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Note on KO-theory of $BO(n)$ and $BU(n)$

By

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§ 0. Introduction

Atiyah and Segal determined the KO-theory of BG, the classifying space of a group G, by its representation rings [3]. In this paper, we describe $KO^*(X)$ for $X=BO(n)$ and $BU(n)$ in words of groups derived from maps $1\pm\tau$ on $K(X)$, where τ is the conjugation map.

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§ 1. Bott exact sequence

Let $KO^*(X)$ and $K^*(X)$ be the real and complex K-theories. The coefficient rings are known as

$$KO^*=\mathbf{Z}[\eta, \alpha, \beta, \beta^{-1}]/(2\eta, \eta^3, \alpha^2-4\beta),$$
$$K^*=\mathbf{Z}[t, t^{-1}],$$

$\deg\eta=-1$, $\deg\alpha=-4$, $\deg\beta=-8$, $\deg t=-2$.

Let $c_*: KO^*(X)\to K^*(X)$ and $r_*: K^*(X)\to KO^*(X)$ be the complexification and the real restriction. It is well known that

$$r_i c_i=2: KO^i(X)\longrightarrow KO^i(X),$$

$$c_i r_i=1+(\tau)^i:\ \begin{array}{ccc} K^i(X) & \longrightarrow & K^i(X) \\ \cong\downarrow & & \cong\downarrow \\ K(X) & \longrightarrow & K(X), \end{array}$$

where τ is the conjugation map.

Consider the *Bott exact sequence* [2]:

(1.1) $$\cdots\longrightarrow KO^i(X)\xrightarrow{c_i} K^i(X)\xrightarrow{\rho_i} KO^{i+2}(X)\xrightarrow{\eta_{i+2}} KO^{i+1}(X)\longrightarrow\cdots,$$

where $\eta_{i+2}: KO^{i+2}(X)\to KO^{i+1}(X)$ is multiplication by $\eta\in KO^{-1}$, and $\rho_i: K^i(X)\to KO^{i+2}(X)$ is the composite: $K^i(X)\xrightarrow{t^{-1}} K^{i+2}(X)\xrightarrow{r_{i+2}} KO^{i+2}(X)$.

Let $D_1^i=KO^i(X)$, $E_1^i=K^i(X)$, then we get the exact triangle:

$(1.2)_1$ $$\begin{array}{ccc} D_1^* & \xrightarrow{\ \eta\ } & D_1^* \\ {\scriptstyle\rho}\nwarrow & & \swarrow{\scriptstyle c} \\ & E_1^* & \end{array}$$

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From this, we have a spectral sequence [6] such that:

$$(1.2)_r \qquad \begin{array}{ccc} D_r^* & \longrightarrow & D_r^* \\ & \nwarrow \quad \swarrow & \\ & E_r^* & \end{array}$$

is the derived exact triangle, where

$$D_r^i = \mathrm{Im}\,[\eta^{r-1} : KO^{i+r-1}(X) \longrightarrow KO^i(X)],$$

and the degree of differential is given by

$$(1.3) \qquad d_r^i : E_r^i \longrightarrow E_r^{i-r+3}.$$

Especially when $r=1$ and $i=2j$, the next diagram is commutative:

$$\begin{array}{ccc} E_1^{2j} & \xrightarrow{d_1} & E_1^{2j+2} \\ \| & & \| \\ K^{2j}(X) & \longrightarrow & K^{2j+2}(X) \\ \cong\downarrow & & \cong\downarrow \\ K(X) & \xrightarrow{1+(-1)^{j+1}\tau} & K(X). \end{array}$$

Thus, let $H^{even}(K(X))=\mathrm{Ker}(1-\tau)/\mathrm{Im}(1+\tau)$ and $H^{odd}(K(X))=\mathrm{Ker}(1+\tau)/\mathrm{Im}(1-\tau)$, then

$$(1.4) \qquad E_2^{2j} \cong \begin{cases} H^{even}(K(X)) & (\textit{if } j \textit{ even}) \\ H^{odd}(K(X)) & (\textit{if } j \textit{ odd}). \end{cases}$$

As $\eta^3=0$, we have $D_4^*=0$ and the spectral sequence collapses. This implies

$$(1.5) \qquad \cdots \xrightarrow{d_3} E_3^{i-4} \xrightarrow{d_3} E_3^i \xrightarrow{d_3} E_3^{i+4} \xrightarrow{d_3} \cdots \qquad (\textit{exact}).$$

For many spaces X we can easily check $K^{odd}(X)=0$, for example when $X=BG$, the classifying space of group or X is a CW complex with cells only in even dimensions. We suppose the assumption through this paper.

Then by Bott sequence, we have

$$(1.6)_j \quad 0 \longrightarrow KO^{2j+1}(X) \xrightarrow{\eta_{2j+1}} KO^{2j}(X) \xrightarrow{c_{2j}} K^{2j}(X) \xrightarrow{\rho_{2j}} KO^{2j+2}(X) \xrightarrow{\eta_{2j+2}} KO^{2j+1}(X) \longrightarrow 0$$

is exact.

From this, we have many exact sequences.

Lemma 1. *Suppose* $K^{odd}(X)=0$, *then*

$$\cdots \longrightarrow KO^{2j+1}(X) \xrightarrow{\eta^2} KO^{2j-1}(X) \longrightarrow H^j(K(X)) \longrightarrow KO^{2j+3}(X) \xrightarrow{\eta^2} KO^{2j+1}(X) \longrightarrow \cdots$$

is exact.

Proof. By (1.6), we have

$$D_2^{2j} = \eta KO^{2j+1}(X) \cong KO^{2j+1}(X),$$

and

$$D_2^{2j-1}=\eta KO^{2j}(X)=KO^{2j-1}(X).$$

Consider $(1.2)_2$, The next is commutative:

$$\begin{array}{ccccccc}
\cdots\longrightarrow & D_2^{2j} & \longrightarrow & D_2^{2j-1} & \longrightarrow & E_2^{2j} \\
& \eta\uparrow\cong & & \| & & \| \\
\cdots\longrightarrow & KO^{2j+1}(X) & \xrightarrow{\eta^2} & KO^{2j-2}(X) & \longrightarrow & H^j(K(X))
\end{array}$$

$$\begin{array}{cccccc}
\longrightarrow & D_2^{2j+2} & \longrightarrow & D_2^{2j} & \longrightarrow\cdots \\
& \| & & \eta\uparrow\cong & \\
\longrightarrow & KO^{2j+3}(X) & \xrightarrow{\eta^2} & KO^{2j+1}(X) & \longrightarrow\cdots
\end{array}$$

The upper line is exact, so is the lower one.

Again by (1.6), we get

$$KO^{2j+1}(X)\cong \mathrm{Ker}\,[KO^{2j}(X)\xrightarrow{c_{2j}} K^{2j}(X)].$$

Moreover, if $K^{even}(X)$ is 2-torsion free, we have

$$KO^{2j+1}(X)\cong \mathbf{Z}/2 \text{ part of } KO^{2j}(X),$$

(see Lemma 2.1 [5]), but here we consider different assumptions, and investigate the kernel and the cokernel of the maps c, r and η.

Theorem 2. *If $K^{odd}(X)=0$, then the conditions are equivalent:*

(A) $$H^{odd}(K(X))=0,$$

(B) $$KO^{4k-3}(X)=0 \qquad (\textit{for all } k),$$

and if either of them is satisfied, then the followings are exact sequences for all k:

(1) **Complexification** c:

(i) $$0\longrightarrow KO^{4k}(X)\xrightarrow{c_{4k}} K^{4k}(X)\xrightarrow{\rho_{4k}} KO^{4k+2}(X)\longrightarrow 0.$$

(ii) $$0\longrightarrow KO^{4k-1}(X)\xrightarrow{\eta_{4k-1}} KO^{4k-2}(X)\xrightarrow{c_{4k-2}} K^{4k-2}(X)\longrightarrow \mathrm{Coker}\,(1-\tau)\longrightarrow 0.$$

(2) **Realization** r:

(iii) $$0\longrightarrow KO^{4k}(X)\xrightarrow{t^{-1}c_{4k}} K^{4k+2}(X)\xrightarrow{r_{4k+2}} KO^{4k+2}(X)\longrightarrow 0,$$

(iv) $$0\longrightarrow \mathrm{Ker}\,(1+\tau)\longrightarrow K^{4k}(X)\xrightarrow{r_{4k}} KO^{4k}(X)\xrightarrow{\eta_{4k}} KO^{4k-1}(X)\longrightarrow 0.$$

(3) **The multiplication by** η:

(v) $$0\longrightarrow KO^{4k-1}(X)\xrightarrow{\eta_{4k-1}} KO^{4k-2}(X)\longrightarrow \mathrm{Ker}\,(1+\tau)\longrightarrow 0.$$

(vi) $$0\longrightarrow \mathrm{Im}\,(1+\tau)\longrightarrow KO^{4k}(X)\xrightarrow{\eta_{4k}} KO^{4k-1}(X)\longrightarrow 0.$$

(4) **Other groups derived from** $1\pm\tau$:

(vii) $0 \longrightarrow KO^{4k-1}(X) \longrightarrow \mathrm{Coker}(1+\tau) \longrightarrow KO^{4k+2}(X) \longrightarrow 0.$

(viii) $0 \longrightarrow KO^{4k-4}(X) \longrightarrow \mathrm{Ker}(1-\tau) \longrightarrow KO^{4k-1}(X) \longrightarrow 0.$

(ix) $0 \longrightarrow KO^{4k-1}(X) \longrightarrow H^{even}(K(X)) \longrightarrow KO^{4k+3}(X) \longrightarrow 0.$

Proof. By Lemma 1, (A) implies

$$0 \longrightarrow KO^{4k-3}(X) \xrightarrow{\eta^2} KO^{4k-4}(X) \longrightarrow H^{even}(K(X)) \longrightarrow KO^{4k-1}(X) \xrightarrow{\eta^2} KO^{4k-3}(X) \longrightarrow 0$$

is exact.

As $\eta^4=0$, we have $KO^{4k-3}(X)=0$. When (B) is sutisfied, (A) is obtained directly from Lemma 1.

Suppose the either of them is sutisfied. By $(1.6)_{2k}$, we have (i) and (iii). Consider $(1.6)_{2k-1}$. As ρ_{4j-4} is surjective and c_{4k} is injective, $\mathrm{Im}\, c_{4k-2}=\mathrm{Im}\, d_2^{4k-4}=\mathrm{Im}(1-\tau)=\mathrm{Ker}(1+\tau)=\mathrm{Ker}\, d_2^{4k-2}=\mathrm{Ker}\, \rho_{4k-2}\cong\mathrm{Ker}\, r_{4k}$. Hence we have (iv) and (v). Besides, $\mathrm{Im}\, \rho_{4j-2}\cong\mathrm{Coker}\, c_{4k-2}=\mathrm{Coker}(1-\tau)\cong\mathrm{Coim}(1+\tau)\cong\mathrm{Im}(1+\tau)$. This leads (ii) and (vi). Take the pushout of (i) by

$$K^{4k-2}(X) \xrightarrow{\rho_{4k-2}} KO^{4k}(X),$$

then we get (vii). Take the pullback of (i) by

$$KO^{4k+2}(X) \xrightarrow{c_{4k+2}} K^{4k+2}(X),$$

then we get (viii). (ix) follows from Lemma 1.

§2. $KO^*BO(n)$

Theorem 3. *If a space X satisfies the next conditions*

$$K^i(X)=\begin{cases} 2\text{-tortion free} & (\text{for } i=\text{even}) \\ 0 & (\text{for } i=\text{odd}), \end{cases}$$

$$c: KO(X) \longrightarrow K(X) \qquad \text{is surjective,}$$

then the following isomorphisms hold:

(a) $KO^0(X) \underset{\cong}{\xrightarrow{c_0}} K^0(X)$,

(b) $KO^{-1}(X)\cong KO^{-2}(X)\cong \eta KO^0(X)\cong K^0(X)\otimes Z/2$,

(c) $KO^{-4}(X) \underset{\cong}{\xleftarrow{r_{-4}}} K^{-4}(X)$,

(d) $KO^i(X)\cong 0 \quad (i=-3, -5, -6, -7)$.

Proof. Surjectivity of c implies $\tau=1$ and $1+\tau=2$ is monic, as $K(X)$ is 2-torsion free. Therefore $\mathrm{Ker}(1+\tau)\cong H^{odd}(K(X))\cong 0$, and the assumption (A) of Theorem 1 is satisfied. Thus by (B) we have $KO^{-3}(X)\cong KO^{-7}(X)\cong 0$. Again from surjectivity of c_0, we have $KO^{-6}(X)\cong 0$ by (i), and $KO^{-5}(X)\cong 0$ by (ii). The others can be easily checked.

Corollary 4 . *For* $X=pt$ *or* $BO(n)$, (a), (b), (c) *and* (d) *hold.*

Proof. It is a well known fact for $X=pt$. For $X=BO(n)$. The complexification: $RO(O(n))\xrightarrow{c}R(O(n))$ is an isomorphism [7]. After completion, the assumption of Theorem 3 is satisfied.

§3. Atiyah-Hirzebruch spectral sequence for $KO^*BU(n)$

In this section we compute the spectral sequece for $KO^*(BU(n))$ and see the condition (B) of Theorem 1 is satisfied.

In Atiyah-Hirzebruch spectral sequece for KO:

$$H^*(X: KO^*)\Rightarrow KO^*(X),$$

the first differential d_2 is given as following [4]:

$$d_2^{p,*}=\begin{cases} Sq^2\pi_2 & (\text{if } p\equiv 0\ (8)) \\ Sq^2 & (\text{if } p\equiv -1\ (8)) \\ 0 & (\text{otherwise}), \end{cases} \tag{3.1}$$

where $\pi_2: H^*(X: \mathbf{Z})\to H^*(X: \mathbf{Z}/2)$ is modulo 2 reduction.

Let $X=BU(n)$, then

$$H^*(BU(n): R)=R[c_1, c_2, \cdots, c_n], \qquad \deg c_i=2i,$$

where c_i is the i-th Chern class, with any ring R. Thus for $p(\equiv -1\ (8))$, we have

$$E_3^{p,*}=H(H^*(BU(n): \mathbf{Z}/2), Sq^2).$$

By Wu formula, we know that

$$Sq^2c_i=\begin{cases} c_1c_i & (\text{if } i=odd) \\ c_{i+1}+c_1c_i & (\text{if } i=even). \end{cases} \tag{3.2}$$

(Remark $c_{i+1}=0$ for $i+1>n$.)

Lemma 5.

$$H(H^*(BU(n): \mathbf{Z}/2), Sq^2)\cong \mathbf{Z}/2[c_2^2, c_4^2, \cdots, c_{2[n/2]}^2].$$

Proof. Let $A=H^*(BO(n): \mathbf{Z}/2)\cong \mathbf{Z}/2[c_1, c_2, \cdots, c_n]$ and $d=Sq^2$. Then (A, d) is a differential algebra. Define $\bar{c}_{odd}$ by

$$\bar{c}_{2n+1}=c_{2n+1}+c_1c_{2n}, \tag{3.3}$$

$$\bar{c}_1=c_1.$$

and subalgebras M_k $(2k+1\leq n)$ and N by

$$M_k=\mathbf{Z}/2[c_{2k},\ \bar{c}_{2k+1}],$$

$$N=\begin{cases}\mathbf{Z}/2[c_1,\ c_n] & (\text{if } n=even),\\ \mathbf{Z}/2[c_1] & (\text{if } n=odd).\end{cases}$$

Then by (3.3), we get

$$dc_{2n}=\bar{c}_{2n+1},$$

$$d\bar{c}_{2n+1}=0,$$

and M_k and N are the sub differential algebra of M. A is split as

$$A\cong M_1\otimes M_2\otimes\cdots\otimes M_{[n/2]-1}\otimes N$$

and it is easy to check that

$$H(M_k)=\mathbf{Z}/2[c_{2k}^2],$$

$$H(N)=\begin{cases}\mathbf{Z}/2[c_n^2] & (\text{if } n=even)\\ \mathbf{Z}/2 & (\text{if } n=odd).\end{cases}$$

Therefore, by Künneth formula, we have

$$H(A)\cong H(M_1)\otimes H(M_2)\otimes\cdots\otimes H(M_{[n/2]-1})\otimes H(N)$$

$$\cong\mathbf{Z}/2[c_2^2,\ c_4^2,\ \cdots,\ c_{2[n/2]}^2].$$

Consider the maps derived from inclusions.

$$q: BU(n)\longrightarrow BSp(n)$$

$$c': BSp(n)\longrightarrow BSU(2n)\longrightarrow BU(2n)$$

We know that

$$H^*(BSp(n):\mathbf{Z}/2)=\mathbf{Z}/2[q_1,\ q_2,\ \cdots,\ q_n],\qquad \deg q_i=4i,$$

and

$$q^*q_i=c_i^2,$$

(3.4) $$c'^*c_i=\begin{cases}q_{i/2} & (\text{if } i=even)\\ 0 & (\text{if } i=odd).\end{cases}$$

Proposition 6. *The Atiyah-Hirzebruch spectral sequence $E_r^{*,*}$ for $KO^*(BU(n))$ collapses for $r\geqq 3$ and is strongly covergent.*

Proof. The Atiyah-Hirzebruch spectral sequence for $KO^*(BSp(n))$ collapses. To see this it is enough to show the elements in $E_2^{*,0}$ are permanent cycles and, by degree reason, it can be easily checked.

Consider the maps between the Atiyah-Hirzebruch spectral sequences:

$$E_3^{*,q}(q): E_3^{*,q}(BSp(n))\longrightarrow E_3^{*,q}(BU(n)),$$

$$E_3^{*,q}(c'): E_3^{*,q}(BU(n))\longrightarrow E_3^{*,q}(BSp(n)).$$

If $q\equiv -1$ (8), by (3.4), the elements of $E_3^{*,q}(BU(n))$ are in the image of $E_3^{*,q}(q)$, and

$E_3^{*,q}(c')$ is an monomrphism. Hence the triviality of $E_3^{*,q}(BSp(n))$ implies $E_r^{*,q}(BU(n)) \cong E_3^{*,q}(BU(n))$ $(r \geqq 3)$. Therefore the nontrivial candidates of sources or targets of d_r are in $E_r^{*,q}$, with $q \equiv 0, -2, -4$ (8). They concentrate in even degrees, so we conclude that $d_r = 0$ for $r \geqq 3$.

Consequently the Atiyah-Hirzebruch spectral sequence is finitely convergent, so it is strongly convergent (Proposition 9, [1]).

§4. $KO^*BU(n)$

Theorem 7. *Let τ be the conjugation map of K-theory. We have following isomorphisms:*

(a) $KO^0(BU(n)) \cong \mathrm{Ker}(1-\tau)$,

(b) $KO^{-1}(BU(n)) \cong \mathrm{Ker}(1-\tau)/\mathrm{Im}(1+\tau)$,

(c) $KO^{-2}(BU(n)) \cong \mathrm{Coker}(1+\tau)$,

(d) $KO^{-4}(BU(n)) \cong \mathrm{Coker}(1-\tau) \cong \mathrm{Im}(1+\tau)$,

(e) $KO^{-6}(BU(n)) \cong \mathrm{Im}(1-\tau) \cong \mathrm{Ker}(1+\tau)$,

(f) $KO^q(BU(n)) \cong 0 \quad (q = -3, -5, -7)$.

Especially, $KO(BU(n))$ is isomorphic to the τ-invariant elements of $K(BU(n))$.

Proof. From the results of the Atiyah-Hirzebruch spectral sequece (Lemma 5, Proposition 6), the elments which have odd degrees are in $E_3^{*,-1} \cong \mathbf{Z}/2\langle \eta \rangle\ [c_2^2, c_4^2, \cdots, c_{2[n/2]}^2]$, and the degrees are all -1 modulo (8). Thus we get (f) and the condition of Theorem 1 is satisfied. Moreover $KO^{4k+3}(BU(n)) \cong 0$. Thus (viii), (ix), (vii), (vi) and (v) imply (a), (b), (c), (d) and (e), respectively.

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