, putting $\Gamma_\delta=\{\gamma\in\Gamma;\ A_\gamma^\delta\ne\emptyset\}$, we have for every $\beta\in B_\delta$,

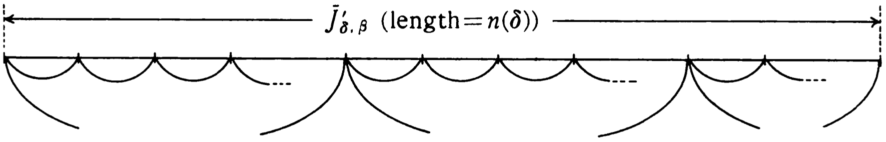
$$\bar{J}'_{\delta,\beta}=\bigcup_\gamma\bigcup_\alpha\bar{J}_{\gamma,\alpha} \qquad (\gamma\in\Gamma_\delta,\ \alpha\in A_\gamma^\delta(\beta)) \tag{3.5}$$

and that $|A_\gamma^\delta(\beta)|=N(\gamma, \delta)$, independent of $\beta\in B_\delta$.

To visualize these relations, let us illustrate the inclusion of $\bar{J}_{\gamma,\alpha}$ ($\alpha\in A_\gamma$) into $\bar{J}'_{\delta,\beta}$

($\beta\in B_\delta$). In the figure below, $\gamma_i$ runs over $\Gamma_\delta$.

**Figure 3.1.**




| | | |
|---|---|---|
| $\bar{J}_{\gamma_1,\alpha}$'s ($\alpha\in A_{\gamma_1}^\delta(\beta)$) | $\bar{J}_{\gamma_2,\alpha}$'s ($\alpha\in A_{\gamma_2}^\delta(\beta)$) | ··· |
| (each length$=n(\gamma_1)$) | (each length$=n(\gamma_2)$) | ··· |
| # of $J_{\gamma_1,\alpha}$'s$=N(\gamma_1, \delta)$ | # of $J_{\gamma_2,\alpha}$'s$=N(\gamma_2, \delta)$ | ··· |

Note that $A_\gamma^\delta$ (resp. $A_\gamma^\delta(\beta)$) is the part of $A_\gamma$ corresponding to $B_\delta$ (resp. $\beta\in B_\delta$), and that $n(\delta)=\sum_{\gamma\in\Gamma_\delta}N(\gamma, \delta)n(\gamma)$, and so $\Gamma_\delta$ is a finite set, and $|A_\gamma^\delta|=|B_\delta|=\infty$ for $\gamma\in\Gamma_\delta$. The meaning of this definition will be clarified just below.

**Theorem 3.2.** *Let $H=H^{\mathfrak{b}}$, $H'=H^{\mathfrak{b}'}$ be in $\mathfrak{A}$ with $\mathfrak{b}, \mathfrak{b}'\in\mathfrak{B}$ in* (3.1), (3.3) *respectively. Then $H'$ is contained in $H$ if and only if the following two conditions* (I1), (I2) *hold.*

(I1) *$\bar{\mathfrak{b}}$ is a refinement of $\bar{\mathfrak{b}}'$.*

(I2) *Fix $\delta\in\Delta$ and a numbering $\gamma_1, \gamma_2, \cdots, \gamma_m$, $m=|\Gamma_\delta|$, of elements in $\Gamma_\delta$. Then there exists an element $\xi_\delta\in\mathfrak{S}_{n(\delta)}$ satisfying the conditions* (I2i)-(I2iii) *below. Put for $\beta\in B_\delta$,*

$$J_{\delta,\beta}^0=(J_\beta^{\gamma_1}, J_\beta^{\gamma_2}, \cdots, J_\beta^{\gamma_m}) \tag{3.6}$$

*where, for each $\gamma\in\Gamma_\delta$, fixing a numbering $\alpha_i$ ($1\leq i\leq N(\gamma, \delta)$) of elements in $A_\gamma^\delta(\beta)$,*

$$J_\beta^\gamma=(J_{\gamma,\alpha_1}, J_{\gamma,\alpha_2}, \cdots, J_{\gamma,\alpha_N}) \quad \text{with } N=N(\gamma, \delta) \tag{3.6'}$$

(*cf. Figure 3.1*). *Then $\mathfrak{J}_\delta^0=\{J_{\delta,\beta}^0; \beta\in B_\delta\}$ is an infinite set of ordered $n(\delta)$-sets.*

(I2i) *$\xi_\delta\circ\mathfrak{J}'_\delta=\{\xi_\delta\circ J'_{\delta,\beta}; \beta\in B_\delta\}$ is equivalent to $\mathfrak{J}_\delta^0$ under*

$$T(\delta)=\prod_{1\leq i\leq m}((T_{\gamma_i})^{N_i}\times\mathfrak{S}_{N_i}),$$

*where $N_i=N(\gamma_i, \delta)$ and $T(\delta)$ is canonically imbedded into $\mathfrak{S}_{n(\delta)}$ according to* (3.6)-(3.6').

(I2ii) *$\xi_\delta\circ T'_\gamma\subset T(\delta)$, where $\xi_\delta\circ T'_\delta=\xi_\delta T'_\delta\xi_\delta^{-1}$ by definition.*

(I2iii) *If $F'=\bar{J}_{\gamma,\alpha}$ with $\gamma\in\Gamma$, $\alpha\in A_\gamma$, then $T_\gamma=\mathfrak{S}_{n(\gamma)}$.*

### 3.3. Proof of Theorem 3.2

The sufficiency of the conditions (I1)-(I2) can be seen without difficulty. So let us prove the necessity. First put

$$N_\gamma=\bigcup_\alpha \bar{J}_{\gamma,\alpha} \ (\alpha\in A_\gamma), \qquad N'_\delta=\bigcup_\beta \bar{J}'_{\delta,\beta} \ (\beta\in B_\delta). \tag{3.7}$$

STEP 1. Assume that $F'\neq F$. Then $F\setminus F'\neq\emptyset$ or $F'\setminus F\neq\emptyset$. First assume $F\setminus F'\neq\emptyset$. Then $(F\setminus F')\cap\bar{J}'_{\delta,\beta_0}\neq\emptyset$ for some $\delta$ and $\beta_0\in B_\delta$. Put $E=\{\beta\in B_\delta;\ (F\setminus F')\cap\bar{J}'_{\delta,\beta}\neq\emptyset\}$. Since $E$ is finite, we have $\sigma\in\mathfrak{S}_{B_\delta}$ such that $\sigma(E)\cap E=\emptyset$. Then $1\times\sigma\in\mathfrak{S}_{B_\delta}\subset H(\mathfrak{J}'_\delta, T'_\delta)\subset H'$ sends $(F\setminus F')\cap\bar{J}'_{\delta,\beta_0}$ into $\bar{J}'_{\delta,\beta}$ which has no intersection with $F\setminus F'$ nor with $F$. This means that $(1\times\sigma)F\not\subset F$, and so $1\times\sigma\notin H$. Hence $H'\not\subset H$ because $1\times\sigma\in H'$. Thus it should hold that $F\setminus F'=\emptyset$ or $F\subset F'$.

Now assume that $F\subset F'$, $F\neq\emptyset$ and $F'\setminus F\neq\emptyset$. Then we have a $\sigma\in\mathfrak{S}_{F'}$, such that $\sigma(F'\setminus F)\cap F\neq\emptyset$, whence $\sigma(F)\neq F$. This means that $1\times\sigma\in H'$ does not belong to $H$. Hence it should hold that $F=F'$ or $F=\emptyset$, if $H'\subset H$.

STEP 2. Consider the case where $F=\emptyset$, $F'\neq\emptyset$. Assume that, for some $J_{\gamma,\alpha}$, $\bar{J}_{\gamma,\alpha}\cap F'\neq\emptyset$ and $\bar{J}_{\gamma,\alpha}\cap\bar{J}'_{\delta,\beta}\neq\emptyset$ with a $\beta\in B_\delta$. Then we can find $\sigma\in\mathfrak{S}_{B_\delta}$ such that $1\times\sigma\in H'$ sends $\bar{J}'_{\delta,\beta}$ $(\supset\bar{J}_{\gamma,\alpha}\cap\bar{J}'_{\delta,\beta})$ into $\bar{J}'_{\delta,\sigma(\beta)}$ which is disjoint with $\bar{J}_{\gamma,\alpha}$. Since $1\times\sigma$ leaves $\bar{J}_{\gamma,\alpha}\cap F'$ invariant, $1\times\sigma\notin H$ and so $H'\not\subset H$. Thus we see that $F'\supset\bar{J}_{\gamma,\alpha}$ if $F'\cap\bar{J}_{\gamma,\alpha}\neq\emptyset$, whence $F'$ is the union of $\bar{J}_{\gamma,\alpha}\subset F'$.

Since $\mathfrak{S}_{F'}\cong\{1\}\times\mathfrak{S}_{F'}\subset H'$, the inclusion $H'\subset H$ means that $\{1\}\times\mathfrak{S}_{F'}\subset H$. This occurs only when $F'=\bar{J}_{\gamma,\alpha}$ with $T_\gamma=\mathfrak{S}_{n(\gamma)}$ and $\alpha\in A_\gamma$, or $F'=\bigcup_{1\leq i\leq p}\bar{J}_{\gamma,\alpha_i}$ with $n(\gamma)=1$ and $\alpha_1, \alpha_2, \cdots, \alpha_p\in A_\gamma$. Thus we have proved in particular that the conditions (R1) and (I2iii) hold.

STEP 3. Assume that for some $J_{\gamma,\alpha}$, $\bar{J}_{\gamma,\alpha}\cap\bar{J}'_{\delta_i,\beta_i}\neq\emptyset$ for two different $\delta_1, \delta_2\in\Delta$, where $\beta_i\in B_{\delta_i}$ $(i=1, 2)$. Take a $\sigma\in\mathfrak{S}_L$ with $L=B_{\delta_1}$ such that $\sigma(\beta_1)=\beta$ satisfies that $\bar{J}_{\gamma,\alpha}\cap\bar{J}'_{\delta_1,\beta}=\emptyset$. Then $1\times\sigma\in H'$ does not belong to $H$, because any element $g\in H$ sends in total $\bar{J}_{\gamma,\alpha}$ onto $\bar{J}_{\gamma,\alpha'}$ for some $\alpha'\in A_\gamma$, whereas, under $1\times\sigma$, $\bar{J}_{\gamma,\alpha}\cap\bar{J}'_{\delta_2,\beta_2}\neq\emptyset$ is left invariant and $\bar{J}_{\gamma,\alpha}\cap\bar{J}'_{\delta_1,\beta_1}$ is sent into $\bar{J}'_{\delta_1,\sigma(\beta_1)}=\bar{J}'_{\delta_1,\beta}$ which is disjoint with $\bar{J}_{\gamma,\alpha}$. Thus it follows that $H'\not\subset H$. Therefore, when $H'\subset H$, there exists, for any $J_{\gamma,\alpha}$, a unique $\delta\in\Delta$ such that $\bar{J}_{\gamma,\alpha}\subset\bigcup_{\beta\in B_\delta}\bar{J}'_{\delta,\beta}$.

STEP 4. Assume that for some $J_{\gamma,\alpha}$, $\bar{J}_{\gamma,\alpha}\cap\bar{J}'_{\delta,\beta_i}\neq\emptyset$ for two different $\beta_1, \beta_2\in B_\delta$. Take $\sigma\in\mathfrak{S}_{B_\delta}$ such that $\sigma(\beta_2)=\beta_2$ and that $\sigma(\beta_1)=\beta$ satisfies $\bar{J}_{\gamma,\alpha}\cap\bar{J}'_{\delta,\beta}=\emptyset$. Then, under $1\times\sigma\in H'$, $\bar{J}_{\gamma,\alpha}\cap\bar{J}'_{\delta,\beta_2}$ is left invariant, whereas $\bar{J}_{\gamma,\alpha}\cap\bar{J}'_{\delta,\beta_1}$ is sent into $\bar{J}'_{\delta,\sigma(\beta_1)}=\bar{J}'_{\delta,\beta}$ which is disjoint with $\bar{J}_{\gamma,\alpha}$. So we see that $1\times\sigma\notin H$ whence $H'\not\subset H$, similarly as in Step 3. Thus we have proved that for every $J_{\gamma,\alpha}$ there exists a unique $J'_{\delta,\beta}$ such that $\bar{J}_{\gamma,\alpha}\subset\bar{J}'_{\delta,\beta}$.

STEP 5. Fix $\gamma\in\Gamma$ and let $\bar{J}_{\gamma,\alpha_0}\subset\bar{J}'_{\delta,\beta_0}$ for some $\alpha_0\in A_\gamma$, $\beta_0\in B_\delta$. Put

$$A_\gamma^\delta(\beta)=\{\alpha\in A_\gamma;\ \bar{J}_{\gamma,\alpha}\subset\bar{J}'_{\delta,\beta}\},\qquad N(\gamma, \delta;\beta)=|A_\gamma^\delta(\beta)|. \tag{3.7}$$

Then we shall prove that $N(\gamma, \delta;\beta)=N(\gamma, \delta;\beta_0)$ for every $\beta\in B_\delta$. Assume the contrary: for some $\beta\in B_\delta$, $N(\gamma, \delta;\beta)\neq N(\gamma, \delta;\beta_0)$. Take $\sigma\in\mathfrak{S}_{B_\delta}$ which sends $\beta_0$ to $\beta$: $\sigma(\beta_0)=\beta$. Then $1\times\sigma\in H'$ sends $\bar{J}'_{\delta,\beta_0}$ to $\bar{J}'_{\delta,\sigma(\beta_0)}=\bar{J}'_{\delta,\beta}$. On the other hand, the numbers of elements in $\bar{J}'_{\delta,\beta_0}$ and in $\bar{J}'_{\delta,\beta}$ coming from $N_\gamma$ are different from each other. Indeed, they are respectively equal to $n(\gamma)N(\gamma, \delta;\beta_0)$ and $n(\gamma)N(\gamma, \delta;\beta)$. Therefore, under $1\times\sigma$, $N_\gamma\cap\bar{J}'_{\delta,\beta_0}$ can not be sent to $N_\gamma\cap\bar{J}'_{\delta,\beta}$ bijectively. This means that $1\times\sigma\notin H$, whence $H'\not\subset H$. Thus, if $H'\subset H$, we have $N(\gamma, \delta;\beta)=N(\gamma, \delta;\beta_0)$ $(=N(\gamma, \delta)$

(put)) for any $\beta\in B_\delta$.

Now put

$$A_\gamma^\delta=\bigcup_\beta A_\gamma^\delta(\beta) \qquad (\beta\in B_\delta). \tag{3.8}$$

Then we get a partition $A_\gamma=\bigcup_{\delta\in\Delta}A_\gamma^\delta$ of $A_\gamma$ for which $|A_\gamma^\delta|=\infty$ or 0, and in turn, for non-empty $A_\gamma^\delta$, (3.8) gives its partition into infinite number of subsets $A_\gamma^\delta(\beta)$ with exactly $N(\gamma, \delta)$ elements. Thus we have proved that the condition (R2) holds.

STEP 6. Fix $\delta\in\Delta$ and $\beta_0\in B_\delta$, and put

$$\Gamma_\delta=\{\gamma\in\Gamma\,;\ A_\gamma^\delta(\beta_0)\neq\emptyset\}. \tag{3.9}$$

Then, for any $\beta\in B_\delta$, we have

$$\bar{J}'_{\delta,\beta}=\bigcup_\gamma\bigcup_\alpha \bar{J}_{\gamma,\alpha} \qquad (\gamma\in\Gamma_\delta,\ \alpha\in A_\gamma^\delta(\beta)). \tag{3.10}$$

Note that

$$n(\delta)=\textstyle\sum_\gamma n(\gamma)N(\gamma, \delta) \qquad (\gamma\in\Gamma_\delta). \tag{3.11}$$

Now fix numberings $\gamma_1, \gamma_2, \cdots, \gamma_m$, $m=|\Gamma_\delta|$, of elements of $\Gamma_\delta$, and $\alpha_1, \alpha_2, \cdots, \alpha_N$, $N=N(\gamma, \delta)$, of elements of $A_\gamma^\delta(\beta_0)$ for each $\gamma\in\Gamma_\delta$. Define

$$J^0_{\delta,\beta_0}=(J^{\gamma_1}_{\beta_0}, J^{\gamma_2}_{\beta_0}, \cdots, J^{\gamma_m}_{\beta_0}), \tag{3.12}$$

where, for every $\gamma\in\Gamma_\delta$,

$$J^\gamma_{\beta_0}=(J_{\gamma,\alpha_1}, J_{\gamma,\alpha_2}, \cdots, J_{\gamma,\alpha_N}). \tag{3.13}$$

Then there exists a unique $\xi_\delta\in\mathfrak{S}_{n(\delta)}$ such that

$$\xi_\delta\circ J'_{\delta,\beta_0}=J^0_{\delta,\beta_0}. \tag{3.14}$$

We know from Lemma 2.1 that $H(\mathfrak{J}'_\delta, T'_\delta)=H(\xi_\delta\circ\mathfrak{J}'_\delta, \xi_\delta\circ T'_\delta)$. On the other hand, take an arbitrary $\beta\in B_\delta$, and then $\sigma\in\mathfrak{S}_{B_\delta}$ such that $\sigma(\beta_0)=\beta$. Then the element $1\times\sigma$ in $H(\mathfrak{J}'_\delta, T'_\delta)=H(\xi_\delta\circ\mathfrak{J}'_\delta, \xi_\delta\circ T'_\delta)\subset H'$ sends $\xi_\delta\circ J'_{\delta,\beta_0}$ onto $\xi_\delta\circ J'_{\delta,\sigma(\beta_0)}=\xi_\delta\circ J'_{\delta,\beta}$ preserving the orders in both of them. Since $H'\subset H$ by assumption, $1\times\sigma$ must send each set of elements $\bar{J}_{\gamma,\alpha}$ in $J^0_{\delta,\beta_0}=\xi_\delta\circ J'_{\delta,\beta_0}$ bijectively onto a set $\bar{J}_{\gamma,\alpha'}$ in $\xi_\delta\circ J'_{\delta,\beta}$. This makes us possible to define a numbering $\alpha'_1, \alpha'_2, \cdots, \alpha'_N$ in $A_\gamma^\delta(\beta)$ corresponding to $\alpha_1, \alpha_2, \cdots, \alpha_N$ in $A_\gamma^\delta(\beta_0)$ by

$$(1\times\sigma)(\bar{J}_{\gamma,\alpha_j})=\bar{J}_{\gamma,\alpha'_j} \qquad (1\leqq j\leqq N=N(\gamma, \delta)). \tag{3.15}$$

Now put for $\beta\in B_\delta$,

$$J^0_{\delta,\beta}=(J^{\gamma_1}_\beta, J^{\gamma_2}_\beta, \cdots, J^{\gamma_m}_\beta), \qquad m=|\Gamma_\delta|, \tag{3.16}$$

where, for every $\gamma\in\Gamma_\delta$,

$$J^\gamma_\beta=(J_{\gamma,\alpha'_1}, J_{\gamma,\alpha'_2}, \cdots, J_{\gamma,\alpha'_N}), \qquad N=N(\gamma, \delta), \tag{3.16'}$$

and put $\mathfrak{J}^0_\delta=\{J^0_{\delta,\beta}\,;\ \beta\in B_\delta\}$ an infinite set of ordered $n(\delta)$-sets.

The relation (3.15) means that there exists an $s(\gamma, \beta, i)\in\mathfrak{S}_{n(\gamma)}$ such that, in $(1\times\sigma)J^0_{\delta,\beta_0}=(1\times\sigma)(\xi_\delta\circ J'_{\delta,\beta_0})=\xi_\delta\circ J'_{\delta,\beta}$ and in $J^0_{\delta,\beta}$, $\beta=\sigma(\beta_0)$, we have

$$(1\times\sigma)J_{\gamma,\alpha_i}=s(\gamma, \beta, i)\circ J_{\gamma,\alpha'_i} \qquad \text{(elementwise)}.$$

Since $1\times\sigma\in H'\subset H$ by assumption, we get $s(\gamma, \beta, i)\in T_\gamma$. Put

$$s(\beta)=(s(\gamma_1, \beta), s(\gamma_2, \beta), \cdots, s(\gamma_m, \beta))$$

with

$$s(\gamma_j, \beta)=(s(\gamma_j, \beta, 1), s(\gamma_j, \beta, 2), \cdots, s(\gamma_j, \beta, N_j))\in\prod_{1\leq k\leq N_j} T_k,$$

where $T_k=T_{\gamma_j}$, $N_j=N(\gamma_j, \delta)$. Then we have proved that

$$\xi_\delta\circ J'_{\delta,\beta}=s(\beta)\circ J^0_{\delta,\beta}. \tag{3.17}$$

This means that $\xi_\delta\circ\mathfrak{J}'_\delta$ is equivalent to $\mathfrak{J}^0_\delta=\{J^0_{\delta,\beta}\,;\,\beta\in B_\delta\}$ under

$$T'(\delta)=(T_{\gamma_1})^{N_1}\times(T_{\gamma_2})^{N_2}\times\cdots\times(T_{\gamma_m})^{N_m}, \tag{3.18}$$

with $N_j=N(\gamma_j, \delta)$. This group $T'(\delta)$ is imbedded canonically into $\mathfrak{S}_{n(\delta)}$ according to (3.16)-(3.16′).

If we take an arbitrary numbering $\alpha'_1, \alpha'_2, \cdots, \alpha'_{N_j}$ of elements in $A^\delta_{\gamma_j}(\beta)$, then the relation (3.15) does not necessarily hold, and therefore $\xi_\delta\circ\mathfrak{J}'_\delta$ is equivalent to $\mathfrak{J}^0_\delta$ under the bigger group

$$T(\delta)=\prod_{1\leq i\leq m}((T_{\gamma_i})^{N_i}\rtimes\mathfrak{S}_{N_i})\supset T'(\delta), \tag{3.19}$$

where $T(\delta)$ is imbedded into $\mathfrak{S}_{n(\delta)}$ according to (3.16)-(3.16′). This gives the condition (I2i) in the theorem.

STEP 7. We see as in Lemma 2.1 that $H'=H^{\mathfrak{b}'}$ is also expressed as $H^{\mathfrak{b}^0}$ with

$$\mathfrak{b}^0=\{F', (\xi_\delta\circ\mathfrak{J}'_\delta, \xi_\delta\circ T'_\delta)\,;\,\delta\in\Delta\}.$$

Then, since $H'\subset H=H^{\mathfrak{b}}$, we see from (3.16)-(3.17) and the last statement in Step 6 that the condition (I2ii) holds:

$$\xi_\delta\circ T'_\delta\subset T(\delta). \tag{3.20}$$

Thus the proof of the necessity is now complete, and so is the proof of Theorem 3.2.

### 3.4. Conjugacy between standard subgroups

We can deduce from Lemma 3.1 and Theorem 3.2 a property of our parametrization by $\mathfrak{B}$ of standard subgroups and a conjugacy criterion for these subgroups.

**Theorem 3.3.** *Let* $H=H^{\mathfrak{b}}$, $H'=H^{\mathfrak{b}'}$ *be two subgroups in* $\mathfrak{A}$ *with* $\mathfrak{b}, \mathfrak{b}'\in\mathfrak{B}$ *in* (3.1), (3.3).

(i) $H=H'$ *if and only if the following conditions* (E1) *and* (E2) *hold for* $\mathfrak{b}, \mathfrak{b}'$:

(E1) $F=F'$;

(E2) *there exist* (1) *a bijection* $\kappa$ *of* $\Gamma$ *onto* $\Delta$, *and* (2) *for every* $\gamma\in\Gamma$, *an element*

$\xi_\gamma \in \mathfrak{S}_{n(\gamma)}$, *such that* $n(\gamma)=n(\delta)$ *with* $\delta=\kappa(\gamma)\in\Delta$ *and* $\xi_\gamma \circ T_\gamma = T'_\delta$, *and that* $\xi_\gamma \circ \mathfrak{J}_\gamma$ *is equivalent to* $\mathfrak{J}'_\delta$ *under* $\xi_\gamma \circ T_\gamma = T'_\delta$.

(ii) *$H$ is conjugate to $H'$ under $G$ if and only if the following conditions* (C1) *and* (C2) *hold for* $\mathfrak{b}$ *and* $\mathfrak{b}'$:

(C1) $|F|=|F'|$;

(C2) *there exists* (1) *a bijection* $\kappa$ *of* $\Gamma$ *onto* $\Delta$, *and* (2) *for every* $\gamma\in\Gamma$, *an element* $\xi_\gamma \in \mathfrak{S}_{n(\gamma)}$, *such that* $n(\gamma)=n(\delta)$ *with* $\delta=\kappa(\gamma)$ *and* $\xi_\gamma \circ T_\gamma = T'_\delta$ *for* $\gamma\in\Gamma$ *except a finite number of* $\gamma$, *and that, for exceptional* $\gamma$, *after removing the same finite number of ordered* $n(\gamma)$*-sets* (*here* $n(\gamma)=n(\delta)$) *from each of* $\xi_\gamma \circ \mathfrak{J}_\gamma$ *and* $\mathfrak{J}'_\delta$, *they are equivalent to each other under* $\xi_\gamma \circ T_\gamma = T'_\delta$.

**Note 3.4.** Put $\mathfrak{b}^0=\}F, (\xi_\gamma\circ\mathfrak{J}_\gamma, \xi_\gamma\circ T_\gamma); \gamma\in\Gamma\}$. Then, by Theorem 3.3(i) above, we have $H=H^{\mathfrak{b}}=H^{\mathfrak{b}^0}$. Replace $\mathfrak{b}$ by $\mathfrak{b}^0$. Then (C1), (C2) in Theorem 3.3(ii) are rewritten in a single form as

(C12) *there exists a bijection* $\kappa$ *of* $\Gamma$ *onto* $\Delta$ *such that, for* $\gamma\in\Gamma$, *put* $\delta=\kappa(\gamma)\in\Delta$, *then,* $n(\gamma)=n(\delta)$, $T_\gamma=T'_\delta$, *and* $\mathfrak{J}_\gamma$ *and* $\mathfrak{J}'_\delta$ *are equivalent under* $T_\gamma=T'_\delta$ *after removing the same finite number* (*zero for almost all* $\gamma$) *of* $n(\gamma)$*-sets* ($=n(\delta)$*-sets*) *from both of* $\mathfrak{J}_\gamma$, $\mathfrak{J}'_\delta$.

The condition (C1), or $|F|=|F'|$, follows from (C12) automatically.

### 3.5. The property (GRP1) for $\mathfrak{A}$

Now let us prove a property of $H=H^{\mathfrak{b}}$ in $\mathfrak{A}$ which consists a principal part of (GRP1) and plays a crucial role in proving the irreducibility of representations of $G$ induced from $H$.

**Theorem 3.5.** *Let $H$ be a standard subgroup of $G=\mathfrak{S}_\infty$ in $\mathfrak{A}$ and $g\in G$. If $gHg^{-1}\subset H$, then we have $g\in H$. In particular, the normalizer $N_G(H)$ of $H$ in $G$ coincides with $H$ itself.*

*Proof.* Take $\mathfrak{b}=\{F, (\mathfrak{J}_\gamma, T_\gamma); \gamma\in\Gamma\}\in\mathfrak{B}$ such that $H=H^{\mathfrak{b}}$. Then $gHg^{-1}=H^{g\mathfrak{b}}$ with ${}^g\mathfrak{b}=\{{}^gF, ({}^g\mathfrak{J}_\gamma, T_\gamma); \gamma\in\Gamma\}$. Note that, in ${}^g\mathfrak{J}_\gamma=\{{}^gJ_{\gamma,\alpha}; \alpha\in A_\gamma\}$, ${}^gJ_{\gamma,\alpha}=J_{\gamma,\alpha}$ for almost all $\alpha\in A_\gamma$. Taking $gHg^{-1}=H^{g\mathfrak{b}}$ as $H'$, we apply Theorm 3.2. Then the necessary and sufficient conditions (I1), (I2) for $H'\subset H$ say that (1) ${}^gF=F$, and (2) if ${}^gJ_{\gamma,\alpha}\neq J_{\gamma,\alpha}$, then ${}^gJ_{\gamma,\alpha}=t_\alpha\circ J_{\gamma,\sigma(\alpha)}$ with some $t_\alpha\in T_\gamma$ and $\sigma\in\mathfrak{S}_{A_\gamma}$. Since the total number of $(\gamma, \alpha)$ for which ${}^gJ_{\gamma,\alpha}\neq J_{\gamma,\alpha}$ is finite, we see that $g\in H$. Thus we have proved that $gHg^{-1}\subset H$ means $g\in H$ whence $gHg^{-1}=H$, and therefore we get $N_G(H)=H$ as a result.

Q. E. D.

**Note 3.6.** By this theorem, every $H\in\mathfrak{A}$ satisfies the condition (ii) in (GRP1) in § 1.1.4.

### 3.6. The properties (GRP1) and (GRP2) for $\mathfrak{A}$

Here we prove a property of the set $\mathfrak{A}$ of subgroups $H$, which plays an important role in studying the equivalency among representations of $G$ induced from $H$'s. Let

$H=H^{\mathfrak{b}}\in\mathfrak{A}$ with $\mathfrak{b}=\{F, (\mathfrak{J}_{\gamma}, T_{\gamma});\ \gamma\in\Gamma\}$ in $\mathfrak{B}$. Then by definition,

$$H=\mathfrak{S}_F\times\prod_{\gamma\in\Gamma}' H(\mathfrak{J}_{\gamma}, T_{\gamma}), \qquad H(\mathfrak{J}_{\gamma}, T_{\gamma})=D(\mathfrak{J}_{\gamma}, T_{\gamma})\rtimes\mathfrak{S}_{A_{\gamma}}. \tag{2.21}$$

Let us define subgroups $H_D$, $H_{\mathfrak{S}}$ of $H$ as

$$H_D=\mathfrak{S}_F\times\prod_{\gamma\in\Gamma}' D(\mathfrak{J}_{\gamma}, T_{\gamma}), \qquad H_{\mathfrak{S}}=\{1\}\times\prod_{\gamma\in\Gamma}'(\{1\}\times\mathfrak{S}_{A_{\gamma}}), \tag{3.22}$$

which we call the $D$-part and the $\mathfrak{S}$-part of $H$ respectively.

**Theorem 3.7.** *Let $H$, $H'$ be two standard subgroups in $\mathfrak{A}$, and let $g\in G$. Assume that $[H: H\cap gH'g^{-1}]<\infty$ and $[gH'g^{-1}: H\cap gH'g^{-1}]<\infty$. Then $H=gH'g^{-1}$.*

*Proof.* Since $gH'g^{-1}$ belongs to $\mathfrak{A}$ again, we may and do assume from the beginning that $g=e$, the identity element of $G$. Let us assume that $H'\not\subset H$. Then, checking carefully each step of the proof of Theorem 3.2, we see that in each step there exist an infinite number of elements $h'\in H'$ which belong to different right cosets of $H$ (hence of $H'\cap H$). In fact, in the cases of Step 1 to Step 5, there exist infinite number of such elements of the form $1\times\sigma$ in $\{1\}\times\mathfrak{S}_{B_{\delta}}\subset H(\mathfrak{J}'_{\delta}, T'_{\delta})\subset H'_{\mathfrak{S}}$ for some $\delta\in\Delta$. In Steps 6 and 7, we get such elements $h'$ from the $D$-part $H'_D$ of $H'$. Q.E.D.

The above theorem says that the condition (GRP2) holds for $\mathfrak{A}$, and therefore so does (i) in (GRP1). Thus, together with Theorem 3.5, we get the following

**Theorem 3.8.** *For the set $\mathfrak{A}=\{H^{\mathfrak{b}};\ \mathfrak{b}\in\mathfrak{B}\}$ of standard subgroups of $G=\mathfrak{S}_{\infty}$, there hold the conditions* (GRP1) *and* (GRP2) *in* § 1.1.4.

**Remark 3.9.** When we treat representations of $G$ induced from finite-dimensional irreducible representations of $H\in\mathfrak{A}$, Theorem 3.8 is sufficient for us because for the set $\mathfrak{R}_f$ of such representations of $H$'s, there holds the conditions (REP), as is remarked in § 1.1.4, and we can apply Corollary 1.4, getting Theorem 5.1. Note that such a representation of $H$ reduces to a character on the infinite part $H_{\infty}$ of $H$ (by Lemma 4.2).

When we study induced representations from infinite-dimensional representations of $H$, the situation is not simple. In particular, the condition (REP) does not hold in general for $\{\pi_1, \pi_2\}$, $\pi_i\in\mathfrak{R}$. These cases will be studied from § 6 on. We prove the irreducibility in Theorem 7.1 by appealing to the boundedness conditions ($B_x$) and ($C_x$) to determine the dimension of the space of intertwining operators, and as a result we know that (REP) holds at least for $\pi_1=\pi_2=\pi$. To study the equivalence relation among the standard induced representations of $G$, we can not appeal to (REP) but apply the results in [DG] essentially. Thus we come to our criterion of equivalence in Theorem 8.9. In between the commutativity of two induced processes (in Theorem 1.9) plays an important role.

## § 4. Finite-dimensional representations of $H\in\mathfrak{A}$

Let $H=H^{\mathfrak{b}}\in\mathfrak{A}$ with $\mathfrak{b}\in\mathfrak{B}$ in (3.1). Take a finite-dimensional, irreducible unitary representation (=IUR) $\pi$ of $H$. Then, according to the direct product decomposition

$$(4.1)\qquad H=H_f H_\infty \quad \text{with } H_f=\mathfrak{S}_F,\ H_\infty={\prod}'_{\gamma\in\Gamma} H_\gamma,$$

where $H_\gamma=H(\mathfrak{J}_\gamma, T_\gamma)=D(\mathfrak{J}_\gamma, T_\gamma)\rtimes\mathfrak{S}_{A_\gamma}\cong\mathfrak{S}_{A_\gamma}(T_\gamma)$, $\pi$ is a tensor product of representations $\pi_F$ of $\mathfrak{S}_F$ and $\pi_\gamma$ of $H_\gamma$ $(\gamma\in\Gamma)$, of which almost all must be one-dimensional (i.e., $\pi_\gamma$ is a character). We denote this as

$$(4.2)\qquad \pi=\pi_F\otimes(\otimes_{\gamma\in\Gamma}\pi_\gamma).$$

Let us study $\pi_\gamma$ of $H_\gamma$, isomorphic to the wreath product $\mathfrak{S}_{A_\gamma}(T_\gamma)$ of $T_\gamma$ with $\mathfrak{S}_{A_\gamma}$. Since $\mathfrak{S}_{A_\gamma}\cong\mathfrak{S}_\infty$, we first give the following

**Lemma 4.1.** *A finite-dimensional irreducible representation of $\mathfrak{S}_\infty$ is necessarily a character, and it is equal to the trivial character* **1** *or to the sign character* sgn : sgn$(\sigma)$ $=1$ *or* $-1$ *according as $\sigma\in\mathfrak{S}_\infty$ is even or odd.*

*Proof.* Let $E$ be the kernel of the representation $\rho$, which is normal in $\mathfrak{S}_\infty$. We see easily that $E\neq\{1\}$. For any $n$, $E\cap\mathfrak{S}_n$ is normal in $\mathfrak{S}_n$. Therefore $E\cap\mathfrak{S}_n=\{1\}$, $\mathfrak{A}_n$ or $\mathfrak{S}_n$ for $n>4$, where $\mathfrak{A}_n$ denotes the alternating group of order $n$. From this, the non-trivial normal subgroup $E$ should be equal to $\mathfrak{A}_\infty$ or $\mathfrak{S}_\infty$. Q.E.D.

Now proceed to $H_\gamma$, then we have the following

**Lemma 4.2.** *Let $|A_\gamma|=\infty$, and $\pi'$ be a finite-dimensional irreducible representation of $H(\mathfrak{J}_\gamma, T_\gamma)=D(\mathfrak{J}_\gamma, T_\gamma)\rtimes\mathfrak{S}_{A_\gamma}\cong\mathfrak{S}_\infty(T_\gamma)$. Then it is necessarily a character, and is given as*

$$(4.3)\qquad \pi'((t_\alpha)\cdot\sigma)=\chi_{T_\gamma}\Big(\prod_{\alpha\in A_\gamma}\varphi_\alpha^{-1}(t_\alpha)\Big)\cdot\chi_{A_\gamma}(\sigma)$$

*for $(t_\alpha)=(t_\alpha)_{\alpha\in A_\gamma}\in D(\mathfrak{J}_\gamma, T_\gamma)$ with $t_\alpha\in T_\alpha=\varphi_\alpha(T_\gamma)$, and $\sigma\in\mathfrak{S}_{A_\gamma}$, where $\chi_{T_\gamma}$ and $\chi_{A_\gamma}$ are characters of $T_\gamma\subset\mathfrak{S}_{n(\gamma)}$ and $\mathfrak{S}_{A_\gamma}\cong\mathfrak{S}_\infty$ respectively.*

*Proof.* Since the dimension of representation is finite, $\pi'|\mathfrak{S}_{A'}(T_\gamma)$ is irreducible for some finite $A'\subset A_\gamma$. Therefore $\pi'(h)$ is a scalar operator for any $h\in\mathfrak{S}_{A''}(T_\gamma)$ with $A''=A_\gamma\setminus A'$. Hence $\pi'(\sigma h\sigma^{-1})=\pi'(\sigma)\pi'(h)\pi'(\sigma)^{-1}$ is also a scalar operator for any $\sigma\in\mathfrak{S}_{A_\gamma}$. On the other hand, $\mathfrak{S}_{A_\gamma}(T_\gamma)$ is the union of $\sigma\cdot\mathfrak{S}_{A''}(T_\gamma)\cdot\sigma^{-1}$ over $\sigma\in\mathfrak{S}_{A_\gamma}$. This proves that $\pi'$ is itself a character.

The expression (4.3) follows from the definition of the product $\sigma\cdot(t_\alpha)\cdot\sigma^{-1}$. Q.E.D.

Thus we have proved the following

**Theorem 4.3.** *Let $\pi$ be a finite-dimensional irreducible representation of $H=H^{\mathfrak{b}}$ in $\mathfrak{A}$. Let the canonical decomposition of $H$ be as in* (4.1). *Then there exist* (1) *an irreducible representation $\pi_F$ of $\mathfrak{S}_F$, and* (2) *for every $\gamma\in\Gamma$, a character $\chi_{T_\gamma}$ of $T_\gamma\subset\mathfrak{S}_{n(\gamma)}$ and $\chi_{A_\gamma}$ of $\mathfrak{S}_{A_\gamma}$, such that*

$$\pi(\tau)=\pi_F(\tau) \quad \textit{for } \tau\in\mathfrak{S}_F\,,$$

$$\pi((t_\alpha)\cdot\sigma)=\chi_{T_\gamma}(\prod_{\alpha\in A_\gamma}\varphi_\alpha{}^{-1}(t_\alpha))\cdot\chi_{A_\gamma}(\sigma)\cdot I\,,$$

$$\textit{for } (t_\alpha)\cdot\sigma\in H(\mathfrak{J}_\gamma,\, T_\gamma)=D(\mathfrak{J}_\gamma,\, T_\gamma)\rtimes\mathfrak{S}_{A_\gamma}\,,$$

*where $I$ denotes the identity operator, and $\varphi_\alpha$ is as in* § 2.2.

*Conversely any datum $\{\pi_F, (\chi_{T_\gamma}, \chi_{A_\gamma});\ \gamma\in\Gamma\}$ gives a finite-dimensional IUR $\pi$ of $H$ as above.*

## § 5. Representations of $G$ induced from finite-dimensional IURs of $H$ in $\mathfrak{A}$

Let $\mathfrak{A}$ be as before the set of standard subgroups of $G=\mathfrak{S}_\infty$: $\mathfrak{A}=\{H^{\mathfrak{b}};\ \mathfrak{b}\in\mathfrak{B}\}$, where

$$\mathfrak{b}=\{F, (\mathfrak{J}_\gamma, T_\gamma);\ \gamma\in\Gamma\} \qquad \text{with } \mathfrak{J}_\gamma=\{J_{\gamma,\alpha};\ \alpha\in A_\gamma\},$$

$$H^{\mathfrak{b}}=\mathfrak{S}_F\times\textstyle\prod'_{\gamma\in\Gamma}H_\gamma \qquad \text{with } H_\gamma=D(\mathfrak{J}_\gamma,\, T_\gamma)\rtimes\mathfrak{S}_{A_\gamma}\,.$$

Further let $\mathfrak{R}_f$ be the set of all finite-dimensional IURs of $H$ in $\mathfrak{A}$. Then such a representation $\pi=\pi(\mathfrak{b}, Q_f)$ of $H=H^{\mathfrak{b}}\in\mathfrak{A}$ is given as in Theorem 4.3 by a datum

$$Q_f=\{\pi_F, (\chi_{T_\gamma}, \chi_{A_\gamma});\ \gamma\in\Gamma\}, \tag{5.1}$$

where $\pi_F$ is an IUR of $\mathfrak{S}_F$, and $\chi_{T_\gamma}$ and $\chi_{A_\gamma}$ are characters of $T_\gamma$ and $\mathfrak{S}_{A_\gamma}\cong\mathfrak{S}_\infty$ respectively.

As is remarked in § 1.1.4, the condition (REP) holds for $\mathfrak{R}_f$. Further we see from Theorem 3.8 that the set of subgroups $\mathfrak{A}$ satisfies the conditions (GRP1)-(GRP2). Hence we can apply Corollary 1.4 and get the following

**Theorme 5.1.** *Let $\mathfrak{A}$ be the set of standard subgroups of $G=\mathfrak{S}_\infty$, and $\mathfrak{R}_f$ the set of finite-dimensional IURs of $H\in\mathfrak{A}$.*

(i) *Induced representations $\rho(\mathfrak{b}, Q_f)=\mathrm{Ind}_H^G\pi(\mathfrak{b}, Q_f)$ are all irreducible.*

(ii) *Let $\rho=\mathrm{Ind}_H^G\pi$ and $\rho'=\mathrm{Ind}_{H'}^G\pi'$ be two such IURs of $G$. Then they are mutually equivalent if and only if there exists an $x\in G$ such that*

$$H'=H^x \quad \textit{and} \quad \pi'\cong\pi^x\,. \tag{5.2}$$

Let us rewrite the above conjugacy condition (5.2) by means of data $(\mathfrak{b}, Q_f)$ for $\pi$ and $(\mathfrak{b}', Q'_f)$ for $\pi'$, where

$$\begin{aligned} &\mathfrak{b}'=\{F', (\mathfrak{J}'_\delta, T'_\delta);\ \delta\in\Delta\} \qquad \text{with } \mathfrak{J}'_\delta=\{J'_{\delta,\beta};\ \beta\in B_\delta\},\\ &Q'_f=\{\pi'_{F'}, (\chi'_{T'_\delta}, \chi'_{B_\delta});\ \delta\in\Delta\}, \end{aligned} \tag{5.3}$$

with $\pi'_{F'}$ an IUR of $\mathfrak{S}_{F'}$, and $\chi'_{T'_\delta}$ and $\chi'_{B_\delta}$ characters of $T'_\delta$ and $\mathfrak{S}_{B_\delta}$ respectively.

We apply Theorem 3.3. Since $H=H^{\mathfrak{b}}$ and $H'=H^{\mathfrak{b}'}$ are conjugate, we have (a) $|F|=|F'|$, and (b) a bijection $\kappa$ of $\Gamma$ onto $\Delta$ and $\xi_\gamma\in\mathfrak{S}_{n(\gamma)}$ for every $\gamma\in\Gamma$ such that $n(\gamma)=n(\kappa(\gamma))$ and $\xi_\gamma\circ T_\gamma=T'_{\kappa(\gamma)}$, and that the condition in (C2) in Theorem 3.3(ii) holds. Moreover, taking into account Note 3.4, we have the following

**Lemma 5.2.** (i) *Assume that* $H'=x^{-1}Hx$ *for* $H', H\in\mathfrak{A}$. *Take* $\mathfrak{b}'\in\mathfrak{B}$ *such that* $H'=H^{\mathfrak{b}'}$, *then there exists an appropriate* $\mathfrak{b}\in\mathfrak{B}$ *giving the subgroup* $H$ *as* $H=H^{\mathfrak{b}}$ *such that* (1) $|F|=|F'|$, *and* (2) *under a bijection* $\kappa$ *of* $\Gamma$ *onto* $\Delta$, *we have, for* $\gamma\in\Gamma$ *and* $\delta=\kappa(\gamma)$, $n(\gamma)=n(\delta)$, $T_\gamma=T'_\delta$, *and* $\mathfrak{J}_\gamma$ *and* $\mathfrak{J}'_\delta$ *are mutually equivalent under* $T_\gamma=T'_\delta$ *after removing the same finite number (zero for almost all* $\gamma\in\Gamma$*) of ordered* $n(\gamma)$*-sets* ($=n(\delta)$*-sets) from both of* $\mathfrak{J}_\gamma$, $\mathfrak{J}'_\delta$.

*Conversely if* $\mathfrak{b}, \mathfrak{b}'\in\mathfrak{B}$ *satisfy the above conditions* (1) *and* (2), *then we have* $H^{\mathfrak{b}'}=x^{-1}H^{\mathfrak{b}}x$ *for some* $x\in G$.

(ii) *Assume* $H'=x^{-1}Hx$ *and take* $\mathfrak{b}$, $\mathfrak{b}'$ *as above. Then the representations* $\pi'(h')$ *and* $\pi(xh'x^{-1})$ ($h'\in H'$) *are mutually equivalent if and only if, in the data* $Q_f$ *and* $Q'_f$, *there hold that* (1) $\pi_F$ *and* $\pi'_{F'}$ *are mutually equivalent, both considered as representations of* $\mathfrak{S}_N$ *with* $N=|F|=|F'|$ *canonically, and* (2) $\chi_{T_\gamma}=\chi'_{T'_\delta}$, $\chi_{A_\gamma}=\chi'_{B_\delta}$ ($=\mathbf{1}$ *or* sgn) *for* $\gamma\in\Gamma$, $\delta=\kappa(\gamma)$.

Note that the statement (1) in (ii) above has well-defined meaning because $\mathfrak{S}_N\cong\mathrm{Int}(\mathfrak{S}_N)$ for $N\geqq 3$.

**Remark 5.3.** To get an IUR $\rho$ of $G=\mathfrak{S}_\infty$ of the above type, we can proceed as follows. First, take a finite symmetric group $\mathfrak{S}_N$, and any finite or countably infinite number of finite groups $\{T_\gamma;\gamma\in\Gamma\}$, and consider the restricted direct product group $K=\mathfrak{S}_N\times\prod'_{\gamma\in\Gamma}\mathfrak{S}_\infty(T_\gamma)$, where $\mathfrak{S}_\infty(T_\gamma)$ is the wreath product of $T_\gamma$ with $\mathfrak{S}_\infty$. Second, we take an IUR $\pi_N$ of $\mathfrak{S}_N$, and any set $\{(\chi_{T_\gamma},\chi_{\infty,\gamma});\gamma\in\Gamma\}$ of pairs of characters of $T_\gamma$ and $\mathfrak{S}_\infty$. Then define canonically an IUR of $K$ from them. Third, we take, for every $\gamma\in\Gamma$, a faithful permutation representation of $T_\gamma$, viz., an isomorphism $\tau_\gamma$ of $T_\gamma$ into a finite symmetric group $\mathfrak{S}_{n(\gamma)}$.

Now, take an injective isomorphism of $K$ into $\mathfrak{S}_\infty=\mathfrak{S}_{\boldsymbol{N}}$ by means of the system $\{\tau_\gamma;\gamma\in\Gamma\}$ and a two-step partition of $\boldsymbol{N}$ into ordered sets as $\{F,\mathfrak{J}_\gamma;\gamma\in\Gamma\}$, where $|F|=N$, and $\mathfrak{J}_\gamma=\{J_{\gamma,\alpha};\alpha\in A_\gamma\}$ is an infinite set of ordered $n(\gamma)$-sets $J_{\gamma,\alpha}$ of integers. Finally the representation of the image $H\subset G$ of $K$ obtained canonically under this imbedding is to be induced up to $G$.

In the succeeding sections, we shall treat the case where the system $\{\chi_{T_\gamma};\gamma\in\Gamma\}$ of characters is replaced by any system of IURs $\{\pi_{T_\gamma};\gamma\in\Gamma\}$ of finite groups $T_\gamma$. However, in this case, the condition (REP) does not hold in general, and so the situation becomes much more complicated.

**Remark 5.4.** It is worthwhile to note here the following. The author got a strong impression, during his study, on certain similarity between the standard subgroups $H=H^{\mathfrak{b}}$ (whose essential parts are the wreath products of type $\mathfrak{S}_\infty(T)$) for $G=\mathfrak{S}_\infty$ on the one hand, and the parabolic subgroups $P$ for a reductive Lie group $G^0$ on the other hand. This impression goes beyond the following rather formal similarities.

(i) First Similarity. In both cases, the normalizer in $G$ or in $G^0$ of such a subgroup coincides with itself, and further there is a similarity about the properties (GRP1) and (GRP2) in § 1.1.4.

(ii) Second Simirarity. This concerns the methods of construction of IURs of

$G$ and $G^0$ by induction from these subgroups. On the one hand, $H$ in $\mathfrak{A}$ is expressed as

$$(5.4)\qquad H=H_D\rtimes H_{\mathfrak{S}}\qquad \text{with } H_D=\mathfrak{S}_F\times\prod_{\gamma\in\Gamma}{}' D(\mathfrak{Z}_\gamma, T_\gamma),\ H_{\mathfrak{S}}=\prod_{\gamma\in\Gamma}{}'\mathfrak{S}_{A_\gamma},$$

and we take an IUR $\xi_D$ of $H_D$ and a character $\chi_{\mathfrak{S}}$ of $H_{\mathfrak{S}}$, and then construct an IUR $\rho$ of $G$ by

$$(5.5)\qquad \rho=\mathrm{Ind}_H^G(\xi_D\otimes\chi_{\mathfrak{S}}))\,.$$

On the other hand, let

$$(5.6)\qquad P=LN=L\ltimes N\quad \text{with } L \text{ a Levi subgroup, } N \text{ the unipotent radical of } P,$$

be a Langlands decomposition of $P$. Take an IUR $\xi_L$ of $L$ and construct

$$(5.7)\qquad \rho=\mathrm{Ind}_P^{G^0}(\xi_L\otimes\mathbf{1}_N)\,,$$

which gives in general an IUR of $G^0$.

(iii) The third point is not a similarity but a contrast between canonical decompositions (5.4) and (5.6). Eventhough $H_D$ and $H_{\mathfrak{S}}$ correspond to $L$ and $N$ respectively in the similarity between (5.4) and (5.6), we see that in (5.4) $H_D$ is normal whereas in (5.6) $N$ is normal.

From these observations, we ask if there is any general characterization of such subgroups as those in $\mathfrak{A}$, which is comparable to the characterization (or definition) for parabolic subgroups in a general setting, or to that for Cartan subgroups for any abstract groups due to Chevalley [2, Chap. VI, §4].

**Note 5.5.** In Obata's recent works [16], he utilized subgroups $H(\theta)$ of $\mathfrak{S}_\infty$ which are given as centralizers of special $\theta\in\mathrm{Aut}(\mathfrak{S}_\infty)$. These subgroups are more similar to Cartan subgroups of Chevalley than parabolic subgroups, at a first glance, but they are also a special subclass of our family $\mathfrak{A}$ of standard subgroups and are essentially equal to restricted direct products of wreath products $\mathfrak{S}_{A_\gamma}(T_\gamma)$ with $T_\gamma$ cyclic and $|A_\gamma|=\infty$. Let $\mathfrak{A}'$ be the set of all such $H(\theta)$'s and $\mathfrak{R}'$ that of elementary IURs of $H\in\mathfrak{A}'$. Then $\mathfrak{R}'$ consists of characters and so $\mathfrak{R}'\subset\mathfrak{R}_f$. In this case, the conditions (GRP1)-(GRP2) hold for $\mathfrak{A}'$, and the condition (REP) holds for $\mathfrak{R}'$, and so Corollary 1.4 can be applied. There does not appear the role of reference vectors.

## §6. A family $\mathfrak{R}$ of irreducible representations of subgroups in $\mathfrak{A}$

In §§4-5, we have studied finite-dimensional IURs of subgroups $H$ in $\mathfrak{A}$ of $G=\mathfrak{S}_\infty$, and their induced representations. These results are in a sense a preparatory step to treat infinite-dimensional IURs of standard subgroups $H$. In this section, we give a certain family $\mathfrak{R}$ of factorizable IURs of $H\in\mathfrak{A}$, which will be utilized to construct IURs of $G$. The reason why we take such a family $\mathfrak{R}$ will be explained at the end of this section.

**6.1. Standard subgroups of $G$**

Let $H=H^{\mathfrak{b}}$ be a standard subgroup of $G$ corresponding to $\mathfrak{b}\in\mathfrak{B}$:

$$\mathfrak{b}=\{F, (\mathfrak{J}_\gamma, T_\gamma);\ \gamma\in\Gamma\} \qquad \text{with } \mathfrak{J}_\gamma=\{J_{\gamma,\alpha};\ \alpha\in A_\gamma\}, \tag{6.1}$$

$$H=\mathfrak{S}_F\times\prod'_{\gamma\in\Gamma}H_\gamma, \qquad H_\gamma=H(\mathfrak{J}_\gamma, T_\gamma)\equiv D(\mathfrak{J}_\gamma, T_\gamma)\rtimes\mathfrak{S}_{A_\gamma}. \tag{6.2}$$

Note that $\mathfrak{b}$ satisfies the conditions (B1)-(B3) in § 2.4. Here the finite group $T_\gamma$ is assumed to be faithfully represented in $\mathfrak{S}_{n(\gamma)}$. The set $\mathfrak{J}_\gamma$ of ordered $n(\gamma)$-sets gives a datum to imbed the wreath product $\mathfrak{S}_{A_\gamma}(T_\gamma)$ into $G$ as follows.

For $\alpha\in A_\gamma$, let $J_{\gamma,\alpha}=(p_1, p_2, \cdots, p_n)$ with $n=n(\gamma)$, and let $\iota_\alpha$ be the correspondence of $J_{\gamma,\alpha}$ to $I_n=(1, 2, \cdots, n)$ such that $\iota_\alpha(p_j)=j$ $(1\leqq j\leqq n)$. By means of $\iota_\alpha$, $\mathfrak{S}_n$ and hence $T_\gamma$ is imbedded into $\mathfrak{S}_{\bar{J}_{\gamma,\alpha}}$ as

$$\varphi_\alpha: \mathfrak{S}_n\supset T_\gamma\ni t\longmapsto \varphi_\alpha(t)=\iota_\alpha^{-1}\circ t\circ\iota_\alpha\in\mathfrak{S}_{\bar{J}_{\alpha,\gamma}}. \tag{6.3}$$

Then by definition,

$$D(\mathfrak{J}_\gamma, T_\gamma)\equiv\prod'_{\alpha\in A_\gamma}T_\alpha \qquad \text{with } T_\alpha=\varphi_\alpha(T_\gamma), \tag{6.4}$$

giving the canonical imbedding

$$\psi_\gamma: \mathfrak{S}_{A_\gamma}(T_\gamma)=D_{A_\gamma}(T_\gamma)\rtimes\mathfrak{S}_{A_\gamma}\longrightarrow H_\gamma=D(\mathfrak{J}_\gamma, T_\gamma)\rtimes\mathfrak{S}_{A_\gamma}\subset G, \tag{6.5}$$

and thus $H$ is the image of a saturated (in the sense of (B3)) imbedding $\Psi=\psi_F\otimes(\otimes_{\gamma\in\Gamma}\psi_\gamma)$ into $G$ of a restricted direct product group $\underline{H}$ given as

$$\underline{H}=\mathfrak{S}_N\times\prod'_{\gamma\in\Gamma}\mathfrak{S}_{A_\gamma}(T_\gamma) \qquad \text{with } N=|F|, \tag{6.6}$$

where $\psi_F$ denotes the natural isomorphism of $\mathfrak{S}_N$ onto $\mathfrak{S}_F$ according to the order in $F$.

Through this canonical imbedding $\Psi$, IURs of $H$ may be given as those of $\underline{H}$ when it is convenient to do so.

**6.2. A family $\mathfrak{N}$ of IURs of subgroups in $\mathfrak{A}$**

Now let us give a family $\mathfrak{N}_H$ of IURs of each $H\in\mathfrak{A}$. Then a family $\mathfrak{N}$ is defined as the union of $\mathfrak{N}_H$ over $H\in\mathfrak{A}$. We give a $\pi$ in $\mathfrak{N}_H$ as a factorizable one:

$$\pi=\pi_F\otimes(\otimes^b_{\gamma\in\Gamma}\Pi_\gamma), \tag{6.7}$$

where $\pi_F$ is an IUR of $\mathfrak{S}_F$, and for each $\gamma\in\Gamma$, $\Pi_\gamma$ is an IUR of $H_\gamma$, and $b=(b_\gamma)$, $b_\gamma\in V(\Pi_\gamma)$, $\|b_\gamma\|=1$, is the reference vector for the (possibly) infinite tensor product of $\Pi_\gamma$'s.

In turn, each $\Pi_\gamma$ of $H_\gamma$ is given by a datum $(\rho^\gamma_{T_\gamma}, \chi_{A_\gamma}, a(\gamma))$ with $\rho^\gamma_{T_\gamma}$ an IUR of $T_\gamma$, $\chi_{A_\gamma}$ a character of $\mathfrak{S}_{A_\gamma}$ and $a(\gamma)=(a_{\gamma,\alpha})_{\alpha\in A_\gamma}$ a reference vector. We put

$$\Pi_\gamma=\Pi(Q_\gamma)\circ\psi_\gamma^{-1} \tag{6.8}$$

by transferring through $\psi_\gamma: \mathfrak{S}_{A_\gamma}(T_\gamma)\rightarrow H_\gamma$, the IUR $\Pi(Q_\gamma)$ of $\mathfrak{S}_{A_\gamma}(T_\gamma)$ in § 1.2.3 corresponding to the datum

$$Q_\gamma=(A_\gamma, \rho^\gamma_{T_\gamma}, \chi_\gamma, a(\gamma)) \qquad \text{with } \chi_\gamma=\chi_{A_\gamma}. \tag{6.9}$$

More exactly, for $(t_\alpha)_{\alpha\in A}\cdot\sigma\in D(\mathfrak{J}_\gamma, T_\gamma)\rtimes\mathfrak{S}_{A_\gamma}$ with $t_\alpha\in T_\alpha$, $\sigma\in\mathfrak{S}_{A_\gamma}$, (6.10) $$\Pi_\gamma((t_\alpha)\cdot\sigma)=(\pi_{(\gamma)}((t_\alpha))\cdot I_\sigma)\otimes\chi_{A_\gamma}(\sigma) \qquad \text{with } \pi_{(\gamma)}((t_\alpha))=\otimes_{\alpha\in A_\gamma}^{a(\gamma)}\pi_\alpha(t_\alpha),$$

where $I_\sigma$ is the operator permuting the factors (of decomposable vectors) on $V(\pi_{(\gamma)})=\otimes_{\alpha\in A_\gamma}^{a(\gamma)}V(\pi_\alpha)$, and $\pi_\alpha(t_\alpha)=\rho_{T_\gamma}^\gamma(\varphi_\alpha^{-1}(t_\alpha))$. The reference vector $a(\gamma)$ for the infinite tensor product is given as

(6.11) $$a(\gamma)=(a_{\gamma,\alpha})_{\alpha\in A_\gamma} \qquad \text{with } a_{\gamma,\alpha}\in V(\pi_\alpha)=V(\rho_{T_\gamma}^\gamma),\ \|a_{\gamma,\alpha}\|=1.$$

In the above sense, each element $\pi$ in $\mathfrak{R}_H$ is given by the datum $(\mathfrak{b}, Q)$ and is denoted by $\pi(\mathfrak{b}, Q)$. Here $\mathfrak{b}$ in (6.1) determines $H=H^{\mathfrak{b}}$ together with the imbedding $\Psi$ of $\underline{H}$ into $\mathfrak{S}_\infty$ with image $H$, and

(6.12) $$Q=\{\pi_F,\ (\rho_{T_\gamma}^\gamma,\ \chi_{A_\gamma},\ a(\gamma))_{\gamma\in\Gamma},\ b=(b_\gamma)_{\gamma\in\Gamma}\}$$

determines an IUR of $H$ as in (6.7)-(6.11). The set of all possible $Q$ corresponding to $\mathfrak{b}$ is denoted by $\mathfrak{Q}(\mathfrak{b})$, whence $\mathfrak{R}_H=\{\pi(\mathfrak{b}, Q);\ Q\in\mathfrak{Q}(\mathfrak{b})\}$.

The set $\mathfrak{R}$ of IURs of $H$'s in $\mathfrak{A}$ is given as

(6.13) $$\mathfrak{R}=\bigcup_{H\in\mathfrak{A}}\mathfrak{R}_H=\{\pi(\mathfrak{b}, Q);\ \mathfrak{b}\in\mathfrak{B},\ Q\in\mathfrak{Q}(\mathfrak{b})\}.$$

### 6.3. On the choice of families $\mathfrak{A}$ of subgroups and $\mathfrak{R}$ of IURs

Here we give some remarks on the choice of the families $\mathfrak{A}$ and $\mathfrak{R}$, which evoke certain open problems.

**Remark 6.1.** The choice of the family $\mathfrak{A}$ of standard subgroups comes principally from the demand for properties (GRP1) and (GRP2). Especially the reason why $|A_\gamma|=\infty$ for every $\gamma\in\Gamma$, and why we take the biggest $\mathfrak{S}_F$ but not $T\subsetneqq\mathfrak{S}_F$ for the finite part, is the following. If we are in the contrary case, then in inducing up to $G=\mathfrak{S}_\infty$, we have anyhow an intermediate step such as $\mathrm{Ind}_{T'}^{\mathfrak{S}_{F'}}\tau$ with $\tau$ an IUR of $T'$, a finite $F'\supset F$ and a $T'\subset\mathfrak{S}_{F'}$. In such a situation, it is a shortcut to take $\mathfrak{S}_{F'}$ and an IUR $\pi_{F'}$ of $\mathfrak{S}_{F'}$ from the beginning. Here we also have to take into account Lemma DG3.16. Moreover it is worthwhile to recall the condition (DIFfin) for irreducibility in Theorem DG3.6.

**Remark 6.2.** To construct an IUR $\pi$ of $H$ in $\mathfrak{R}$, we take for every factor $H_\gamma$ of $H=\mathfrak{S}_F\times\prod'_{\gamma\in\Gamma}H_\gamma$ a factorizable IUR $\pi_{(\gamma)}=\otimes_{\alpha\in A_\gamma}^{a(\gamma)}\pi_\alpha$ of $D(\mathfrak{J}_\gamma, T_\gamma)\subset H_\gamma$. The reason why, is that we know little about infactorizable IUR $\pi_\gamma$ of the restricted direct product group $\prod'_{\alpha\in A_\gamma}T_\alpha$, especially nothing about the stationary subgroup $\mathfrak{S}_{A_\gamma}(\pi_\gamma)\subset\mathfrak{S}_{A_\gamma}$ and its second cohomology group (cf. Appendix in [DG]).

**Remark 6.3.** For each component $H_\gamma=D(\mathfrak{J}_\gamma, T_\gamma)\rtimes\mathfrak{S}_{A_\gamma}$ of $H$, we take only one IUR $\rho_{T_\gamma}^\gamma$ of $T_\gamma$. Contrary to this, let us take some number of IURs $\rho^m$ of $T_\gamma$ not necssarily different from each other but indexed by $m\in I_\gamma$, and construct a standard IUR of $H_\gamma$, starting from them. This means the following. We divide $\mathfrak{J}_\gamma$ into subsets $\mathfrak{J}^m=\{J_{\gamma,\alpha};\ \alpha\in A^m\}$, $\coprod_{m\in I_\gamma}A^m=A_\gamma$, and define for each $m$ an elementary IUR of $D^m=D(\mathfrak{J}^m, T_\gamma)$ as

$$\pi^m((t_\alpha)_{\alpha\in A^m})=\otimes_{\alpha\in A^m}^{c^m}\pi_\alpha(t_\alpha)$$

with $\pi_\alpha(t_\alpha)=\rho^m(\varphi_\alpha{}^{-1}(t_\alpha))$ for $\alpha\in A^m$ and a reference vector $c_m=(c_{m,\alpha})_{\alpha\in A^m}$. Then define an IUR $\pi_{(\gamma)}$ of $D(\mathfrak{J}_\gamma, T_\gamma)=\prod'_{m\in I_\gamma}D^m$ as a tensor product of $\pi^m$'s with respect to a reference vector $d=(d_m)_{m\in I_\gamma}$. At last, taking a subgroup $\prod'_{m\in I_\gamma}\mathfrak{S}_{A^m}\subset\mathfrak{S}_{A_\gamma}(\pi_{(\gamma)})$ and its character, we get an IUR of $H_\gamma\cong\mathfrak{S}_{A_\gamma}(T_\gamma)$ by the standard inducing method in Theorem DG3.6, that is, we induce an IUR $\otimes^d_{m\in I_\gamma}(\pi^m\otimes\chi^m)$ with characters $\chi^m$ of $\mathfrak{S}_{A^m}$ from the subgroup $H'_\gamma\equiv\prod'_{m\in I_\gamma}(D(\mathfrak{J}^m, T_\gamma)\rtimes\mathfrak{S}_{A^m})\cong\prod'_{m\in I_\gamma}\mathfrak{S}_{A^m}(T_\gamma)$ to $H_\gamma$.

In case $|I_\gamma|<\infty$, since the role of the reference vector $d$ is superfluous, it is clear that we can start from the subgroup $H'_\gamma$ instead of $H_\gamma$, and take as the starting point the datum $\{A^m, (\mathfrak{J}^m, T_\gamma), \rho^m\}_{m\in I_\gamma}$ instead of $\{A_\gamma, (\mathfrak{J}_\gamma, T_\gamma), \rho^m\text{'s}\}$, $A_\gamma=\coprod_{m\in I_\gamma}A^m$. Thus the situation is reduced to our standard case. In this connection, cf. also Theorem DG3.6 and Examples in §DG3.9.

However, in case $|I_\gamma|=\infty$, we see that the role of the reference vector $d$ is essential in general and can not be neglected. Therefore, repeating this inducing process, we get, at least formally, an unlimited hierarchy of induced representations of $\mathfrak{S}_\infty$, counting the number of accumulation or steps of essential necessity of reference vectors. In our construction of $\rho(\mathfrak{b}, Q)$, this accumulation is two-fold: first one is $a(\gamma)$'s and second one is $b$. Whereas in the above discussion the accumulation is three-fold $c_m$'s and $d$'s (both depending on $\gamma\in\Gamma$) and $b=(b_\gamma)_{\gamma\in\Gamma}$.

We will return to this point in another occasion.

Another point left to discuss is the possibility to start with not necessarily irreducible, but cyclic, representations $\rho^{I}_{T_\gamma}$'s.

## §7. Irreducible unitary representations of $\mathfrak{S}$ induced from those in $\mathfrak{R}$

### 7.1. Irreducibility of induced representations

Let $H=H^{\mathfrak{b}}$ be a standard subgroup with $\mathfrak{b}\in\mathfrak{B}$ in (6.1), and $\pi(\mathfrak{b}, Q)\in\mathfrak{R}$ an IUR of $H$ with the parameter $Q$ in (6.12). We define a unitary representation of $G=\mathfrak{S}_\infty$ as

$$\rho(\mathfrak{b}, Q)=\mathrm{Ind}_H^G\pi(\mathfrak{b}, Q)\,. \tag{7.1}$$

Here we give one of our main results as

**Theorem 7.1.** *For any IUR $\pi(\mathfrak{b}, Q)\in\mathfrak{R}$ of an $H=H^{\mathfrak{b}}\in\mathfrak{A}$, the induced representation $\rho(\mathfrak{b}, Q)=\mathrm{Ind}_H^G\pi(\mathfrak{b}, Q)$ is always irreducible.*

Before proving the theorem we explain the meaning of the set of IURs of $G$ thus obtained.

**Nete 7.2.** To get such IURs, we can start with an arbitrary but at most countable set of finite groups $\{E_\gamma\}_{\gamma\in\Gamma}$, not necessarily mutually non-isomorphic. First take any faithful permutation representation $E_\gamma\to\mathfrak{S}_{n(\gamma)}$ for each $E_\gamma$. Then starting from this datum, we can choose (in many ways) a standard subgroup $H=H^{\mathfrak{b}}\in\mathfrak{A}$ as follows. Take $\mathfrak{J}_\gamma=\{J_{\gamma,\alpha}\,;\ \alpha\in A_\gamma\}$, $|A_\gamma|=\infty$, a set of ordered $n(\gamma)$-sets. We assume that $\bar{J}_{\gamma,\alpha}$ ($\gamma\in\Gamma$, $\alpha\in A_\gamma$) are mutually disjoint and $F\equiv\boldsymbol{N}\setminus(\coprod_{\gamma\in\Gamma}\mathrm{supp}\,(\mathfrak{J}_\gamma))$ is finite, where $\mathrm{supp}\,(\mathfrak{J}_\gamma)=\bigcup_{\alpha\in A_\gamma}\bar{J}_{\gamma,\alpha}$. Then we get a $\mathfrak{b}\in\mathfrak{B}$ by $\mathfrak{b}=\{F, (\mathfrak{J}_\gamma, T_\gamma)\,;\ \gamma\in\Gamma\}$, and so $H=H^{\mathfrak{b}}\in\mathfrak{A}$, where

$T_\gamma$ is the image of $E_\gamma$ in $\mathfrak{S}_{n(\gamma)}$.

Now, take a system of IURs $\rho^\gamma_{T_\gamma}$ of $T_\gamma$ and characters $\chi_{A_\gamma}=\mathrm{sgn}$ or $\mathbf{1}$ of $\mathfrak{S}_{A_\gamma}\cong\mathfrak{S}_\infty$. Then they give a datum $Q$ if we take in addition an IUR $\pi_F$ of $\mathfrak{S}_F$ and reference vectors $a(\gamma)$ and (when $|\Gamma|=\infty$) a reference vector $b=(b_\gamma)_{\gamma\in\Gamma}$. Thus we get uncountably many IURs $\rho(\mathfrak{b}, Q)$ of $G$ starting from an arbitrary set $\{E_\gamma\}$.

### 7.2. Proof of Theorem 7.1 (Step 1).

To prove the irreducibility of $\rho(\mathfrak{b}, Q)=\mathrm{Ind}_H^G\pi(\mathfrak{b}, Q)$, we apply Theorem 1.2(i). As we saw in §3.6, the set $\mathfrak{A}$ of standard subgroups has properties (GRP1) and (GRP2). Therefore it is enough for the irreducibility that the condition (REP) is satisfied for $\pi=\pi(\mathfrak{b}, Q)$ with itself. Since the proof here is simple, we do not appeal exactly to the condition (REP), but rather prove directly that if an $L\in\mathrm{Hom}(\pi, \pi^x; H\cap H^x)$ satisfies the conditions $(\mathrm{B}_x)$ and $(\mathrm{C}_x)$, then $L=0$, for any representative $x\notin H$ of $H\backslash G/H$.

To do so, put $\pi'=\pi^x$, $H'=H^x$, and apply the conditions $(\mathrm{B}_e)$ and $(\mathrm{C}_e)$ in §1.1.4 for the identity element $e\in G$, to an $L\in\mathrm{Hom}(\pi, \pi'; H\cap H')$. These conditions are written as

$$(\mathrm{B}_e)\qquad \sum_h \|L\pi(h)v\|^2\leqq M\|v\|^2 \qquad (h\in(H\cap H')\backslash H),$$

$$(\mathrm{C}_e)\qquad \sum_{h'} \|L^*\pi'(h')w\|^2\leqq M\|w\|^2 \qquad (h'\in(H\cap H')\backslash H'),$$

for $v\in V(\pi)$, $w\in V(\pi')$, where $M$ is a positive constant.

For $H=H^{\mathfrak{b}}$, the subgroup $H'=H^x$ is expressed by Lemma 3.1 as $H'=H^{\mathfrak{b}'}$ with $\mathfrak{b}'={}^y\mathfrak{b}\in\mathfrak{B}$, $y=x^{-1}$, where

$$(7.2)\qquad \mathfrak{b}'=\{F', (\mathfrak{J}'_\gamma, T_\gamma); \gamma\in\Gamma\} \qquad \text{with}$$

$$F'={}^yF\equiv\{y(i); i\in F\},\ \mathfrak{J}'_\gamma=\{J'_{\gamma,\alpha}; \alpha\in A_\gamma\},\ J'_{\gamma,\alpha}\equiv{}^yJ_{\gamma,\alpha}.$$

Note that, since $x\in G$ is a finite permutation, there exists a finite subset $\Gamma_1$ of $\Gamma$ such that $\mathfrak{J}'_\gamma=\mathfrak{J}_\gamma$ except for $\gamma\in\Gamma_1$, and further for each $\gamma\in\Gamma_1$ there exists a finite subset $A_{\gamma1}$ of $A_\gamma$ such that $J'_{\gamma,\alpha}=J_{\gamma,\alpha}$ except for $\alpha\in A_{\gamma1}$. Let $E$ be the union of $F$ and $\bar{J}_{\gamma,\alpha}$ ($\gamma\in\Gamma_1$, $\alpha\in A_{\gamma1}$). Then $E\supset F'$, and any element $m\in E''=\boldsymbol{N}\backslash E$ is invariant under $x^{-1}$, and so $\mathrm{supp}(x^{-1})\equiv\{m\in\boldsymbol{N}; x^{-1}(m)\neq m\}=\mathrm{supp}(x)$ is contained in $E$.

Put $H''=H\cap\mathfrak{S}_{E''}$. Then $H''=H'\cap\mathfrak{S}_{E''}\subset H\cap H'$, and $\pi|H''=\pi'|H''$ because $x$ commutes with any $h\in H''$. Note that

$$(7.3)\qquad H''=(\prod_{\gamma\in\Gamma_1} H''_\gamma)\times(\prod_{\gamma\in\Gamma''}{}' H_\gamma) \qquad \text{with } H''_\gamma=(\prod_{\alpha\in A''_\gamma}{}' T_\alpha)\rtimes\mathfrak{S}_{A''_\gamma}\subset H_\gamma,$$

where $\Gamma''=\Gamma\backslash\Gamma_1$, $A''_\gamma=A_\gamma\backslash A_{\gamma1}$, and $T_\alpha=\varphi_\alpha(T_\alpha)$ for $\alpha\in A''_\gamma$. Further $H''$ contains the subgroup

$$(7.4)\qquad D''=(\prod_{\gamma\in\Gamma_1}(\prod_{\alpha\in A''_\gamma}{}' T_\alpha))\times(\prod_{\gamma\in\Gamma''}{}' D(\mathfrak{J}_\gamma, T_\gamma)),$$

and the $D$-parts $H_D$ and $H'_D$ of $H$ and $H'$ defined in (3.22) are expressed as

(7.5) $$H_D=H_{D1}\times D'',\quad H'_D=H'_{D1}\times D'',\quad \text{with}$$

$$H_{D1}=\mathfrak{S}_F\times\prod_{\gamma\in\Gamma_1}(\prod_{\alpha\in A_{\gamma 1}}T_\alpha),\quad H'_{D1}=x^{-1}H_{D1}x=\mathfrak{S}_{F'}\times\prod_{\gamma\in\Gamma_1}(\prod_{\alpha\in A_{\gamma 1}}x^{-1}T_\alpha x).$$

On the other hand, put $b''=(b_\gamma)_{\gamma\in\Gamma''}$, $a''(\gamma)=(a_{\gamma,\alpha})_{\alpha\in A''_\gamma}$ for $\gamma\in\Gamma_1$. Then we see easily that the representation space $V(\pi)=V(\pi_F)\otimes(\otimes^b_{\gamma\in\Gamma}V(\Pi_\gamma))$ of $\pi$ is decomposed as

(7.6) $$V(\pi)=V_1\otimes V''\ \ (=V(\pi'))\quad \text{with}$$

(7.7) $$\begin{cases} V_1=V(\pi_F)\otimes(\otimes_{\gamma\in\Gamma_1}(\otimes_{\alpha\in A_{\gamma 1}}V(\pi_\alpha))), & \dim V_1<\infty, \\ V''=(\otimes_{\gamma\in\Gamma_1}(\otimes^{a''(\gamma)}_{\alpha\in A''_\gamma}V(\pi_\alpha))\otimes(\otimes^{b''}_{\gamma\in\Gamma''}V(\Pi_\gamma)) \end{cases}$$

recalling $V(\Pi_\gamma)=\otimes^{a(\gamma)}_{\alpha\in A_\gamma}V(\pi_\alpha)$ in (6.8)-(6.11).

Corresponding to (7.6), both the restrictions $\pi|D''$ and $\pi'|D''$ are expressed as

(7.8) $$\pi(d'')=I_{V_1}\otimes\tau(d''),\quad \pi'(d'')=I_{V_1}\otimes\tau(d'')\qquad (d''\in D''),$$

where $\tau$ is an IUR of $D''$ realized on $V''$.

Take an $L\in\mathrm{Hom}(\pi,\pi';H\cap H')$. Then $L$ intertwines $\pi|D''$ and $\pi'|D''$, and so it is expressed according to (7.6) as

(7.9) $$L=L_1\otimes I_{V''}\quad \text{with an } L_1\in\boldsymbol{B}(V_1).$$

Using this expression, we shall prove that if both the conditions ($\mathrm{B}_e$) and ($\mathrm{C}_e$) hold for $L\neq 0$, then the element $x$ should belong to $H$. This is an assertion stronger than (REP) and proves the irreducibility of $\mathrm{Ind}^G_H\pi$ in Theorem 7.1.

### 7.3. Proof of Theorem 7.1 (Conditions ($\mathrm{B}_e$) and ($\mathrm{C}_e$))

Take an $x\in G$. We study ($\mathrm{B}_e$) and ($\mathrm{C}_e$) for $L\in\mathrm{Hom}(\pi,\pi';H\cap H')$ in several cases step by step. Here for the convenience for later calculations, we use temporarily the following notation:

$$gC=\{g(i);\,i\in C\}\ (={}^gC),\qquad gJ=(g(p_1),g(p_2),\cdots,g(p_n))\ (={}^gJ)$$

for $g\in G$, a subset $C$ of $\boldsymbol{N}$, and an ordered $n$-set $J$ of integers.

CASE 1. Assume that $\mathrm{supp}(\mathfrak{J}_\gamma)\cap\mathrm{supp}(\mathfrak{J}'_{\gamma'})=\mathrm{supp}(\mathfrak{J}_\gamma)\cap x^{-1}\,\mathrm{supp}(\mathfrak{J}_{\gamma'})\neq\emptyset$ for different $\gamma,\gamma'\in\Gamma$. Then, take $\alpha_1\in A_\gamma$, $\alpha_2\in A_{\gamma'}$ such that $\bar{J}_{\gamma,\alpha_1}\cap x^{-1}(\bar{J}_{\gamma',\alpha_2})\neq\emptyset$. Here we consider the condition ($\mathrm{B}_e$). We choose an infinite subset $\Sigma$ of $\mathfrak{S}_{A_\gamma}\hookrightarrow H$ which gives a system of representatives of $(H\cap H')\backslash H$ such that the partial sum of $\|L\pi(h)v\|^2$ over $h\in\Sigma$ in ($\mathrm{B}_e$) already does not satisfy the inequality in ($\mathrm{B}_e$) if $L\neq 0$. From the definition of $\Gamma_1$ and $A_{\gamma 1}$, it is clear that $\gamma,\gamma'\in\Gamma_1$, $\alpha_1\in A_{\gamma 1}$, $\alpha_2\in A_{\gamma' 1}$. So let $\Sigma$ consist of transpositions $\sigma_\alpha=(\alpha_1,\alpha)$ in $\mathfrak{S}_{A_\gamma}$ with $\alpha\in A''_\gamma=A_\gamma\setminus A_{\gamma 1}$. Then for $\alpha\neq\beta$, $\sigma_\alpha\sigma_\beta^{-1}=(\alpha_1,\beta,\alpha)$ does not belong to $H'$, whence not to $H\cap H'$. In fact, $x(\bar{J}_{\gamma,\beta})=\bar{J}_{\gamma,\beta}$ since $\beta\in A''_\gamma$, and so we have

$$\emptyset\neq(x\cdot\sigma_\alpha\sigma_\beta^{-1})[\bar{J}_{\gamma,\alpha_1}\cap x^{-1}(\bar{J}_{\gamma',\alpha_2})]=x(\bar{J}_{\gamma,\beta})\cap(x\cdot\sigma_\alpha\sigma_\beta^{-1}\cdot x^{-1})(\bar{J}_{\gamma',\alpha_2})$$
$$=\bar{J}_{\gamma,\beta}\cap(x\cdot\sigma_\alpha\sigma_\beta^{-1}\cdot x^{-1})(\bar{J}_{\gamma',\alpha_2}).$$

This means that $y=x\cdot\sigma_\alpha\sigma_\beta^{-1}\cdot x^{-1}$ does not belong to $H$ and so $\sigma_\alpha\sigma_\beta^{-1}\notin x^{-1}Hx=H'$ as desired.

Now let us consider the partial sum. We know that $L=L_1\otimes I_{V''}$ with $L_1\in\boldsymbol{B}(V_1)$. Assume $L\neq 0$. Then there exists a unit decomposable vector $v_1\in V_1$ such that $L_1v_1\neq 0$. Let $v_1=v_F\otimes(\otimes_{\gamma''\in\Gamma_1}(\otimes_{\alpha\in A_{\gamma 1}}v_\alpha))$ with $v_F\in V(\pi_F)$, $v_\alpha\in V(\pi_\alpha)$. Fix an integer $N>0$. Take a subset $\Sigma_N$ of $N$-elements of $\Sigma\subset\mathfrak{S}_{A_\gamma}$ and choose a decomposable $v''\in V''$ whose component in $V(\pi_\beta)=V(\rho^\gamma_{T_\gamma})$ is just equal to $v_{\alpha_1}\in V(\pi_{\alpha_1})=V(\rho^\gamma_{T_\gamma})$ for every $\sigma_\beta\in\Sigma_N$. Then for $v=v_1\otimes v''\in V(\pi)$, we have $\pi(\sigma_\beta)v=v$ for any $\sigma_\beta\in\Sigma_N$, and so

$$\sum_{h\in(H\cap H')\backslash H}\|L\pi(h)v\|^2\geqq\sum_{\sigma\in\Sigma}\|(L_1\otimes I_{V''})\pi(\sigma)v\|^2\geqq\sum_{\sigma_\beta\in\Sigma_N}\|(L_1v_1)\otimes v''\|^2$$
$$=N\cdot\|L_1v_1\|^2\|v''\|^2=(N\cdot\|L_1v_1\|^2)\cdot\|v\|^2\,.$$

Since $N\cdot\|L_1v_1\|^2\to\infty$ as $N\to\infty$, the condition $(\mathrm{B}_e)$ does not hold for $L\neq 0$.

CASE 2. Assume that $\mathrm{supp}(\mathfrak{J}_\gamma)\cap F'=\mathrm{supp}(\mathfrak{J}_\gamma)\cap x^{-1}F\neq\emptyset$ for some $\gamma\in\Gamma$. In this case too, we consider the condition $(\mathrm{B}_e)$. The argument goes on the same line as in Case 1, and thus we see that any non-zero $L\in\mathrm{Hom}(\pi,\pi';H\cap H')$ does not satisfy $(\mathrm{B}_e)$.

CASE 3. Assume that $F\cap\mathrm{supp}(\mathfrak{J}'_{\gamma'})=F\cap x^{-1}\,\mathrm{supp}(\mathfrak{J}_{\gamma'})\neq\emptyset$ for some $\gamma'\in\Gamma$. In this case, we consider the condition $(\mathrm{C}_e)$. Accordingly we replace $H$ by $H'$ in the argument in Case 2, and arrive at the similar conclusion.

From Cases 1~3, we see that it is ncessary that $x$ leaves each $F$, $\mathrm{supp}(\mathfrak{J}_\gamma)$ $(\gamma\in\Gamma)$ stable, for $L\neq 0$ to satisfy $(\mathrm{B}_e)$ and $(\mathrm{C}_e)$.

CASE 4. Assume that $\bar{J}_{\gamma,\alpha_1}\cap\bar{J}'_{\gamma,\alpha_2}\neq\emptyset$, $\bar{J}_{\gamma,\alpha_1}\neq\bar{J}'_{\gamma,\alpha_2}\equiv x^{-1}(\bar{J}_{\gamma,\alpha_2})$. Let $\Sigma\subset\mathfrak{S}_{A_\gamma}$ be as in Case 1 the set of $\sigma_\alpha=(\alpha_1,\alpha)$ with $\alpha\in A''_\gamma=A_\gamma\setminus A_{\gamma 1}$. Note that $x^{-1}(\bar{J}_{\gamma,\alpha})=\bar{J}_{\gamma,\alpha}$ for any $\alpha\in A''_\gamma$. Then, applying $x\cdot\sigma_\alpha\sigma_\beta^{-1}$ to both sides of the above two equations, we get for $y=x\cdot\sigma_\alpha\sigma_\beta^{-1}\cdot x^{-1}$,

$$\bar{J}_{\gamma,\beta}\cap y(\bar{J}_{\gamma,\alpha_2})\neq\emptyset\,,\qquad \bar{J}_{\gamma,\beta}\neq y(\bar{J}_{\gamma,\alpha_2})\,.$$

This means that $y$ does not belong to $H$, and so $\Sigma=\{\sigma_\alpha\,;\,\alpha\in A''_\gamma\}$ is a system of representatives of $(H\cap H')\setminus H$. Starting from this point the argument for the condition $(\mathrm{B}_e)$ is the same as that in Case 1.

Thus we have shown that $x$ should satisfy that $x(\bar{J}_{\gamma,\alpha})=\bar{J}_{\gamma,\eta(\alpha)}=\eta(\bar{J}_{\gamma,\alpha})$ $(\alpha\in A_\gamma)$ for some $\eta\in\mathfrak{S}_{A_\gamma}\hookrightarrow H_\gamma\subset\mathfrak{S}_\infty$. In this situation, we have for every $\alpha\in A_\gamma$ a permutation $\xi_\alpha\in\mathfrak{S}_{n(\gamma)}$ such that

$$x(J_{\gamma,\alpha})=\xi_\alpha\circ J_{\gamma,\eta(\alpha)}\,;\tag{7.10}$$

where $\xi\circ J$ is defined in (2.6) for $\xi\in\mathfrak{S}_n$ and $J$ an ordered $n$-set. From the definition, we have $\xi\circ(\sigma J)=\sigma(\xi\circ J)$ for $\xi\in\mathfrak{S}_n$, $\sigma\in\mathfrak{S}_\infty$. In particular, (7.10) is equivalent to

$$\xi_\alpha^{-1}\circ J_{\gamma,\alpha}=x^{-1}(J_{\gamma,\eta(\alpha)})\,.\tag{7.11}$$

CASE 5. Assume that $\xi_{\alpha_1}\notin T_\gamma\subset\mathfrak{S}_{n(\gamma)}$ for some $\gamma\in\Gamma$ and $\alpha_1\in A_{\gamma 1}$. We consider again the condition $(B_e)$.

For any $\alpha\in A''_\gamma=A_\gamma\setminus A_{\gamma 1}$, we have $\eta(\alpha)=\alpha$ and $\xi_\alpha=1$. Let $\Sigma\subset\mathfrak{S}_{A_\gamma}$ be as in Case 1: $\Sigma=\{\sigma_\alpha=(\alpha_1, \alpha)\,;\ \alpha\in A''_\gamma\}$. Let us prove that $\Sigma$ gives a system of representatives of $(H\cap H')\setminus H$. We apply $y=x\cdot\sigma_\alpha\sigma_\beta^{-1}\cdot x^{-1}$ $(\sigma_\alpha, \sigma_\beta\in\Sigma)$ to $J_{\gamma,\eta(\alpha_1)}$, then by (7.11),

$$y(J_{\gamma,\eta(\alpha_1)})=(x\cdot\sigma_\alpha\sigma_\beta^{-1})(\xi_{\alpha_1}^{-1}\circ J_{\gamma,\alpha_1})=\xi_{\alpha_1}^{-1}\circ((x\cdot\sigma_\alpha\sigma_\beta^{-1})(J_{\gamma,\alpha_1}))$$
$$=\xi_{\alpha_1}^{-1}\circ(x\cdot J_{\gamma,\beta})=\xi_{\alpha_1}^{-1}\circ J_{\gamma,\beta}$$

because $\sigma(J_{\gamma,\alpha})=J_{\alpha,\sigma(\alpha)}$ by definition of the action of $\sigma\in\mathfrak{S}_{A_\gamma}$ in § 2.2, and $x(J_{\gamma,\beta})=J_{\gamma,\beta}$ since $\beta\in A''_\gamma=A_\gamma\setminus A_{\gamma 1}$. The fact that $y(J_{\gamma,\eta(\alpha_1)})=\xi_{\alpha_1}^{-1}\circ J_{\gamma,\beta}$ with $\xi_{\alpha_1}\notin T_\gamma$, means that $y\notin H$, whence $\sigma_\alpha\sigma_\beta^{-1}\notin H'=x^{-1}Hx\cdot$ Thus any $\sigma_\alpha$, $\sigma_\beta\in\Sigma$ represent different classes in $(H\cap H')\setminus H$. (These calculations can be much visualized by means of the matrix expression in § 2.2 of elements in $H_\gamma$.)

From this point on, the argument about the condition $(B_e)$ is the same as in Case 1.

Thus, by Cases 1~5, we see finally that for any $x\in G$ not in $H$ itself, any non-zero $L\in\mathrm{Hom}(\pi, \pi^x\,;\ H\cap H^x)$ does not satisfy $(B_e)$ or $(C_e)$. Hence we get for $\rho=\mathrm{Ind}_H^G\pi$,

$$\dim \mathrm{Hom}_G(\rho, \rho)=\dim \mathrm{Hom}_H(\pi, \pi)=1\,.$$

This proves that $\rho$ is irreducible, and the proof of Theorem 7.1 is now complete.

## § 8. Equivalence relations among standard irreducible unitary representations of $\mathfrak{S}_\infty$

Let $\rho(\mathfrak{b}, Q)$ and $\rho(\mathfrak{b}', Q')$ be two standard IURs of $G=\mathfrak{S}_\infty$ given as

(8.1) $$\rho(\mathfrak{b}, Q)=\mathrm{Ind}_{H^{\mathfrak{b}}}^G\pi(\mathfrak{b}, Q)\,,\qquad \rho(\mathfrak{b}', Q')=\mathrm{Ind}_{H^{\mathfrak{b}'}}^G\pi(\mathfrak{b}', Q')\,.$$

Here $(\mathfrak{b}, Q)$ and $(\mathfrak{b}', Q')$ are respectively

(8.2) $$\begin{cases}\mathfrak{b}=\{F, (\mathfrak{J}_\gamma, T_\gamma)\,;\ \gamma\in\Gamma\} & \text{with } \mathfrak{J}_\gamma=\{J_{\gamma,\alpha}\,;\ \alpha\in A_\gamma\},\\ Q=\{\pi_F, (\rho^\gamma_{T_\gamma}, \chi_{A_\gamma}, a(\gamma))_{\gamma\in\Gamma}, b=(b_\gamma)_{\gamma\in\Gamma}\}\,; \end{cases}$$

(8.3) $$\begin{cases}\mathfrak{b}'=\{F', (\mathfrak{J}'_\delta, T'_\delta)\,;\ \delta\in\Delta\} & \text{with } \mathfrak{J}'_\delta=\{J'_{\delta,\beta}\,;\ \beta\in B_\delta\},\\ Q'=\{\pi'_{F'}, (\rho'^\delta_{T'_\delta}, \chi'_{B_\delta}, a'(\delta))_{\delta\in\Delta}, b'=(b'_\delta)_{\delta\in\Delta}\}, \end{cases}$$

with $\{F, \bar{J}_{\gamma,\alpha}\,;\ \gamma\in\Gamma, \alpha\in A_\gamma\}$, $\{F', \bar{J}'_{\delta,\beta}\,;\ \delta\in\Delta, \beta\in B_\alpha\}$, two-step partitions of $\boldsymbol{N}$.

We study here a necessary and sufficient conditions for unitary equivalence $\rho(\mathfrak{b}, Q)\cong\rho(\mathfrak{b}', Q')$, and give it in Theorem 8.9 as our final main result. We see that, apart from elementary equivalences coming from inner automorphisms of $\mathfrak{S}_\infty$, there exist non-elementary equivalences. The latter corresponds to the similar equivalence relations between standard IURs of wreath product groups $\mathfrak{S}_A(T)$, studied in [DG, §§ 4-8].

### 8.1. The boundedness conditions $(\mathbf{B}_x)$ and $(\mathbf{C}_x)$

Put $\pi_1=\pi(\mathfrak{b}, Q)$, $\pi_2=\pi(\mathfrak{b}', Q')$, and

(8.4) $$H_1=H^{\mathfrak{b}}\equiv\mathfrak{S}_F\times{\prod_{\gamma\in\Gamma}}' H_\gamma\,,\qquad H_2=H^{\mathfrak{b}'}\equiv\mathfrak{S}_{F'}\times{\prod_{\delta\in\Delta}}' H'_\delta\,,\qquad\text{with}$$

$$H_\gamma=H(\mathfrak{J}_\gamma, T_\gamma)\equiv D(\mathfrak{J}_\gamma, T_\gamma)\rtimes\mathfrak{S}_{A_\gamma},\quad H'_\delta=H(\mathfrak{J}'_\delta, T'_\delta)\equiv D(\mathfrak{J}'_\delta, T'_\delta)\rtimes\mathfrak{S}_{B_\delta}\,.$$

As we know, the study of $\mathrm{Hom}_G(\rho_1, \rho_2)$ for $\rho_j=\mathrm{Ind}_{H_j}^G\pi_j$, $j=1, 2$, is reduced to the study of $L\in\mathrm{Hom}(\pi_1, \pi_2{}^x; H_1\cap H_2{}^x)$ which satisfies the boundedness conditions ($\mathrm{B}_x$) and ($\mathrm{C}_x$).. Here $x\in G$ varies over representatives of $H_1\backslash G/H_2$.

Note that if we replace $(\pi_2, H_2)$ by $(\pi_2{}^x, H_2{}^x)$, the condition ($\mathrm{B}_x$) (resp. ($\mathrm{C}_x$)) turns out to be the condition ($\mathrm{B}_e$) (resp. ($\mathrm{C}_e$)) corresponding to the identity element $x=e$. On the other hand, we know that $x^{-1}H^{\mathfrak{b}'}x=H^{\mathfrak{b}''}$ with $\mathfrak{b}''={}^y\mathfrak{b}'$, $y=x^{-1}$, given as

(8.5) $$\mathfrak{b}''=\{F'', (\mathfrak{J}_\delta'', T_\delta')_{\delta\in\Delta}\} \quad \text{with}$$
$$F''={}^yF', \quad \mathfrak{J}_\delta''=\{J_{\delta,\beta}''; \beta\in B_\delta\}, \quad J_{\delta,\beta}''={}^y(J_\delta'),$$

so that

(8.6) $$H_2{}^x=\mathfrak{S}_{F''}\times\prod_{\omega\in\Delta}{}' H_\delta'' \quad \text{with } H_\delta''=D(\mathfrak{J}_\delta'', T_\delta')\rtimes\mathfrak{S}_{B_\delta}.$$

In this way, we see that our study is essentially reduced to discuss the conditions ($\mathrm{B}_e$) and ($\mathrm{C}_e$) in case $x=e$.

### 8.2. First step to apply the boundedness conditions

Let us now study in what situation there can exist a non-zero $L\in\mathrm{Hom}(\pi_1, \pi_2; H_1\cap H_2)$ satisfying ($\mathrm{B}_e$) and ($\mathrm{C}_e$). First consider ($\mathrm{B}_e$) and denote by $I(u_1)$ the sum for $u_1\in V(\pi_1)$ appearing in it:

(8.7) $$I(u_1)=\sum_{h_1\in(H_1\cap H_2)\backslash H_1}\|L\pi_1(h_1)u_1\|^2.$$

Fix a $\gamma\in\Gamma$ and consider the $\gamma$-component $H_\gamma=H(\mathfrak{J}_\gamma, T_\gamma)=D(\mathfrak{J}_\gamma, T_\gamma)\rtimes\mathfrak{S}_{A_\gamma}$. For a subset $A'$ of $A_\gamma$, consider, through the isomorphism $\phi_\gamma: \mathfrak{S}_{A_\gamma}(T_\gamma)\to H_\gamma$, the following subgroup of $H_\gamma$:

(8.8) $$H_{A'}=\phi_\gamma(\mathfrak{S}_{A'}(T_\gamma))=D_{A'}\rtimes\mathfrak{S}_{A'} \quad \text{with } D_{A'}=\phi_\gamma(D_{A'}(T_\gamma)).$$

Let $I_{A'}(u_1)$ be a partial sum of $I(u_1)$ for which $h_1$ runs over $H_{A'}$ modulo $H_1\cap H_2$ from the left. Then $I(u_1)\geqq I_{A'}(u_1)$ and

(8.9) $$I_{A'}(u_1)=\sum_{h_1\in(H_{A'}\cap H_2)\backslash H_{A'}}\|L\pi_1(h_1)u_1\|^2.$$

Further fix an element $\sigma\in\mathfrak{S}_{A'}$ and consider a partial sum $I_{A',\sigma}(u_1)$ for which $h_1\in D_{A'}\cdot\sigma$. Note that the relation $d\cdot\sigma\sim d'\cdot\sigma$ ($d, d'\in D_{A'}$) modulo $H_{A'}\cap H_2$ from the left, is equivalent to $d\sim d'$ modulo $D_{A'}\cap H_2$. Put $T=D_{A'}$ and $S=D_{A'}\cap H_2\subset T$. Then $I_{A',\sigma}(u_1)$ is expressed as

(8.10) $$I_{A',\sigma}(u_1)=\sum_{t\in S\backslash T}\|L\pi_1(t\cdot\sigma)u_1\|^2.$$

Applying Lemma 1.12 we get the following

**Lemma 8.1.** *Suppose $A'\subset A_\gamma$ be finite. Then, for any $\sigma\in\mathfrak{S}_{A'}$.*

$$I_{A',\sigma}(u_1)=I_{A',e}(u_1),$$

*where $e$ denotes the identity element in $\mathfrak{S}_{A'}$. Further*

$$\sup_{\|u_1\|\leq 1} I_{A',e}(u_1) \geqq \|L\|^2.$$

*Proof.* Let $\rho$ be the IUR of $T \cong (T_\gamma)^{A'}$ acting on $V_1 = \otimes_{\alpha \in A'} V(\rho_\alpha)$ with $\rho_\alpha = \rho^\gamma_{T_\gamma}$. Then, since $A'$ is finite, the space $V(\pi_1)$, considered as a $T$-module, is expressed as $V_1 \otimes W_1$ with $W_1$ the tensor product of other factors of $V(\pi_1)$. Further let $U$ be the restriction of $\pi_1(\sigma)$ on $V_1$, then $\pi_1(\sigma) = U \otimes I_{W_1}$. By Lemma 1.12,

$$I_{A',\sigma}(u_1) = \sum_{t \in S \backslash T} \|L\rho(t)Uu_1\|^2 = \sum_{t \in S \backslash T} \|L\rho(t)u_1\|^2 \geqq \|L\|^2.$$

Using this lemma, we obtain the following criterion for vanishing of an $L \in \mathrm{Hom}(\pi_1, \pi_2; H_1 \cap H_2)$ which satisfies $(\mathrm{B}_e)$.

**Lemma 8.2.** *Let $\gamma \in \Gamma$. Assume that there exists an infinite subset $\Sigma$ of $\mathfrak{S}_{A_\gamma}$ such that, for different $\sigma, \sigma' \in \Sigma$, any elements $d \cdot \sigma$ and $d' \cdot \sigma'$ with $d, d' \in D_{A_\gamma} = \phi_\gamma(D_{A_\gamma}(T_\gamma))$, are not mutually equivalent modulo $H_1 \cap H_2$ from the left. Then an $L \in \mathrm{Hom}(\pi_1, \pi_2; H_1 \cap H_2)$ is zero if it satisfies the condition* $(\mathrm{B}_e)$.

*Proof.* Let $A'$ be a finite subset of $A_\gamma$ and put $\Sigma_{A'} = \Sigma \cap \mathfrak{S}_{A'}$. We apply Lemma 8.1. First, for $u_1 \in V(\pi_1)$,

$$I(u_1) \geqq I_{A'}(u_1) \geqq \sum_{\sigma \in \Sigma_{A'}} I_{A',\sigma}(u_1) \geqq |\Sigma_{A'}| \cdot I_{A',e}(u_1).$$

Then $M \geqq \sup\{I(u_1); \|u_1\| \leqq 1\} \geqq |\Sigma_{A'}| \cdot \|L\|^2$. Since $|\Sigma_{A'}| \to \infty$ as $A' \to A$, we get $L = 0$.

Q.E.D.

### 8.3. Relations between two subgroups $H_1$ and $H_2$

Let us now apply the criterion in Lemma 8.2. To do so, we construct an infinite subset $\Sigma \subset \mathfrak{S}_{A_\gamma}$ satisfying the condition there, according to the cases.

CASE 1. Assume that there exist $(\gamma, \alpha_1)$ and $(\delta, \beta_1)$ such that

$$\bar{J}_{\gamma,\alpha_1} \cap \bar{J}'_{\delta,\beta_1} \neq \emptyset \quad \text{and} \quad \bar{J}_{\gamma,\alpha_1} \not\supset \bar{J}'_{\delta,\beta_1}.$$

Put $C = \bar{J}'_{\delta,\beta_1}$, then every element in $H_2$ sends $C$ to one of $\bar{J}'_{\delta,\beta}$, $\beta \in B_\delta$, in total. Take an infinite subset $A'_\gamma$ of $A_\gamma$ such that $\bar{J}_{\gamma,\alpha} \cap C = \emptyset$ for any $\alpha \in A'_\gamma$. Put $\sigma_\alpha = (\alpha_1, \alpha)$, the transposition of $\alpha_1$ and $\alpha$. Then, for different $\alpha, \alpha' \in A'_\gamma$, the element $y = \sigma_\alpha \sigma_{\alpha'}^{-1} = (\alpha_1, \alpha', \alpha) \in \mathfrak{S}_{A_\gamma} \hookrightarrow H_1$ does not belong to $H_2$. In fact, $y\bar{J}_{\gamma,\alpha_1} = \bar{J}_{\gamma,\alpha'}$ and $y$ fixes every element in $C \setminus \bar{J}_{\gamma,\alpha_1} \neq \emptyset$, and so $yC \cap C \neq \emptyset$ but $yC \neq C$. Hence $\sigma_\alpha$ and $\sigma_{\alpha'}$ represent different classes in $(H_1 \cap H_2) \backslash H_1$. Here $\sigma \in \mathfrak{S}_{A_\gamma}$ is imbedded into $D(\mathfrak{Z}_\gamma, T_\gamma) \rtimes \mathfrak{S}_{A_\gamma} \subset H_1$ as $1 \times \sigma$. Thus $\Sigma = \{\sigma_\alpha; \alpha \in A'_\gamma\}$ gives an infinite system of representatives of $(H_1 \cap H_2) \backslash H_1$. Further take $d, d' \in D_{A_\gamma} \hookrightarrow H_1$. Then, since $(d \cdot \sigma_\alpha)(d' \cdot \sigma_{\alpha'})^{-1} = d \cdot y \cdot d'^{-1}$, we see that $d \cdot \sigma_\alpha \not\sim d' \cdot \sigma_{\alpha'}$ if $\alpha \neq \alpha'$.

CASE 2. Assume that some $\bar{J}_{\gamma,\alpha_1}$ meets $F'$: $\bar{J}_{\gamma,\alpha_1} \cap F' \neq \emptyset$. Put $C = F'$, then every element in $H_2$ sends $C$ onto $C$. Similarly as in Case 1, put $A'_\gamma = \{\alpha \in A_\gamma; \bar{J}_{\gamma,\alpha_1} \cap C \neq \emptyset\}$ and $\Sigma = \{\sigma_\alpha; \alpha \in A'_\gamma\}$. Then this gives an infinite subset of $\mathfrak{S}_{A_\gamma}$ with the desired property.

According to Cases 1 and 2, we come to the situation where every $\bar{J}_{\delta,\beta}$ is contained in some of $\bar{J}_{\gamma,\alpha}$ and $F\supset F'$. Now take into account the condition $(C_e)$ and a version of Lemma 8.2. Then every $J_{\gamma,\alpha}$ should be equal to some $J'_{\delta,\beta}$.

CASE 3. Assume that, for two different $\delta_1, \delta_2\in\Delta$, we have $\bar{J}_{\gamma,\alpha_j}\subset\mathrm{supp}(\mathfrak{J}'_{\delta_j})$ with $\alpha_j\in A_\gamma$, $j=1, 2$. In this case, we may assume that $A'_\gamma=\{\alpha\in A_\gamma;\ \bar{J}_{\gamma,\alpha}\cap\mathrm{supp}(\mathfrak{J}'_{\delta_1})=\emptyset\}$ is infinite. Note that every element in $H_2$ leaves the set $\mathrm{supp}(\mathfrak{J}'_{\delta_1})$ stable. Then we see that $\Sigma=\{\sigma_\alpha=(\alpha_1, \alpha);\ \alpha\in A'_\delta\}$ gives an infinite subset of $\mathfrak{S}_{A_\gamma}\hookrightarrow H_1$ having the desired property for the condition $(B_e)$.

**Lemma 8.3.** *Assume that a non-zero* $L\in\mathrm{Hom}(\pi_1, \pi_2;\ H_1\cap H_2)$ *satisfies the conditions* $(B_e)$ *and* $(C_e)$. *Then, for any* $\gamma\in\Gamma$, *there exists a unique* $\delta\in\Delta$ *such that every* $\bar{J}_{\gamma,\alpha}$, $\alpha\in A_\gamma$, *is equal to some* $\bar{J}'_{\delta,\beta}$, $\beta\in B_\delta$.

Thus, as a first consequence of the boundedness conditions, we obtain

**Theorem 8.4.** *Let* $H_1=H^{\mathfrak{b}}$, $H_2=H^{\mathfrak{b}'}$ *with* $\mathfrak{b}$ *in* (8.2) *and* $\mathfrak{b}'$ *in* (8.3), *and, for* $i=1, 2$, *let* $\pi_i$ *be an IUR of* $H_i$ *given above. Assume that there exists a non-zero* $L\in\mathrm{Hom}(\pi_1, \pi_2;\ H_1\cap H_2)$ *which satisfies both the conditions* $(B_e)$ *and* $(C_e)$. *Then* $\mathfrak{b}$ *and* $\mathfrak{b}'$ *satisfy the following:* $F=F'$, *and there exists a bijective correspondence* $\kappa$ *between* $\Gamma$ *and* $\Delta$ *such that, for* $\delta=\kappa(\gamma)$,

$$\bar{J}_{\gamma,\alpha}=\bar{J}'_{\delta,\beta} \quad \text{for any } \alpha\in A_\gamma \text{ with some } \beta\in B_\delta, \tag{8.11}$$

*giving a bijection* $\alpha\mapsto\beta$ *of* $A_\gamma$ *onto* $B_\delta$.

### 8.4. More explicit relations between $H_1$ and $H_2$

We consider the situation where a non-zero $L\in\mathrm{Hom}(\pi_1, \pi_2;\ H_1\cap H_2)$ satisfies the conditions $(B_e)$ and $(C_e)$. By Theorem 8.4, we can identify $\Delta$ with $\Gamma$ through $\kappa$ and, for each $\gamma\in\Gamma$, $B_{\kappa(\gamma)}$ with $A_\gamma$ through (8.11). Then the data $(\mathfrak{b}', Q')$ for $(\pi_2, H_2)$ is rewritten as follows:

$$\mathfrak{b}'=\{F, (\mathfrak{J}'_\gamma, T'_\gamma);\ \gamma\in\Gamma) \quad \text{with } \mathfrak{J}'_\gamma=\{J'_{\gamma,\alpha};\ \alpha\in A_\gamma\} \tag{8.12}$$

in such a way that $\bar{J}'_{\gamma,\alpha}=\bar{J}_{\gamma,\alpha}$ and

$$Q'=\{\pi'_F, (\rho'^{\gamma}_{T'_\gamma}, \chi'_{A_\gamma}, a'(\gamma))_{\gamma\in\Gamma},\quad b'=(b'_\gamma)_{\gamma\in\Gamma}\}. \tag{8.13}$$

By the definition in §6.1, we have

$$H_\gamma=\psi_\gamma(\mathfrak{S}_{A_\gamma}(T_\gamma)), \qquad H'_\gamma=\psi'_\gamma(\mathfrak{S}_{A_\gamma}(T'_\gamma)), \tag{8.14}$$

where the imbeddings $\psi_\gamma$ and $\psi'_\gamma$ into $\mathfrak{S}_{C_\gamma}\subset\mathfrak{S}_\infty$ with $C_\gamma=\mathrm{supp}(\mathfrak{J}_\gamma)=\mathrm{supp}(\mathfrak{J}'_\gamma)$ are determined respectively by the families of ordered $n(\gamma)$-sets $\mathfrak{J}_\gamma=\{J_{\alpha,\gamma};\ \alpha\in A_\gamma\}$ and $\mathfrak{J}'_\gamma=\{J'_{\alpha,\gamma};\ \alpha\in A_\gamma\}$. Further $H_1=H^{\mathfrak{b}}$ and $H_2=H^{\mathfrak{b}'}$ are given respectively as the images of

$$\underline{H}=\mathfrak{S}_N\times\prod_{\gamma\in\Gamma}{}'\ \mathfrak{S}_{A_\gamma}(T_\gamma), \qquad \underline{H}'=\mathfrak{S}_N\times\prod_{\gamma\in\Gamma}{}'\ \mathfrak{S}_{A_\gamma}(T'_\gamma), \tag{8.15}$$

with $N=|F|$, through the imbeddings into $\mathfrak{S}_\infty$

(8.16) $$\Psi=\psi_F\otimes(\otimes_{\gamma\in\Gamma}\psi_\gamma)\,,\qquad \Psi'=\psi_F\otimes(\otimes_{\gamma\in\Gamma}\psi'_\gamma)\,,$$

where $\psi_F$ is the natural isomorphism of $\mathfrak{S}_N$ onto $\mathfrak{S}_F$.

For the moment we restrict ourselves to the $\gamma$-factors and clarify the difference between two imbeddings $\psi_\gamma$ and $\psi'_\gamma$ and so on. Let $\sigma_{\gamma\alpha}$ be a unique element in $\mathfrak{S}_\infty$ such that $\sigma_{\gamma\alpha}J_{\gamma,\alpha}=J'_{\gamma,\alpha}$ and that $\sigma_{\gamma\alpha}p=p$ for any other $p\in N\setminus\bar{J}_{\gamma,\alpha}$, and denote by $\sigma_\gamma$ the product of $\sigma_{\gamma\alpha}$ over $\alpha\in A_\gamma$. Then $\sigma_\gamma$ is not necessarily in $\mathfrak{S}_\infty$ but in the bigger group $\tilde{\mathfrak{S}}_N$ of all permutations of $N$, which is isomorphic to $\mathrm{Aut}(\mathfrak{S}_\infty)$.

Note that $\psi_\gamma$ in (6.5) can be extended naturally to an imbedding of $Y\equiv\mathfrak{S}_{A_\gamma}(\mathfrak{S}_n)\supset\mathfrak{S}_{A_\gamma}(T_\gamma)$ into $\mathfrak{S}_\infty$, and similarly for $\psi'_\gamma$, where $n=n(\gamma)$. Then by (6.3)-(6.4), we have for $y\in Y$,

(8.17) $$\psi'_\gamma(y)=\sigma_\gamma\cdot\psi_\gamma(y)\cdot\sigma_\gamma^{-1}=(\iota(\sigma_\gamma)\circ\psi_\gamma)(y)\,,$$

where $\iota(\sigma_\gamma)$ denotes the automorphism of $\mathfrak{S}_\infty$ induced by $\sigma_\gamma$. In particular, we have the same image $\psi_\gamma(Y)=\psi'_\gamma(Y)$.

The above relation can be rewritten with an automorphism of $Y$ as follows. Let $\xi_{\gamma\alpha}$ be an element of $\mathfrak{S}_n$ such that

(8,18) $$\xi_{\gamma\alpha}\circ J_{\gamma,\alpha}=J'_{\gamma,\alpha}\,,$$

and consider $\tilde{\xi}_\gamma=(\xi_{\gamma\alpha})_{\alpha\in A_\gamma}$ as an element of the direct product $\Xi=\prod_{\alpha\in A_\gamma}\Xi_\alpha$ with $\Xi_\alpha=\mathfrak{S}_n$ $(\alpha\in A_\gamma)$. Every element $\tilde{\xi}$ of $\Xi$ acts on $Y$ as $\iota(\tilde{\xi})y=\tilde{\xi}y\tilde{\xi}^{-1}$ $(y\in Y)$. We assert that

(8.19) $$\psi'_\gamma(y)=(\psi_\gamma\circ\iota(\tilde{\xi}_\gamma)^{-1})(y)=\psi_\gamma(\tilde{\xi}_\gamma^{-1}y\tilde{\xi}_\gamma)\qquad(y\in Y)\,.$$

In fact, in the notations in (2.1) and (2.6), we have for $\xi\in\mathfrak{S}_n$ and $J_\alpha=(p_1,p_2,\cdots,p_n)$,

$$\xi\circ J_\alpha=(p_{\xi^{-1}(1)},p_{\xi^{-1}(2)},\cdots,p_{\xi^{-1}(n)})=\varphi_\alpha(\xi^{-1})J_\alpha\,,$$

and therefore $\sigma_{\gamma\alpha}=\varphi_\alpha(\xi_{\gamma\alpha}^{-1})$. Hence (8.19) follows from (8.17).

**Lemma 8.5.** (i) *The subgroups $H_\gamma$ and $H'_\gamma$ are both contained in $\psi_\gamma(Y)=\psi'_\gamma(Y)$ and*

(8.20) $$\psi_\gamma^{-1}(H_\gamma)=\mathfrak{S}_{A_\gamma}(T_\gamma)\,,\qquad \psi_\gamma^{-1}(H'_\gamma)=\tilde{\xi}^{-1}\mathfrak{S}_{A_\gamma}(T'_\gamma)\tilde{\xi}\,.$$

(ii) *An element $t\times\sigma\in\mathfrak{S}_{A_\gamma}(T_\gamma)$ with $t=(t_\alpha)_{\alpha\in A_\gamma}$, $t_\alpha\in T_\gamma$ and $\sigma\in\mathfrak{S}_{A_\gamma}$, belongs to $\psi_\gamma^{-1}(H_\gamma\cap H'_\gamma)$ if and only if*

(8.21) $$t_\alpha\in T_\gamma\cap\xi_\alpha^{-1}T'_\gamma\xi_{\sigma^{-1}(\alpha)}\qquad(\alpha\in A_\gamma)\,.$$

Now put for every $\xi\in\mathfrak{S}_n$, $A_{\gamma\xi}=\{\alpha\in A_\gamma\,;\,\xi_{\gamma\alpha}=\xi\}$. Then $A_\gamma=\coprod_{\xi\in\mathfrak{S}_n}A_{\gamma\xi}$ is a partition of $A_\gamma$. Our aim at this step is to prove the relation (8.22) in the following

**Proposition 8.6.** *Assume there exists a non-zero $L\in\mathrm{Hom}(\pi_1,\pi_2\,;\,H_1\cap H_2)$ satisfying* ($\mathrm{B}_e$) *and* ($\mathrm{C}_e$). *Then we may assume that the data for $\pi_1$ and $\pi_2$ are given by* ($\mathfrak{b}$, $Q$) *in* (8.2) *and* ($\mathfrak{b}'$, $Q'$) *in* (8.12)-(8.13). *Fix $\gamma\in\Gamma$. If $\xi,\eta\in\mathfrak{S}_n$, $n=n(\gamma)$, satisfy $|A_{\gamma\xi}|=|A_{\gamma\eta}|=\infty$, then*

(8.22) $$T'_\gamma\xi T_\gamma=T'_\gamma\eta T_\gamma\,.$$

The proof of this proposition continues until § 8.9.

At first, we consider subgroups of $Y=\mathfrak{S}_{A_\gamma}(\mathfrak{S}_n)$ as

$$
\begin{aligned}
&Y_0=\prod_{\xi\in\mathfrak{S}_n}\mathfrak{S}_{A_{\gamma\xi}}(\mathfrak{S}_n), \qquad Y_1=\prod_{\xi\in\mathfrak{S}_n}\mathfrak{S}_{A_{\gamma\xi}}(T_\gamma),\\
&Y_2=\prod_{\xi\in\mathfrak{S}_n}\mathfrak{S}_{A_{\gamma\xi}}(T_\gamma'^{\xi}) \quad \text{with } T_\gamma'^{\xi}=\xi^{-1}T_\gamma'\xi,\\
&Z_2=\prod_{\xi\in\mathfrak{S}_n}\mathfrak{S}_{A_{\gamma\xi}}(T_\gamma')\subset\mathfrak{S}_{A_\gamma}(T_\gamma').
\end{aligned}
\tag{8.23}
$$

Then $Y_1\subset\phi_\gamma^{-1}(H_\gamma)$, $Y_2\subset\phi_\gamma^{-1}(H_\gamma')$, and $Y_1\cap Y_2\subset\phi_\gamma^{-1}(H_\gamma\cap H_\gamma')$ by Lemma 8.5(ii). Moreover we have $\phi_\gamma(Y_2)=\phi_\gamma'(Z_2)$, since $Z_2=\iota(\tilde{\xi})Y_2$, and

$$\phi_\gamma(Y_2)=\prod_{\xi\in\mathfrak{S}_n}H(\mathfrak{J}_{\gamma\xi},T_\gamma'^{\xi}), \qquad \phi_\gamma'(Z_2)=\prod_{\xi\in\mathfrak{S}_n}H(\mathfrak{J}_{\gamma\xi}',T_\gamma'),$$

with $\mathfrak{J}_{\gamma\xi}=\{J_{\gamma,\alpha}\,;\,\alpha\in A_{\gamma\xi}\}$, $\mathfrak{J}_{\gamma\xi}'=\{J_{\gamma,\alpha}'\,;\,\alpha\in A_{\gamma\xi}\}$, because $\xi\circ\mathfrak{J}_{\gamma\xi}=\mathfrak{J}_{\gamma\xi}'$.

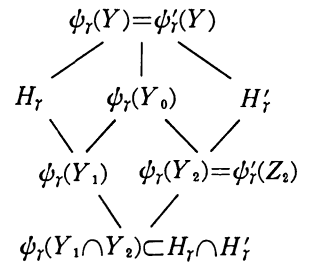



**Diagram 8.1.** Inclusion Relations

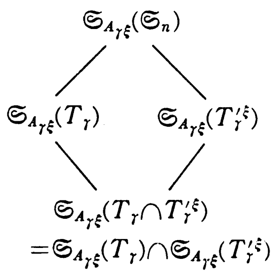



**Diagram 8.2.** $\xi$-components of $Y_j$'s

**8.5. Applications of results in [DG]**

We can reduce the present situation to the case of infinite wreath product groups studied in [DG, § 7] as follows.

(1) Assume there exists a non-zero $L\in\mathrm{Hom}(\pi_1,\pi_2\,;\,H_1\cap H_2)$ which satisfies the boundedness conditions $(\mathrm{B}_e)$ and $(\mathrm{C}_e)$ for $H_1$, $H_2$ and $H_1\cap H_2$. Then, for any $\gamma\in\Gamma$, we have a non-zero element $L^\gamma\in\mathrm{Hom}(\Pi_\gamma,\Pi_\gamma'\,;\,H_\gamma\cap H_\gamma')$ satisfying the conditions $(\mathrm{B}_e)$ and $(\mathrm{C}_e)$ for $H_\gamma$, $H_\gamma'$ and $H_\gamma\cap H_\gamma'$, where $\Pi_\gamma=\Pi(Q_\gamma)\circ\phi_\gamma^{-1}$ of $H_\gamma$ in (6.8)-(6.9) is the $\gamma$-component of $(\pi_1,H_1)$, and $\Pi_\gamma'$ is that of $(\pi_2,H_2)$.

(2) In Diagram 8.1, restricting $\Pi_\gamma$ from $H_\gamma$ to $\phi_\gamma(Y_1)$, and $\Pi_\gamma'$ from $H_\gamma'$ to $\phi_\gamma(Y_2)$, we get a non-zero element in $\mathrm{Hom}(\Pi_\gamma|\phi_\gamma(Y_1),\Pi_\gamma'|\phi_\gamma(Y_2)\,;\,\phi_\gamma(Y_1\cap Y_2))$ satisfying $(\mathrm{B}_e)$ and $(\mathrm{C}_e)$ for $\phi_\gamma(Y_1)$, $\phi_\gamma(Y_2)$ and $\phi_\gamma(Y_1\cap Y_2)$.

(3) Pull back the above relations to $Y_1$, $Y_2$ and $Y_1\cap Y_2$ through $\phi_\gamma^{-1}$, and take its $\xi$-component for each $\xi\in\mathfrak{S}_n$ with $A_{\gamma\xi}\neq\emptyset$. Then we get a non-zero element

$$L_\xi\in\mathrm{Hom}(\pi_{1\xi},\pi_{2\xi}\,;\,\mathfrak{S}_{A_{\gamma\xi}}(T_\gamma)\cap\mathfrak{S}_{A_{\gamma\xi}}(T_\gamma'^{\xi})) \tag{8.24}$$

satisfying $(\mathrm{B}_e)$ and $(\mathrm{C}_e)$ for $\mathfrak{S}_{A_{\gamma\xi}}(T_\gamma)$, $\mathfrak{S}_{A_{\gamma\xi}}(T_\gamma'^{\xi})$ and $\mathfrak{S}_{A_{\gamma\xi}}(T_\gamma\cap T_\gamma'^{\xi})$. Here $\pi_{1\xi}$ and $\pi_{2\xi}$ denote respectively $\xi$-components of the pull-backs $\Pi_\gamma\circ\phi_\gamma^{-1}|Y_1$ and $\Pi_\gamma'\circ\phi_\gamma^{-1}|Y_2$.

Now let us study the situation arrived in (3) above, and apply the results in § DG7. First we specify from what data the representations $\pi_{1\xi}$ and $\pi_{2\xi}$ of wreath product groups $\mathfrak{S}_{A_{\gamma\xi}}(T_\gamma)$ and $\mathfrak{S}_{A_\xi}(T_\gamma'^{\xi})$ are determined. Put $\rho_\gamma=\rho_{T_\gamma}^\gamma$, $\rho_\gamma'=\rho_{T_\gamma'}'^\gamma$, and define an IUR $\rho_\gamma'^{\xi}$ of $T_\gamma'^{\xi}$ by

(8.25) $$\rho_\gamma'^\xi(s)=\rho_\gamma'(\xi s\xi^{-1}) \qquad (s\in T_\gamma'^\xi).$$

Further put $a_{\gamma\xi}=(a_{\gamma,\alpha})_{\alpha\in A_{\gamma\xi}}$, $a'_{\gamma\xi}=(a'_{\gamma,\alpha})_{\alpha\in A_{\gamma\xi}}$, and $\chi_{A_{\gamma\xi}}=\chi_{A_\gamma}|\mathfrak{S}_{A_{\gamma\xi}}$, $\chi'_{A_{\gamma\xi}}=\chi'_{A_\gamma}|\mathfrak{S}_{A_{\gamma\xi}}$. Then $\pi_{1\xi}$ and $\pi_{2\xi}$ are determined respectively by the data

(8.26) $$((\rho_\gamma, T_\gamma), \chi_{A_{\gamma\xi}}, a_{\gamma\xi}), \qquad ((\rho_\gamma'^\xi, T_\gamma'^\xi), \chi'_{A_{\gamma\xi}}, a'_{\gamma\xi}),$$

where $a_{\gamma\xi}$ is the reference vector for the tensor product

$$\otimes_{\alpha\in A_{\gamma\xi}}^{a_{\gamma\xi}}\rho_\alpha \text{ on } \otimes_{\alpha\in A_{\gamma\xi}}^{a_{\gamma\xi}}V_\alpha \qquad \text{with } \rho_\alpha=\rho_\gamma,\ V_\alpha=V(\rho_\alpha),$$

and $a'_{\gamma\xi}$ is the similar one.

Assume $|A_{\gamma\xi}|=\infty$. Then we can apply Theorem DG7.8 (cf. Theorem 1.10) and obtain an explicit expression for $L_\xi$ in (8.24). It is proved there that $L_\xi$ is unique up to scalar multiples. To apply Theorem DG7.8, we put in it

(8.27) $$A=A_{\gamma\xi},\quad T=\mathfrak{S}_n,\quad T_1=T_\gamma,\quad T_2=T_\gamma'^\xi,\quad \pi_1=\pi_{1\xi},\quad \pi_2=\pi_{2\xi}.$$

Then we get the following results.

(I) There exist a unique (up to equivalence) IUR $\tau_\xi$ of $T_{(\xi)}=T_\gamma\cap T_\gamma'^\xi$ and a reference vector $c_\xi=(c_{\xi\alpha})_{\alpha\in A_{\gamma\xi}}$, $c_{\xi\alpha}\in V(\tau_\xi)$, $\|c_{\xi\alpha}\|=1$, such that

(8.28) $$\rho_\gamma\cong\mathrm{Ind}\,(\tau_\xi\,;\,T_{(\xi)}\uparrow T_\gamma), \qquad \rho_\gamma'^\xi\cong\mathrm{Ind}\,(\tau_\xi\,;\,T_{(\xi)}\uparrow T_\gamma'^\xi);$$

(8.28′) $$a_{\gamma\xi}\cong K_1(\mathrm{Ind}\,(c_\xi\,;\,T_{(\xi)})\uparrow T_\gamma)), \qquad a'_{\gamma\xi}\cong K_2(\mathrm{Ind}\,(c_\xi\,;\,T_{(\xi)}\uparrow T_\gamma'^\xi)),$$

where $K_i$, $i=1, 2$, are respectively the tensor products over $\alpha\in A_{\gamma\xi}$ of unitary intertwining operators for the equivalences in (8.28). Moreover we have $\chi_{A_{\gamma\xi}}=\chi'_{A_{\gamma\xi}}$.

(II) Let $J_{(\xi)}$ be a unitary $T_{(\xi)}$-isomorphism of the $\tau_\xi$-part of $V(\rho_\gamma)$ onto that of $V(\rho_\gamma'^\xi)$, and extend it trivially as a partial isometry. Then there exist $(\lambda_\alpha)_{\alpha\in A_{\gamma\xi}}$, $\lambda_\alpha\in\boldsymbol{C}$, $|\lambda_\alpha|=1$, and a constant $d_\xi$ such that

(8.29) $$L_\xi=d_\xi\cdot\otimes_{\alpha\in A_{\gamma\xi}}(\lambda_\alpha J_\alpha) \qquad \text{with } J_\alpha=J_{(\xi)}.$$

### 8.6. Interpretation of the preceeding results

For simplification of notations, we put

$$Y_\xi=\mathfrak{S}_{A_{\gamma\xi}}(\mathfrak{S}_n), \qquad Y_{1\xi}=\mathfrak{S}_{A_{\gamma\xi}}(T_\gamma), \qquad Y_{2\xi}=\mathfrak{S}_{A_{\gamma\xi}}(T_\gamma'),$$

$$Z_\xi=D_{A_{\gamma\xi}}(\mathfrak{S}_n), \qquad Z_{1\xi}=D_{A_{\gamma\xi}}(T_\gamma), \qquad Z_{2\xi}=D_{A_{\gamma\xi}}(T_\gamma'),$$

where $D_{A_{\gamma\xi}}(S)=\prod'_{\alpha\in A_{\gamma\xi}}S_\alpha$ with $S_\alpha=S$. Then, since $Y_\xi=Z_\xi\rtimes\mathfrak{S}_{A_{\gamma\xi}}$, $Y_{i\xi}=Z_{i\xi}\rtimes\mathfrak{S}_{A_{\gamma\xi}}$, we have

$$Y_{i\xi}\backslash Y_\xi\cong Z_{i\xi}\backslash Z_\xi, \qquad (Y_{1\xi}\cap Y_{2\xi})\backslash Y_{i\xi}\cong(Z_{1\xi}\cap Z_{2\xi})\backslash Z_{i\xi}.$$

Interpreting in the notation in (8.27), we know by Theorem 1.7 (cf. Theorem DC4.2) that the representations

$$P_{i\xi}=\mathrm{Ind}_{Y_{i\xi}}^{Y_\xi}\pi_{i\xi} \qquad (i=1, 2)$$

are both irreducible, and hence $\dim\mathrm{Hom}\,(P_{1\xi}, P_{2\xi}\,;\,Y_\xi)\leqq 1$. Through standard discussions, we know by Theorem 1.10 (cf. Theorem DG4.5) that the result (I) above gives a sufficient condition for $P_{1\xi}\cong P_{2\xi}$ (and gives easily a necessary and sufficient one) and that the result (II) gives explicitly a non-zero intertwining operator $K_\xi$ for the equivalence $P_{1\xi}\cong P_{2\xi}$.

Let us recall from § 1.1 how $K_\xi$ is determined from $L_\xi\in\mathrm{Hom}(\pi_{1\xi}, \pi_{2\xi}; Y_{1\xi}\cap Y_{2\xi})$ in (8.29). The representation space $V(P_{i\xi})$, $i=1$ or 2, is the space of $V(\pi_{i\xi})$-valued functions on $Y_\xi$ such that

$$
f(hy)=\pi_{i\xi}(h)f(y) \quad (h\in Y_{i\xi},\ y\in Y_\xi), \qquad \|f\|^2\equiv\sum_{y\in Y_{i\xi}\backslash Y_\xi}\|f(y)\|^2<\infty . \tag{8.30}
$$

The action of $y_0\in Y_\xi$ on $f$ is given by $P_{i\xi}(y_0)f(y)=f(yy_0)$. The opertor $K_\xi$ is written with a kernel, denoted again by the same symbol, as

$$
(K_\xi f)(y)=\sum_{y'\in Y_{1\xi}\backslash Y_\xi}K_\xi(y, y')f(y') \quad (y\in Y_\xi), \tag{8.31}
$$

and the kernel itself is given as

$$
K_\xi(y, y')=\pi_{2\xi}(h_2)\circ L_\xi\circ\pi_{1\xi}(h_1) \quad \text{if } yy'^{-1}=h_2h_1 \text{ with } h_i\in Y_{i\xi}, \tag{8.32}
$$

$$
K_\xi(y, y')=0 \quad \text{otherwise.} \tag{8.32'}
$$

Note that $f\in V(P_{i\xi})$ is uniquely determined by its restriction on $Z_\xi$. Hence we may and do consider $V(P_{i\xi})$ consisting of functions on $Z_\xi$. Then the kernel itself is given on $Z_\xi\times Z_\xi$, and the summation in (8.31) is actually over $Z_{1\xi}\backslash Z_\xi$. Put $k_\xi(z)=K_\xi(z, e)$ for $z\in Z_\xi$. Then $K_\xi(z, z')=k_\xi(zz'^{-1})$. It follows from (8.29) and (8.32)-(8.32′) the following

**Lmma 8.7.** *For* $z=(z_\alpha)_{\alpha\in A_{\gamma\xi}}\in Z_\xi$,

$$
k_\xi(z)=d_\xi\cdot\otimes_{\alpha\in A_{\gamma\xi}}k_\alpha(z_\alpha), \tag{8.33}
$$

*where* $d_\xi$ *is a constant and* $k_\alpha(\cdot)$ *on* $\mathfrak{S}_n$ *is given by*

$$
k_\alpha(\xi^{-1}z_{2\alpha}\xi z_{1\alpha})=\rho'_\gamma(z_{2\alpha})\circ(\lambda_\alpha J_{(\xi)})\circ\rho_\gamma(z_{1\alpha}) \quad \textit{for } z_{1\alpha}\in T_\gamma,\ z_{2\alpha}\in T'_\gamma, \tag{8.34}
$$

$$
k_\alpha(z_\alpha)=0 \quad \textit{outside } T'^{\xi}_\gamma T_\gamma=\xi^{-1}(T'_\gamma\xi T_\gamma). \tag{8.34'}
$$

### 8.7. Twsits by $\xi\in\mathfrak{S}_n$

By Theorem 1.9 (cf. Theorem DG3.13), $P_{2\xi}=\mathrm{Ind}_{Y_{2\xi}}^{Y_\xi}\pi_{2\xi}$ is given by the datum

$$
(A_{\gamma\xi},\ \mathrm{Ind}(\rho'^{\xi}_\gamma;\ T'^{\xi}_\gamma\uparrow\mathfrak{S}_n),\ \chi_{A_{\gamma\xi}},\ \mathrm{Ind}(a'_{\gamma\xi};\ T'^{\xi}_\gamma\uparrow\mathfrak{S}_n)) \quad \text{with}
$$

$$
\mathrm{Ind}(a'_{\gamma\xi};\ T'^{\xi}_\gamma\uparrow\mathfrak{S}_n)=(\mathrm{Ind}(a'_{\gamma,\alpha};\ T'^{\xi}_\gamma\uparrow\mathfrak{S}_n))_{\alpha\in A_{\gamma\xi}} \quad \text{for } a'_{\gamma\xi}=(a'_{\gamma,\alpha})_{\alpha\in A_{\gamma\xi}}.
$$

In this datum, $\mathrm{Ind}(\rho'^{\xi}_\gamma;\ T'^{\xi}_\gamma\uparrow\mathfrak{S}_n)$ is equivalent to $\mathrm{Ind}(\rho'_\gamma;\ T'_\gamma\uparrow\mathfrak{S}_n)=\mathrm{Ind}_{T'_\gamma}^{\mathfrak{S}_n}\rho'_\gamma$, because $(\rho'^{\xi}_\gamma, T'^{\xi}_\gamma)$ is a twist of $(\rho'_\gamma, T'_\gamma)$ by $\xi\in\mathfrak{S}_n$.

Let us give an equivalence map explicitly. Similarly as in (8.30), an element $\varphi\in V(\mathrm{Ind}_{T'_\gamma}^{\mathfrak{S}_n}\rho'_\gamma)$ is given as a $V(\rho'_\gamma)$-valued function on $\mathfrak{S}_n$.

**Lemma 8.8.** *For* $\varphi\in V(\mathrm{Ind}_{T'_\gamma}^{\mathfrak{S}_n}\rho'_\gamma)$, *put*

$$
(\Phi_\xi\varphi)(s)=\varphi(\xi s) \quad (s\in\mathfrak{S}_n). \tag{8.35}
$$

*Then* $\Phi_\xi$ *gives a natural* $\mathfrak{S}_n$*-isomorphism of* $\mathrm{Ind}_{T'_\gamma}^{\mathfrak{S}_n}\rho'_\gamma$ *with* $\mathrm{Ind}(\rho'^{\xi}_\gamma;\, T'^{\xi}_\gamma\uparrow\mathfrak{S}_n)$.

### 8.8. Equivalence relation for $\gamma$-parts

To study the relation between $\xi,\ \eta\in\mathfrak{S}_n$ such that $|A_{\gamma\xi}|=|A_{\gamma\eta}|=\infty$, we should study the situation for $\gamma$-parts in § 8.5(1). For $Y=\mathfrak{S}_{A_\gamma}(\mathfrak{S}_n)$, we consider subgroups

$$Z=D_{A_\gamma}(\mathfrak{S}_n),\quad Y_1^\gamma=\phi_\gamma{}^{-1}(H_\gamma)=\mathfrak{S}_{A_\gamma}(T_\gamma),\quad Z_1^\gamma=D_{A_\gamma}(T_\gamma),$$

$$Y_2^\gamma=\phi_\gamma{}^{-1}(H'_\gamma)=\iota(\tilde{\xi})^{-1}(\mathfrak{S}_{A_\gamma}(T'_\gamma)),\quad Z_2^\gamma=\iota(\tilde{\xi})^{-1}(D_{A_\gamma}(T'_\gamma)).$$

Take IURs $\Pi(Q_\gamma)$ of $Y_1^\gamma$ and $\Pi(Q'_\gamma)\circ\iota(\tilde{\xi})$ of $Y_2^\gamma$ and consider their induced representations

$$P_1^\gamma=\mathrm{Ind}_{Y_1^\gamma}^{Y}\Pi(Q_\gamma),\qquad P_2^\gamma=\mathrm{Ind}_{Y_2^\gamma}^{Y}(\Pi(Q'_\gamma)\circ\iota(\tilde{\xi})).$$

Then they are both irreducible and mutually equivalent because of the existence of $L^\gamma\in\mathrm{Hom}(\Pi_\gamma,\ \Pi'_\gamma;\ H_\gamma\cap H'_\gamma)$ (consider rather its $\phi_\gamma{}^{-1}$-version).

The space $V(P_1^\gamma)$ consists of $V(\Pi(Q_\gamma))$-valued functions $f$ on $Y$ such that

$$f(hy)=(\Pi(Q_\gamma)(h))f(y)\qquad(h\in Y_1^\gamma,\ y\in Y),$$

$$\|f\|^2\equiv\textstyle\sum_{y\in Y_1^\gamma\backslash Y}\|f(y)\|^2<\infty,$$

Noting that $f$ is uniquely determined by $f|Z$ and replacing $f$ by $f|Z$, we may and do consider $V(P_1^\gamma)$ as a space of functions on $Z$ satisfying

$$f(hz)=(\Pi(Q_\gamma)(h))f(z)\qquad(h\in Z_1^\gamma,\ z\in Z),$$

and the $L^2$-condition on $Z_1^\gamma\backslash Z$ $(\cong Y_1^\gamma\backslash Y)$. Moreover, let $z_0\in Z$ and $\sigma\in\mathfrak{S}_{A_\gamma}$, then

(8.36) $$P_1^\gamma(z_0)f(z)=f(zz_0),\qquad P_1^\gamma(\sigma)f(z)=(\Pi(Q_\gamma)(\sigma))f(z^\sigma),$$

with $z^\sigma=(z_{\sigma(\alpha)})_{\alpha\in A_\gamma}$, and for a decomposable element $f=\otimes_{\alpha\in A_\gamma}f_\alpha,\ f_\alpha\in V(\mathrm{Ind}_{T_\gamma}^{\mathfrak{S}_n}\rho_\gamma)$,

(8.37) $$P_1^\gamma(\sigma)f=\otimes_{\alpha\in A_\gamma}g_\alpha\qquad\text{with } g_\alpha=f_{\sigma^{-1}(\alpha)}.$$

Note that here we have taken into account Theorem 1.9 (cf. Theorem DG3.13).

Similarly we can write down $P_2^\gamma$ as follows. The space $V(P_2^\gamma)$ consists of functions $f$ on $Z$ with values in $V(\Pi(Q'_\gamma)\circ\iota(\tilde{\xi}))=V(\Pi(Q'_\gamma))$ satisfying

$$f(\tilde{\xi}^{-1}h\tilde{\xi}\cdot z)=(\Pi(Q'_\gamma)(h))f(z)\qquad(h\in D_{A_\gamma}(T'_\gamma),\ z\in Z),$$

and the $L^2$-condition on $Z_2^\gamma\backslash Z$. Further, for $z_0\in Z$ and $\sigma\in\mathfrak{S}_{A_\gamma}$,

(8.38) $$P_2^\gamma(z_0)f(z)=f(zz_0),\qquad P_2^\gamma(\sigma)f(z)=(\Pi(Q'_\gamma)(\sigma))f(z'),$$

with $z'=(z'_\alpha)_{\alpha\in A_\gamma}$, $z'_\alpha=\xi_\alpha{}^{-1}\xi_{\sigma(\alpha)}z_{\sigma(\alpha)}$. In particular, for a decomposable $f=\otimes_{\alpha\in A_\gamma}f_\alpha$, $f_\alpha\in V(\mathrm{Ind}\,((\rho'_\gamma)^{\xi_\alpha};\,(T'_\gamma)^{\xi_\alpha}\uparrow\mathfrak{S}_n))$,

(8.39) $$P_2^\gamma(\sigma)f=\otimes_{\alpha\in A_\gamma}g_\alpha\qquad\text{with } g_\alpha(z_\alpha)=f_{\sigma^{-1}(\alpha)}((\xi_{\sigma^{-1}(\alpha)})^{-1}\xi_\alpha z_\alpha).$$

Let us prove the second formula in (8.38):

$$P_2^\gamma(\sigma)f(z)=f(z\cdot\sigma)=f(\sigma\cdot z^\sigma)$$
$$=f(\tilde{\xi}^{-1}\sigma\tilde{\xi}\cdot\tilde{\xi}^{-1}\tilde{\xi}^\sigma z^\sigma)=(\Pi(Q'_\gamma)(\sigma))f(\tilde{\xi}^{-1}(\tilde{\xi}z)^\sigma).$$

As remarked before, $P_1^\gamma$ and $P_2^\gamma$ are mutually equivalent and the intertwining operator $K^\gamma$ between them is described by a kernel $K^\gamma(z, z')$, $z, z' \in Z$, determined from $L^\gamma$ similarly as $K_\xi$ from $L_\xi$ in §8.6. Take $\sigma \in \mathfrak{S}_{A_\gamma}$ in particular, then

$$P_2^\gamma(\sigma) \circ K^\gamma = K^\gamma \circ P_1^\gamma(\sigma). \tag{8.40}$$

Applying (8.37), (8.39) and the results in §8.6 on $K_\xi$, we can prove (8.22) in Proposition 8.6.

### 8.9. Proof of Proposition 8.6

Assume that $|A_{\gamma\xi}| = |A_{\gamma\eta}| = \infty$ for two different $\xi, \eta \in \mathfrak{S}_n$. Then we have the intertwining operators $K_\xi$ and $K_\eta$ for $A_{\gamma\xi}$-part and $A_{\gamma\eta}$-part, that is, for the representations of $Y_\xi$ and $Y_\eta$. Put $B = A_\gamma \setminus (A_{\gamma\xi} \cup A_{\gamma\eta})$. Then the irreducibility for $Y_\xi$ and $Y_\eta$ tells us that $K^\gamma$ can be expressed as a tensor product $K^\gamma = K_\xi \otimes K_\eta \otimes K_B^\gamma$ with $B$-factor $K_B^\gamma$ of $K^\gamma$ according to $Y_\xi \times Y_\eta \times \mathfrak{S}_B(\mathfrak{S}_n) \subset Y$. Furthermore, by the results in §8.6, $K_\xi$ and $K_\eta$ are factorized as

$$K_\xi = d_\xi \cdot \otimes_{\alpha \in A_{\gamma\xi}} K_\alpha, \qquad K_\eta = d_\eta \cdot \otimes_{\alpha \in A_{\gamma\eta}} K_\alpha,$$

where $K_\alpha$ has the kernel $K_\alpha(z_\alpha, z'_\alpha) = k_\alpha(z_\alpha z'^{-1}_\alpha)$, $z_\alpha, z'_\alpha \in Z_\alpha = \mathfrak{S}_n$, and $k_\alpha(z_\alpha) = 0$ outside $\xi_\alpha^{-1} T'_\gamma \xi_\alpha T_\gamma$ ($= \xi^{-1} T'_\gamma \xi T_\gamma$ or $\eta^{-1} T'_\gamma \eta T_\gamma$ according as $\alpha \in A_{\gamma\xi}$ or $\alpha \in A_{\gamma\eta}$).

Let us now apply (8.40) to a transposition $\sigma = (\alpha_0, \beta_0)$ with $\alpha_0 \in A_{\gamma\xi}$, $\beta_0 \in A_{\gamma\eta}$. Take as $f \in V(P_1^\gamma)$ an element of the form

$$f = f_{\alpha_0} \otimes f_{\beta_0} \otimes f_C, \qquad C = A_\gamma \setminus \{\alpha_0, \beta_0\},$$

with $\alpha$-components $f_\alpha$ for $\alpha = \alpha_0, \beta_0$, and $C$-component $f_C$. Accordingly, we decompose $K^\gamma$ as $K^\gamma = K_{\alpha_0} \otimes K_{\beta_0} \otimes K_C^\gamma$. By (8.37) and (8.39), we get

$$(K^\gamma \circ P_1^\gamma(\sigma)) f = g_{\alpha_0}^{(1)} \otimes g_{\beta_0}^{(1)} \otimes (K_C^\gamma f_C), \qquad (P_2^\gamma(\sigma) \circ K^\gamma) f = g_{\alpha_0}^{(2)} \otimes g_{\beta_0}^{(2)} \otimes (K_C^\gamma f_C),$$

where

$$g_{\alpha_0}^{(1)}(z_{\alpha_0}) = (K_{\alpha_0} f_{\beta_0})(z_{\alpha_0}), \qquad g_{\beta_0}^{(1)}(z_{\beta_0}) = (K_{\beta_0} f_{\alpha_0})(z_{\beta_0});$$

$$g_{\alpha_0}^{(2)}(z_{\alpha_0}) = (K_{\beta_0} f_{\beta_0})(\eta^{-1}\xi z_{\alpha_0}), \qquad g_{\beta_0}^{(2)}(z_{\beta_0}) = (K_{\alpha_0} f_{\alpha_0})(\xi^{-1}\eta z_{\beta_0}).$$

Therefore (8.40) for $\sigma = (\alpha_0, \beta_0)$ is equivalent to

$$(K_{\alpha_0}\varphi)(s) = (K_{\beta_0}\varphi)(\eta^{-1}\xi s) \qquad (s \in \mathfrak{S}_n) \tag{8.41}$$

for $\varphi \in V(\mathrm{Ind}_{T_\gamma}^{\mathfrak{S}_n} \rho_\gamma)$. Hence, with a constant $d$,

$$d \cdot \sum_{s' \in T_\gamma \setminus \mathfrak{S}_n} k_{\alpha_0}(s s'^{-1}) \varphi(s') = \sum_{s' \in T_\gamma \setminus \mathfrak{S}_n} k_{\beta_0}(\eta^{-1}\xi s s'^{-1}) \varphi(s').$$

On the other hand, we know by Lemma 8.7 that $k_{\alpha_0}(s)$ is zero outside $\xi^{-1}(T'_\gamma \xi T_\gamma)$, and similarly $k_{\beta_0}(\eta^{-1}\xi s)$ is zero outside $(\eta^{-1}\xi)^{-1} \cdot \eta^{-1} T'_\gamma \eta T_\gamma = \xi^{-1}(T'_\gamma \eta T_\gamma)$. Thus we should have $T'_\gamma \xi T_\gamma = T'_\gamma \eta T_\gamma$ as asserted in Proposition 8.6.

### 8.10. Generation of unitary equivalence in the set of standard IURs

From our study until now, we can deduce three operations on the set of data $(\mathfrak{b}, Q)$ with $\mathfrak{b} \in \mathfrak{B}$ and $Q \in \mathfrak{Q}(\mathfrak{b})$ for standard IURs of $G = \mathfrak{S}_\infty$, and also two criterions for unitary equivalences. These five operations and criterions will generate altogether all the unitary equivalences in the set of standard IURs of $G$.

In our terminology, an operation on a datum $(\mathfrak{b}, Q)$ means a specified replacement of some parts of the datum which again gives a datum corresponding to a standard IUR equivalent to the original $\rho(\mathfrak{b}, Q)$.

(OPERATION 1) In $(\mathfrak{b}, Q)$ in (8.2) or rather in $Q$, we admit repetitions of the following replacements:

(1i) $\pi_F$ by $\pi'_F$ such that $\pi'_F \cong \pi_F$;

(1ii) $b$ by $b'=(b'_\gamma)_{\gamma\in\Gamma}$ such that $b'\cong b$;

(1iii) $(\rho^\gamma_{T_\gamma}, a(\gamma))$ by $(\rho'^\gamma_{T_\gamma}, a'(\gamma))$, where $\rho'^\gamma_{T_\gamma} \cong \rho^\gamma_{T_\gamma}$, and $a'(\gamma)\cong(M_\gamma a_{\gamma,\alpha})_{\alpha\in A_\gamma}$ with a unitary $M_\gamma \in \mathrm{Hom}_{T_\gamma}(\rho^\gamma_{T_\gamma}, \rho'^\gamma_{T_\gamma})$ and $a(\gamma)=(a_{\gamma,\alpha})_{\alpha\in A_\gamma}$ (the reference vector $b$ should be replaced accordingly).

Note that here we have taken into account Moore's criterion for equivalence of tensor products of IURs.

(EQUIVALENCE 1) Let $x\in G$. For $(\pi, H)$ with $H\in\mathfrak{A}$ and $\pi\in\mathfrak{R}$ an IUR of $H$, put $\pi^x(h)=\pi(xhx^{-1})$ for $h\in H^x=x^{-1}Hx$. Then

$$\mathrm{Ind}_H^G \pi \cong \mathrm{Ind}_{H^x}^G \pi^x .$$

Note that the replacement of $(\pi, H)$ by $(\pi^x, H^x)$ can be easily written down by means of the datum $(\mathfrak{b}, Q)$. Cf. also Lemma 5.2.

(OPERATION 2) For every $\gamma\in\Gamma$, take $t_\alpha\in T_\gamma\subset\mathfrak{S}_{n(\gamma)}$ for $\alpha\in A_\gamma$. Then, in $(\mathfrak{b}, Q)$, replace $\mathfrak{J}_\gamma$, $a(\gamma)$ and also $b$ by

$$J'_\gamma=\{J'_{\gamma,\alpha}\,;\,\alpha\in A_\gamma\} \qquad \text{with } J'_{\gamma,\alpha}=t_\alpha\circ J_{\gamma,\alpha},$$

$$a'(\gamma)=(a'_{\gamma,\alpha})_{\alpha\in A_\gamma} \qquad \text{with } a'_{\gamma,\alpha}=\rho^\gamma_{T_\gamma}(t_\alpha)a_{\gamma,\alpha},$$

$$b'=(b'_\gamma)_{\gamma\in\Gamma} \qquad \text{with } b'_\gamma=\Big(\bigotimes_{\alpha\in A_\gamma}\rho^\gamma_{T_\gamma}(t_\alpha)\Big)b_\gamma .$$

(OPERATION 3) For every $\gamma\in\Gamma$, take $\xi_\gamma\in\mathfrak{S}_{n(\gamma)}$ and replace $\mathfrak{J}_\gamma$, $T_\gamma$ and $\rho^\gamma_{T_\gamma}$ respectively by

$$\mathfrak{J}'_\gamma=\{\xi_\gamma\circ J_{\gamma,\alpha}\,;\,\alpha\in A_\gamma\},\quad T'_\gamma=T_\gamma^{\xi_\gamma} \quad\text{and}\quad \rho'^\gamma_{T'_\gamma}=(\rho^\gamma_{T_\gamma})^{\xi_\gamma}.$$

Note that Operations 2 and 3 are, so to speak, normalizations in the set of data $(\mathfrak{b}, Q)$ or equivalence relations in it to reduce the degree of freedom of choosing $(\mathfrak{b}, Q)$ for essentially the same $(\pi, H)$, $H\in\mathfrak{A}$ and $\pi\in\mathfrak{R}$ of $H$. In fact, in each case, eventhough $\psi'_\gamma=\psi_\gamma\circ\iota(\tilde{\xi})^{-1}$, $\tilde{\xi}=(t_\alpha)_{\alpha\in A_\gamma}$ or $\tilde{\xi}=(\xi_\alpha)_{\alpha\in A_\gamma}$ with $\xi_\alpha=\xi_\gamma$, the group $H_\gamma=\psi_\gamma(\mathfrak{S}_{A_\gamma}(T_\gamma))$ coincides with $\psi'_\gamma(\mathfrak{S}_{A_\gamma}(T'_\gamma))$, and its representations to be induced up are mutually equivalent in Operation 2 and are the same in Operation 3. We can see the meanings of these operations also from the discussions in § 2.3, especially from Lemmas 2.1 and 2.4. See also Theorem 3.3.

The following equivalence criterion gives a quite new feature to our study, which has first been encountered in the case of infinite wreath product groups (cf. Theorem

1.10 or Theorem DG4.5).

(EQUIVALENCE 2) For two data $(\mathfrak{b}, Q)$ and $(\mathfrak{b}', Q')$, introduce the relation consisting of the following:

(2i) $\mathfrak{b}'=\mathfrak{b}$, and $\pi'_F=\pi_F$ $(F'=F)$;

(2ii) for every $\gamma\in\Gamma$, put $S_\gamma=T_\gamma\cap T'_\gamma$, then there exist an IUR $\tau_\gamma$ of $S_\gamma$ and a reference vector $c(\gamma)=(c_{\gamma\alpha})_{\alpha\in A_\gamma}$, $c_{\gamma\alpha}\in V(\tau_\gamma)$, $\|c_{\gamma\alpha}\|=1$, such that

$$\rho^\gamma_{T_\gamma}=\mathrm{Ind}^{T_\gamma}_{S_\gamma}\tau_\gamma, \qquad \rho'^{\gamma}_{T'_\gamma}\cong\mathrm{Ind}^{T'_\gamma}_{S_\gamma}\tau_\gamma, \tag{8.42}$$

$$a(\gamma)\cong(K_\alpha(\mathrm{Ind}^{T_\gamma}_{S_\gamma}c_{\gamma\alpha}))_{\alpha\in A_\gamma}, \qquad a'(\gamma)\cong(K'_\alpha(\mathrm{Ind}^{T'_\gamma}_{S_\gamma}c_{\gamma\alpha}))_{\alpha\in A_\gamma},$$

where $K_\alpha$ (resp. $K'_\alpha$) denotes a fixed $T_\gamma$- (resp. $T'_\gamma$-) isomorphism for (8.42);

(2iii) for every $\gamma\in\Gamma$, let $H_{0\gamma}=\psi_\gamma(\mathfrak{S}_{A_\gamma}(S_\gamma))$ and define its standard IUR $\pi_{0\gamma}$ from the datum $((\tau_\gamma, S_\gamma), \chi_{A_\gamma}, c(\gamma))$, then there exists a unit vector $e_\gamma\in V(\pi_{0\gamma})$ such that

$$b\cong(K_\gamma(\mathrm{Ind}^{H_\gamma}_{H_{0\gamma}}e_\gamma))_{\gamma\in\Gamma}, \qquad b'\cong(K'_\gamma(\mathrm{Ind}^{H'_\gamma}_{H_{0\gamma}}e_\gamma))_{\gamma\in\Gamma},$$

where $K_\gamma=\psi_\gamma\circ(\otimes_{\alpha\in A_\gamma}K_\alpha)\circ\psi_\gamma^{-1}$, $K'_\gamma=\psi_\gamma\circ(\otimes_{\alpha\in A_\gamma}K'_\alpha)\circ\psi_\gamma^{-1}$,

Note that in (2iii) we take into account Theorem 1.9, and $\psi'_\gamma=\psi_\gamma$ coming from $\mathfrak{b}'=\mathfrak{b}$ in (2i).

### 8.11. Unitary equivalences in the set of standard IURs

We can give finally one of our main results, the most important one.

**Theorem 8.9.** *In the set of all standard IURs* $\rho(\mathfrak{b}, Q)=\mathrm{Ind}^G_{H^\mathfrak{b}}\pi(\mathfrak{b}, Q)$ *the relation of unitary equivalence is generated by Operations* 1, 2 *and* 3, *and Equivalences* 1 *and* 2.

*Proof.* We give here a sketch of our proof since some parts of it are a kind of repetitions of discussions in §§DC4-DG8 in the case of infinite wreath product groups studied in detail in [DG] and also since another essential part of it has already been given in Proposition 8.6.

Assume $\rho(\mathfrak{b}, Q)\cong\rho(\mathfrak{b}', Q')$. Then the proof goes along the following line.

(i) We apply in §8.1 Equivalence 1 and reduce the discussion to the case of the boundedness conditions $(\mathrm{B}_e)$ and $(\mathrm{C}_e)$ for $x=e$.

(ii) In Proposition 8.6, we get $T'_\gamma\xi T_\gamma=T'_\gamma\eta T_\gamma$ if $|A_{\gamma\xi}|=|A_{\gamma\eta}|=\infty$. Fix such a $\xi$ and represent $\xi_{\gamma\alpha}$ in (8.18) as $\xi_{\gamma\alpha}=t'_\alpha\xi t_\alpha$ with $t'_\alpha\in T'_\gamma$, $t_\alpha\in T_\gamma$. Again applying Equivalence 1 if necessary, we may replace a finite number of $\xi_{\gamma\alpha}$'s appropriately and hence may assume that the above expression is possible for any $\alpha\in A_\gamma$.

(iii) We apply Operation 2 twice: once for $(\mathfrak{b}, Q)$ and $(t_\alpha)_{\alpha\in A_\gamma}$ and once for $(\mathfrak{b}', Q')$ in (8.12)-(8.13) and $(t'^{-1}_\alpha)_{\alpha\in A_\gamma}$.

(iv) Thus we come to the case where $A_{\gamma\xi}=A_\gamma$ for some $\xi\in\mathfrak{S}_{n(\gamma)}$. Then we can apply Operation 3 and then arrive to the case $\xi=e$ (for each $\gamma\in\Gamma$), and hence to $\mathfrak{b}=\mathfrak{b}'$.

(v) Under the condition (2i) in Equivalence 2, we can apply the results given in

§DG4 for infinite wreath product groups (cf. Theorem 1.10). Then we get (2ii) and (2iii). Note that the condition (2iii) says that the reference vectors $b$ and $b'$ are mutually equivalent in the sense formulated in Definition 4.6′ in [DG]. Thus we have arrived at Equivalence 2.

The converse way is easy to follow, that is, we see easily that Operations 1-3 and Equivalences 1-2 give rise to unitary equivalences between standard IURs. Q.E.D.

### 8.12. Final remarks

**Remark 8.10.** Restrict ourselves to the set $\mathfrak{R}_f$ of finite-dimensional IURs of subgroups in $\mathfrak{A}$. Then Equivalence 2 has no place to apply since there appear only characters $\rho^{\gamma}_{T_\gamma}$'s in this case, which can not be obtained as induced representations. Further the reference vectors play no role. Operations 2 and 3 have no essential meaning because the subgroups $H_\gamma$ and $H'_\gamma$ there coincide with each other. Thus the only thing essential is Equivalence 1 as we saw in Theorem 5.1.

**Remark 8.11.** In the case of infinite wreath product group $\mathfrak{S}_A(T)$ studied in [DG], we have formulated the result so as to exclude Operation 2 from the biginning, and moreover Operation 3 is almost trivial. However Equivalences 1 and 2 both play essential roles (cf. Theorems DG4.2, DG4.5 and DG4.7).

**Remark 8.12.** The families $\mathfrak{A}$ and $\mathfrak{R}$ for $G=\mathfrak{S}_\infty\equiv\mathfrak{S}_{\boldsymbol{N}}$ are invariant under $\mathrm{Aut}(G)=\tilde{\mathfrak{S}}_{\boldsymbol{N}}$, the group of all permutations on $\boldsymbol{N}$. Since the equivalence relation in the family of all standard IURs of $G$ is completely known in Theorem 8.9, we know how the outer automorphisms of $G$ act on the family.

**Remark 8.13.** Our method here consists of (a) saturated imbeddings of wreath product groups and (b) inducing up their standard IURs. This method can be applied to other types of infinite discrete groups such as $GL(\infty, \boldsymbol{F}_q)$ or $SL(\infty, \boldsymbol{F}_q)$.

## Appendix. On the equivalence for the tensor products of representations

Let $A$ be a set of indices and, for each $\alpha\in A$, $G_\alpha$ be a topological group and $K_\alpha$ its open compact subgroup. Let $\pi_\alpha$ be a continuous irreducible unitary representation (=IUR) of $G_\alpha$ which has a non-zero $K_\alpha$-invariant vector. Put $V_\alpha=V(\pi_\alpha)$ and let $a=(a_\alpha)_{\alpha\in A}$ be a reference vector consisting of $K_\alpha$-invariant $a_\alpha\in V_\alpha$, $\|a_\alpha\|=1$, and consider the tensor product of $\pi_\alpha$'s with respect to $a$:

$$\pi^a=\otimes^a_{\alpha\in A}V_\alpha \qquad \text{on } V^a=\otimes^a_{\alpha\in A}V_\alpha .$$

This is an IUR of the restricted direct product $G_A=\prod'_{\alpha\in A}(G_\alpha, K_\alpha)$ of $(G_\alpha)_{\alpha\in A}$ with respect to $(K_\alpha)_{\alpha\in A}$ (for the definition, cf. [14]). This contains the case of discrete groups where we take $K_\alpha$ as the trivial subgroup consisting of the identity element.

We give here a simple proof of the following criterion for mutual equivalence due to C.C. Moore [14].

**Theorem A1.** *Let* $a=(a_\alpha)_{\alpha\in A}$ *and* $b=(b_\alpha)_{\alpha\in A}$ *be two reference vectors with* $K_\alpha$-*invariant* $a_\alpha$, $b_\alpha$, *and let* $\{\pi^a, V^a\}$ *and* $\{\pi^b, V^b\}$ *be the tensor products of* $\pi_\alpha$, $\alpha\in A$, *with respect to them respectively. Then* $\pi^a\cong\pi^b$ *(unitary equivalent) if and only if* $a\cong b$ *(Moore-equivalent), i.e.,*

(A1) $$\sum_{\alpha\in A}(1-|\langle a_\alpha, b_\alpha|)<\infty .$$

*Proof.* For any subset $B\subset A$, put

$$G_B=\prod'_{\alpha\in B}(G_\alpha, K_\alpha),\quad a_B=(a_\alpha)_{\alpha\in B},\quad V_B^{a_B}=\otimes_{\alpha\in B}^{a_B}V_\alpha,\quad \pi_B^{a_B}=\otimes_{\alpha\in B}^{a_B}\pi_\alpha ,$$

Then $\pi_B^{a_B}$ is an IUR of $G_B$ on $V_B^{a_B}$. Consider a vector in $V_B^{a_B}$ given by $\underline{a}_B=\otimes_{\alpha\in B}a_\alpha$. Under the condition (A1), the vector $\underline{a}_A$ can be considered as a decomposable element in $V^b$, and similarly for $\underline{b}_A$ and $V^a$. This proves the "if" part of the theorem.

Now let us prove the "only if" part. Assume that $\pi^a\cong\pi^b$. Take a non-zero $L\in \mathrm{Hom}(\pi^a, \pi^b; G_A)$. For any finite subset $F$ of $A$, put $\pi_F=\otimes_{\alpha\in F}\pi_\alpha$, $V_F=\otimes_{\alpha\in F}V_\alpha$ and $B=A\setminus F$, then

$$\pi^a\cong\pi_F\otimes\pi_B^{a_B},\quad V^a\cong V_F\otimes V_B^{a_B};\quad \pi^b\cong\pi_F\otimes\pi_B^{b_B},\quad V^b\cong V_F\otimes V_B^{b_B},$$

From the irreducibility of $\pi_F$ on $V_F$, we see easily that $L$ can be expressed as $L=I_{V_F}\otimes L_B$ with an $L_B\in\mathrm{Hom}(\pi_B^{a_B}, \pi_B^{b_B}; G_B)$, where $I_{V_F}$ denotes the identity operator on $V_F$. Put $w=L\underline{a}_A$, then $w\neq 0$, and for any $F\subset A$, $w$ can be expressed as

(A2) $$w=\underline{a}_F\otimes x_B \qquad \text{with } x_B=L_B\underline{a}_B\in V_B^{a_B} .$$

On the other hand, it follows from the definition of tensor product space $V^b$ that every vector in it can be approximated by elements of the form $y_F\otimes\underline{b}_B$, where $F\subset A$ is finite and $y_F\in V_F$, $B=A\setminus F$. Hence, for any $\varepsilon>0$, we find a finite $F\subset A$ and a $y_F\in V_F$ such that

(A3) $$\|w-y_F\otimes\underline{b}_B\|<\varepsilon .$$

By (A2)-(A3), we get

$$\varepsilon^2\geqq\|\underline{a}_F\otimes x_B-y_F\otimes\underline{b}_B\|^2=\|w\|^2+\|y_F\|^2-2\,\mathrm{Re}\,(\langle\underline{a}_F, y_F\rangle\langle x_B, \underline{b}_B\rangle) .$$

Therefore, if $\varepsilon<\|w\|$, then $\langle x_B, \underline{b}_B\rangle\neq 0$. In turn, for any finite $F'\subset B$, we have $x_B=\underline{a}_{F'}\otimes x_{B'}$ with $B'=B\setminus F'$, and so

$$\langle x_B, \underline{b}_B\rangle=(\prod_{\alpha\in F'}\langle a_\alpha, b_\alpha\rangle)\cdot\langle x_{B'}, \underline{b}_{B'}\rangle .$$

Since $|\langle x_{B'}, \underline{b}_{B'}\rangle|\leqq\|x_{B'}\|=\|w\|$, the product $\prod_{\alpha\in B}|\langle a_\alpha, b_\alpha\rangle|$ should converge. Hence we have $\sum_{\alpha\in B}(1-|\langle a_\alpha, b_\alpha\rangle|)<\infty$, which is equivalent to the condition (A1) in the theorem.

**Added in Proof.** Reference [8] has appeared in Japan. J. Math., **16** (1990), 197-268.

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