J. Math. Kyoto Univ. (JMKYAZ)
31-2 (1991) 419-466

Logarithmic Enriques surfaces

By

De-Qi ZHANG

Introduction

Normal projective surfaces with only quotient singularities appear in studies of threefolds and semi-stable degenerations of surfaces (cf. Kawamata [5], Miyanishi [6], Tsunoda [11]). We have been interested in such singular surfaces with logarithmic Kodaira dimension $-\infty$ (cf. Miyanishi-Tsunoda [8], Zhang [12, 13]). In the present paper, we shall study the case of logarithmic Kodaira dimension 0.

Let $\bar{V}$ be a normal projective rational surface with only quotient singularities but with no rational double singular points. Let $K_{\bar{V}}$ be the canonical divisor of $\bar{V}$ as a Weil divisor. We call $\bar{V}$ a logarithmic Enriques surface if $H^1(\bar{V}, \mathcal{O}_{\bar{V}})=0$ and $K_{\bar{V}}$ is a trivial Cartier divisor for some positive integer N. The smallest one of such integers N is called the index of $K_{\bar{V}}$ and denoted by Index($K_{\bar{V}}$) or simply by I. Since $IK_{\bar{V}}$ is trivial, there is a $\boldsymbol{Z}/I\boldsymbol{Z}$-covering $\pi: \bar{U}\to\bar{V}$, which is unique up to isomorphisms and ètale outside Sing$\bar{V}$. Then $\bar{U}$, called the canonical covering of $\bar{V}$, is a Gorenstein surface, and the minimal resolution of singularities of $\bar{U}$ is an abelian surface or a $K3$-surface.

Let $f: V\to\bar{V}$ be a minimal resolution of singularities of $\bar{V}$ and set $D:=f^{-1}(\mathrm{Sing}\bar{V})$. We often confuse $\bar{V}$ deliberately with (V, D) or (V, D, f).

§1 is a preparation and contains a proof of an inequality (cf. Proposition 1.6) which plays an important role in the whole theory; in particular, if $I\geqq3$ then $c:=\#(\mathrm{Sing}\bar{V})\leqq(D, K_V)\leqq c-1-(K_V^2)$, and it $I\geqq4$ then $c<-3(K_V^2)$. In §2, it is proved that if a positive integer p is a factor of I then $\bar{U}/(\boldsymbol{Z}/p\boldsymbol{Z})$ is a logarithmic Enriques surface, as well. We also prove that $I\leqq66$; this result is originally due to S. Tsunoda. Moreover, $I\leqq19$ if I is a prime number. §§3-5 are devoted to the proofs of the following three theorems:

Theorem 3.6. *Let $\bar{V}$ or synonymously (V, D) be a logarithmic Enriques surface with* Index($K_{\bar{V}}$)$=2$. *Then there is a logarithmic Enriques surface $\bar{W}$ or (W, B) with* Index($K_{\bar{W}}$) $=2$ *and* $\#(\mathrm{Sing}\bar{W})=1$ *such that V is obtained from W by blowing up all singular points of B (i.e., intersection points of irreducible components of B) and then blowing down several (-1)-curves on the blown-up surface.*

Moreover, $\#(\mathrm{Sing}\bar{U})=\#(\mathrm{Sing}\bar{V})\leqq\#\{$*irreducible component of* $D\}\leqq10$ (*cf. Lemma* 3.1). *The case with* $\#(\mathrm{Sing}\bar{V})=10$ *occurs* (*see Example* 3.2) *and, in this case, there is a* (-2)-*rod of Dynkin type* A_{19} *on* U (*cf. Cor.* 3.10).

Communicated by Prof. Nagata.
Received June 23, 1989

Theorem 4.1. *Let (V, D) be a logarithmic Enriques surface such that the canonical covering $\bar{U}$ is an abelian surface. Then* Index$(K_{\bar{V}})=3$ *or* 5, *and the configuration of D is explicitly given.*

Theorem 5.1. *Let (V, D) be a logarithmic Enriques surface such that $I(=\text{Index}(K_{\bar{V}}))$ is a prime number and the canonical covering $\bar{U}$ is a $K3$-surface. Then $I\neq 2, 13$. Moreover, the singularity type of $\bar{V}$ is explicitly given. In particular, $(D, K_{\bar{V}})=c-1-(K_{\bar{V}}^2)$.*

In §6, we consider the remaining case where the canonical covering $\bar{U}$ of $\bar{V}$ is singular. Possible types of singularities of $\bar{V}$ and $\bar{U}$ are given when $I:=\text{Index}(K_{\bar{V}})=3$ or 5. As a corollary, we see that if there is a singularity of Dynkin type $E_k(k=6, 7$ or $8)$ on $\bar{U}$ then $I=5, 25, 7, 11, 13, 17$ or 19. It remains to consider possible combinations of singularities on $\bar{V}$. We obtain the following theorem (cf. Proposition 6.6 and Lemma 6.14):

Theorem *Let (V, D) be a logarithmic Enriques surface such that I is an odd prime number and $\text{Sing}\bar{U}\neq\emptyset$. Then $c:=\#(\text{Sing}\bar{V})\leqq\text{Min}\{16, 23-I\}$, $\#(\text{Sing}\bar{U})\leqq(24-I)/2$ and $-1\leqq\rho(\bar{V})-c\leqq 4$, where $\rho(\bar{V})$ is the Picard number of $\bar{V}$. Moreover, if $c=16$ or $\rho(\bar{V})-c=4$, then $I=5$ or 3, respectively and $\text{Sing}\bar{V}$ is precisely described in Proposition* 6.6 *(cf. Examples* 6.12 *and* 6.8*); particularly, $(D, K_V)=c-1-(K_{\bar{V}}^2)$.*

Example 6.11 gives a logarithmic Enriques surface (V, D) with $(c, I)=(15, 3)$. Moreover, there is a (-2)-fork Γ of Dynkin type D_{19} on the minimal resolution U of the canonical covering $\bar{U}$ of (V, D). By contracting Γ on U we get the canonical covering $\bar{U}'$ of a new log Enriques surface (V', D'). In particular, U is a $K3$-surface with $\rho(U)=20$. Such a $K3$-surface is probably new. Note that $\bar{U}'$ can not be a quartic surfaces of $\boldsymbol{P}^3$ (cf. Kato-Naruki [4]).

The author heartily thanks Professor M. Miyanishi for careful reading of the present article and giving valuable advice. He also thanks Professor S. Tsunoda who kindly reminded him the reference [10] for the proof of Lemma 2.3 and suggested Lemma 2.2.

Terminology. We refer to [8; §§ 1.1-1.5] or [9; § 2] for the definitions of (admissible rational) rods, twigs and forks, and the definition of $B^{\#}$ for a reduced effective divisor B. A $(-n)$-curve on a nonsingular projective surface is a nonsingular rational curve of self-intersection number $-n$. A (-2)-rod (resp. fork) is a rod (resp. fork) whose irreducible components are all (-2)-curves.

Notation. Let V be a nonsingular projective surface and let D, D_1 and D_2 be divisors on V.

K_V: Canonical divisor of V,

$\kappa(V)$: Kodaira dimension of V,

$\bar{\kappa}(X)$: Logarithmic Kodaira dimension of a non-complete surface X,

$\rho(V)$: Picard number of V,

$h^i(V, D)$: $=\dim H^i(V, D)$,

$\#(D)$: The number of all irreducible components of $\mathrm{Supp}(D)$,
f^*D: Total transform of D,
$f'D$: Proper transform of D,
$D_1 \sim D_2$: D_1 and D_2 are linearly equivalent,
$D_1 \equiv D_2$: D_1 and D_2 are numerically equivalent,
$e(D)$: Euler number of D,
$\sum_n$: Hirzebruch surface of degree n.

§ 1. Preliminaries

We work over the complex number field $\boldsymbol{C}$. Let $\bar{V}$ be a normal projective algebraic surface defined over $\boldsymbol{C}$ and let $f: V \to \bar{V}$ be a minimal resolution of $\mathrm{Sing}(\bar{V})$. Denote by D the reduced effective divisor whose support is $f^{-1}(\mathrm{Sing}\bar{V})$.

Definition 1.1. $\bar{V}$ is said to be *a log (=logarithmic) Enriques surface* if the following three conditions are satisfied:

(1) $\bar{V}$ has only quotient singularities and $\mathrm{Sing}(\bar{V}) \neq \emptyset$,
(2) $NK_{\bar{V}}$ is a trivial Cartier divisor for some positive integer N,
(3) $q(\bar{V}) := \dim H^1(\bar{V}, \mathcal{O}_{\bar{V}}) = 0$.

Let Δ be a connected component of D. Then Δ is an admissible rational rod or an admissible rational fork, which are defined in [9; § 2] (cf. Brieskorn [2; Satz 2.11]). $f(\Delta)$ is a rational double singular point if and only if Δ is a (-2)-rod or a (-2)-fork. We can define the direct image f_*F for each divisor F on V as in the case where f is a morphism between nonsingular surfaces. Then the property of linear equivalence "$\sim$" between divisors on V is preserved under f_*. By [8; Lemma 2.4], there exists a positive integer P such that for each Weil divisor $\bar{F}$ on $\bar{V}$, $P\bar{F}$ is linearly equivalent to a Cartier divisor. Let $\bar{F}_1$ and $\bar{F}_2$ be two Weil divisor on $\bar{V}$, we define the intesection number of $\bar{F}_1$ and $\bar{F}_2$ by $(\bar{F}_1, \bar{F}_2) := (1/P^2)(f^*(P\bar{F}_1), f^*(P\bar{F}_2))$.

We often identify $\bar{V}$ with (V, D, f) or (V, D).

Lemma 1.2. *Let $\bar{V}$ be a log Enriques surface. Then the following assertions hold:*

(1) $q(V)=0$.

(2) *We have $f_*K_V = K_{\bar{V}}$. There exists a $\boldsymbol{Q}$-divisor $D^\#$ on V, such that $f^*(NK_{\bar{V}}) \equiv N(D^\# + K_V)$ and* $\mathrm{Supp}D^\# \subseteq \mathrm{Supp}D$ *and that if α_i is the coefficient in $D^\#$ of an irreducible component D_i of D then $0 \leqq \alpha_i < 1$. Here, N is a positive integer such that $NK_{\bar{V}}$ is a Cartier divisor. In particular, we have $D^\# + K_V \equiv 0$. Moreover,* $\mathrm{Supp}(D) - \mathrm{Supp}(D^\#)$ *consists of exactly those connected components of D which are contracted to rational double singular points on $\bar{V}$.*

(3) *Let N be a positive integer. Then $NK_{\bar{V}}$ is a Cartier divisor if and only if $ND^\#$ is an integral divisor. If this is the case, then $f^*(NK_{\bar{V}}) \sim N(D^\# + K_V)$ and $NK_{\bar{V}} \sim f_*N(D^\# + K_V)$. Hence $NK_{\bar{V}} \sim 0$ if and only if $N(D^\# + K_V) \sim 0$.*

Proof. (1) Since $\bar{V}$ has only rational singularities, we have $q(V) = q(\bar{V}) = 0$. For (2), we refer to [8; § 1.5 & § 2.5].

(3) Suppose that $NK_{\bar{V}}$ is a Cartier divisor. Then $E:=f^*(NK_{\bar{V}})-NK_V$ is a Cartier divisor and supported by Supp$D$. By the assertion (2), we see $E-ND^{\#}\equiv 0$. Since Supp$D^{\#}\cup$Supp$E$ is contained in Supp$D$ which has negative intersection matrix, we must have $ND^{\#}=E$. Hence $ND^{\#}$ is an integral divisor and $f^*(NK_{\bar{V}})=N(D^{\#}+K_V)$.

Suppose that $ND^{\#}$ is an integral divisor. Since $(N(D^{\#}+K_V), D_i)=(f^*(NK_{\bar{V}}), D_i)=0$ for each irreducible component D_i of D, $N(D^{\#}+K_V)$ is linearly equivalent to a divisor $\varDelta$ which is disjoint from D (cf. Artin [1; Cor. 2.6]). Note that $NK_{\bar{V}}=f_*NK_V=f_*N(D^{\#}+K_V)\sim f_*\varDelta$ which is a Cartier divisor. Hence $NK_{\bar{V}}$ is a Cartier divisor.

Q. E. D.

Proposition 1.3. *Let* (V, D) *be a log Enriques surface. Then* $\kappa(V)\leqq\bar{\kappa}(V-D)=0$. *Moreover, if* $\kappa(V)=0$, *then* $\bar{V}$ *has only rational double singular points and either* V *is a* $K3$*-surface or* V *is an Enriques surface.*

Proof. By virtue of [8; Lemma 1.10], we have $h^0(V, n(D+K_V))=h^0(V, n(D^{\#}+K_V))=1$, for each positive integer n satisfying $n(D^{\#}+K_V)\sim 0$ (cf. Lemma 1.2). Therefore, $\bar{\kappa}(V-D)=0$.

Suppose that $\kappa(V)=0$. Then there exists a positive integer N such that $ND^{\#}$ is an integral divisor and NK_V is linearly equivalent to an effective divisor $\varDelta$. Since $0\equiv N(D^{\#}+K_V)\sim ND^{\#}+\varDelta$, we have $D^{\#}=\varDelta=0$. $D^{\#}=0$ means that D consists of (-2)-rods and (-2)-forks (cf. [8; § 1.5]). Namely, $\bar{V}$ has only rational double singular points. Note that V is a minimal surface, for $NK_V\sim 0$. By the classification theory of non-singular surfaces and by the hypothesis that $\kappa(V)=q(V)=0$, we see that V is a $K3$ surface or an Enriques surface.

Q. E. D.

Let (V, D) be a log Enriques surface. Denote by $\tilde{D}$ the reduced divisor Supp$D^{\#}$. Then $D-\tilde{D}$ consists of exactly those connected components of D which are contracted to rational double singular points on $\bar{V}$. Therefore, $(V, \tilde{D})$ is also a log Enriques surface with the same index as (V, D) (cf. Definition 1.4 below).

In view of Proposition 1.3 and the above argument, we assume, until the end of the present article, the following two conditions:

(1) $\kappa(V)=-\infty$, *hence* V *is a rational surface,*

(2) Supp$(D^{\#})$=Supp$(D)\neq\emptyset$.

Definition 1.4. Let $\bar{V}$ be a log Enriques surface. We denote by Index$(K_{\bar{V}})$ or simply by I, the smallest positive integer such that $IK_{\bar{V}}$ is a Cartier divisor.

Actually, $IK_{\bar{V}}\sim 0$ which is proved in the following lemma.

Lemma 1.5. (1) $(K_{\bar{V}}^2)\leqq -1$, $I\geqq 2$, $IK_{\bar{V}}\sim 0$ *and* $I(D^{\#}+K_V)\sim 0$.

(2) *Let* N *be a positive integer. Then* $h^0(V, -NK_V)\neq 0$ *if and only if* I *is a divisor of* N.

Proof. (1) Since $K_V\equiv -D^{\#}$, Supp$D^{\#}$=Supp$D\neq\emptyset$ and D has negative definite intersection matrix, we have $(K_{\bar{V}}^2)\leqq -1$. If $I=1$, then $\bar{V}$ is Gorenstein. Hence $K_V=f^*K_{\bar{V}}$

and $D^{\#}=0$ because $\bar{V}$ has only rational singularities. This contradicts the assumptions that $\mathrm{Sing}(\bar{V})\neq\emptyset$ and $\mathrm{Supp}D^{\#}=\mathrm{Supp}D$. Hence $I\geqq 2$. Note that $I(D^{\#}+K_V)\equiv 0$. Hence $I(D^{\#}+K_V)\sim 0$ and $IK_{\bar{V}}\sim 0$ by the additional assumption that V is rational. In particular, $h^0(V, -IK_V)\neq 0$.

(2) Suppose that $h^0(V, -NK_V)\neq 0$. Then $-NK_V$ is linearly equivalent to an effective divisor Δ. Note that $ND^{\#}-\Delta\sim N(D^{\#}+K_V)\equiv 0$. Since $D^{\#}$ has negative definite intersection matrix, we have $ND^{\#}=\Delta$. Hence $ND^{\#}$ is an integral divisor. So, $NK_{\bar{V}}$ is a Cartier divisor by Lemma 1.2. Then, N is divisible by I by the definition of I.

Q. E. D.

The inequality (**) in the following proposition is very helpful in proving Theorem 5.1 and Proposition 6.6.

Proposition 1.6. *Let* (V, D) *be a log Enriques surface and let* c *be the number of connected components of* D. *Let* p *and* q *be integers satisfying* $1\leqq q<p\leqq I-1$ $(I:=\mathrm{Index}(K_{\bar{V}}))$. *Then we have:*

$$(*)\qquad c\leqq (D, K_V)<\frac{2c(p-q)^2+(p-p^2)(K_{\bar{V}}^2)}{(p-q)(p+q-1)},$$

and

$$(**)\qquad (D, K_V)\leqq c-1-(K_{\bar{V}}^2) \quad \text{if } I\geqq 3.$$

If $I\geqq 4$ *then* $c<-3(K_{\bar{V}}^2)$. *If* $c=1$ *then* $I=2$ *and* D *has the configuration to be given in Lemma* 1.8 *below.* (*The case* $c=1$ *has been treated in* [10; Proposition 2.2]).

Proof. Let p, q be the same as in the statement. We claim first that $h^2(V, (p-q)D+pK_V)=h^0(V, -(p-q)D-(p-1)K_V)=0$. Indeed, suppose that $h^0(V, -(p-q)D-(p-1)K_V)\neq 0$. Then $h^0(V, -(p-1)K_V)\neq 0$. Hence I is a divisor of $(p-1)$ and $I\leqq p-1$ by Lemma 1.5. This contradicts the assumption $p\leqq I-1$.

Next, we claim that $h^0(V, (p-q)D+pK_V)=0$. Suppose, on the contrary, that $h^0(V, (p-q)D+pK_V)\neq 0$. Then $h^0(V, [pD^{\#}]+pK_V)=h^0(V, pD+pK_V)\neq 0$ (cf. [8; Lemma 1.10]). Here, $[pD^{\#}]$ is the maximal effective integral divisor such that $pD^{\#}-[pD^{\#}]$ is effective. Let Δ be an effective divisor such that $[pD^{\#}]+pK_V\sim\Delta$. Then $p(D^{\#}+K_V)\sim\Delta+(pD^{\#}-[pD^{\#}])$. Since $D^{\#}+K_V\equiv 0$, we have $\Delta=0$ and $pD^{\#}=[pD^{\#}]$ which is an integral divisor. Hence I is a factor of p and $I\leqq p$. This contradicts the assumption $p\leqq I-1$.

Write $D=\sum_{i=1}^{n} D_i$ where D_i's are irreducible components of D. Note that D consists of rational trees. Hence we have $\sum_{i<j}(D_i, D_j)=n-c$. Therefore, $2p_a(D)-2=(D, D+K_V)=\sum_i(D_i^2)+\sum_i(D_i, K_V)+2\sum_{i<j}(D_i, D_j)=\sum_i(2p_a(D_i)-2)+2(n-c)=-2c$. Hence, $p_a(D)=1-c$.

Applying the Riemann-Roch theorem, we obtain:

$$0\geqq -h^1(V, (p-q)D+pK_V)=\frac{1}{2}\{[(p-q)D+pK_V][(p-q)D+(p-1)K_V]\}+1.$$

Hence we have:

$$\begin{aligned}0&>[(p-q)D+pK_V][(p-q)D+(p-1)K_V]\\&=(p-q)^2(D^2)+(2p-1)(p-q)(D,\ K_V)+(p^2-p)(K_V^2)\\&=(p-q)^2[-2c-(D,\ K_V)]+(2p-1)(p-q)(D,\ K_V)+(p^2-p)(K_V^2)\\&=-2c(p-q)^2+(p-q)(p+q-1)(D,\ K_V)+(p^2-p)(K_V^2).\end{aligned}$$

Thence follows the second half of the inequality (*). Setting $p=2$ and $q=1$, we obtain the inequality (**).

Since $\mathrm{Supp}D^{\#}=\mathrm{Supp}D$, each connected component Δ_i of D contains an irreducible component D_i with $(D_i^2)\leqq-3$. Hence $(\Delta_i,\ K_V)\geqq(D_i,\ K_V)=-2-(D_i^2)\geqq1$. Therefore, $(D,\ K_V)\geqq c$.

Suppose $I\geqq4$. Setting $p=3$ and $q=2$ in the inequality (*), we obtain $c<(2c-6(K_V^2))/4$, i. e., $c<-3(K_V^2)$.

Consider the case $c=1$. Suppose $I\geqq3$. Then $(D,\ K_V)\leqq-(K_V^2)$ by the inequality (**). Hence $(D-D^{\#},\ K_V)=(D+K_V,\ K_V)\leqq0$ because $D^{\#}+K_V\equiv0$. Since $D-D^{\#}\geqq0$ by Lemma 1.2, we have $(D-D^{\#},\ K_V)=0$. Hence $D-D^{\#}$, whose support coincides with $\mathrm{Supp}D$ by Lemma 1.2, consists of (-2)-curves. Hence $D^{\#}=0$, $\mathrm{Supp}D=\mathrm{Supp}D^{\#}=\emptyset$ and $\mathrm{Sing}\bar{V}=\emptyset$. This is a contradiction. Q. E. D.

In the subsequent Lemmas 1.7, 1.8 and 1.9, we shall prove that $c\leqq-3(K_V^2)$ even when I $(=\mathrm{Index}(K_{\bar{V}}))=2$ or 3, where c is the number of connected components of D.

Lemma 1.7. *Let $(V,\ D)$ be a log Enriques surface. Write $D=\sum_{i=1}^{n}D_i$ and $D^{\#}=\sum_i\alpha_iD_i$, where D_i's are irreducible. Then we have:*

(1) g. c. d. $(I\alpha_1,\ \cdots,\ I\alpha_n)=1$. *In particular, if $\alpha_1=\cdots=\alpha_n$, then $\alpha_i=1/I(1\leqq i\leqq n)$.*

(2) *$\alpha_i\leqq1/2$ for at least one index i.*

Proof. (1) Denote by $s=$g. c. d.$(I\alpha_1,\ \cdots,\ I\alpha_n)$. Since $(K_V^2)=(D^{\#})^2<0$, there is a (-1)-curve E on V. Note that $1=-(E,\ K_V)=(E,\ D^{\#})=s/I\sum_i(I\alpha_i/s)(E,\ D_i)$. Hence I/s is an integer. On the other hand, $(I/s)D^{\#}=\sum_i(I\alpha_i/s)D_i$ is an integral divisor. Hence, we have $s=1$.

(2) Suppose that $\alpha_i>1/2(1\leqq i\leqq n)$. Let E be a (-1)-curve on V. Then $0=(E,\ D^{\#}+K_V)=-1+\sum_i\alpha_i(E,\ D_i)>-1+(1/2)\sum_i(E,\ D_i)$. Hence $\sum_i(E,\ D_i)\leqq1$ and $0=(E,\ D^{\#}+K_V)\leqq-1+\max\{\alpha_1,\ \cdots,\ \alpha_n\}<0$ by Lemma 1.2. This is a contradiction. Q. E. D.

Lemma 1.8. *Let $(V,\ D)$ be a log Enriques surface and let Δ be a connected component of D. Suppose that each irreducible component of Δ has the same coefficient in $D^{\#}$, say α. Then either Δ consists of a single curve with self-intersection number $-2/(1-\alpha)$, or Δ is a linear chain such that two tips of Δ have self-intersection numbers $(\alpha-2)/(1-\alpha)$ and the others have self-intersection numbers -2.*

Suppose that $D^{\#}=\alpha D$. Then $\alpha=1/I$, $I=2$ or 3 and $c=-(K_V^2)$ or $-3(K_V^2)$, accordingly. Moreover, $D^{\#}=(1/3)D$ if and only if D consists of isolated (-3)-curves.

Remark. (1) If $I=2$ then $D^\#=(1/2)D$ (cf. Lemma 1.2).

(2) If $D^\#=(1/3)D$, we shall prove in Corollary 5.2 that $c=3$ or 9.

Proof. We claim that Δ is a rod. Suppose, on the contrary, that Δ is a fork. Then one of three tips of Δ, say D_1, is a (-2)-curve. Since $(D_1, \Delta-D_1)=1$, we have $(D_1, D^\#+K_V)=\alpha+\alpha(D_1^2)+(D_1, K_V)=\alpha-2\alpha=-\alpha\neq 0$. This contradicts $D^\#+K_V\equiv 0$. Therefore, Δ is a rod. Then the first assertion of Lemma 1.8 follows from the observation that the intersection number of $D^\#+K_V$ with each irreducible component of Δ vanishes.

Suppose that $D^\#=\alpha D$. Then $\alpha=1/I$ by Lemma 1.7. Put $n_1:=-2/(1-\alpha)$ and $n_2:=(\alpha-2)/(1-\alpha)$. Since n_1 or n_2 is the self-intersection number of a tip of a connected component Δ of D, we see that n_1 or n_2 must be an integer. Hence $I=2$ or 3. Let t be the number of all isolated irreducible components of D. Note that $K_V\equiv -D^\#=-\alpha D$. Hence, $-(K_V^2)/\alpha=(D, K_V)=t(-2+2/(1-\alpha))+2(c-t)(-2+(2-\alpha)/(1-\alpha))=2\alpha c/(1-\alpha)$. Hence $c=(\alpha-1)(K_V^2)/2\alpha^2$. So, we obtain $c=-(K_V^2)$ or $-3(K_V^2)$ according as $I=2$ or 3. If $I=3$, then D consists of isolated (-3)-curves. Conversely, if D consists of isolated (-3)-curves, then $D^\#=(1/3)D$ because $(D^\#+K_V, D_i)=0$ for each component D_i of D. Q. E. D.

Lemma 1.9. *Let (V, D) be a log Enriques surface with $I=3$. Then $c\leqq -3(K_V^2)$, and the equality holds if and only if $D^\#=(1/3)D$.*

Proof. If $h^0(V, (p-q)D+pK_V)=0$ for $p=3$ and $q=2$, we have $c<-3(K_V^2)$ by the same proof as in Proposition 1.6. Suppose $h^0(V, D+3K_V)\neq 0$. Then $D+3K_V$ is linearly equivalent to an effective divisor Δ. The hypothesis $\mathrm{Supp}D^\#=\mathrm{Supp}D$ implies that $0\leqq 3D^\#-D\sim -3K_V-D\sim -\Delta\leqq 0$. Hence $\Delta=0$ and $D^\#=(1/3)D$. By Lemma 1.8, we have $c=-3(K_V^2)$. So, $c\leqq -3(K_V^2)$. If $c=-3(K_V^2)$ then $h^0(V, D+3K_V)\neq 0$ and $D^\#=(1/3)D$. If $D^\#=(1/3)D$ then Lemma 1.8 shows $c=-3(K_V^2)$. Q. E. D.

We end this section by proving the following lemma.

Lemma 1.10. *Let (V, D) be a log Enriques surface. Write $D=\sum_{i=1}^{n} D_i$ and $D^\#=\sum_{i=1}^{n}\alpha_i D_i$, where D_i's are irreducible components of D.*

(1) *Let E be a $(-m)$-curve on V which is not contained in D. Then $m\leqq 2$, and $m=2$ if and only if $E\cap D=\emptyset$.*

(2) *Take r irreducible components of D, say $D_1, \cdots, D_r$ $(r\leqq n)$. Define rational numbers β_i's by the condition:*

$$\Big(\sum_{i=1}^{r}\beta_i D_i+K_V,\ D_j\Big)=0 \qquad (1\leqq j\leqq r).$$

Then, $0\leqq\beta_i\leqq\alpha_i<1$ $(1\leqq i\leqq r)$.

(3) *Furthermore, we assign a virtual curve B_i to each i $(1\leqq i\leqq r)$, so that $(D_i^2)\leqq(B_i^2)\leqq -2$, $(B_i, K_V)=-2-(B_i^2)$ and $(B_i, B_j)=(D_i, D_j)$ $(j\neq i)$. Define γ_i by the condition:*

$$\Big(\sum_{i=1}^{r}\gamma_i B_i+K_V,\ B_j\Big)=0 \qquad (1\leqq j\leqq r).$$

Then, $0\leqq\gamma_i\leqq\beta_i\leqq\alpha_i$ $(1\leqq i\leqq r)$.

Proof. (1) results from the observation:

$$0=(E, D^{\#}+K_V)=(E, D^{\#})+m-2\geqq m-2.$$

(2) Since $\sum_{i=1}^{r} D_i$ has negative definite intersection matrix, we have $\beta_i\leqq\alpha_i$ because

$$\left(\sum_{i=1}^{r}(\alpha_i-\beta_i)D_i,\ D_j\right)\leqq 0 \qquad (0\leqq j\leqq r).$$

Indeed,

$$\left(\sum_{i=1}^{r}(\alpha_i-\beta_i)D_i,\ D_j\right)=\left(\sum_{i=1}^{n}\alpha_iD_i+K_V,\ D_j\right)-\left(\sum_{i=r+1}^{n}\alpha_iD_i,\ D_j\right)-\left(\sum_{i=1}^{r}\beta_iD_i+K_V,\ D_j\right)$$

$$=-\left(\sum_{i=r+1}^{n}\alpha_iD_i,\ D_j\right)\leqq 0, \quad \text{if} \quad 1\leqq j\leqq r.$$

We also have $\beta_i\geqq 0$ $(1\leqq i\leqq r)$ because

$$\left(\sum_{i=1}^{r}\beta_iD_i,\ D_j\right)=-(K_V,\ D_j)\leqq 0 \qquad (1\leqq j\leqq r).$$

(3) Note that $\sum_{i=1}^{r} B_i$ has negative definite intersection matrix. We have $\gamma_i\leqq\beta_i$ $(1\leqq i\leqq r)$ because:

$$\left(\sum_{i=1}^{r}(\beta_i-\gamma_i)B_i,\ B_j\right)=\left(\sum_{i=1}^{r}\beta_iB_i+K_V,\ B_j\right)-\left(\sum_{i=1}^{r}\gamma_iB_i+K_V,\ B_j\right)$$

$$=\left(\sum_{i=1}^{r}\beta_iB_i+K_V,\ B_j\right)=\left(\sum_{i=1}^{r}\beta_iD_i+K_V,\ D_j\right)+\beta_j(B_j^2)-\beta_j(D_j^2)$$

$$-2-(B_j^2)+2+(D_j^2)=(1-\beta_j)((D_j^2)-(B_j^2))\leqq 0 \qquad (1\leqq j\leqq r).$$

We also have $\gamma_i\geqq 0$ $(1\leqq i\leqq r)$ because

$$\left(\sum_{i=1}^{r}\gamma_iB_i,\ B_j\right)=-(K_V,\ B_j)=(B_j^2)+2\leqq 0 \qquad (1\leqq j\leqq r).$$

Q. E. D.

§2. Canonical coverings of logarithmic Enriques surfaces

Let $\bar{V}$ (or synonymously (V, D, f)) be a log Enriques surface. Denote by V^0 the smooth part $\bar{V}-(\mathrm{Sing}\bar{V})=V-D$. By the relation $\mathcal{O}(ID^{\#})\cong\mathcal{O}(-K_V)^{\otimes I}$ $(I:=\mathrm{Index}(K_{\bar{V}}))$ and a nonzero global section of $\mathcal{O}(ID^{\#})$, we can define a $\boldsymbol{Z}/I\boldsymbol{Z}$-covering $\hat{\pi}: \hat{U}\to V$ such that $\hat{U}$ is normal and the restriction π^0 of $\hat{\pi}$ to $U^0:=\hat{\pi}^{-1}(V^0)$ is finite and étale. By Lemma 1.7, $\hat{U}$ is connected. Actually, $\hat{\pi}^{-1}(D)$ is contractible to quotient singular points on a normal projective surface $\bar{U}$ (cf. [13; Cor. 5.2]). Let $\pi: \bar{U}\to\bar{V}$ be the finite morphism induced by $\hat{\pi}$. Note that π^0 is induced by the relation $I(-K_{V^0})\sim 0$ and $\bar{U}$ is the normalization of $\bar{V}$ in the function field $\boldsymbol{C}(U^0)$. Note that $K_{U^0}\sim\pi^{0*}(K_{V^0}+(I-1)(-K_{V^0}))\sim 2\pi^{0*}K_{V^0}\sim 2K_{U^0}$ and $K_{U^0}\sim 0$. Hence $K_{\bar{U}}\sim 0$ and there are only rational double singular points on $\bar{U}$. Let $g: U\to\bar{U}$ be a minimal resolution of singularities of $\bar{U}$. Then $K_U\sim 0$. Hence U is an abelian surface or a $K3$-surface. Note that $\bar{U}=U$ when U is an abelian surface.

Definition 2.1. The surface $\bar{U}$ (resp. the map $\pi: \bar{U}\to\bar{V}$) defined above is called *the canonical covering* (resp. *the canonical map*) *of* $\bar{V}$.

Assume $I=pq$ with $p<I$ and $q<I$. Set $\bar{U}_1=\bar{U}/(\boldsymbol{Z}/p\boldsymbol{Z})$ where $\boldsymbol{Z}/p\boldsymbol{Z}$ is considered as a subgroup of $\boldsymbol{Z}/I\boldsymbol{Z}$ which acts on $\bar{U}$. Then $\bar{V}=\bar{U}/(\boldsymbol{Z}/I\boldsymbol{Z})=\bar{U}_1/(\boldsymbol{Z}/q\boldsymbol{Z})$ where the action of $\boldsymbol{Z}/q\boldsymbol{Z}\cong(\boldsymbol{Z}/I\boldsymbol{Z})/(\boldsymbol{Z}/p\boldsymbol{Z})$ on $\bar{U}_1$ is induced by the action of $\boldsymbol{Z}/I\boldsymbol{Z}$ on $\bar{U}$. Let $\pi_1: \bar{U}\to\bar{U}_1$ and $\pi_2: \bar{U}_1\to\bar{V}$ be the natural quotient morphisms. Let $U_1^0=\pi_2^{-1}(V^0)$, $\pi_1^0=\pi_{1|U^0}$ and $\pi_2^0=\pi_{2|U_1^0}$. Note that π_1^0 and π_2^0 are étale and π_2^0 is constructed by means of the relation $q(-pK_{V^0})\sim 0$. We have $K_{U_1^0}\sim\pi_2^{0*}(K_{V^0}-(q-1)pK_{V^0})\sim(p+1)\pi_2^{0*}K_{V^0}\sim(p+1)K_{U_1^0}$. Hence $pK_{U_1^0}\sim 0$ and $pK_{\bar{U}_1}\sim 0$. Note also that π_1^0 is constructed by means of the relation $p(-K_{U_1^0})\sim 0$. Let $g_1: U_1\to\bar{U}_1$ be a minimal resolution and let $B=g_1^{-1}(\mathrm{Sing}\bar{U}_1)$. As in Lemma 1.2, we have $p(B^{\#}+K_{U_1})\sim g_1^{*}(pK_{\bar{U}_1})\sim 0$.

Lemma 2.2. *Let J be a positive integer. Then $JK_{\bar{U}_1}$ is a Cartier divisor if and only if p is a divisor of J. Moreover, $\bar{U}_1$ is a rational log Enriques surface with* $\mathrm{Index}(K_{\bar{U}_1})=p$. *If $\bar{U}$ is nonsingular then 2 is not a divisor of I.*

Proof. We have proved that $pK_{\bar{U}_1}$ is a trivial Cartier divisor. Conversely, suppose that $JK_{\bar{U}_1}$ is a Cartier divisor. In order to show that p is a divisor of J, we have only to show that $qJK_{\bar{V}}$ is a Cartier divisor, or equivalently that a divisorial sheaf $\mathcal{O}(qJK_{\bar{V}})$ is invertible.

Consider the case where q is a prime number. Let y be a singular point of $\bar{V}$. Then $\pi_2^{-1}(y)$ consists of one or q points because π_2 is a finite Galois morphism of degree q between normal surfaces. Moreover, if $\pi_2^{-1}(y)$ consists of q points $\{x_i\}$, we have $\hat{\mathcal{O}}_{\bar{V},y}\cong\hat{\mathcal{O}}_{\bar{U}_1,x_i}$, where "$\wedge$" means the completion. Hence $JK_{\bar{V}}$ is a Cartier divisor near y. Now we assume that $\pi_2^{-1}(y)$ consists of a single point x. Let ξ be a generator of $\mathcal{O}(JK_{\bar{U}_1})$ at an affine neighbourhood N of x. Note that x is fixed under the $\boldsymbol{Z}/q\boldsymbol{Z}$-action. We may assume that N is stable under the action of $\boldsymbol{Z}/q\boldsymbol{Z}$ by replacing N by $\cap gN$ where g moves in $\boldsymbol{Z}/q\boldsymbol{Z}$. Since $K_{U_1^0}=\pi_2^{*}K_{V^0}$, there is a natural $\boldsymbol{Z}/q\boldsymbol{Z}$-action on $\mathcal{O}(JK_{U_1^0})$ compatible with the action of $\boldsymbol{Z}/q\boldsymbol{Z}$ on O_{U_1}. The action extends naturally to an action on $O(JK_{\bar{U}_1})$. Note that for each $g\in\boldsymbol{Z}/q\boldsymbol{Z}$, $g(\xi)=\chi(g)\xi$ with a unit $\chi(g)$. Note that $\mathcal{O}(qJK_{\bar{U}_1})$ is an invertible sheaf over N which has a generator ξ^q and on which $\boldsymbol{Z}/q\boldsymbol{Z}$ also acts. Set $\eta=\prod_g g(\xi)$, where g moves in $\boldsymbol{Z}/q\boldsymbol{Z}$. Since $\eta=u\xi^q$ with a unit u, η is a generator of $\mathcal{O}(qJK_{\bar{U}_1})$ over N. Since η is $\boldsymbol{Z}/q\boldsymbol{Z}$-invariant, η is viewed as an element of $\Gamma(\pi_2(N)-y, \mathcal{O}(qJK_{V^0}))=\Gamma(\pi_2(N'), \mathcal{O}(qJK_{\bar{V}}))$. We claim that η is a generator of $\mathcal{O}(qJK_{\bar{V}})$ over $\pi_2(N)$. For any $\alpha\in\Gamma(\pi_2(N), \mathcal{O}(qJK_{\bar{V}}))=\Gamma(\pi_2(N)-y, \mathcal{O}(qJK_{V^0}))\hookrightarrow\Gamma(N-x, \mathcal{O}(qJK_{U_1^0}))=\Gamma(N, \mathcal{O}(qJK_{\bar{U}_1}))$, α is written as $\alpha=v\eta$ with a section v of $\mathcal{O}_N$. Since α and η are $\boldsymbol{Z}/q\boldsymbol{Z}$-invariant, v is $\boldsymbol{Z}/q\boldsymbol{Z}$-invariant. Hence v comes from a section of $\mathcal{O}_{\pi_2(N)}$. Therefore η is a generator of $\mathcal{O}(qJK_{\bar{V}})$ and $\mathcal{O}(qJK_{\bar{V}})$ is invertible over $\pi_2(N)$.

In a general case, let q_1 be a prime divisor of q. We consider the natural morphism $\bar{U}_1\to\bar{U}_2:=\bar{U}_1/(\boldsymbol{Z}/q_1\boldsymbol{Z})$ instead of the morphism π_2. By the same arguments as above, we can prove that $q_1JK_{\bar{U}_2}$ is a Cartier divisor. Continuing this process, we see that $qJK_{\bar{V}}$ is a Cartier divisor.

Hence $p(=I/q)$ is a divisor of J by the definition of I. In particular, $K_{\bar{U}_1}$ is not a Cartier divisor. Hence $\bar{U}_1$ has at least one singularity of multiplicity greater than 2 and $B^{\#}\neq 0$. So, $\kappa(U_1)=-\infty$ because $p(B^{\#}+K_{U_1})\sim 0$. If U_1 is a ruled surface with $q(U_1)\geqq 1$, there is a $\boldsymbol{P}^1$-fibration Φ: $U_1\to C$ with a nonsingular curve C of genus equal to $q(U_1)$. Hence B is contained in singular fibers of Φ. Let L be a general fiber of Φ. Then $-2=(L, K_{U_1})=(L, B^{\#}+K_{U_1})=0$. This is absurd. So, U_1 is a rational surface and $\bar{U}_1$ is a log Enriques surface.

Suppose that 2 is a divisor of I and $\bar{U}$ is nonsingular. Let $\bar{U}_1:=\bar{U}/(\boldsymbol{Z}/2\boldsymbol{Z})$. Then $\bar{U}_1$ has only rational double singular points and $K_{\bar{U}_1}$ is a Cartier divisor. This is a contradiction. Q. E. D.

In view of the above lemma, we assume that I $(=\mathrm{Index}(K_{\bar{V}}))$ is a prime number in order to obtain the information about $\bar{U}$, e. g., the singularity type of $\bar{U}$. Possible divisors of I are given in the following lemma. The idea of the proof is found in [10; p. 108].

Lemma 2.3. *Let $\bar{V}$ be a log Enriques surface. Then $\varphi(I)\leqq b_2(U)-\rho(U)\leqq 21$, where $\varphi(I)$ is the Euler function and $b_2(U)$ is the second Betti number. Hence each prime divisor of I is not greater than* 19 *and the following assertions hold true.*

(1) *If $J|I$ with $J=13$, 17 or 19, then $I=2^i\cdot J\,(i=0, 1)$.*

(2) *If $11|I$, then $I=2^i\cdot 11\,(i=0, 1, 2)$ or $2^i\cdot 3\cdot 11\,(i=0, 1)$.*

(3) *If $7|I$, then $I=2^i\cdot 7\,(i=0, 1, 2)$ or $2^i\cdot 3\cdot 7\,(i=0, 1)$.*

(4) *If $5|I$, then $I=2^i\cdot 5\,(0\leqq i\leqq 3)$, $2^i\cdot 5^2\,(i=0, 1)$ or $2^i\cdot 3\cdot 5\,(0\leqq i\leqq 2)$.*

(5) *If there are no prime divisors in I other than* 2 *or* 3, *and if $3|I$, then $I=2^i\cdot 3$ $(0\leqq i\leqq 4)$, $2^i\cdot 3^2(0\leqq i\leqq 2)$ or $2^i\cdot 3^3(i=0, 1)$.*

(6) *If $I=2^i$ then $1\leqq i\leqq 5$.*

In particular, $2\leqq I\leqq 66$, and if I is not a prime number then $2|I$, $3|I$ or $5|I$.

Proof. We use the same notations as set before Lemma 2.2. Note the $\boldsymbol{Z}/I\boldsymbol{Z}$ acts on U biregularly because it acts on $\bar{U}$ biregularly and U is a minimal resolution of singularities of $\bar{U}$. Hence $\boldsymbol{Z}/I\boldsymbol{Z}$ acts on $H:=H^2(U\,;\boldsymbol{Q})/NS(U)\otimes_{\boldsymbol{Z}}\boldsymbol{Q}$ and $\dim H=b_2(U)-\rho(U)\leqq 21$ because $b_2(U)=6$ if U is an abelian surface and $b_2(U)=22$ if U is a $K3$-surface.

Claim. $\boldsymbol{Z}/I\boldsymbol{Z}$ acts effectively on H, i. e., the natural map η: $\boldsymbol{Z}/I\boldsymbol{Z}\to GL(H)$ is injective.

Denote by $G_0=\mathrm{Ker}\eta$ and $\bar{U}_1=\bar{U}/G_0$. Note that G_0 acts trivially on $H\otimes_{\boldsymbol{Q}}\boldsymbol{C}=H^0(U, K_U)\oplus H^2(U, \mathcal{O}_U)\oplus H^1(U, \Omega_U^1)/NS(U)\otimes_{\boldsymbol{Z}}\boldsymbol{C}$ and hence acts trivially on $H^0(U, K_U)=H^0(\bar{U}, K_{\bar{U}})=H^0(U^0, K_{U^0})\cong\boldsymbol{C}$. Hence $H^0(\bar{U}_1, K_{U_1})=H^0(U_1^0, K_{U_1^0})\cong H^0(U^0, K_{U^0})\neq 0$ and $K_{\bar{U}_1}$ is linearly equivalent to an effective divisor. This, together with $|G_0|K_{\bar{U}_1}\sim 0$ (cf. Lemma 2.2), implies $K_{\bar{U}_1}\sim 0$. By the same Lemma 2.2 we have $G_0=(0)$. The claim is proved.

Note that a generator A of $\eta(\boldsymbol{Z}/I\boldsymbol{Z})$ satisfies the equation $T^I-1=0$ and that, as an element of $GL(H\otimes_{\boldsymbol{Q}}\boldsymbol{C})$, A is conjugate to a diagonal matrx $[\xi_1, \cdots, \xi_h]$ where $h=\dim H$. Then $\xi_i^I=1$ $(1\leqq i\leqq h)$ and we may assume that ξ_1 is a primitive I-th root of the unit by the same arguments as in the proof of the above claim. Let $f(T)$ and $g(T)$ be the minimal polynomials of A and ξ_1 over $\boldsymbol{Q}$, respectively. Then $f(A)=0$ implies $f(\xi_i)=0$ $(1\leqq i\leqq h)$. Hence $g(T)|f(T)$ in $\boldsymbol{Q}[T]$. In particular, $\varphi(I)=\deg g(T)\leqq$

$\deg f(T) \leqq \dim H$. The first assertion of Lemma 2.3 is now proved. The remaining assertions follow by a straightforward computation. Q. E. D.

The following two lemmas will be used in the subsequent sections.

Lemma 2.4. *Let $\bar{V}$ be a log Enriques surface. Let $I := \mathrm{Index}(K_{\bar{V}})$ and let c and $\tilde{c}$ be the numbers of all connected components of* $\mathrm{Sing}\bar{V}$ *and* $\pi^{-1}(\mathrm{Sing}\bar{V})$, *respectively. We use the notations π: $\bar{U} \to \bar{V}$ and g: $U \to \bar{U}$ as set at the beginning of* § 2. *Then we have:*

$$e(U)+\rho(\bar{U})-\rho(U)-\tilde{c}=I(\rho(\bar{V})-c+2),$$

where $e(U)$ is the Euler number.

Suppose further that $\tilde{c}=c$ (this hypothesis is satisfied if I is a prime number) and that U is a $K3$-surface. Then we have:

$$c \leqq 21+\rho(\bar{U})-\rho(U) \leqq 21 \quad \textit{and} \quad 1 \leqq \rho(\bar{V})-c+2 \leqq 23/I.$$

Proof. Let $y_1, \cdots, y_c$ be all singular points of $\bar{V}$. Then $e(g^{-1}\pi^{-1}(\mathrm{Sing}\bar{V}))=\tilde{c}+\rho(U)-\rho(\bar{U})$ because $g^{-1}(\mathrm{Sing}\bar{U})$ consists of rational trees. Since D consists of rational trees, we have $e(D)=c+\#(D)$, where $\#(D)$ signifies the number of all irreducible components of D. By noting that π is étale over V^0, we obtain:

$$e(U)-e(g^{-1}\pi^{-1}(\mathrm{Sing}\bar{V}))=I(e(V)-e(D)).$$

By Noether's formula, we have $e(V)=12-(K_V^2)=\rho(V)+2=\rho(\bar{V})+\#(D)+2$. So, the first assertion of Lemma 2.4 follows.

Suppose that $\tilde{c}=c$ and U is a $K3$-surface. By the first assertion of Lemma 2.4, we have:

$$\begin{aligned} c &= \frac{1}{I-1}(2I+I\rho(\bar{V})+\rho(U)-\rho(\bar{U})-24) \\ &= 2+\frac{1}{I-1}(I\rho(\bar{V})+\rho(U)-\rho(\bar{U})-22) \\ &\leqq 2+\frac{1}{I-1}(I\rho(\bar{U})+\rho(U)-\rho(\bar{U})-22) \\ &= 2+\rho(U)-\rho(U)+\rho(\bar{U})+\frac{1}{I-1}(\rho(U)-22) \\ &< 22+\rho(\bar{U})-\rho(U). \end{aligned}$$

We also have $I(\rho(\bar{V})-c+2)=24+\rho(\bar{U})-\rho(U)-c \geqq 24+\rho(\bar{V})-20-c=(\rho(\bar{V})-c+2)+2$. Hence we obtain $\rho(\bar{V})-c+2 \geqq 2/(I-1)>0$. On the other hand, we have $\rho(\bar{V})-c+2=(24+\rho(\bar{U})-\rho(U)-c)/I \leqq 23/I$.

Consider the case where I is a prime number. Then $\pi^{-1}(y_i)$ consists of one or I points. If $\pi^{-1}(y_i)$ consists of I points $\{x_{ij}\}$ for some i, then $\hat{\mathcal{O}}_{\bar{U},x_{ij}} \cong \hat{\mathcal{O}}_{\bar{V},y_i}$. Hence y_i is a rational double singular point. This contradicts our assumption. Therefore, $\pi^{-1}(y_i)$ $(1 \leqq i \leqq c)$ consists of a single point and $\tilde{c}=c$. Q. E. D.

Lemma 2.5. *Let $\bar{V}$ be a log Enriques surface. Suppose that $\bar{U}$ is nonsingular and*

I (=Index$(K_{\bar{V}})$) is a prime number. Then for each singular point y of $\bar{V}$, $\pi^{-1}(y)$ consists of a single smooth point, and $\hat{O}_{\bar{V},y}\cong\boldsymbol{C}[[X, Y]]/C_{I,q}$ with a cyclic subgroup $C_{I,q}$ of $GL(2, \boldsymbol{C})$, where $1\leqq q\leqq I-2$ and g. c. d. *$(q, I)=1$. The action of $C_{I,q}$ is given by: $gX=\xi X$ and $gY=\xi^q Y$, where g is a generator of $C_{I,q}$ and ξ is a primitive I-th root of the unity.*

Proof. This follows from the argument at the end of the previous lemma, the smoothness of U and the assumption that y is not a rational double singular point.
Q. E. D.

§3. The case where the bi-canonical divisor is trival

Let $\bar{V}$ (or synonymously (V, D)) be a log Enriques surface with Index$(K_{\bar{V}})=2$. Then $D^\#=(1/2)D$ and the configuration of D is described by Lemma 1.8. Let $G_i(1\leqq i\leqq c)$ be all connected components of D and set $n_i=\#(G_i)$. Let $\tau: \tilde{V}\to V$ be the blowing-up of all singular points of D (intersection points of irreducible components of D). Denote by $\tilde{D}$ the proper transform of D. Then $\tilde{D}$ consists of isolated (-4)-curves. Since $2(D^\#+K_V)=D+2K_V\sim 0$, we have $\tilde{D}+2K_{\tilde{V}}\sim 0$. Hence $(\tilde{V}, \tilde{D})$ is again a log Enriques surface and if $\tilde{f}: \tilde{V}\to V^*$ is the contraction of $\tilde{D}$ then Index$(K_{V^*})=2$. As in §2, using the relation $\tilde{D}\sim -2K_{\tilde{V}}$, we can find a finite morphism $\tilde{\pi}: \tilde{U}\to\tilde{V}$, which is étale over $\tilde{V}-\tilde{D}$ and totally ramified over $\tilde{D}$. Then $\tilde{U}$ is nonsingular and $(\tau\circ\tilde{\pi})^{-1}(G_i)$ consists of $2n_i-1$ (-2)-curves which are contractible to a rational double singular point of Dynkin type A_{2n_i-1}. Indeed, if $\pi: \bar{U}\to\bar{V}$ is the canonical covering and if $f: V\to\bar{V}$ and $g: U\to\bar{U}$ are minimal resolutions, then $\tilde{U}=U$ and $\pi\circ g=f\circ\tau\circ\tilde{\pi}$. Note that U is a $K3$-surface because there are rational curves on U.

Lemma 3.1. *Let (V, D) be a log Enriques surface with* Index$(K_{\bar{V}})=2$. *Then the minimal resolution U of the canonical covering $\bar{U}$ of (V, D) is a $K3$-surface. Moreover, $\#(D)\leqq 10$, and if G_i $(1\leqq i\leqq c)$ is a connected component of D with $n_i:=\#(G_i)$, then $\pi^{-1}(f(G_i))$ is a singular point of Dynkin type A_{2n_i-1} on $\bar{U}$ and $\pi^{-1}(f(G_i))$ $(1\leqq i\leqq c)$ exhausts all singular points of $\bar{U}$.*

In particular, $\#(\mathrm{Sing}\bar{U})=\#(\mathrm{Sing}\bar{V})=c\leqq\#(D)\leqq 10$.

Proof. We have only to show that $\#(D)\leqq 10$. By Lemma 1.8, we have $-(K_{\tilde{V}})^2=\#(\tilde{D})=\#(D)$. Note that $20\geqq\rho(\tilde{U})\geqq\rho(\tilde{V})=10-(K_{\tilde{V}})^2=10+\#(\tilde{D})=10+\#(D)$. So, $\#(D)\leqq 10$.
Q. E. D.

The upper bound 10 for $\#(\mathrm{Sing}\bar{V})$ is the best possible one in view of the following example:

Example 3.2. Let $\pi: \Sigma_1\to\boldsymbol{P}^1$ be the $\boldsymbol{P}^1$-fibration on a Hirzebruch surface Σ_1, let L be a general fiber and let M be the (-1)-curve of Σ_1. Take a nonsingular irreducible member A in $|2M+2L|$. Then there are exactly two ramification points P_i $(i=1, 2)$ for a double covering $\pi_{|A}: A\to\boldsymbol{P}^1$. Let L_i be the fiber with $P_i\in L_i$ and let L_3 $(\neq L_1, L_2)$ be an arbitrary fiber. Then A meets L_3 in two distinct points. Since $\dim|M+L|=2$, there is an irreducible member C in $|M+L|$ so that $P_1, P_2\in C$. Denote by $P_3:=$

$M\cap L_1$ and $P_4:=C\cap L_3$ and denote one of the points $A\cap L_3$ by P_5. Let $\tau_1\colon V_1\to\Sigma_1$ be the blowing-up of five points P_i's and set $E_j:=\tau_1^{-1}(P_j)$ $(j=1, 2)$. Let $\tau_2\colon V_2\to V_1$ be the blowing-up of two points $Q_3:=\tau_1'(A)\cap E_1$ and $Q_4:=\tau_1'(A)\cap E_2$ and set $E_k:=\tau_2^{-1}(Q_k)$ $(k=3, 4)$. Let $\tau_3\colon V\to V_2$ be the blowing-up of two points $\tau_2'\tau_1'(A)\cap E_3$ and $\tau_2'\tau_1'(A)\cap E_4$. Set $\tau:=\tau_1\circ\tau_2\circ\tau_3$, $E_j':=\tau_3'\tau_2'(E_j)$, $E_k':=\tau_3'(E_k)$, $L_p':=\tau'(L_p)$, $A':=\tau'(A)$, $C':=\tau'(C)$, $M':=\tau'(M)$ and $D:=\sum_{n=1}^{4}E_n'+\sum_{p=1}^{3}L_p'+A'+C'+M'$. Then D is a rod with two (-3)-curves as tips and eight (-2)-curves in between. By noting that $\sum_{p=1}^{3}L_p+A+C+M\sim-2K_{\Sigma_1}$, we can check that $D\sim-2K_V$. Hence (V, D) is a log Enriques surface with $\mathrm{Index}(K_{\tilde V})=2$ and with $\#(D)=10$. Let $\tau\colon\tilde V\to V$ be the blowing-up of all nine singular points of D and and let $\tilde D:=\tau'(D)$. Then $(\tilde V, \tilde D)$ is a log Enriques surface such that $\tilde D+2K_{\tilde V}\sim 0$ and $\tilde D$ consists of ten isolated (-4)-curves.

Now we are going to state and prove Theorem 3.6 which is a main result of the present section. For this purpose, we need several lemmas.

Lemma 3.3. *Let* (V, D) *be a log Enriques surface such that* $\mathrm{Index}(K_{\tilde V})=2$ *and* D *consists of isolated* (-4)-*curves. Let* $\Phi\colon V\to\boldsymbol{P}^1$ *be a* $\boldsymbol{P}^1$-*fibration. Suppose that* S *is a singular fiber containing at least one component of* D *and that* D_u $(1\leqq u\leqq r+1)$ *are all components of* D *contained in* S. *Then either* $r=0$ *or there are* (-1)-*curves* E_v $(1\leqq v\leqq r)$ *such that* $(E_v, D_v)=(E_v, D_{v+1})=1$. *More precisely, one of the following cases occurs:*

Case (1). *We have* $r=0$. *There are integers* $s\geqq 1$, $a_i\geqq 0$ *and irreducible components* $C_i(j)$ $(1\leqq i\leqq s;\ 0\leqq j\leqq a_i)$ *of* S *such that* $C_i(0)$ *is a* (-1)-*curve and* $C_i(j)$ *is a* (-2)-*curve if* $j\geqq 1$. *Moreover,* $\sum_{i=1}^{s}(1+a_i)=4$, $(D_1, C_i(0))=(C_i(j), C_i(j+1))=1$ $(0\leqq j<a_i)$ *and* $\mathrm{Supp}S=D_1+\sum_{i,j}C_i(j)$.

Case (2). *We have* $r\geqq 1$. *There are integers* $s\geqq 1$, $t\geqq 1$, $a_i\geqq 0$, $b_j\geqq 0$ *and irreducible components* $C_i(m)$ $(1\leqq i\leqq s;\ 0\leqq m\leqq a_i)$ *and* $C_{s+j}(n)$ $(1\leqq j\leqq t;\ 0\leqq n\leqq b_j)$ *of* S *such that* $C_p(0)$ $(1\leqq p\leqq s+t)$ *is a* (-1)-*curve and* $C_p(q)$ *is a* (-2)-*curve if* $q\geqq 1$. *Moreover,* $\sum_{i=1}^{s}(1+a_i)=\sum_{j=1}^{t}(1+b_j)=2$, $(D_1, C_i(0))=(D_{r+1}, C_{s+j}(0))=(C_p(q), C_p(q+1))=1$ *and* $\mathrm{Supp}S=\sum D_u+\sum E_v+\sum_{p,q}C_p(q)$ *for all possible* i, j, p *and* q.

Case (3). *We have* $r=2$. *There are* (-1)-*curves* F_i $(1\leqq i\leqq 3)$ *such that* $(F_i, D_i)=1$ *and* $\mathrm{Supp}S=\sum D_u+\sum E_v+\sum F_i$.

Case (4). *We have* $r=3$. *There are* (-1)-*curves* F_i $(i=1, 2)$ *such that* $(F_1, D_1)=(F_2, D_3)=1$ *and* $\mathrm{Supp}S=\sum D_u+\sum E_v+\sum F_i$.

Proof. Let E_i $(1\leqq i\leqq m)$ and C_j $(1\leqq j\leqq n)$ be all (-1)-curves and (-2)-curves in S, respectively. Then $\mathrm{Supp}S=\sum D_u+\sum E_i+\sum C_j$ by Lemma 1.10, (1). Note that $(E_i, E_k)=0$ $(i\neq k)$ and the dual graph of S is a connected tree. We shall show that $\sum D_u+\sum E_i$ is a connected tree. We have only to consider the case where there are (-2)-curves in S. Let C be a connected component of $\sum C_j$. Noting that $(C, D)=0$ by (1) of Lemma 1.10, that S is connected and that $\sum E_i+\sum C_j$ has negative definite intersection matrix, we can find a (-1)-curve in S, say E_1, such that E_1+C is a rod, $(E_1,$

$\sum D_u)\geqq 1$ and $(E_1+C, E_i)=(E_1+C, C_k)=0$ for each $i\neq 1$ and each $C_k\leqq\sum C_i-C$. Hence C looks like a twig in S. Therefore, $\sum D_u+\sum E_i$ is a connected tree. So, S is as in the case (1) of Lemma 3.3 if $r=0$, i. e., if there is only one component of D in S.

Suppose $r\geqq 1$. Take (-1)-curves in S, say E_v $(1\leqq v\leqq r')$, such that $\sum D_u+\sum E_v$ is connected while $\sum D_u+\sum_{v\neq k} E_v$ is not connected for every $1\leqq k\leqq r'$. We shall prove that $r'=r$ and E_v's satisfy the requirement of the first assertion of Lemma 3.3. Suppose that $\sum D_u+\sum E_v$ is not a rod. Then, there is a (-4)-curve in S, say D_1, such that D_1 meets three (-1)-curves, say E_k $(k=1, 2, 3)$ because S is contractible to a nonsingular rational curve and $(D_i, D_j)=0$ $(i\neq j)$. By our assumption, $\sum D_u+\sum_{v\neq k} E_v$ is not connected. Hence E_k meets a component H_k of $\sum D_u$. Then, Supp S $(\supseteqq \mathrm{Supp}(D_1+\sum H_k+\sum E_k))$ is not contractible to a nonsingular curve. We reach a contradiction. Therefore, $\sum D_u+\sum E_v$ is a rod. Note that $(E_k, \sum D_u)=2$ $(1\leqq k\leqq r')$, for otherwise $(E_k, \sum D_u)=1$ and $\sum D_u+\sum_{v\neq k} E_v$ is connected, which contradicts our assumption. Hence $r'=r$ and $\sum E_v$ meets $\sum D_u$ as described in Lemma 3.3.

If each (-1)-curve other than E_v's in S meets only D_1 or D_{r+1} among D_u's, then S drops in the case (2) of Lemma 3.3 by the above arguments. Suppose that there are (-1)-curves F_k $(1\leqq k\leqq s)$, other than E_v's, meeting one of $D_2, \cdots, D_r$. Then $s=1$ and F_1 meets only one component of Supp $S-F_1$ because $\sum_{u\geqq 2} D_u+\sum E_v+\sum F_k$ has negative definite intersection matrix and S is contractible to a nonsingular curve. Thus S drops in the case (3) or (4) of Lemma 3.3. Q. E. D.

Lemma 3.4. *Let (V, D) be a log Enriques surface with* $\mathrm{Index}(K_{\bar V})=2$. *Then $\boldsymbol{P}^2$ is a relatively minimal model of V.*

Proof. Since $(K_{\bar V}^2)=-c<9$ by Lemma 1.8, c being the number of all connected components of D, there is a birational morphism $\eta: V\rightarrow\sum_n(0\leqq n\leqq 4)$ by Lemma 1.10, (1). Let $\pi: \sum_n\rightarrow \boldsymbol{P}^1$ be a $\boldsymbol{P}^1$-fibration of $\sum_n$ and let M be a minimal section of π.

Consider first the case where $\eta'(M)$ is not a component of D. Then $-2\leqq(\eta'M)^2\leqq(M^2)=-n\leqq 0$ by Lemma 1.10, (1). Lemma 3.4 is clear if $n=1$. Suppose $n=0$ or 2. Since $(K_{\bar V}^2)\leqq 7$, there is a blowing-up $\eta_1: V_1\rightarrow\sum_n$ of a point P in a fiber L of π and a birational morphism $\eta_2: V\rightarrow V_1$ such that $\eta=\eta_1\circ\eta_2$. If $n=2$, then P is not contained in M for we must have $(\eta'M)^2\geqq -2$. Let $\eta_3: V_1\rightarrow \boldsymbol{P}^2$ be the blowing-down of $\eta_1'(L)$ and $\eta_1'(M)$. Then we obtain a birational morphism $\eta_3\circ\eta_2: V\rightarrow\boldsymbol{P}^2$. If $n=0$, let $\eta_3: V_1\rightarrow\boldsymbol{P}^2$ be the blowing-down of $\eta_1'(L)$ and $\eta_1'(M_1)$ where M_1 is the minimal section with $P\in M_1$.

Assume $\eta'(M)$ is a component of D. If $n\leqq 1$, Lemma 3.4 can be proved by the same argument as above. So, we assume $n\geqq 2$. Let $\tau: \tilde V\rightarrow V$ be the blowing-up of all singular points of D. Set $\tilde D:=\tau'(D)$ and $\tilde M:=\tau'\eta'(M)$. Then $(\tilde V, \tilde D)$ is a log Enriques surface and $\tilde D$ consists of isolated (-4)-curves. Set $\Phi:=\pi\circ\eta$ and $\tilde\Phi:=\Phi\circ\tau: \tilde V\rightarrow\boldsymbol{P}^1$. Then $\tilde M$ is a cross-section of $\tilde\Phi$. Let $S_1, \cdots, S_k$ be all singular fibers of Φ and let $\tilde S_i:=\tau^*(S_i)$.

Suppose $k\geqq 3$. Then, there are blowing-up $\eta_1: V_1\rightarrow\sum_n$ of three points P_i of $\eta(S_i)$ $(i=1, 2, 3)$ and a birational morphism $\eta_2: V\rightarrow V_1$ such that $\eta=\eta_1\circ\eta_2$. Note that $-4\leqq$

$(\eta_1'M)^2 \leqq (M^2) = -n \leqq -2$. Let $n' = -(\eta_1'M)^2$. Let $\eta_3 \colon V_1 \to \boldsymbol{P}^2$ be the blowing-down of, $n'-1$ (-1)-curves contained in $\sum \eta_1^{-1}\eta(S_i)$ and meeting $\eta_1'(M)$, $4-n'$ (-1)-curves contained in $\sum \eta_1^{-1}\eta(S_i)$, not meeting $\eta_1'M$ and disjoint from the previous (-1)-curves, and then the curve $\eta_1'(M)$. Thus we obtain a birational morphism $\eta_3 \circ \eta_2 \colon V \to \boldsymbol{P}^2$.

Suppose $k \leqq 2$. If S_i contains a component of D, then $\tilde{S}_i$ looks as in one of the cases (1)-(4) of Lemma 3.3 and τ contracts no (-4)-curves of $\tilde{S}_i$. If S_i contains no components of D, then S_i is a rod consisting of several (-2)-curves and two (-1)-curves E_1 and E_2 as tips with $(E_1, \tau(\tilde{M}))=1$ by (1) of Lemma 1.10 and because $(\tau(\tilde{M}), S_i)=1$. We have $8=(K_{\Sigma_n}^2)=(K_V^2)+\sum(\#(S_i)-1)$ and $\sum$ $(\#(S_i)-1)=8+c \geqq 9$. Note that $-4 \leqq (\tau(\tilde{M}))^2 \leqq -2$. Note also that if $k=1$ and S_1 contains components of D, then $\tilde{S}_1$ is in the case (2) of Lemma 3.3 with $\#(\tilde{S}_1) \geqq \#(S_1) \geqq 10$ and $\tilde{M}$ meets a (-1)-curve of $\tilde{S}_1$ with coefficient one in $\tilde{S}_1$. Therefore, in the case $k=1$, we can find a birational morphism $\eta_1 \colon V \to \Sigma_1$ such that $(\eta_1\tau(\tilde{M}))^2=-1$ because $\#(S_1) \geqq 10$. This implies Lemma 3.4. Suppose $k=2$. It is impossible that both $\tilde{S}_1$ and $\tilde{S}_2$ belong to the case (1) of Lemma 3.3 by virtue of the inequality $\sum(\#(S_i)-1) \geqq 9$. So, in the case $k=2$, by using the above inequality, we can find a birational $\eta_1 \colon V \to \Sigma_1$ such that $(\eta_1\tau(\tilde{M}))^2=-1$ and conclude Lemma 3.4. Q. E. D.

Lemma 3.5. *Let* (V, D) *be a log Enriques surface with* $\mathrm{Index}(K_V)=2$ *and* c $(=\#\{$*connected component of* $D\}) \geqq 2$. *Let* $\eta \colon V \to \boldsymbol{P}^2$ *be a birational morphism. Then there are exceptional curves* E_v $(1 \leqq v \leqq c-1)$ *of* η *such that* E_v *is a* (-1)*-curve and the dual graph of* $D+\sum E_v$ *is a connected tree.*

Proof. Let E_i $(1 \leqq i \leqq m)$ be all exceptional curves of η such that E_i is a (-1)-curve on V. Let C_j $(1 \leqq j \leqq n)$ be all exceptional curves of η such that $(C_j^2) \leqq -2$ and C_j is not contained in D. By (1) of Lemma 1.10, we have $(C_j^2)=-2$ and $(C_j, D)=0$. Note that $(E_i, E_k)=0$ $(i \neq k)$. Since $(E_i, D)=(E_i, -2K_V)=2>0$, we have $\eta(E_i) \in \eta(D)$.

We assert that $\eta^{-1}\eta(D)=D+\sum E_i+\sum C_j$ and that $D+\sum E_i+\sum C_j$ is connected if and only if so is $D+\sum E_i$. Let C be a connected conmponent of $\sum C_j$. Since $(C, D)=0$ and $\sum E_i+\sum C_j$ is an exceptional divisor of η, there is a curve among E_i's, say E_1, such that $C+E_1$ is a rod and $(C+E_1, E_i)=(C+E_1, C_k)=0$ for each $i \neq 1$ and each $C_k \leqq \sum C_j - C$. Thus, $\eta(C)=\eta(E_1) \in \eta(D)$ and C looks like a twig in $D+\sum E_i+\sum C_j$. This proves our assertion.

We now claim that $D+\sum E_i$ is connected. Suppose the claim is false. Then $D+\sum E_i+\sum C_j (=\eta^{-1}\eta(D))$ and $\eta(D)$ are not connected. So, there is a union $\varDelta$ of connected components of D such that $\eta(\varDelta)$ consists of a single point, $\eta(D-\varDelta) \neq \emptyset$ and $\eta(\varDelta) \cap \eta(D-\varDelta)=\emptyset$ because $\rho(\boldsymbol{P}^2)=1$. Hence $\eta^{-1}\eta(\varDelta) \cap \eta^{-1}\eta(D-\varDelta)=\emptyset$. So, if we write $\eta^{-1}\eta(\varDelta)=\varDelta+\sum E_i'+\sum C_j'$ and $\eta^{-1}\eta(D-\varDelta)=D-\varDelta+\sum E_i''+\sum C_j''$ with $E_i', E_i'' \in \{E_i;\ 1 \leqq i \leqq m\}$ and $C_j', C_j'' \in \{C_j;\ 1 \leqq j \leqq n\}$, then $\sum E_i'+\sum E_i''=\sum E_i$ and $\sum C_j'+\sum C_j''=\sum C_j$. Since $\eta(\varDelta)$ is a smooth point of $\boldsymbol{P}^2$, there are (-1)-curves F_p's in $\{E_i'\}$ such that $\varDelta+\sum F_p$ is a linear chain while $\varDelta+\sum F_p-F_q$ is not connected for each $F_q \leqq \sum F_p$. Let $\eta_1 \colon V \to \tilde{V}$ be the contraction of $\sum F_p$, let $\tilde{D}=\eta_1(D)$ and let $\tilde{\varDelta}=\eta_1(\varDelta)$. Then $(\tilde{V}, \tilde{D})$ is a log Enriques surface with $\tilde{D}+2K_{\tilde{V}} \sim 0$, as well. Clearly, η is factored as $\eta=\eta_2 \circ \eta_1$ with a birational morphism $\eta_2 \colon \tilde{V} \to \boldsymbol{P}^2$. Since $\eta_2(\tilde{\varDelta})=\eta(\varDelta)$ is a smooth point of $\boldsymbol{P}^2$, there is a (-1)-curve $\tilde{G}_r$ in $\{\eta_1(E_i')\}$ or $\{\eta_1(C_j')\}$ such that $(\tilde{G}_r, \tilde{\Delta})\geqq 1$. Then $(\tilde{G}_r, \tilde{\Delta})=(\tilde{G}_r, \tilde{D})=2$ and it is impossible that $\eta_2(\tilde{\Delta})=\eta_2(\tilde{\Delta}+\tilde{G}_r)$ is a smooth point of $\boldsymbol{P}^2$. Therefore, the claim is true.

Restrict E_i's to a subset $\{E_i;\ 1\leqq i\leqq r\}$, relabelled suitably, where $r\leqq m$, so that $D+\sum_{v=1}^{r} E_v$ is connected while $D+\sum_{v\neq j} E_v$ is not connected for each $1\leqq j\leqq r$. We shall show that $r=c-1$ and E_v's satisfy the requirement of Lemma 3.5. If $(E_j, \Delta)=2$ for some $1\leqq j\leqq r$ and some connected component Δ of D, then $(E_j, D-\Delta+\sum_{v\neq j} E_v)=0$ for $(E_j, D)=2$. Then $D+\sum_{v\neq j} E_v$ is connected, which contradicts our assumption. Thus each E_v meets exactly two connected components of D. Hence there are no three components of $D+\sum E_v$ passing through one and the same point because D has only simple normal crossings and $(E_i, E_j)=0$ $(i\neq j)$. Therefore $D+\sum E_v$ has only simple normal crossings. Suppose $D+\sum E_v$ contains a loop. Then there are (-1)-curves, say E_k $(1\leqq k\leqq s;\ s\leqq r)$, and rods Δ_k such that $\Delta_k\leqq D$ and $(\Delta_{k-1}, E_k)=(E_k, \Delta_k)=1$ $(\Delta_0:=\Delta_s)$ because D contains no loops. Then $(E_1, D-\mathrm{Supp}(\Delta_1+\Delta_s)+\sum_{v\neq 1} E_v)=0$ and $D+\sum_{v\neq 1} E_v$ is connected. This contradicts our assumption. Therefore, the dual graph of $D+\sum_{v=1}^{r} E_v$ is a tree. By noting that $(E_i, E_j)=0$ $(i\neq j)$ and E_v meets exactly two connected components of D, we have $r=c-1$. Q. E. D.

Theorem 3.6. *Let (V, D) be a log Enriques surface such that* $\mathrm{Index}(K_V)=2$ *and D consists of exactly c $(\geqq 2)$ isolated (-4)-curves. Then there are (-1)-curves F_j $(1\leqq j\leqq c-1)$ of V such that $D+\sum F_j$ is a linear chain. More precisely, we can write $D=\sum D_i$ with irreducible components D_i's of D such that $(D_j, F_j)=(F_j, D_{j+1})=1$, $1\leqq j<c$. Hence, if $\varphi\colon V\to W$ is the blowing-down of F_j's, then $\varphi(D)$ is a rod consisting of two (-3)-curves as tips and $c-2$ (-2)-curves and $(W, \varphi(D))$ is a log Enriques surface with* $\mathrm{Index}(K_W)=2$.

Remark. Let (V, D) be an arbitrary log Enriques surface with $\mathrm{Index}(K_V)=2$. Let $(\tilde{V}, \tilde{D})$ be the log Enriques surface which is associated with (V, D) and defined at the beginning of §3. Then we can apply Theorem 3.6 to $(\tilde{V}, \tilde{D})$ and obtain Theorem 3.6′ which is stated in the Introduction.

Proof. Suppose that there are (-4)-curves $\tilde{R}_i$ $(1\leqq i\leqq r)$ of D and (-1)-curves F_j $(1\leqq j\leqq r-1)$ of V such that $(F_j, \tilde{R}_j)=(F_j, \tilde{R}_{j+1})=1$. Let $\sigma\colon V\to X$ be the blowing-down of F_j's and let $G=\sigma(D)$. Then (X, G) is a log Enriques surface with $\mathrm{Index}(K_X)=2$. Set $R_i:=\sigma(\tilde{R}_i)$ and $R=\sum R_i$. Then R is a rod and $(R_i^2)=-3$ if $i=1$ or r and $(R_i^2)=-2$ otherwise. The divisor G consists of R and several isolated (-4)-curves. Denote by Σ the set of all morphisms σ of the above type. Then Σ is not empty. Indeed, by Lemma 3.5, there are (-1)-curves E_j $(1\leqq j\leqq c-1)$ of V such that $D+\sum E_j$ is a tree. Then, the blowing-down of E_1 belongs to Σ. Theorem 3.6 is equivalent to asserting that there is a $\sigma\in\Sigma$ such that $\sigma(D)$ contains no isolated (-4)-curves. It suffices to prove the following:

CLAIM 1. For any $\sigma\in\Sigma$ such that $\sigma(D)$ contains at least one isolated (-4)-curve, there is a $\tau\in\Sigma$ such that $\tau(D)$ contains less isolated (-4)-curves than $\sigma(D)$.

We shall prove the claim 1 by using the following three lemmas. We use the above notations $\sigma: V\rightarrow X$, $G=\sigma(D)$ and $R=\sum_{i=1}^{r}R_i$.

Lemma 3.7. *If there is a* (-1)-*curve* E *of* X *such that* E *meets one isolated* (-4)-*curve of* G *and one* (-3)-*curve of* R, *then the claim* 1 *holds with a morphism* τ *which is the composite of* σ *and the blowing-down of* E.

Proof. Obvious.

Lemma 3.8. *If there are two disjoint* (-1)-*curves* E_1 *and* E_2 *of* X *such that* $(E_1, R_1)=(E_1, R_r)=(E_2, R_q)=(E_2, G_1)=1$ *for some* $2\leqq q\leqq r-1$ *and some isolated* (-4)-*curve* G_1 *of* G, *then the claim* 1 *holds.*

Proof. Blowing down E_1 and E_2 and blowing up one of the intersection points of two divisors R_q and $R-R_q$. We obtain a new surface Y from X. Evidently, there is a birational morphism $\tau: V\rightarrow Y$ such that $\tau\in\Sigma$. Then τ satisfies the condition of the claim 1. Q. E. D.

Lemma 3.9. *If there is a* (-1)-*curve* E *of* X *such that* $(E, R_q)=(E, G_1)=1$ *for some* $3\leqq q\leqq r-2$ *and some isolated* (-4)-*curve* G_1 *of* G *then the claim* 1 *holds.*

Proof. Relabelling $R=\sum R_i$ anew if necessary, we may assume $q\leqq r-q+1$. Let $S_0=2(E+R_q)+R_{q-1}+R_{q+1}$ and $\Phi: X\rightarrow \boldsymbol{P}^1$ the $\boldsymbol{P}^1$-fibration defined by $|S_0|$. Then R_{q-2} and R_{q+2} are cross-sections.

Assume $r=5$. Then $q=3$. Since $(K_X^2)<0$, there is a singular fiber $S(\neq S_0)$. Then there is a (-1)-curve F_1 in S such that $(F_1, R_1)=1$ (cf. Lemma 1.10, (1)). Since $(F_1, G)=2$, F_1 meets a (-4)-curve in G or R_5. Accordingly, the claim 1 follows from Lemma 3.7 with $E:=F_1$ or Lemma 3.8 with $E_1:=F_1$ and $E_2:=E$.

Assume $r\geqq 6$. Let S_1 be the singular fiber of Φ containing $R_{q+3}+\cdots+R_r$. Suppose that S_1 contains at least one (-4)-curve of G. As shown in the proof of Lemma 3.3, the divisor consisting of all (-1)-curves in S_1 and all components of G in S_1 is a connected tree. Suppose further that there is a (-1)-curve F_1 and a (-4)-curve H_1 in S_1 such that $(F_1, H_1)=(F_1, R_t)=1$ for some $q+3\leqq t\leqq r$. Since S_1 is contractible to a nonsingular rational curve, $t=q+3$ or r. We have $t=r$ because $(R_{q+2}, S_1)=1$ and R_{q+3} has coefficient one in S_1. Thus the claim 1 follows from Lemma 3.7 with $E:=F_1$. If there is no such a (-1)-curve F_1 as above connecting a (-4)-curve and a linear chain $R_{q+3}+\cdots+R_r$, then S_1 contains a linear chain $R_1+\cdots+R_{q-3}$ and there exists a (-1)-curve F_1 connecting a (-4)-curve H_1 and the linear chain $R_1+\cdots+R_{q-3}$. Then we are done by the same argument as above. So, we may assume that S_1 contains no (-4)-curves.

If $q\geqq 4$, we may assume that $R_1+\cdots+R_{q-3}$ is not contained in S_1. Indeed, since the divisor consisting of all (-1)-curves in S_1 and all components of G in S_1 is a connected tree (cf. Lemma 3.3), in the case where $R_1+\cdots+R_{q-3}$ is contained in S_1, we can find integers $1 \leqq s \leqq q-3$ and $q+3 \leqq t \leqq r$ such that there is a (-1)-curve F_1 in S_1 satisfying $(F_1, R_s)=(F_1, R_t)=1$. By Lemma 3.8 with $E_1:=F_1$ and $E_2:=E$, we may assume that $(F_1, R_1+R_r) \leqq 1$. In the case $(F_1, R_1)=0$, we have $q \geqq 5$ and $s=q-3$ because S_1 is contractible to a nonsingular curve. Then R_{q-3} has coefficient greater than one in S_1. This is a contradiction to $(R_{q-2}, S_1)=1$. Similarly, we are led to a contradiction if $(F_1, R_r)=0$. So, we assume that $R_1+\cdots+R_{q-3}$ is not contained in S_1.

Now we are reduced to considering the case where S_1 consists of one (-3)-curve R_r and several (-1)-curves and (-2)-curves. Such a degenerate fiber S_1 is described in [12; Lemma 1.6]. If there is only one (-1)-curve F_1 in S_1 then F_1 has coefficient greater than two in S_1. This is impossible for the 2-section G_1 of Φ meets only F_1 in S_1 by Lemma 1.10, (1). So, S_1 contains at least two (-1)-curves. Suppose that there are more than two (-1)-curves F_i's in S_1, then two of them, say F_1 and F_2, meet R_r. We may assume that $(F_1, R_{q-2})=0$. Then F_1 meets a (-4)-curve in G because $(F_1, G)=2$. Then the claim 1 follows from Lemma 3.7 with $E:=F_1$. Suppose that there are exactly two (-1)-curves F_1 and F_2 in S_1. Then one of them, say F_1, meets R_r. Since $(F_1, G)=2$, F_1 meets the cross-section R_{q-2} or a (-4)-curve of G. If F_1 meets a (-4)-curve of G then we are done by Lemma 3.7 with $E:=F_1$. So, we assume that $(F_1, R_{q-2})=1$. Hence $(F_2, R_{q-2})=0$, F_1 has coefficient one in S_1 and F_2 meets one component of $R_{q+3}+\cdots+R_r$. Applying the same argument to F_2, we may assume that $(F_2, R_r)=0$. Then we can show that $r=q+5$, $(F_2, R_{r-1})=1$ and $S_1=2(F_2+R_{q+4})+F_1+R_{q+3}+R_{q+5}$. If $q=3$, in particular, the claim 1 follows from Lemma 3.8 with $E_1:=F_1$ and $E_2:=E$. Suppose $q \geqq 4$. Let S_2 be the singular fiber of Φ containing $R_1+\cdots+R_{q-3}$. Applying the same argument for S_1 to the fiber S_2, we can prove the claim 1 except for the following case: $q=6$, $r=11$, $\#(S_2)=\#(S_1)=5$ and S_1 and S_2 have the same configuration. In the exceptional case, we have $\#(G) \geqq 12$, which is a contradiction to Lemma 3.1. Q. E. D.

We resume the proof of the claim 1. Consider the case where G contains at least two isolated (-4)-curves. By Lemma 3.5, there are (-1)-curves E_i's of X such that $G+\sum E_i$ is a connected tree. In view of Lemma 3.7, we may assume that there are two (-4)-curves G_1 and G_2 and two (-1)-curves, say E_1 and E_2, such that one of the following two cases occurs.

Case (1). $(G_i, E_i)=(G_1, E_2)=(R_q, E_1)=1$ $(i=1, 2)$ for some $2 \leqq q \leqq r-1$.

Case (2). $(G_i, E_i)=(R_q, E_1)=(R_p, E_2)=1$ $(i=1, 2)$ for some $2 \leqq q \leqq p \leqq r-1$.

Assume the case (1) occurs. Labelling $R=\sum R_i$ anew if necessary, we may assume $q \leqq r-q+1$. If $q \geqq 3$, the claim 1 follows from Lemma 3.9 with $E:=E_1$. Suppose $q=2$. Blowing down E_1 and E_2 and blowing up the point $R_1 \cap R_2$, we obtain a new surface Y from X. Clearly, there is a birational morphism τ: $V \rightarrow Y$ such that $\tau \in \Sigma$ and τ satisfies the condition of the claim 1.

Assume the case (2) occurs. Let $S_0:=E_1+R_q+\cdots+R_p+E_2$ and Φ: $X \rightarrow \boldsymbol{P}^1$ be the $\boldsymbol{P}^1$-fibration defined by $|S_0|$. Then R_{q-1}, R_{p+1}, G_1 and G_2 are cross-sections of Φ. By the same argument as in Lemma 3.9 applied to a singular fiber S_1 of Φ containing $R_1+\cdots+R_{q-2}$ or a singular fiber S_2 containing $R_{p+2}+\cdots+R_r$, it suffices to consider the case where $q=5$ and $S_1:=2(F_2+R_2)+F_1+R_1+R_3$ is a singular fiber of Φ with two

(-1)-curves F_1 and F_2 such that $(F_1, R_1)=(F_1, R_{p+1})=(F_2, R_2)=1$. Then $(S_1, G_1)=1$ implies $(F_1, G_1)=1$. This leads to $(F_1, G)\geqq 3$, a contradiction.

Next, we consider the case where $G=R+G_1$ with a unique isolated (-4)-curve G_1. By Lemma 3.5, there is a (-1)-curve E such that $(E, G_1)=(E, R_q)=1$ for some $1\leqq q\leqq r$. In view of Lemma 3.7, we may assume $2\leqq q\leqq r-1$. Labelling $R=\sum R_i$ anew if necessary, we may assume $q\leqq r-q+1$. In view of Lemma 3.9, it suffices to consider the case $q=2$. In this case, we have $r\geqq 2q-1=3$.

Assume $r\geqq 5$. Let ϕ: $X\to \boldsymbol{P}^1$ be the $\boldsymbol{P}^1$-fibration such that $f_0:=3E+3R_2+2R_3+R_1+R_4$ is a singular fiber of ϕ. Since $(K_X^2)=-2<4$, there is a singular fiber f_1 other than f_0. By (1) of Lemma 1.10 and since $(f_1, G_1)=3$, there is a (-1)-curve E_1 in f_1 such that $(E_1, G_1)=1$ or 3. Since $(E_1, G_1)\leqq(E_1, G)=2$, we have $(E_1, G_1)=1$. Moreover, $(E_1, R_p)=1$ for some $5\leqq p\leqq r$. By Lemma 3.7, we may assume $p\neq r$. Let $S_0:=E+R_2+\cdots+R_p+E_1$ and Φ: $X\to \boldsymbol{P}^1$ the $\boldsymbol{P}^1$-fibration defined by $|S_0|$. Using the same arguments as in Lemma 3.9, we can prove the claim 1.

Assume $r=4$. We shall show that there is a (-1)-curve E_1 of X such that $(E_1, R_4)=(E_1, G_1)=1$. This will imply the claim 1 by Lemma 3.7. Indeed, let ξ_1: $X\to X_1$ be the blowing-down of E, R_2, R_3 and R_1, let ξ_2: $X_1\to Y$ be the blowing-down of $\xi_1(R_4)$ and set $\xi:=\xi_2\circ\xi_1$. Then $\xi(G)=\xi(G_1)$ and it has only one singular point P. Note that $(K_Y^2)=(K_X^2)+5=3<9$. Hence there is a nonsingular rational curve l of Y such that $P\in l$ and $(l^2)\leqq 0$. By noting that $3\leqq(l, \xi(G_1))=-2(l, K_Y)$, we have $(l, K_Y)=-2$, $(l^2)=0$ and $(l, \xi(G_1))=4$. Hence $(\xi_2'(l), \xi_1(G_1))=(\xi_2^*(l), \xi_1(G_1))-(\xi_1(R_4), \xi_1(G_1))=(l, \xi(G_1))-3=1$. So, $\xi_2'(l)$ does not pass through the unique singular point of $\xi_1(G_1)$. Note also that $(\xi_2'(l), \xi_1(R_4))=1$. Hence $E_1:=\xi'(l)$ satisfies the requirement.

To complete the proof of the claim 1, it remains to consider the case $r=3$. Let ξ: $X\to Y$ be the blowing-down of E and R_2. Since $(K_Y^2)=0<9$, there is a nonsingular rational curve l such that $(l^2)\leqq 0$ and l contains the point $\xi(G_1)\cap\xi(R_1)\cap\xi(R_3)$. We have $(l, K_Y)=-2$, $(l^2)=0$ and $(l, \xi(G_1+R_1+R_3))=4$ because $3\leqq(l, \xi(G_1+R_1+R_3))=(l, \xi(G))=(l, -2K_Y)$. Interchanging the roles of R_1 and R_3 if necessary, we may assume that $(l, \xi(R_3))=1$. Since $(K_Y^2)<8$, there is a singular fiber f_1 of the $\boldsymbol{P}^1$-fibration $\Phi_{|l|}$: $Y\to \boldsymbol{P}^1$. Then there is a (-1)-curve $\tilde{E}_1$ in f_1 sech that $(\tilde{E}_1, \xi(R_3))=1$ (cf. Lemma 1.10, (1)). Since $(\tilde{E}_1, \xi(G))=2$, we have $(\tilde{E}_1, \xi(G_1+R_1))=1$. Then $E_1:=\xi'(\tilde{E}_1)$ is a (-1)-curve of X with $(E_1, R_3)=(E_1, G_1+R_1)=1$. Then the claim 1 follows from Lemma 3.7 with $E:=E_1$ or Lemma 3.8 with $E_1:=E_1$ and $E_2:=E$.

This completes the proof of Theorem 3.6.

Corollary 3.10. *Let (V, D) be a log Enriques surface with* Index$(K_{\bar{V}})=2$ *and let U be a minimal resolution of singularities of the cauonical covering $\bar{U}$ of $\bar{V}$. Then there is a (-2)-rod R on U with $\#(R)=2(\#(D))-1$. In particular, U is a $K3$-surface with $\rho(U)\geqq 2(\#(D))$. Moreover, if $\#(D)=10$ then $\rho(U)=20$ and U is a singular $K3$-surface.*

Proof. Set $\tilde{c}:=\#(D)$. If $\tilde{c}=1$, then the inverse image of D is a (-2)-curve on U. Suppose $\tilde{c}\geqq 2$. Let τ: $\tilde{V}\to V$ be the blowing-up of all singular points of D and let $\tilde{D}:=\tau'(D)$ with the notation at the beginning of § 3. Then $(\tilde{V}, \tilde{D})$ is again a log Enriques surface satisfying the hypothesis of Theorem 3.6. Hence, there are (-1)-

curves $\tilde{F}_j$ $(1 \leqq j \leqq \tilde{c}-1)$ of $\tilde{V}$ such that $\tilde{D}+\sum\tilde{F}_j$ is a linear chain. Note that the canonical coverings of $(\tilde{V}, \tilde{D})$ and (V, D) have the same (up to isomorphisms) minimal resolution U. Then the inverse image of $\tilde{D}+\sum\tilde{F}_j$ is a (-2)-rod on U satisfying the requirement of Corollary 3.10. Q. E. D.

§ 4. The case where the canonical covering is an abelian surface

We shall prove the following theorem in the present section.

Theorem 4.1. *Let (V, D), or synoymously (V, D), be a log Enriques surface whose canonical covering $\bar{U}$ is an abelian surface. Then I* $(=\text{Index}(K_{\bar{V}}))=3$ *or* 5. *More precisely, we have:*

(1) *Suppose $I=3$. Then $\rho(\bar{U})=\rho(\bar{V})=4$ and D consists of nine isolated (-3)-curves. Hence $\bar{U}$ is a singular abelian surface.*

(2) *Suppose $I=5$. Then $\rho(\bar{U})=\rho(\bar{V})=2$, and D consists of five connected components each of which consists of one (-2)-curve and one (-3)-curve.*

Proof. By Lemma 2.2, I is not divisible by 2. By Lemma 2.3, we have $\varphi(I) \leqq b_2(\bar{U})-\rho(\bar{U})=6-\rho(\bar{U}) \leqq 5$. Hence $I=3$ or 5, and we have $\rho(\bar{U}) \leqq 2$ if $I=5$ and $\rho(\bar{U}) \leqq 4$ if $I=3$. By Lemma 2.4, we have $\tilde{c}=c$ and

$$\rho(\bar{V})=c-2-c/I \quad \text{and} \quad I \mid c,$$

where $c=\#(\text{Sing}\bar{V})=\#\{\text{connected component of } D\}$. By noting that $\rho(\bar{V}) \leqq \rho(\bar{U}) \leqq 4$, we obtain:

$$c=I(\rho(\bar{V})+2)/(I-1) \leqq 6+6/(I-1) \leqq 9.$$

Therefore, $(c, I)=(3, 3)$, $(6, 3)$, $(9, 3)$ or $(5, 5)$. Here $(c, I) \neq (3, 3)$ for $\rho(\bar{V}) \geqq 1$.

We consider these cases separately. Employ the same notations q, ξ, $C_{I,q}$, etc. as in Lemma 2.5.

Case $(c, I)=(6, 3)$. Then $q=1$, D consists of six isolated (-3)-curves and $D^{\#}=(1/3)D$. Hence $-(K_V^2)=c/3=2$ by Lemma 1.8. On the other hand $\rho(\bar{V})=\rho(V)-\#(D)=10-(K_V^2)-6=6$, while $\rho(\bar{V})=2$. This is absurd.

Case $(c, I)=(9, 3)$. Then $q=1$, D consists of nine isolated (-3)-curves and $4 \geqq \rho(\bar{U}) \geqq \rho(\bar{V})=c-2-c/I=4$. Hence $\rho(\bar{U})=\rho(\bar{V})=4$.

Case $(c, I)=(5, 5)$. Then $\rho(\bar{U}) \geqq \rho(\bar{V})=c-2-c/I=2$. Since we have shown $\rho(\bar{U}) \leqq 2$, we see $\rho(\bar{U})=\rho(\bar{V})=2$. By replacing the generator ξ of $C_{I,q}$ by a new one and interchanging the coordinates X and Y of $\boldsymbol{C}^2$ if necessary, we may assume that $q=1$ or 2. Let α be the number of all singular points of $\bar{V}$ with $q=1$. Then D consists of α isolated (-5)-curves D_i's and $(5-\alpha)$ connected components Δ_j's, each of which consists of one (-2)-curve B_{1j} and one (-3)-curve B_{2j}. Note that $D^{\#}=(3/5)\sum D_i+(1/5)\sum(B_{1j}+2B_{2j})$ and $(K_V^2)=(D^{\#})^2=-9\alpha/5-2(5-\alpha)/5=-2-7\alpha/5$. Thus, $\alpha=0$ or 5. If $\alpha=5$ then $\rho(\bar{V})=10-(K_V^2)-\#(D)=10+9-5=14 \neq 2$. This is a contradiction. Hence $\alpha=0$.

Q. E. D.

For the case $I=5$, we can not find any example yet. For the case $I=3$, we have the following example.

Example 4.2. Let $E=\boldsymbol{C}/(\boldsymbol{Z}+\boldsymbol{Z}\omega)$ be an elliptic curve, where ω is a primitive third root of the unity. Then E has complex multiplication and the Picard number of the abelian surface $U:=E\times E$ is 4. Since $G:=\{1, \omega, \omega^2\}$ acts on E by the natural multiplication, we can consider the diagonal action of G on U. Denote by $[x, y]$ a point of U represented by two complex numbers x and y. Then all fixed points of G are as follows:

$$[1, 1],\ [1, (1-\omega)/3],\ [1, (2-2\omega)/3],\ [(1-\omega)/3, 1],$$
$$[(1-\omega)/3, (1-\omega)/3],\ [(1-\omega)/3, (2-2\omega)/3],\ [(2-2\omega)/3, 1],$$
$$[(2-2\omega)/3, (1-\omega)/3],\ [(2-2\omega)/3, (2-2\omega)/3].$$

Hence there are exactly nine singular points on $\bar{V}:=U/G$. More precisely, if $f: V\to\bar{V}$ is a minimal resolution of $\mathrm{Sing}\bar{V}$ then $D:=f^{-1}(\mathrm{Sing}\bar{V})$ consists of nine isolated (-3)-curves D_i $(1\leqq i\leqq 9)$. We assert that V is a rational surface. Indeed, since $K_U\sim 0$, $3K_{\bar{V}}$ is a trivial Cartier divisor. Hence $3(D^{\#}+K_V)\sim f^*(3K_{\bar{V}})\sim 0$, where $D^{\#}=(1/3)\sum_i D_i$ (cf. Lemma 1.2). Hence $\kappa(V)=-\infty$. By the argument in the proof of Lemma 2.2, we see that V is a rational surface. Hence (V, D) is a log Enriques surface fitting the case $I=3$ of Theorem 4.1.

§ 5. The case where the canonical covering is a K3-surface

Employ the notations as set at the beginning of § 2. In the present section, we consider log Enriques surfaces $\bar{V}$ satisfying that the canonical covering $\bar{U}$ is a $K3$-surface and the index I of $K_{\bar{V}}$ is a prime number. Since $\bar{U}$ is nonsingular, we can apply Lemma 2.5. Let $m_1, \cdots, m_a$ be integers such that the following three conditions are satisfied:

(1) $1=m_1<m_2<\cdots<m_a<I-1$,

(2) the singularity $(\boldsymbol{C}^2/C_{I,m_i}, 0)$ is not isomorphic to the singularity $(\boldsymbol{C}^2/C_{I,m_j}, 0)$ if $i\neq j$,

(3) for each $1\leqq k\leqq I-2$, the singularity $(\boldsymbol{C}^2/C_{I,k}, 0)$ is isomorphic to a singularity $(\boldsymbol{C}^2/C_{I,m_i}, 0)$ for some m_i with $m_i\leqq k$.

$(m_1, m_2, \cdots, m_a)$ is uniquely determined and easily found (cf. [2; Satz 2.11]). Let n_i be the number of all singular points of $\bar{V}$ which have the same singularity as $(\boldsymbol{C}^2/C_{I,m_i}, 0)$. By our assumption that $\bar{V}$ has no rational double singular points, we have $\sum n_i=c(=\#(\mathrm{Sing}\bar{V}))$. A precise description of $(n_1, n_2, \cdots, n_a)$ is given in the following theorem:

Theorem 5.1. *We use the above notations. Let $\bar{V}$, or synonymously (V, D) be a log Enriques surface. Suppose that the canonical covering $\bar{U}$ is a $K3$-surface and the index I of $K_{\bar{V}}$ is a prime number. Then $\rho(\bar{V})=c-2+(24-c)/I$, and one of the following cases occurs, where $\sum n_i=c$:*

(1) $(c, I)=(3, 3)$. *Then $(m_1, \cdots, m_a)=(1)$, $c=n_1=3$ and $\rho(V)=11$. Hence D consists*

of three isolated (-3)*-curves.*

(2) $(c, I)=(4, 5)$. *Then* $(m_1, \cdots, m_a)=(1, 2)$, $(n_1, n_2)=(1, 3)$ *and* $\rho(V)=13$.

(3) $(c, I)=(3, 7)$. *Then* $(m_1, \cdots, m_a)=(1, 2, 3)$, $(n_1, n_2, n_3)=(0, 1, 2)$ *and* $\rho(V)=12$.

(4) $(c, I)=(2, 11)$. *Then* $(m_1, \cdots, m_a)=(1, 2, 3, 5, 7)$, $(n_1, \cdots, n_5)=(0, 0, 0, 1, 1)$ *and* $\rho(V)=11$.

(5) $(c, I)=(13, 11)$. *Then* $(m_1, \cdots, m_a)=(1, 2, 3, 5, 7)$, $(n_1, \cdots, n_5)=(3, 4, 0, 0, 6)$, $(4, 1, 1, 0, 7)$, $(4, 2, 0, 1, 6)$ *or* $(5, 0, 0, 2, 6)$ *and* $\rho(V)=47, 48, 49$ *or* 51, *respectively.*

(6) $(c, I)=(7, 17)$. *Then* $(m_1, \cdots, m_a)=(1, 2, 3, 4, 5, 8, 10, 11)$ *and* $(n_1, \cdots, n_8)=(1, 0, 1, 1, 0, 0, 2, 2)$, $(1, 0, 0, 1, 1, 0, 3, 1)$, $(0, 2, 1, 0, 0, 0, 3, 1)$, $(0, 2, 0, 0, 1, 0, 4, 0)$, $(1, 1, 1, 0, 0, 0, 0, 4)$, $(1, 1, 0, 0, 1, 0, 1, 3)$, $(1, 0, 1, 0, 0, 1, 4, 0)$, $(2, 0, 0, 0, 0, 2, 1, 2)$, $(1, 2, 0, 0, 0, 1, 0, 3)$, $(1, 1, 0, 2, 0, 0, 0, 3)$, $(1, 1, 0, 1, 0, 1, 2, 1)$, $(1, 0, 0, 3, 0, 0, 2, 1)$, $(0, 3, 0, 1, 0, 0, 1, 2)$, $(0, 3, 0, 0, 0, 1, 3, 0)$ *or* $(0, 2, 0, 2, 0, 0, 3, 0)$.

(7) $(c, I)=(5, 19)$. *Then* $(m_1, \cdots, m_a)=(1, 2, 3, 4, 6, 7, 8, 9, 14)$, $(n_1, \cdots, n_9)=(1, 0, 0, 0, 0, 1, 0, 1, 2)$, $(1, 0, 0, 0, 2, 0, 0, 0, 2)$, $(0, 1, 1, 0, 0, 1, 0, 0, 2)$ *or* $(0, 2, 0, 0, 1, 0, 0, 0, 2)$ *and* $\rho(V)=29, 29, 24$ *or* 26, *respectively.*

In particular, $(D, K_V)=c-1-(K_{\bar{V}}^2)$.

Conversely, if $\bar{V}$ *is a log Enriques surface of which the singularity type belongs to one of the above cases, then the canonical covering* $\bar{U}$ *is a* $K3$*-surface.*

Finally, for each prime number I *with* $3 \leqq I \leqq 19$ *and* $I \neq 13$, *there is a log Enriques surface* $\bar{V}$ *such that* I *is the index of* $K_{\bar{V}}$ *and the canonical covering* $\bar{U}$ *of* $\bar{V}$ *is a* $K3$*-surface* (*cf. Examples* 5.3-5.8).

Proof. At first, we show the converse part. Let $\bar{V}$ be a log Enriques surface of which the singularity type belongs to one of the cases of Theorem 5.1. Every singular point x of $\bar{V}$ has the same singularity as $(\boldsymbol{C}^2/G_x, 0)$ with a cyclic subgroup G_x of $GL(2, \boldsymbol{C})$ of order I. Since the canonical covering $\pi: \bar{U} \rightarrow \bar{V}$ has degree I and is an étale cyclic covering outside Sing$\bar{V}$, we see that $\bar{U}$ is nonsingular. Then $\bar{U}$ is a $K3$-surface in view of Theorem 4.1. Now we shall prove a main part of Theorem 5.1.

By Lemma 2.4, we obtain the first assertion and that $c \leqq 21$. In particular, $I \mid (24-c)$. By Lemma 2.2, we have $I \geqq 3$. Hence $c \geqq 2$ by Proposition 1.6.

Consider the case $I=3$. Then $(m_1, \cdots, m_a)=(1)$ and D consists of c isolated (-3)-curves D_i $(1 \leqq i \leqq c)$. Note that $D^{\#}=(1/3)D$ and $(K_{\bar{V}}^2)=(D^{\#})^2=-c/3$. Hence we have $c/3+10=\rho(V)=\rho(\bar{V})+\#(D)=c-2+(24-c)/3+c$. This implies $c=3$ and $\rho(V)=11$.

Now we assume $I \geqq 5$. Since $2 \leqq c \leqq 21$ and $I \mid (24-c)$, we see that $(c, I)=(4, 5)$, $(9, 5)$, $(14, 5)$, $(19, 5)$, $(3, 7)$, $(10, 7)$, $(17, 7)$, $(2, 11)$, $(13, 11)$, $(11, 13)$, $(7, 17)$ or $(5, 19)$.

Consider the case $I=5$. Then $(m_1, \cdots, m_a)=(1, 2)$. As in Theorem 4.1, we have $(K_{\bar{V}}^2)=(D^{\#})^2=-(9(c-n_2)+2n_2)/5$. Hence $10+9c/5-7n_2/5=\rho(V)=\rho(\bar{V})+\#(D)=(4c+14)/5+(c-n_2+2n_2)$. This implies $n_2=3$ and $n_1=c-3$. We shall prove $c=4$. Indeed, by Proposition 1.6, we obtain:

$3(c-3)+3=(D, K_V) \leqq c-1-(K_{\bar{V}}^2)=c-1+(9c-21)/5$, whence $c \leqq 4$. Since $c \geqq 4$ when $I=5$, we have $c=4$, $(n_1, n_2)=(1, 3)$ and $\rho(V)=13$.

Consider the case $I=7$. Then $(m_1, \cdots, m_a)=(1, 2, 3)$. Note that D consists of the following c connected components:

(1) isolated (-7)-curves A_i $(1 \leqq i \leqq n_1)$,

(2) rods B_j $(n_1+1 \leqq j \leqq n_1+n_2)$, each of which consists of one (-2)-curve B_{1j} and one (-4)-curve B_{2j},

(3) rods C_k $(n_1+n_2+1 \leqq k \leqq n_1+n_2+n_3=c)$, each of which consists of two (-2)-curves C_{1k}, C_{2k} and one (-3)-curve C_{3k} with $(C_{bk}, C_{b+1,k})=1$ $(b=1, 2)$.

Then $D^{\#}=(5/7)\sum A_i+(2/7)\sum(B_{1j}+2B_{2j})+(1/7)\sum(C_{1k}+2C_{2k}+3C_{3k})$ and $-(25(c-n_2-n_3)+8n_2+3n_3)/7=(D^{\#})^2=(K_V^2)=10-\rho(V)=10-\rho(\bar{V})-\#(D)=10-(c-2+(24-c)/7)-(c-n_2-n_3+2n_2+3n_3)$. This implies $5+c=2n_2+3n_3$. Note that $c=n_1+n_2+n_3=3$, 10 or 17. Hence all possible pairs of (n_1, n_2, n_3) are as follows:

$$(0, 1, 2), (5, 0, 5), (4, 3, 3), (3, 6, 1),$$
$$(9, 2, 6), (8, 5, 4), (7, 8, 2), (6, 11, 0).$$

On the other hand, by Proposition 1.6, we have:

$$5n_1+2n_2+n_3=(D, K_V)<c-(K_V^2)=c+(25n_1+8n_2+3n_3)/7.$$

Therefore we have $c=3$, $(n_1, n_2, n_3)=(0, 1, 2)$ and $\rho(V)=12$.

Consider the case $I=11$. Then $(m_1, \cdots, m_a)=(1, 2, 3, 5, 7)$. Note that D consists of the following c connected components:

(1) isolated (-11)-curves A_i $(1 \leqq i \leqq n_1)$,

(2) rods B_j $(n_1+1 \leqq j \leqq n_1+n_2)$, each of which consists of one (-2)-curve B_{1j} and one (-6)-curve B_{2j},

(3) rods C_k $(n_1+n_2+1 \leqq k \leqq n_1+n_2+n_3)$, each of which consists of one (-3)-curve C_{1k} and one (-4)-curve C_{2k},

(4) rods D_r $(n_1+n_2+n_3+1 \leqq r \leqq n_1+\cdots+n_4)$, each of which consists of four (-2)-curves $D_{1r}, \cdots, D_{4r}$ and one (-3)-curve D_{5r} with $(D_{br}, D_{b+1,r})=1$ $(1 \leqq b \leqq 4)$,

(5) rods E_s $(n_1+\cdots+n_4+1 \leqq s \leqq n_1+\cdots+n_5=c)$, each of which consists of three (-2)-curves E_{1s}, E_{2s}, E_{4s} and one (-3)-curve E_{3s} with $(E_{bs}, E_{b+1,s})=1$ $(1 \leqq b \leqq 3)$.

Then $D^{\#}=(9/11)\sum A_i+(4/11)\sum(B_{1j}+2B_{2j})+(1/11)\sum(6C_{1k}+7C_{2k})+(1/11)\sum(D_{1r}+2D_{2r}+3D_{3r}+4D_{4r}+5D_{5r})+(1/11)\sum(2E_{1s}+4E_{2s}+6E_{3s}+3E_{4s})$, and $-(81n_1+32n_2+20n_3+5n_4+6n_5)/11=-7n_1-3n_2-2n_3--n_5+(--4n_1+n_2+2n_3-5n_4+5n_5)/11=(D^{\#})^2=(K_V^2)=10-\rho(V)=10-\rho(\bar{V})-\#(D)=(108-10c)/11-(n_1+2n_2+2n_3+5n_4+4n_5)$. In particular, we have $11|(-4n_1+n_2+2n_3-5n_4+5n_5)$. Hence, if $c=2$ then $(n_1, \cdots, n_5)=(0, 0, 0, 1, 1)$ and $\rho(V)=11$.

Now we suppose that $c=13$. We shall show that $(n_1, \cdots, n_5)=(3, 4, 0, 0, 6)$, $(4, 1, 1, 0, 7)$, $(4, 2, 0, 1, 6)$ or $(5, 0, 0, 2, 6)$. Hence, $\rho(V)=47$, 48, 49 or 51, respectively. By the above computations of $(D^{\#})^2$, we deduce $0=-22+70n_1+10n_2-2n_3-50n_4-38n_5=-22+10c+60n_1-12n_3-60n_4-48n_5$ and thence the following equality:

$$5n_4+4n_5=9-n_3+5n_1. \tag{1}$$

On the other hand, by Proposition 1.6, we obtain $9n_1+4n_2+3n_3+n_4+n_5=(D, K_V) \leqq 13-1-(K_V^2)=12+2+n_1+2n_2+2n_3+5n_4+4n_5$ and hence $0 \leqq 14-8n_1-2n_2-n_3+4n_4+3n_5=14-2c-6n_1+n_3+6n_4+5n_5$. Using the equality (1) to eliminate n_3 in the later inequality, we obtain an inequality:

(2) $$3+n_1 \leqq n_4+n_5 .$$

This, together with the equality (1), implies $0=n_4+4(n_4+n_5)-9+n_3-5n_1 \geqq 3-n_1+n_3+n_4 \geqq 3-n_1$. Hence $n_1 \geqq 3$ and $n_4+n_5 \geqq 3+n_1 \geqq 6$. If $n_4+n_5 \geqq 9$, then $n_1=13-(n_2+\cdots+n_5) \leqq 4$ and $36 \leqq 5n_4+4n_5=9-n_3+5n_1 \leqq 29$ by the equality (1). This is a contradiction. Therefore we have $6 \leqq n_4+n_5 \leqq 8$.

Case $n_4+n_5=6$. Then $n_1+n_2+n_3=7$ and $n_1 \leqq 3$ by the inequality (2). Hence $n_1=3$, and $24+n_4=24-n_3$ by the equality (1). Thus $n_3=n_4=0$ and $(n_1, \cdots, n_5)=(3, 4, 0, 0, 6)$.

Case $n_4+n_5=7$. Then $n_1+n_2+n_3=6$ and $28 \leqq 28+n_4=9-n_3+5n_1 \leqq 9+5n_1$ by the equality (1). Hence $n_1 \geqq 4$. Thus, by the inequality (2), we have $n_1=4$. Hence $n_2+n_3=2$ and $28+n_4=29-n_3$. Therefore, $(n_1, \cdots, n_5)=(4, 1, 1, 0, 7)$ or $(4, 2, 0, 1, 6)$.

Case $n_4+n_5=8$. Then $n_1+n_2+n_3=5$ and $32 \leqq 32+n_4=9-n_3+5n_1 \leqq 9+5n_1$ by the equality (1). So, $n_1=5$ and $(n_1, \cdots, n_5)=(5, 0, 0, 2, 6)$.

Next we shall prove that the case $(c, I)=(11, 13)$ is impossible. Indeed, if the case $(c, I)=(11, 13)$ occurs, then $(m_1, \cdots, m_a)=(1, 2, 3, 4, 5, 6)$, $\rho(\bar{V})=c-2+(24-c)/I=10$, and D consists of the following eleven connected components:

(1) isolated (-13)-curves A_i $(1 \leqq i \leqq n_1)$,

(2) rods B_j $(n_1+1 \leqq j \leqq n_1+n_2)$, each of which consists of one (-2)-curve B_{1j} and one (-7)-curve B_{2j},

(3) rods C_k $(n_1+n_2+1 \leqq k \leqq n_1+n_2+n_3)$, each of which consists of two (-2)-curves C_{1k}, C_{2k} and one (-5)-curve C_{3k} with $(C_{bk}, C_{b+1,k})=1$ $(b=1, 2)$,

(4) rods D_r $(n_1+n_2+n_3+1 \leqq r \leqq n_1+\cdots+n_4)$, each of which consists of three (-2)-curves D_{1r}, D_{2r}, D_{3r} and one (-4)-curve D_{4r} with $(D_{br}, D_{b+1,r})=1$ $(1 \leqq b \leqq 3)$,

(5) rods E_s $(n_1+\cdots+n_4+1 \leqq s \leqq n_1+\cdots+n_5)$, each of which consists of one (-2)-curve E_{1s} and two (-3)-curves E_{2s} and E_{3s} with $(E_{bs}, E_{b+1,s})=1$ $(b=1, 2)$,

(6) rods F_t $(n_1+\cdots+n_5+1 \leqq t \leqq n_1+\cdots+n_6=11)$, each of which consists of five (-2)-curves $F_{1t}, \cdots, F_{5t}$ and one (-3)-curve F_{6t} with $(F_{bt}, F_{b+1,t})=1$ $(1 \leqq b \leqq 5)$.

Then $D^{\#}=(11/13)\sum A_i+(5/13)\sum(B_{1j}+2B_{2j})+(3/13)\sum(C_{1k}+2C_{2k}+3C_{3k})+(2/13)\sum(D_{1r}+2D_{2r}+3D_{3r}+4D_{4r})+(1/13)\sum(4E_{1s}+8E_{2s}+7E_{3s})+(1/13)\sum(F_{1t}+2F_{2t}+3F_{4t}+4F_{4t}+5F_{5t}+6F_{6t})$ and $-(121n_1+50n_2+27n_3+16n_4+15n_5+16n_6)/13=(D^{\#})^2=(K_V^2)=10-\rho(V)=10-\rho(\bar{V})-\#(D)=-(n_1+2n_2+3n_3+4n_4+3n_5+6n_6)$. This implies $0=-9n_1-2n_2+n_3+3n_4+2n_5+6n_6=c-10n_1-3n_2+2n_4+n_5+5n_6=11-10n_1-3n_2+2n_4+n_5+5n_6$. On the other hand, by Proposition 1.6, we obtain $11n_1+5n_2+3n_3+2n_4+2n_5+n_6=(D, K_V)<11-(K_V^2)=11+n_1+2n_2+3n_3+4n_4+3n_5+6n_6$ and hence $0<11-10n_1-3n_2+2n_4+n_5+5n_6$. This contradicts the above equality. Therefore the case $(c, I)=(11, 13)$ is impossible.

Consider the case $(c, I)=(7, 17)$. Then $(m_1, \cdots m_a)=(1, 2, 3, 4, 5, 8, 10, 11)$. Note that $\rho(\bar{V})=c-2+(24-c)/I=6$ and D consists of seven connected components of the following type:

(1) isolated (-17)-curves A_i $(1 \leqq i \leqq n_1)$,

(2) rods B_j $(n_1+1 \leqq j \leqq n_1+n_2)$, each of which consists of one (-2)-curve B_{1j} and one (-9)-curve B_{2j},

(3) rods C_k $(n_1+n_2+1 \leqq k \leqq n_1+n_2+n_3)$, each of which consists of one (-3)-curve C_{1k} and one (-6)-curve C_{2k},

(4) rods D_r $(n_1+n_2+n_3+1 \leqq r \leqq n_1+\cdots+n_4)$, each of which consists of three (-2)-

curves D_{1r}, D_{2r}, D_{3r} and one (-5)-curve D_{4r} with $(D_{br}, D_{b+1,r})=1$ $(1\leqq b\leqq 3)$,

(5) rods E_s $(n_1+\cdots+n_4+1\leqq s\leqq n_1+\cdots+n_5)$, each of which consists of one (-3)-curve E_{1s}, one (-2)-curve E_{2s} and one (-4)-curve E_{3s} with $(E_{bs}, E_{b+1,s})=1$ $(b=1, 2)$,

(6) rods F_t $(n_1+\cdots+n_5+1\leqq t\leqq n_1+\cdots+n_6)$, each of which consists of seven (-2)-curves $F_{1t}, \cdots, F_{7t}$ and one (-3)-curve F_{8t} with $(F_{bt}, F_{b+1,t})=1$ $(1\leqq b\leqq 7)$,

(7) rods G_u $(n_1+\cdots+n_6+1\leqq u\leqq n_1+\cdots+n_7)$, each of which consists of three (-2)-curves G_{1u}, G_{2u}, G_{4u} and one (-4)-curve G_{3u} with $(G_{bs}, G_{b+1,s})=1$ $(1\leqq b\leqq 3)$,

(8) rods H_v $(n_1+\cdots+n_7+1\leqq v\leqq n_1+\cdots+n_8=7)$, each of which consists of five (-2)-curves $H_{1v}, \cdots, H_{4v}, H_{6v}$ and one (-3)-curve H_{5v} with $(H_{bv}, H_{b+1,v})=1$ $(1\leqq b\leqq 5)$.

Then $D^{\#}=(15/17)\sum A_i+(7/17)\sum(B_{1j}+2B_{2j})+(1/17)\sum(10C_{1k}+13C_{2k})+(3/17)\sum(D_{1r}+2D_{2r}+3D_{3r}+4D_{4r})+(1/17)\sum(9E_{1s}+10E_{2s}+11E_{3s})+(1/17)\sum(F_{1t}+2F_{2t}+3F_{3t}+4F_{4t}+5F_{5t}+6F_{6t}+7F_{7t}+8F_{8t})+(2/17)\sum(2G_{1u}+4G_{2u}+6G_{3u}+3G_{4u})+(1/17)\sum(2H_{1v}+4H_{2v}+6H_{3v}+8H_{4v}+10H_{5v}+5H_{6v})$. Note that $-(225n_1+98n_2+62n_3+36n_4+31n_5+8n_6+24n_7+10n_8)/17=(D^{\#})^2=(K_V^2)=10-\rho(V)=10-\rho(\bar{V})-\#(D)=4-(n_1+2n_2+2n_3+4n_4+3n_5+8n_6+4n_7+6n_8)$. This implies $0=17+52n_1+16n_2+7n_3-8n_4-5n_5-32n_6-11n_7-23n_8=17-5c+57n_1+21n_2+12n_3-3n_4-27n_6-6n_7-18n_8$. Hence we obtain:

$$19n_1+7n_2+4n_3=6+n_4+9n_6+2n_7+6n_8\,. \tag{3}$$

In particular, $\sum_{i\leqq 3} n_i\geqq 1$. On the other hand, by Proposition 1.6, we obtain $15n_1+7n_2+5n_3+3n_4+3n_5+n_6+2n_7+n_8=(D, K_V)\leqq 7-1-(K_V^2)=2+n_1+2n_2+2n_3+4n_4+3n_5+8n_6+4n_7+6n_8$. By using the equality (3), we eliminate n_4 in the above inequality and obtain $4+2n_6+n_8\leqq 5n_1+2n_2+n_3$. Multiplying both sides of the later inequality by 4 and using the equality (3), we obtain $16+8n_6+4n_8\leqq n_1+n_2+(19n_1+7n_2+4n_3)=6+n_1+n_2+n_4+9n_6+2n_7+6n_8$ and hence

$$10\leqq n_1+n_2+n_4+n_6+2n_7+2n_8\leqq c+n_7+n_8\,. \tag{4}$$

So, $3=10-c\leqq n_7+n_8\leqq c-(n_1+n_2+n_3)\leqq 6$.

Case $n_7+n_8=6$. Then $\sum_{i\leqq 6} n_i=1$ and $19n_1+7n_2+4n_3=18+n_4+9n_6+4n_8\geqq 18$ by virtue of the equality (3). This leads to $(n_1, \cdots, n_6)=(1, 0, \cdots, 0)$ and $19=18+4n_8\equiv 0 \pmod 2$, a contradiction.

Case $n_7+n_8=5$. Then $\sum_{i\leqq 6} n_i=2$ and $19n_1+7n_2+4n_3=16+n_4+9n_6+4n_8\geqq 16$ by the equality (3). If $\sum_{i\leqq 3} n_i\leqq 1$, then $(n_1, n_2, n_3)=(1, 0, 0)$ and $3=n_4+9n_6+4n_8$. Hence we must have $(n_4, n_6, n_8)=(3, 0, 0)$, which contradicts $\sum_{i\leqq 6} n_i=2$. Therefore, $\sum_{i\leqq 3} n_i=2$ and $n_4=n_5=n_6=0$. Then the equality (3) becomes $15n_1+3n_2=8+4n_8$. Hence $n_1+n_2\geqq 1$ and $4|(n_1+n_2)$, contradicting $\sum_{i\leqq 3} n_i=2$. So, it is impossible that $n_7+n_8=5$.

Case $n_7+n_8=4$. Then $\sum_{i\leqq 6} n_i=3$ and $n_1+n_2+n_4+n_6\geqq 2$ by the inequality (4). On the other hand, we have $n_1+n_2+n_4+n_6=-14+20n_1+8n_2+4n_3-8n_6-4n_8\equiv 0 \pmod 2$ by the equality (3). Hence $n_1+n_2+n_4+n_6=2$, $n_3+n_5=1$ and $3\leqq 4-n_3+n_8=5n_1+2n_2-2n_6$. All seven solutions of $(n_1, \cdots, n_8)$ are given in the assertion (6) of Theorem 5.1.

Case $n_7+n_8=3$. Then $\sum_{i\leqq 6} n_i=4$ and $n_1+n_2+n_4+n_6\geqq 4$ by the inequality (4). Hence

$n_1+n_2+n_4+n_6=4$ and $n_3=n_5=0$. By virtue of the equality (3), we have $12\geqq 4-2n_2+2n_6=5n_1-n_8$. In particular, $n_1\leqq 3$ and $2|(n_1+n_8)$. We can show that $(n_1, n_8)=(2, 2)$, $(1, 3)$, $(1, 1)$, $(0, 2)$ or $(0, 0)$. All eight solutions of $(n_1, \cdots, n_8)$ are given in the assertion (6).

Consider the case $(c, I)=(5, 19)$. Then $(m_1, \cdots, m_a)=(1, 2, 3, 4, 6, 7, 8, 9, 14)$. Note that $\rho(\bar{V})=c-2+(24-c)/I=4$, and D consists of five connected components of the following type:

(1) isolated (-19)-curves A_i $(1\leqq i\leqq n_1)$,

(2) rods B_j $(n_1+1\leqq j\leqq n_1+n_2)$, each of which consists of one (-2)-curve B_{1j} and one (-10)-curve B_{2j},

(3) rods C_k $(n_1+n_2+1\leqq k\leqq n_1+n_2+n_3)$, each of which consists of two (-2)-curves C_{1k}, C_{2k} and one (-7)-curve C_{3k} with $(C_{bk}, C_{b+1,k})=1$ $(\mathrm{b}=1, 2)$,

(4) rods D_r $(n_1+n_2+n_3+1\leqq r\leqq n_1+\cdots+n_4)$, each of which consists of one (-4)-curve D_{1r} and one (-5)-curve D_{2r},

(5) rods E_s $(n_1+\cdots+n_4+1\leqq s\leqq n_1+\cdots+n_5)$, each of which consists of five (-2)-curves $E_{1s}, \cdots, E_{5s}$ and one (-4)-curve E_{6s} with $(E_{bs}, E_{b+1,s})=1$ $(1\leqq b\leqq 5)$,

(6) rods F_t $(n_1+\cdots+n_5+1\leqq t\leqq n_1+\cdots+n_6)$, each of which consists of one (-2)-curve F_{1t}, one (-4)-curve F_{2t} and one (-3)-curve F_{3t} with $(F_{bt}, F_{b+1,t})=1$ $(b=1, 2)$,

(7) rods G_u $(n_1+\cdots+n_6+1\leqq u\leqq n_1+\cdots+n_7)$, each of which consists of two (-2)-curves G_{1u}, G_{3u} and two (-3)-curves G_{2u}, G_{4u} with $(G_{bu}, G_{b+1,u})=1$ $(1\leqq b\leqq 3)$,

(8) rods H_v $(n_1+\cdots+n_7+1\leqq v\leqq n_1+\cdots+n_8)$, each of which consists of eight (-2)-curves $H_{1v}, \cdots, H_{8v}$ and one (-3)-curve H_{9v} with $(H_{bv}, H_{b+1,v})=1$ $(1\leqq b\leqq 8)$,

(9) rods J_w $(n_1+\cdots+n_8+1\leqq w\leqq n_1+\cdots+n_9=5)$, each of which consists of five (-2)-curves J_{1w}, J_{2w}, J_{3w}, J_{5w}, J_{6w} and one (-3)-curve J_{4w} with $(J_{bw}, J_{b+1,w})=1$ $(1\leqq b\leqq 5)$.

Then

$$D^{\#}=\frac{17}{19}\Sigma A_i+\frac{8}{19}\Sigma(B_{1j}+2B_{2j})+\frac{5}{19}\Sigma(C_{1k}+2C_{2k}+3C_{3k})$$
$$+\frac{1}{19}\Sigma(13D_{1r}+14D_{2r})+\frac{2}{19}\Sigma(E_{1s}+2E_{2s}+3E_{3s}+4E_{4s}+5E_{5s}+6E_{6s})$$
$$+\frac{1}{19}\Sigma(7F_{1t}+14F_{2t}+11F_{3t})+\frac{1}{19}\Sigma(6G_{1u}+12G_{2u}+11G_{3u}+10G_{4u})$$
$$+\frac{1}{19}\Sigma(H_{1v}+2H_{2v}+3H_{3v}+4H_{4v}+5H_{5v}+6H_{6v}+7H_{7v}+8H_{8v}+9H_{9v})$$
$$+\frac{1}{19}\Sigma(3J_{1w}+6J_{2w}+9J_{3w}+12J_{4w}+8J_{5w}+4J_{6w}).$$

Note that

$$-(289n_1+128n_2+75n_3+68n_4+24n_5+39n_6+22n_7+9n_8+12n_9)/19=(D^{\#})^2=(K_V^2)$$
$$=10-\rho(V)=10-\rho(\bar{V})-\#(D)=6-(n_1+2n_2+3n_3+2n_4+6n_5+3n_6+4n_7+9n_8+6n_9).$$

This implies

$$0=19+45n_1+15n_2+3n_3+5n_4-15n_5-3n_6-9n_7-27n_8-17n_9$$

$$=19-3c+48n_1+18n_2+6n_3+8n_4-12n_5-6n_7-24n_8-14n_9.$$

Hence we obtain:

$$(5) \qquad 2+24n_1+9n_2+3n_3+4n_4=6n_5+3n_7+12n_8+7n_9.$$

In particular, $3|(n_9-n_4-2)$. On the other hand, by Proposition 1.6, we obtain $17n_1+8n_2+5n_3+5n_4+2n_5+3n_6+2n_7+n_8+n_9=(D, K_V)\leqq 5-1-(K_V^{\#2})=-2+n_1+2n_2+3n_3+2n_4+6n_5+3n_6+4n_7+9n_8+6n_9$. Eliminating n_7 from the above inequality by means of the equality (5), we obtain $n_9-n_4-2\geqq 0$. This inequality and the equality (5) will be used below to show that $(n_1, \cdots, n_9)=(1, 0, 0, 0, 0, 1, 0, 1, 2)$, $(1, 0, 0, 0, 2, 0, 0, 0, 2)$, $(0, 1, 1, 0, 0, 1, 0, 0, 2)$ or $(0, 2, 0, 0, 1, 0, 0, 0, 2)$. Hence $\rho(V)=29$, 29, 24 or 26, respectively.

Since $3|(n_9-n_4-2)$ and $n_9\leqq c=5$, we see that $n_9-n_4-2=0$ or 3. If $n_9-n_4-2=3$, then $n_9=5$ and $n_i=0$ $(i\neq 9)$. This is impossible by the equality (5). So, $n_9=n_4+2$. Since $2+2n_4=n_4+n_9\leqq c=5$, $n_4\leqq 1$. If $n_4=1$, then $n_9=3$ and $\sum_{i\neq 4,9} n_i=1$. Hence $8n_1+3n_2+n_3=5+2n_5+n_7+4n_8$ by the equality (5). This is impossible because $n_1+n_2+n_3+n_5+n_7+n_8\leqq 1$. Thus, $n_4=0$, $n_9=2$ and $\sum_{i\neq 4,9} n_i=3$. The equality (5) becomes

$$(5)' \qquad 8n_1+3n_2+n_3=4+2n_5+n_7+4n_8.$$

In particular, $n_1\leqq 1$ and $n_8\leqq 1$. If $n_8=1$ then $n_1=1$ and $(n_1, \cdots, n_9)=(1, 0, 0, 0, 0, 1, 0, 1, 2)$. Now suppose $n_8=0$. If $n_1=1$ then $n_5=2$ and $(n_1, \cdots, n_9)=(1, 0, 0, 0, 2, 0, 0, 0, 2)$. Next, suppose $n_1=n_8=0$. Then $n_2+n_3+n_5+n_6+n_7=3$, and $3n_2+n_3=4+2n_5+n_7\geqq 4$ by virtue of the equality (5)'. In particular, $n_2+n_3\geqq 2$. If $n_2+n_3=3$ then $n_5=n_6=n_7=0$ and the equality (5)' implies $n_2=1/2$. This is a contradiction. So, $n_2+n_3=2$. Hence $n_5+n_6+n_7=1$, and $2n_2=2+2n_5+n_7$ by the equality (5)'. Therefore, $(n_1, \cdots, n_9)=(0, 1, 1, 0, 0, 1, 0, 0, 2)$ or $(0, 2, 0, 0, 1, 0, 0, 0, 2)$. Q. E. D.

Corollary 5.2. *Let (V, D) be a log Enriques surface such that $D+3K_V\sim 0$, i.e., $D^{\#}=(1/3)D$. Then the canonical covering $\bar{U}$ is a $K3$-surface or an abelian surface, and D consists of three or nine isolated (-3)-curves, accordingly.*

Proof. Suppose that $D^{\#}=(1/3)D$. By Lemma 1.8, D consists of c isolated (-3)-curves. We use the notations as set at the beginning of § 2. Note that $\bar{\pi}^{-1}(D)$ consists of c (-1)-curves. Hence $\bar{U}$ is nonsingular. Now we can apply Theorems 4.1 and 5.1 to obtain the result. Q. E. D.

The following example is due to S. Tsunoda.

Example 5.3. Denote by X, Y, Z the homogeneous coordinates of $\boldsymbol{P}^2$. Consider three cuspidal cubic curves C_1, C_2 and C_3 of $\boldsymbol{P}^2$:

$$C_1:\ X^3=Y^2Z, \quad C_2:\ Y^3=Z^2X, \quad C_3:\ Z^3=X^2Y.$$

Let ξ be a primitive 7-th root of the unity. Then $C_1\cap C_2\cap C_3=\{(\xi^{3i}:\xi^i:1);\ 0\leqq i\leqq 6\}$. Let $\tau\colon V\to\boldsymbol{P}^2$ be the blowing-up of $(1:0:0)\in C_2\cap C_3$, $(0:1:0)\in C_3\cap C_1$, $(0:0:1)\in C_1\cap C_2$, and seven points of $C_1\cap C_2\cap C_3$. Denote by $D_i:=\tau'(C_i)$ and $D:=\sum D_i$. Evidently, we have $0\sim\tau^*(\sum C_i+3K_{P^2})=\sum D_i+3K_V$. Hence the surface (V, D) is a log

Enriques surface fitting the case $I=3$ of Theorem 5.1.

Next, we shall give examples for the cases $(c, I)=(4, 5)$, $(3, 7)$ and $(2, 11)$ of Theorem 5.1. We need several notations:

Let $\pi: \Sigma_2 \to \boldsymbol{P}^1$ be the $\boldsymbol{P}^1$-fibration on a Hirzebruch surface Σ_2 and let M be the (-2)-curve of Σ_2. Take an irreducible curve $A \in |-K_{\Sigma_2}|$ so that A has a node P_1. Let L_1 be the fiber of π containing P_1 and let $L_2\,(\neq L_1)$ be a fiber of π so that $P_2 := A \cap L_2$ is a ramification point of $\pi|_A$.

Example 5.4 (for the case $(c, I)=(4, 5)$). Take an irreducible curve C_1 in $|M+2L_1|$ such that $P_1, P_2 \in C_1$ and C_1 has the same tangent as one of those of A at the node P_1 of A. Let C_2 be an irreducible member of $|M+2L_1|$ such that C_2 meets C_1 in two distinct points P_3 and P_4 other than P_1 or P_2. Denote the point $C_2 \cap L_2$ by P_5. Let P_6, P_7, P_8, P_9 be all intersection points of A and C_2, where some of them might be infinitely near to the other. Let $\tau_1: V_1 \to \Sigma_2$ be the blowing-up of nine points P_i's and let $E_j := \tau_1^{-1}(P_j)$ $(j=1, 2)$. Let $\tau_2: V \to V_1$ be the blowing-up of two points $\tau_1'(C_1) \cap E_1$ and $\tau_1'(A) \cap E_2$. Set $\tau := \tau_1 \circ \tau_2$, $E_j' := \tau_2'(E_j)$, $L_2' := \tau'(L_2)$, $M' := \tau'(M)$, $A' := \tau'(A)$, $C_j' := \tau'(C_j)$, $D := \sum E_j' + L_2' + M' + A' + \sum C_j'$. Then D has the same configuration as $f^{-1}(\mathrm{Sing}\overline{V})$ $(\subseteq V)$ in the case $(c, I)=(4, 5)$ of Theorem 5.1. By noting that $M+2L_2+2A+2C_1+3C_2 \sim -5K_{\Sigma_2}$, we see that $E_1'+E_2'+M'+2(L_2'+A'+C_1')+3C_2' \sim -5K_V$. Hence (V, D) is a log Enriques surface fitting the case $(c, I)=(4, 5)$ of Theorem 5.1.

Example 5.5 (for the case $(c, I)=(3, 7)$). Take an irreducible curve C_1 in $|M+2L_1|$ such that C_1 passes through $P_1\ (=A \cap L_1)$, $P_2\ (=A \cap L_2)$ and the third point P_3 of A other than P_1 or P_2. Let C_2 be an irreducible member of $|M+2L_1|$ such that C_2 and C_1 have one and the same tangent at P_3. Denote by P_4, P_5 and P_6 all intersection points of A and C_2 other than P_3, where some of P_r's $(r=4, 5, 6)$ might be infinitely near points of the other. Let $\tau_1: V_1 \to \Sigma_2$ be the blowing-up of six points P_i's and let $E_j := \tau_1^{-1}(P_j)$ $(j=1, 2, 3)$. Let $\tau_2: V \to V_1$ be the blowing-up of two points of $\tau_1'(A) \cap E_1$ and two points $\tau_1'(A) \cap E_2$ and $\tau_1'(C_1) \cap E_3$. Set $\tau := \tau_1 \circ \tau_2$, $E_j := \tau_2'(E_j)$, $L_2 := \tau'(L_2)$, $M' := \tau'(M)$, $A' := \tau'(A)$, $C_k' := \tau'(C_k)$, $D := \sum E_j' + L_2' + M' + A' + \sum C_k'$. Then D has the same configuration as $f^{-1}(\mathrm{Sing}\overline{V})$ $(\subseteqq V)$ in the case $(c, I)=(3, 7)$ of Theorem 5.1. Note that $M+2L_2+4A+2C_1+3C_2 \sim -7K_{\Sigma_2}$. Then we can check that $E_2'+M'+2(E_3'+C_1'+L_2')+3(E_1'+C_2')+4A' \sim -7K_V$. Hence (V, D) is a log Enriques surface fitting the case $(c, I)=(3, 7)$ of Theorem 5.1.

Example 5.6 (for the case $(c, I)=(2, 11)$). Take an irreducible member C_1 in $|M+2L_1|$ such that $P_2(=A \cap L_2) \in C_1$ and C_1 and A have one and the same tangent at a smooth point P_3 of A. Let C_2 be an irreducible curve in $|M+2L_1|$ such that $P_3 \in C_2$ and C_2 has the same tangent as one of those of A at the node P_1 of A. Let $P_4 \in C_1 \cap C_2$ be the point different from P_3 and let L_3 be the fiber containing P_4. Then A meeis L_3 in two distinct points P_5 and P_6 and $P_i \neq P_j$ $(i \neq j,\ 1 \leqq i,\ j \leqq 6)$ because $(A, C_l)=4$ $(l=1, 2)$. Let $\tau_1: V_1 \to \Sigma_2$ be the blowing-up of six points P_i's and let $E_j := \tau_1^{-1}(P_j)$ $(j=1, 2, 3)$. Let $\tau_2: V \to V_1$ be the blowingup of three points $\tau_1'(C_2) \cap E_1$, $\tau_1'(A) \cap E_2$ and $\tau_1'(A) \cap E_3$. Set $\tau := \tau_1 \circ \tau_2$, $E_j := \tau_1'(E_j)$, $L_k' := \tau'(L_k)$ $(k=2, 3)$, $M' := \tau'(M)$, $A' := \tau'(A)$,

$C_l':=\tau'(C_l)$, $D:=\sum E_j'+\sum L_k'+M'+A'+\sum C_l'$. Then D has the same configuration as $f^{-1}(\mathrm{Sing}\bar{V})$ ($\subseteq V$) in the case $(c, I)=(2, 11)$ of Theorem 5.1. Note that $4M+3L_2+5L_3+6A+4C_1+2C_2\sim -11K_{\Sigma_2}$. We can check that $2E_2'+4C_1'+6A'+3E_1'+E_3'+2C_2'+3L_2'+4M'+5L_3'\sim -11K_V$. Hence (V, D) is a log Enriques surface fitting the case $(c, I)=(2, 11)$ of Theorem 5.1.

We complete this section by giving two examples for the cases $(c, I)=(7, 17)$ and $(5, 19)$. We use the following notations:

Let $\pi: \Sigma_2\rightarrow \boldsymbol{P}^1$ be the $\boldsymbol{P}^1$-fibration on a Hirzebruch surface Σ_2 and let M and L be the (-2)-curve of Σ_2 and a general fiber of π, respectively. Let C_1 be an irreducible member in $|M+2L|$.

Example 5.7 (for the case $(c, I)=(7, 17)$ and $(n_1, \cdots, n_8)=(1, 1, 0, 2, 0, 0, 0, 3)$). Since $\dim|M+2L|=3$, there is an irreducible member C_2 in $|M+2L|$ such that C_2 meets C_1 in a single point P_3 with order of contact 2. Take two distinct fibers L_i $(i=1, 2)$ so that P_3 is not contained in L_i. Denote the points $L_i\cap C_i$ $(i=1, 2)$ and $L_2\cap C_1$ by P_i and P_4, respectively. Let τ_1: $V_1\rightarrow\Sigma_2$ be the blowing-up of four points P_i's and set $E_j:=\tau_1^{-1}(P_j)$ $(1\leqq j\leqq 3)$. Let τ_2: $V_2\rightarrow V_1$ be the blowing-up of three points $P_5:=\tau_1'(L_1)\cap E_1$, $P_6:=\tau_1'(C_2)\cap E_2$ and $P_7:=\tau_1'(C_2)\cap E_3$ and set $E_{k-1}:=\tau_1^{-1}(P_k)$. Let τ_3: $V_3\rightarrow V_2$ be the blowing-up of three points $P_8:=\tau_2'\tau_1'(L_1)\cap E_4$, $P_9:=\tau_2'\tau_1'(C_2)\cap E_5$ and $P_{10}:=\tau_2'\tau_1'(C_2)\cap E_6$, and set $E_7:=\tau_3^{-1}(P_8)$ and $E_8:=\tau_3^{-1}(P_{10})$. Let τ_4: $V'\rightarrow V_3$ be the blowing-up of two points $\tau_3'\tau_2'\tau_1'(L_1)\cap E_7$ and $\tau_3'(E_6)\cap E_8$. Denote by E_i' $(1\leqq i\leqq 8)$, M', C_j' and $L_j'(j=1, 2)$ the proper transforms on V' of E_i, M, C_j and L_j, respectively. Set $\tau:=\tau_1\cdots\tau_4$ and $D':=\sum E_i'+\sum C_j'+\sum L_j'+M'$. Noting that $8C_1+14C_2+15L_1+9L_2+12M\sim -17K_{\Sigma_2}$, we can check that $2E_7'+4E_4'+6E_1'+8C_1'+10E_6'+5E_3'+3E_5'+6E_2'+9L_2'+12M'+15L_1'+14C_2'+7E_8'\sim -17K_{V'}$. Hence (V', D') is a log Enriques surface with $(c, I)=(2, 17)$. The dual graph of D' is given in Figure (1), where the corresponding intersection number of each irreducible component of D' is given.

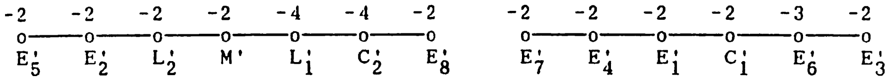


Figure (1)

We can find a sequence of blowing-ups σ: $V\rightarrow V'$ of several singular points of $\Delta':=E_5'+E_2'+L_2'+M'+L_1'+C_2'+E_8'$ such that the dual graph of $\sigma^{-1}(\Delta')$ is given in Figure (2), where $\tilde{E}_i:=\sigma'(E_i')$, $\tilde{C}_j:=\sigma'(C_j')$, $\tilde{L}_k:=\sigma'(L_k')$ and $\tilde{M}:=\sigma'(M')$.

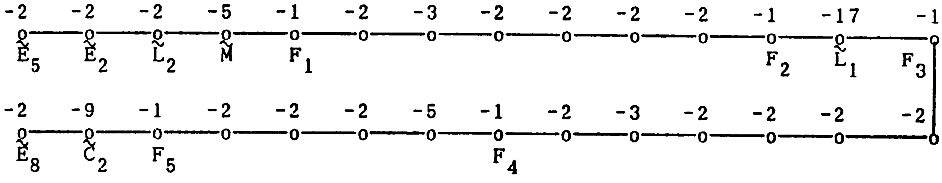


Figure (2)

Let $D:=\sigma^{-1}(D')-\sum_{i=1}^{5} F_i$. Then (V, D) is a log Enriques surface satisfying $(c, I)=(7, 17)$ and $(n_1, \cdots, n_8)=(1, 1, 0, 2, 0, 0, 0, 3)$.

Example 5.8 (for the case $(c, I)=(5, 19)$ and $(n_1, \cdots, n_9)=(0, 1, 1, 0, 0, 1, 0, 0, 2)$). Take an irreducible member C_2 in $|M+2L|$ such that C_2 meets C_1 in two distinct points P_1 and P_3. Take an arbitrary point P_2 $(\neq P_1, P_3)$ of C_1. Let L_i $(i=1, 2)$ be the fiber of π containing P_i. Let $\tau_1: V_1 \to \Sigma_2$ be the blowing-up of three points P_i's and set $E_i:=\tau_1^{-1}(P_i)$. Let $\tau_2: V_2 \to V_1$ be the blowing-up of four points $P_4:=\tau_1'(L_1)\cap E_1$, $P_5:=\tau_1'(C_1)\cap E_1$, $P_6:=\tau_1'(L_2)\cap E_2$ and $P_7:=\tau_1'(C_2)\cap E_3$, and set $E_{j-1}:=\tau_2^{-1}(P_j)$ $(5\leqq j\leqq 7)$. Let $\tau_3: V_3 \to V_2$ be the blowing-up of three points $P_8:=\tau_2'\tau_1'(C_1)\cap E_4$, $P_9:=\tau_2'\tau_1'(L_2)\cap E_5$ and $P_{10}:=\tau_2'\tau_1'(C_2)\cap E_6$, and set $E_7:=\tau_3^{-1}(P_{10})$. Let $\tau_4: V' \to V_3$ be the blowing-up of the point $\tau_3'\tau_2'\tau_1'(C_2)\cap E_7$. Denote by E_i' $(1\leqq i\leqq 7)$, M', C_j' and L_j' $(j=1, 2)$ the proper transforms on V' of E_i, M, C_j and L_j, respectively. Set $\tau:=\tau_1\cdots\tau_4$ and $D':=\sum E_i'+\sum C_j'+\sum L_j'+M'$. Noting that $12C_1+16C_2+5L_1+15L_2+10M\sim-19K_{\Sigma_2}$, we can check that $3E_7'+6E_6'+9E_3'+12C_1'+8E_2'+4E_5'+7E_4'+14E_1'+16C_2'+15L_2'+10M'+5L_1'\sim-19K_{V'}$. Hence (V', D') is a log Enriques surface with $(c, I)=(2, 19)$. The dual graph of D' is given in Figure (3), where the intersection number of each irreducible component of D' is given correspondingly.

Figure (3)

We can find a sequence of blowing-ups $\sigma: V \to V'$ of several singular points of $\Delta':=E_4'+E_1'+C_2'+L_2'+M'+L_1'$ such that the dual graph of $\sigma^{-1}(\Delta')$ is given in Figure (4), where $\tilde{E}_i:=\sigma'(E_i')$, $\tilde{C}_j:=\sigma'(C_j')$, $\tilde{L}_j:=\sigma'(L_j')$ and $\tilde{M}:=\sigma'(M')$.

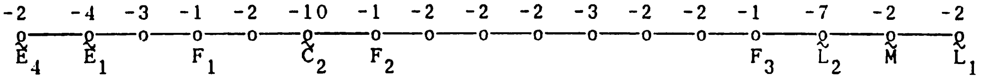


Figure (4)

Let $D:=\sigma^{-1}(D')-\sum_{i=1}^{3} F_i$. Then (V, D) is a log Enriques surface satisfying $(c, I)=(5, 19)$ and $(n_1, \cdots, n_9)=(0, 1, 1, 0, 0, 1, 0, 0, 2)$.

§6. The case where the canonical covering is singular

Let (V, D) or $\bar{V}$ be a log Enriques surface. In the present section, we let $c:=\#(\mathrm{Sing}\bar{V})=\#\{\text{connected component of } D\}$ and $I:=\mathrm{Index}(K_{\bar{V}})$, and use the notations $\pi: \bar{U} \to \bar{V}$, $f: V \to \bar{V}$ and $g: U \to \bar{U}$ as set at the beginning of §2.

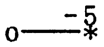
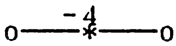
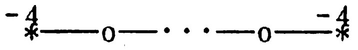
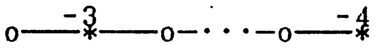
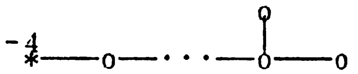
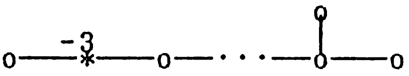
In the following two propositions, we shall give the possible types of singularities of a log Enriques surface $\bar{V}$ with $I=3$ or 5.

Proposition 6.1. *Let $\bar{V}$ be a log Enriques surface with $I=3$. Let y be a singular* *point of* $\bar{V}$ *and set* $\Delta := f^{-1}(y)$ $(\subseteqq V)$. *Then* $\pi^{-1}(y)$ *consists of a single point* x *of* $\bar{U}$ (cf. *Lemma* 6.5), *and the dual graph of* Δ *and the Dynkin type of the singularity* x *are given in Table* 1 *below, where* $\circ$ (*resp.* $\overset{-\alpha}{*}$) *stands for a* (-2)-*curve* (*resp.* $(-\alpha)$-*curve*) *and* $n := \#(\Delta)$. *Moreover,* $n \leqq 9$.

In particular, x *is a cyclic singularity if and only if so is* y.

Remark. We shall see in Example 6.11 that there is a log Enriques surface (V', D') with $I=3$ such that D' consists of one isolated (-3)-curve and one fork Δ' of type No. 9 below with $n=9$. Hence the cannonical covering $\bar{U}'$ of (V', D') has only one singular point x and x is of Dynkin type D_{19}. In particular, the minimal resolution U' of $\bar{U}'$ is a $K3$-surface with $\rho(U')=20$.

Table 1

No	dual graph of Δ	Dynkin type of x
1		smooth
2		A_1
3		A_2
4		A_3
5		$A_{3(n-1)+1}$ (7≥n≥2)
6		$A_{3(n-2)+2}$ (7≥n≥3)
7		$A_{3(n-2)}$ (8≥n≥4)
8		$D_{3(n-2)}$ (8≥n≥4)
9		$D_{3(n-3)+1}$ (9≥n≥4)

Proof. Note that the coefficient in $D^{\#}$ of each component of D is 1/3 of 2/3. Consider first the case where Δ is a rod. Write $\Delta = R_1 + \cdots + R_r$, where R_i's are irreducible components of R and $(R_j, R_{j+1})=1$ $(1 \leqq j \leqq r-1)$. Let α_i be the coefficient in $D^{\#}$ of R_i.

Suppose that $(R_s^2) = -a \leqq -3$ for some $1 \leqq s \leqq r$ and $(R_i^2) = -2$ $(i \neq s)$. Then $\alpha_i = i(a-2)\ (r-s+1)/(r+1+s(a-2)\ (r-s+1))$ when $i \leqq s$, and $\alpha_i = s(a-2)\ (r-i+1)/(r+1+s(a-2)(r-s+1))$ when $i > s$. Note that $\alpha_1 < \alpha_2 < \cdots < \alpha_s$ and $\alpha_s > \alpha_{s+1} > \cdots > \alpha_r$ and that $3\alpha_i = 1$ or 2. Hence $r \leqq 3$. If $r=3$ then $\alpha_1 = \alpha_3 = 1/3$, $\alpha_2 = 2/3$, $a=4$, $s=2$ and Δ is given in the row No. 4 of Table 1. If $r=2$, then $\alpha_s = 2/3$, $\alpha_i = 1/3$ $(i \neq s)$, and Δ is given in the row No. 3 of Table 1. If $r=1$, then $3\alpha_1 = 1$ or 2, and Δ is given in the row No.

1 or No. 2 of Table 1.

Suppose that $(R_q^2)\leqq -3$ and $(R_p^2)\leqq -3$ for some $1\leqq q<p\leqq r$ and $(R_i^2)=-2$ if $i<q$ or $i>p$. By Lemma 1.10, (3), where we set $B_j:=R_j$ $(q\leqq j\leqq p)$, $(B_q^2)=(B_p^2)=-3$ and $(B_i^2)=-2$ $(i\neq q, p)$, we obtain $\alpha_j\geqq 1/2$. Hence we have $\alpha_j=2/3$ $(q\leqq j\leqq p)$. Then $(R_k^2)=-2$ if $q+1\leqq k\leqq p-1$ because $(D^\#+K_V, R_k)=0$. Using $(D^\#+K_V, R_i)=0\,(1\leqq i\leqq r)$ again, we see that Δ is given in the row No. 5, No. 6 or No. 7 of Table 1.

Next we consider the case where Δ is a fork. Write $\Delta=T_0+T_1+T_2+T_3$, where T_0 is the central component of Δ and T_i's are three twigs of Δ. Write $T_i=T_i(1)+\cdots+T_i(n_i)$, where $T_i(j)$'s are irreducible components of T_i and $(T_i(k), T_i(k+1))=(T_0, T_i(1))=1$ $(1\leqq k\leqq n_i-1)$. We may assume that T_1 consists of a single (-2)-curve. Set $r:=n_3$, $G_j:=T_3(r-j+1)$, $G_{r+1}:=T_0$, $G_{r+2}:=T_1$ and $G_{r+2+p}:=T_2(p)$ $(1\leqq p\leqq n_2)$. Let α_i be the coefficient in $D^\#$ of G_i. Then $3\alpha_i=1$ or 2. $(D^\#+K_V, T_1)=0$ implies $\alpha_{r+1}=2/3$ and $\alpha_{r+2}=1/3$. $(D^\#+K_V, T_0)=0$ implies that either $(T_0^2)=-3$ and $\alpha_{r+3}=\alpha_r=1/3$, or $(T_0^2)=-2$ and $\alpha_{r+3}=1/3$, $\alpha_r=2/3$ after twigs T_2 and T_3 are interchanged if necessary. $(D^\#+K_V, T_i(1))=0$ $(i=2, 3)$ implies that T_2 consists of a single (-2)-curve and that if $(T_0^2)=-3$ then Δ is given in the row No. 9 of Table 1 with $n=4$.

Consider the case $(T_0^2)=-2$. Then there is an integer $1\leqq s\leqq r$ such that $(G_s^2)\leqq -3$ and $(G_j^2)=-2$ if $j<s$ by our hypothesis that $\mathrm{Supp}D^\#=\mathrm{Supp}D$. Note that $s=1$ or 2 because $(D^\#+K_V, G_k)=0$ if $k<s$. By Lemma 1.10, where we consider a divisor consisting of $B_k:=G_k$ $(s\leqq k\leqq r+3)$ with $(B_s^2)=-3$ and $(B_l^2)=-2\,(l\neq s)$, we obtain $\alpha_p\geqq 1/2$ $(s\leqq p\leqq r)$. Hence $\alpha_p=2/3$. We also have $(G_s^2)=5-s$ and $(G_q^2)=-2$ $(q>s)$ because $(D^\#+K_V, G_{q-1})=0$. Then Δ is given in the row No. 8 or No. 9 of Table 1.

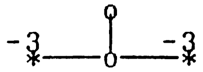
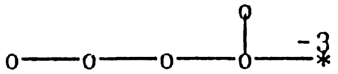
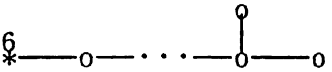
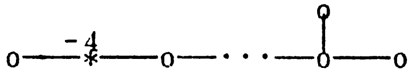
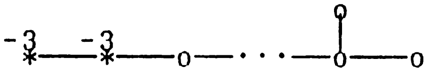
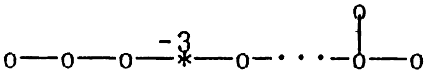
To deduce the Dynkin type of the singularity $x=\pi^{-1}(f(\Delta))$ of $\bar{U}$, we explain our method by treating the No. 3 case where Δ is a rod with one (-2)-curve D_1 and one (-5)-curve D_2. For general cases, we refer to Hirzebruch [3] and Miyanishi-Russell [7]. Let $\tau\colon W\to V$ be the blowing-up of the point $P:=D_1\cap D_2$ and set $E:=\tau^{-1}(P)$. Note that the coefficient in $D^\#$ of D_1, D_2 are 1/3, 2/3, respectively. Hence $\tau'(3D^\#)\sim -3K_W$ because $3D^\#\sim -3K_V$. Let $\tilde{\pi}\colon \tilde{U}\to W$ be the composite of the covering morphism of a $Z/3Z$-covering which is defined by a relation $\mathcal{O}(-K_W)^{\otimes 3}\cong\mathcal{O}(\tau'(3D^\#))$ and a nonzero global section of $\mathcal{O}(\tau'(3D^\#))$ followed by the normalization of the covering surface. We see that $\tilde{\pi}^{-1}\tau^{-1}(\Delta)$ is a rod with one (-1)-curve, one (-2)-curve and one (-3)-curve as the central component. Then the canonical covering $\bar{U}$ of $\bar{V}$ is nothing but the surface obtained from $\tilde{U}$ by contracting $\tilde{\pi}^{-1}\tau^{-1}(\Delta)$. Hence $x=\pi^{-1}(f(\Delta))$ is a rational double singular point of Dynkin type A_2.

Denote the reduced divisor $g^{-1}(x)$ $(\subseteqq U)$ by Γ. Then $\#(\Gamma)\leqq\rho(U)-\rho(\bar{U})\leqq 20-1=19$. Hence $n=\#(\Delta)\leqq 9$ (cf. Table 1). Q. E. D.

Proposition 6.2. *Let $\bar{V}$ be a log Enriques surface with $I=5$. Let y be a singular point of $\bar{V}$ and set $\Delta:=f^{-1}(y)$ $(\subseteqq V)$. Then $\pi^{-1}(y)$ consists of a single point x of $\bar{U}$ (cf. Lemma 6.5). Suppose further that Δ is a fork. Then the dual graph of Δ and the Dynkin type of the singularity x are given in Table 2 below, where $\circ$ (resp. $\overset{-\alpha}{*}$) stands for a (-2)-curve (resp. $(-\alpha)$-curve) and $n:=\#(\Delta)$.*

Furthermore, x is a cyclic singularity if and only if so is y.

Table 2

No	dual graph of Δ	Dynkin type of x
1		E_6
2		E_7
3		$D_{5(n-3)+3}$ (6≥n≥4)
4		$D_{5(n-4)+4}$ (7≥n≥4)
5		$D_{5(n-3)}$ (6≥n≥4)
6		$D_{5(n-5)+1}$ (8≥n≥6)

Proof. Write $\Delta=T_0+T_1+T_2+T_3$ with the central component T_0 and three twigs T_i's. We may assume that T_1 is a (-2)-curve and T_2 is a (-2)-curve, a (-3)-curve or a rod with two (-2)-curves. Write $T_3=\sum_{i=1}^{r} G_i$, $T_0=G_{r+1}$, $T_1=G_{r+2}$ and $T_2=\sum_{j=r+3}^{n} G_j$, where G_q is irreducible and $(G_k, G_{k+1})=(T_0, G_{r+3})=1$ $(1\leqq k\leqq n-1,\ k\neq r+2)$. Let α_q be the coefficient of G_q in $D^\#$. Then $5\alpha_q=1, 2, 3$ or 4. We have $\alpha_{r+1}=2\alpha_{r+2}$ for $(D^\#+K_V, T_1)=0$. Hence $\alpha_{r+1}=2/5$ or 4/5. Denote by $a_q:=-(G_q^2)$.

Assume that T_2 is a (-2)-curve. Then $\alpha_{r+1}=2\alpha_{r+3}$ for $(D^\#+K_V, T_2)=0$. By our hypothesis that $\bar{V}$ contains no rational double singular points, we may assume that $a_m\geqq3$ for some $1\leqq m\leqq r+1$ and $a_q=2$ if $q>m$. Applying Lemma 1.10 to a divisor consisting of $B_q:=G_q$ $(1\leqq q\leqq r+3)$ with $(B_m^2)=-3$ and $(B_q^2)=-2$ $(q\neq m)$, we obtain $\alpha_q\geqq m/(m+1)\geqq1/2(m\leqq q\leqq r+1)$ and $\alpha_q\geqq q/(m+1)(1\leqq q\leqq m)$. In particular, $\alpha_{r+1}=4/5$. Then $\alpha_q=4/5$ $(m\leqq q\leqq r)$ for $(D^\#+K_V, G_q)=0$. If $m=1$, then $a_1=6$ for $(D^\#+K_V, G_1)=0$, and Δ is given in the row No. 3 of Table 2. Suppose $m\geqq2$. Note that $a_m+5\alpha_{m-1}-6=5$ $(D^\#+K_V, G_m)=0$. Hence $5\alpha_{m-1}=6-a_m=1$, 2 or 3. Since $\alpha_{m-1}\geqq(m-1)/(m+1)\geqq1/3$, $\alpha_{m-1}=2/5$ or 3/5. If $m=2$, then $(\alpha_1, a_1, a_2)=(2/5, 2, 4)$ or $(3/5, 3, 3)$ for $(D^\#+K_V, G_1)=0$, and Δ is given in the row No. 4 or No. 5 of Table 2. Suppose $m\geqq3$. Then $m=4$, $\alpha_q=q/5$, $a_q=2$ and $a_4=3$ for $(D^\#+K_V, G_q)=0\ (1\leqq q<m)$. Hence D is given in the row No. 6 of Table 2.

Assume that T_2 is a (-3)-curve. Since $5\alpha_{r+1}+5-15\alpha_{r+3}=5$ $(D^\#+K_V, T_2)=0$, we have $\alpha_{r+1}=4/5$ and $\alpha_{r+3}=3/5$. Applying Lemma 1.10 to a divisor consisting of $B_q:=G_q$ $(r\leqq q\leqq r+3)$ with $(B_{r+3}^2)=-3$ and $(B_q^2)=-2$ $(q\neq r+3)$, we obtain $\alpha_r\geqq1/4$. On the other hand, since $a_{r+1}+5\alpha_r-5=5$ $(D^\#+K_V, T_0)=0$, we have $5\alpha_r=5-a_{r+1}=2$ or 3. If $r=1$, we have $(\alpha_1, a_1, a_2)=(2/5, 2, 3)$ or $(3/5, 3, 2)$ for $(D^\#+K_V, G_1)=0$. Then Δ is given in the row No. 5 or No. 1 of Table 2. Suppose $r\geqq2$. Then $5\alpha_{r-1}+(5-5\alpha_r)a_r$

$-6=5\ (D^{\#}+K_V,\ G_r)=0$ implies $(\alpha_{r-1},\ \alpha_r,\ a_r,\ a_{r+1})=(2/5,\ 3/5,\ 2,\ 2)$ and $(D^{\#}+K_V,\ G_q)=0$ $(1\leqq q<r)$ implies that $r=3$, $\alpha_q=q/5$ and $a_q=2$. Hence D is given in the row No. 2 of Table 2.

Assume that T_2 is a rod with two (-2)-curves. Then $\alpha_{r+1}=3/5$, $\alpha_{r+3}=2/5$ and $\alpha_{r+4}=1/5$ for $(D^{\#}+K_V,\ G_q)=0$ $(q=r+3$ and $r+4)$. This is absurd because $\alpha_{r+1}=2/5$ or $4/5$. Hence this case does not occur.

The Dynkin type of the singularity $x=\pi^{-1}\ (f(\Delta))$ can be determined in the same fashion as in Proposition 6.1. Q. E. D.

Corollary 6.3. *Let $\bar{V}$ be a log Enriques surface.*

(1) *Assume that there is a singularity of Dynkin type E_8 on $\bar{U}$. Then $I=7$, 11, 13, 17 or 19.*

(2) *Assume that there is a singularity of Dynkin type E_k $(k=6$, 7 or 8) on $\bar{U}$. Then $I=5$, 25, 7, 11, 13, 17 or 19.*

Proof. (1) Assume that x is a singularity of Dynkin type E_8 on $\bar{U}$. We assert that I is not divisible by 2, 3 or 5. Then we conclude the assertion (1) by Lemma 2.3. Suppose, on the contrary, that I is divisible by p where $p=2$, 3 or 5. By Lemma 2.2, $\bar{U}_1:=\bar{U}/(\boldsymbol{Z}/p\boldsymbol{Z})$ is a (rational) log Enriques surface such that $\bar{U}$ is the canonical covering of $\bar{U}_1$ and $\mathrm{Index}(K_{U_1})=p$. Applying Lemma 3.1 and Proposition 6.1 or Proposition 6.2 to $\bar{U}_1$, we reach a contradiction.

(2) can be proved similarly. Q. E. D.

The following two lemmas will be used in the proof of Proposition 6.6.

Lemma 6.4. *Let G be a finite subgroup of $GL(2,\ \boldsymbol{C})$. Suppose that G contains no reflections and that the order n of G is not divisible by 4. Then G is a cyclic group. Hence G is conjugate to a group $C_{n,q}$ with* g. c. d. $(n,\ q)=1$ *and $1\leqq q\leqq n-1$; for the definition of $C_{n,q}$, see Lemma 2.5 or [2; Satz 2.9]. Moreover, we have $q\leqq n-2$ when the origin of $\boldsymbol{C}^2/G$ is not a rational double singular point.*

Proof. By [2; Satz 2.9], G is conjugate to one of the groups listed there. In particular, if G is not cyclic then 4 is a factor of n. Q. E. D.

Lemma 6.5. (1) *Let $(V,\ D)$ be a log Enriques surface such that I is an odd prime number. Let y be a singular point of $\bar{V}$. Then $\pi^{-1}(y)$ consists of a single point x of $\bar{U}$, and the singularity of x (resp. y) is isomorphic to $(\boldsymbol{C}^2/G_x,\ 0)$ (resp. $(\boldsymbol{C}^2/G_y,\ 0)$) with a finite subgroup G_x (resp. G_y) of $GL\ (2,\ \boldsymbol{C})$ of order n (resp. nI) which contains no reflections provided $n\geqq 2$. (When $n=1$, x is a smooth point).*

(2) *Suppose further x is a cyclic singularity of Dynkin type A_{n-1}. In the case where $I=3$ or 5 or in the case where 4 is not a factor of n, then y is a cyclic singularity isomorphic to $(\boldsymbol{C}^2/C_{nI,k_{n-1}},\ 0)$ for some $1\leqq k_{n-1}\leqq nI-2$ with* g. c. d. $(nI,\ k_{n-1})=1$.

(3) *By changing coordinates of $\boldsymbol{C}^2$ if necessary, we have:*

(3a) *If $I=3$, then $k_0=k_1=1$, $k_2=2$, $k_3=7$, $k_4=4$, $k_5=5$ and $k_6=13$ (cf. Proposition 6.1).*

(3b) *If* $I=5$, *then* $k_0=1$ *or* 2, $k_1=1$ *or* 3, $k_2=2$ *or* 11, $k_3=3$ *or* 11 *and* $k_4=4$ *or* 9.

(3c) *If* $I=7$,*then* $k_0=1$, 2 *or* 3, $k_1=1$, 3 *or* 9 *and* $k_2=2$, 5 *or* 8.

Proof. (1) By the argument in the proof of Lemma 2.4, $\pi^{-1}(y)$ consists of a single point x. Then the assertion (1) follows if one notes that $\pi: \bar{U} \to \bar{V}$ is a finite morphism of degree I and is étale outside Sing $\bar{V}$.

(2) Assume x is of Dynkin type A_{n-1}. In the case where $I=3$ or 5, then y is a cyclic singularity by Propositions 6.1 and 6.2. In the case where 4 is not a factor of n, then the order nI of G_y is not divisible by 4 and hence y is a cyclic singularity by Lemma 6.5. Thus, in either case, G_y is a cyclic group conjugate to $\tilde{G}_y := C_{nI, k_{n-1}}$ for some $1 \leqq k_{n-1} \leqq nI-2$ with g. c. d. $(nI, k_{n-1})=1$ and y is isomorphic to $(C^2/\tilde{G}_y, 0)$ because y is not a rational double singularity.

The assertion (3) is a consequence of the fact that $ID^{\#}$ is an integral divisor of V.
Q. E. D.

We shall define some notations to be used in the following proposition. Let (V, D) be a log Enriques surface such that I is an odd prime number and Sing $\bar{U} = \sum_{i=1}^{6} m_i A_i$ for some integers $m_i \geqq 0$ $(1 \leqq i \leqq 6)$. The second condition means, by definition, that Sing $\bar{U}$ consists of m_i singularities $\{x_{ij}\}$ $(1 \leqq j \leqq m_i)$ of Dynkin type A_i for each $1 \leqq i \leqq 6$. Let m_0 be the number of all singularities $\{y_{0j}\}$ of $\bar{V}$ such that $x_{0j} := \pi^{-1}(y_{0j})$ is a smooth point of $\bar{U}$. In the case where $I=3$ or 5 or in the case where $m_3=0$, then the singularities $y_{ij} := \pi(x_{ij})$ $(0 \leqq i \leqq 6)$ exhaust Sing $\bar{V}$ and are isomorphic to $(C^2/C_{I(i+1), k_i}, 0)$ for some $1 \leqq k_i \leqq I\ (i+1)-2$ with g. c. d. $(I(i+1), k_i)=1$ by Lemma 6.5. We also have $\sum_{i=1}^{6} m_i = \#(\text{Sing } \bar{U})$ and $\sum_{i=0}^{6} m_i = c$.

In the case $I=5$, let $n_1, \cdots, n_{10}$ be respectively the numbers of all singularities $\{y_{\alpha j}\}$ of $\bar{V}$ such that $(\alpha, k_\alpha)=(0, 1), (0, 2), (1, 1), (1, 3), (2, 2), (2, 11), (3,3), (3,11), (4,4), (4, 9)$. Then $m_i = n_{2i+1} + n_{2i+2}$ $(0 \leqq i \leqq 4)$.

In the case $I=7$, let $n_1, \cdots, n_9$ be the numbers of all singularities $\{y_{\alpha j}\}$ of $\bar{V}$ such that $(\alpha, k_\alpha)=(0, 1), (0, 2), (0, 3), (1, 1), (1, 3), (1, 9), (2,2), (2,5), (2,8)$, respectively. Then $m_i = n_{3i+1} + n_{3i+2} + n_{3i+3}$ $(0 \leqq i \leqq 2)$.

In general, if $I=3$ then Sing $\bar{U} = \sum_{i \geqq 1} m_i A_i + \sum_{j \geqq 4} \delta_j D_j$ for some integers $m_i \geqq 0$ and $\delta_j \geqq 0$, where $\delta_j = 0$ if $j \equiv 2 \pmod 3$ by virtue of Proposition 6.1. Set $m_0 := c - \#(\text{Sing } \bar{U}) = c - \sum_{i \geqq 1} m_i - \sum \delta_j$.

The bounds for c and $\rho(\bar{V}) - c$ are given below.

Proposition 6.6. *Let* (V, D) *be a log Enriques surface such that* I *is an odd prime number and* Sing $\bar{U} \neq \emptyset$. *Then we have* $2 \leqq c \leqq \text{Min}\{16, 23-I\}$ *and* $c-1 \leqq \rho(\bar{V}) \leqq c+4$. *More precisely, we have:*

(1) *Suppose* $I=3$. *Then* $c \leqq 15$ *and* $\rho(\bar{V}) \leqq c+4$. *Moreover, if* $c=15$, *then* $\rho(\bar{V})=14$, $\rho(V)=29$, Sing $\bar{U}=6A_1$ *and* $(m_0, m_1)=(9, 6)$. *If* $\rho(\bar{V})=c+4$, *then* $\sum_{i=0}^{3} m_i + \delta_4 = c$, Sing $\bar{U}$ $= D_4, A_3, A_2$ *or* A_1, $(m_0, \cdots, m_3, \delta_4)=(1, 0, 0, 0, 1), (2, 0, 0, 1, 0), (3, 0, 1, 0, 0)$ *or* (4, 1, 0,

0, 0) *and* $\rho(V)=11$, 12, 13 *or* 14, *respectively.*

(2) *Suppose* $I=5$. *Then* $c\leqq 16$ *and* $\rho(\bar{V})\leqq c+2$. *Moreover, if* $c=16$, *then* $\rho(\bar{V})=15$, $\rho(V)=40$, $\mathrm{Sing}\,\bar{U}=3A_1$, $(m_0, m_1)=(13, 3)$ *and* $(n_1, \cdots, n_4)=(4, 9, 3, 0)$. *If* $\rho(\bar{V})=c+2$, *then* $\sum_{i=0}^{2} m_i=c$, *Sing* $\bar{U}=A_2$ *or* A_1, $(m_0, m_1, m_2)=(1, 0, 1)$ *or* $(2, 1, 0)$, $(n_1, \cdots, n_6)=(0, 1, 0, 0, 0, 1)$ *or* $(0, 2, 0, 1, 0, 0)$ *and* $\rho(V)=11$ *or* 12, *respectively.*

(3) *Suppose* $I=7$. *Then* $c\leqq 15$ *and* $\rho(\bar{V})\leqq c+1$. *Moreover, if* $c=15$, *then* $\rho(\bar{V})=14$, $\mathrm{Sing}\,\bar{U}=2A_1$, $(m_0, m_1)=(13, 2)$, $(n_1, \cdots, n_6)=(0, 11, 2, 2, 0, 0)$, $(1, 8, 4, 2, 0, 0)$, $(2, 5, 6, 2, 0, 0)$ *or* $(3, 2, 8, 2, 0, 0)$ *and* $\rho(V)=44$, 45, 46 *or* 47, *respectively. If* $\rho(\bar{V})=c+1$, *then* $c=2$, $\rho(V)=11$, $\mathrm{Sing}\,\bar{U}=A_1$, $(m_0, m_1)=(1, 1)$ *and* $(n_1, \cdots, n_6)=(0, 0, 1, 0, 0, 1)$.

(4) *Suppose* $I\geqq 11$. *Then* $\rho(\bar{V})=c-1$.

In particular, we have $24-kI\leqq c+\rho(U)-\rho(\bar{U})=24-I(\rho(\bar{V})-c+2)\leqq 24-I$, *where* $k=6$ (*resp.* 4, 3 *or* 1) *if* $I=3$ (*resp.* $I=5$, $I=7$ *or* $I\geqq 11$) (*cf. Lemma* 2.4). *Moreover,* $(D, K_V)=c-1-(K_{\bar{V}}^2)$ *when the upper bound of* c *or* $\rho(\bar{V})-c$ *in* (1), (2) *and* (3) *is attained.*

Proof. Since $I\geqq 3$ we have $c\geqq 2$ by Proposition 1.6. We use the result $1\leqq\rho(\bar{V})-c+2=(24+\rho(\bar{U})-\rho(U)-c)/I\leqq 21/I\leqq 7$ in Lemma 2.4. In particular, we obtain the assertion (4), and $c-1\leqq\rho(\bar{V})\leqq c+5$ and $c=24+\rho(\bar{U})-\rho(U)-I(\rho(\bar{V})-c+2)\leqq 23-I\leqq 20$. Moreover, if $\rho(\bar{V})=c+5$ then $I=3$ and $24+\rho(\bar{U})-\rho(U)-c=21$, whence $c=2$ and $\mathrm{Sing}\,\bar{U}=A_1$. In proving the assertion (1), we will show that this case does not occurs. Therefore, in order to prove Proposition 6.6, we have only to consider the case where $I=3$, 5 or 7 and show the assertions (1), (2) and (3).

(1) Assume $I=3$. Then $0<\rho(U)-\rho(\bar{U})=24-c-I(\rho(\bar{V})-c+2)\leqq 21-c$. In particular, if $c\geqq 15$ we have $\rho(U)-\rho(\bar{U})\leqq 6$ and hence write $\mathrm{Sing}\,\bar{U}=\sum_{i=1}^{6} m_iA_i+\delta_4D_4+\delta_6D_6$. For the time being, we assume that $\mathrm{Sing}\,\bar{U}$ is written this way. Then $D=f^{-1}$ $(\mathrm{Sing}\,\bar{V})$ consists of δ_4 forks Γ_p $(1\leqq p\leqq\delta_4)$, δ_6 forks Δ_q $(1\leqq q\leqq\delta_6)$ and $\sum_{i=0}^{6} m_i$ rods B_d $(1\leqq d\leqq m_0)$, $C_e(m_0+1\leqq e\leqq m_0+m_1)$, $D_f(m_0+m_1+1\leqq f\leqq m_0+m_1+m_2)$, $E_g(m_0+m_1+m_2+1\leqq g\leqq m_0+\cdots+m_3)$, $F_h(m_0+\cdots+m_3+1\leqq h\leqq m_0+\cdots+m_4)$, $G_i(m_0+\cdots+m_4+1\leqq i\leqq m_0+\cdots+m_5)$ and $H_j(m_0+\cdots+m_5+1\leqq j\leqq m_0+\cdots+m_6)$, which are defined as follows (cf. Proposition 6.1):

(i) B_d is a (-3)-curve,

(ii) C_e is a (-6)-curve,

(iii) D_f consists of one (-2)-curve D_{1f} and one (-5)-curve D_{2f},

(iv) E_g consists of two (-2)-curves E_{1g}, E_{3g} and one (-4)-curve E_{2g} with $(E_{bg}, E_{b+1,g})=1$ $(b=1, 2)$,

(v) F_h consists of two (-4)-curves F_{1h} and F_{2h},

(vi) G_i consists of one (-2)-curve G_{1i}, one (-3)-curve G_{2i} and one (-4)-curve G_{3i} with $(G_{bi}, G_{b+1,i})=1$ $(b=1, 2)$,

(vii) H_j consists of two (-2)-curves H_{1j}, H_{4j} and two (-3)-curves H_{2j}, H_{3j} with $(H_{bj}, H_{b+1,j})=1$ $(1\leqq b\leqq 3)$,

(viii) $\Gamma_p=\sum_{r=0}^{3} S_{rp}$, where S_{0p} is the central component and S_{up} $(1\leqq u\leqq 3)$ is a twig and where S_{0p} is a (-3)-curve and S_{up} is a (-2)-curve,

(ix) $\varDelta_q=\sum_{s=0}^{3} T_{sq}$, where T_{0q} is the central component and T_{uq} $(1\leqq u\leqq 3)$ is a twig and where T_{3q} is a (-4)-curve and $T_{vq}(0\leqq v\leqq 2)$ is a (-2)-curve.

Then $D^{\#}=(1/3)\sum B_d+(2/3)\sum C_e+(1/3)\sum(D_{1f}+2D_{2f})+(1/3)\sum(E_{1g}+2E_{2g}+E_{3g})+(2/3)\sum(F_{1h}+F_{2h})+(1/3)\sum(G_{1i}+2G_{2i}+2G_{3i})+(1/3)\sum(H_{1j}+2H_{2j}+2H_{3j}+H_{4j})+(1/3)\sum(2S_{0p}+S_{1p}+S_{2p}+S_{3p})+(1/3)\sum(2T_{0q}+T_{1q}+T_{2q}+2T_{3q})$. Hence $-(m_0+8m_1+6m_2+4m_3+8m_4+6m_5+4m_6+2\delta_4+4\delta_6)/3=(D^{\#})^2=(K_V^2)=10-\rho(V)=10-\rho(\bar{V})-\#(D)=10-\rho(\bar{V})-(m_0+m_1+2m_2+3m_3+2m_4+3m_5+4m_6+4\delta_4+4\delta_6)$. This entails:

$$\text{(1a)}\qquad 3(\rho(\bar{V})-10)+2m_0-5m_1+5m_3-2m_4+3m_5+8m_6+10\delta_4+8\delta_6=0.$$

In particular, $m_1+m_4\geqq(5m_1+2m_4)/5\geqq 3(\rho(\bar{V})-10)/5$. On the other hand, by Proposition 1.6, we have $m_0+4m_1+3m_2+2m_3+4m_4+3m_5+2m_6+\delta_4+2\delta_6=(D, K_V)<c-(K_V^2)=c+\rho(V)-10=c+\rho(\bar{V})-10+\#(D)=c+\rho(\bar{V})-10+(m_0+m_1+2m_2+3m_3+2m_4+3m_5+4m_6+4\delta_4+4\delta_6)$. Hence we obtain:

$$\text{(1b)}\qquad c+\rho(\bar{V})-10>3m_1+m_2-m_3+2m_4-2m_6-3\delta_4-2\delta_6.$$

To prove $\rho(\bar{V})\leqq c+4$, we have only to show that the case $\rho(\bar{V})=c+5$ is impossible. Indeed, in the case $\rho(\bar{V})=c+5$, we have $c=2$ and $\operatorname{Sing}\bar{U}=A_1$. Hence $m_0=m_1=1$ and $\delta_4=\delta_6=m_i=0$ $(i\geqq 2)$, contradicting the above equality (1a).

Assume $\rho(\bar{V})=c+4$. Then $c+\rho(U)-\rho(\bar{U})=24-I(\rho(\bar{V})-c+2)=6$. Hence $(c, \rho(U)-\rho(\bar{U}))=(2, 4), (3, 3), (4, 2)$ or $(5, 1)$. So, the above expression of $\operatorname{Sing}\bar{U}$ is still effective. Note that $\delta_4=\delta_6=0$ when $\rho(U)-\rho(\bar{U})\leqq 3$. We consider these cases separately.

Case $(c, \rho(U)-\rho(\bar{U}))=(2, 4)$. Then $\rho(\bar{V})=6$ and $\operatorname{Sing}\bar{U}=D_4, A_4, A_1+A_3$ or $2A_2$. If $\operatorname{Sing}\bar{U}=A_4, A_1+A_3$ or $2A_2$, then $(m_0, \cdots, m_6, \delta_4, \delta_6)=(1, 0, 0, 0, 1, 0, \cdots, 0), (0, 1, 0, 1, 0, \cdots, 0)$ or $(0, 0, 2, 0, \cdots, 0)$, respectively. This contradicts the above equality (1a). Hence we must have $\operatorname{Sing}\bar{U}=D_4$. Then $(m_0, \cdots, m_6, \delta_4, \delta_6)=(1, 0, \cdots, 0, 1, 0)$ and $\rho(V)=\rho(\bar{V})+\#(D)=11$. This is one of the cases given in the assertion (1).

Case $(c, \rho(U)-\rho(\bar{U}))=(3, 3)$. Then $\rho(\bar{V})=7$ and $\operatorname{Sing}\bar{U}=A_3, A_1+A_2$ or $3A_1$. If $\operatorname{Sing}\bar{U}=A_1+A_2$ or $3A_1$, then $(m_0, \cdots, m_6)=(1, 1, 1, 0, \cdots, 0)$ or $(0, 3, 0, \cdots, 0)$, respectively. This contradicts the above equality (1a). Thus, we must have $\operatorname{Sing}\bar{U}=A_3$. Then $(m_0, \cdots, m_6)=(2, 0, 0, 1, 0, 0, 0)$ and $\rho(V)=\rho(\bar{V})+\#(D)=12$. This is one of the cases given in the assertion (1).

Case $(c, \rho(U)-\rho(\bar{U}))=(4, 2)$. Then $\rho(\bar{V})=8$ and $\operatorname{Sing}\bar{U}=A_2$ or $2A_1$. If $\operatorname{Sing}\bar{U}=2A_1$, then $(m_0, \cdots, m_6)=(2, 2, 0, \cdots, 0)$, which contradicts the above equality (1a). Therefore, $\operatorname{Sing}\bar{U}=A_2$. Then $(m_0, \cdots, m_6)=(3, 0, 1, 0, \cdots, 0)$ and $\rho(V)=\rho(\bar{V})+\#(D)=13$. This is one of the cases given in the assertion (1).

Case $(c, \rho(U)-\rho(\bar{U}))=(5, 1)$. Then $\rho(\bar{V})=9$ and $\operatorname{Sing}\bar{U}=A_1$. Hence $(m_0, \cdots, m_6)=(4, 1, 0, \cdots, 0)$ and $\rho(V)=\rho(\bar{V})+\#(D)=14$. This is one of the cases given in the assertion (1).

Next, we shall prove $c\leqq 15$. We consider the cases $c=20$, 19, 18, 17 and 16, separately.

Assume $c=20$. Then $1\leqq\rho(U)-\rho(\bar{U})=24-c-I(\rho(\bar{V})-c+2)=4-3(\rho(\bar{V})-18)\leqq 1$. Hence $\rho(\bar{V})=19$ and $\operatorname{Sing}\bar{U}=A_1$. Then $(m_0, \cdots, m_6, \delta_4, \delta_6)=(19, 1, 0, \cdots, 0)$, which contradicts the above equality (1a).

Assume $c=19$. Then $0<\rho(U)-\rho(\bar{U})=24-c-I(\rho(\bar{V})-c+2)=5-3(\rho(\bar{V})-17)\leqq 2$. Hence $\rho(\bar{V})=18$ and $\operatorname{Sing}\bar{U}=A_2$ or $2A_1$. In particular, $m_1+m_4=m_1\leqq 2$. On the other hand, we have $m_1+m_4\geqq 3(\rho(\bar{V})-10)/5=24/5$. We thus have a contradiction.

Assume $c=18$. Then $0<\rho(U)-\rho(\bar{U})=24-c-I(\rho(\bar{V})-c+2)=6-3(\rho(\bar{V})-16)\leqq 3$. Hence $\rho(\bar{V})=17$ and $\operatorname{Sing}\bar{U}=A_3$, A_1+A_2 or $3A_1$. This leads to a contradiction as in the case $c=19$.

Assume $c=17$. Then $0<\rho(U)-\rho(\bar{U})=24-c-I(\rho(\bar{V})-c+2)=7-3(\rho(\bar{V})-15)\leqq 4$. Then either $\rho(\bar{V})=17$ and $\operatorname{Sing}\bar{U}=A_1$, or $\rho(\bar{V})=16$ and $\operatorname{Sing}\bar{U}=D_4$, A_4, A_1+A_3, $2A_2$, $2A_1+A_2$ or $4A_1$. Since $m_1+m_4\geqq 3(\rho(\bar{V})-10)/5$, we have $\rho(\bar{V})=16$ and $\operatorname{Sing}\bar{U}=4A_1$. Then $(m_0, \cdots, m_6, \delta_4, \delta_6)=(13, 4, 0, \cdots, 0)$, which contradicts the above equality (1a).

Assume $c=16$. Then $0<\rho(U)-\rho(\bar{U})=24-c-I(\rho(\bar{V})-c+2)=8-3(\rho(\bar{V})-14)\leqq 5$. Then either $\rho(\bar{V})=16$ and $\operatorname{Sing}\bar{U}=A_2$ or $2A_1$, or $\rho(\bar{V})=15$ and $\operatorname{Sing}\bar{U}=A_1+D_4$, A_5, A_1+A_4, A_2+A_3, $2A_1+A_3$, A_1+2A_2, $3A_1+A_2$ or $5A_1$. Since $m_1+m_4\geqq 3\ (\rho(\bar{V})-10)/5$, we have $\rho(\bar{V})=15$ and $\operatorname{Sing}\bar{U}=3A_1+A_2$ or $5A_1$. Then $(m_0, \cdots, m_6, \delta_4, \delta_6)=(12, 3, 1, 0, \cdots, 0)$ or $(11, 5, 0, \cdots, 0)$. This contradicts the above equality (1a).

We have thus proved $c\leqq 15$. We now consider the case $c=15$. Then $0<\rho(U)-\rho(\bar{U})=24-c-I\,(\rho(\bar{V})-c+2)=9-3(\rho(\bar{V})-13)\leqq 6$. Hence, either $\rho(\bar{V})=15$ and $\operatorname{Sing}\bar{U}=A_3$, A_1+A_2 or $3A_1$, or $\rho(\bar{V})=14$ and $\operatorname{Sing}\bar{U}=D_6$, A_2+D_4, $2A_1+D_4$, A_6, A_1+A_5, A_2+A_4, $2A_1+A_4$, $2A_3$, $A_1+A_2+A_3$, $3A_1+A_3$, $3A_2$, $2A_1+2A_2$, $4A_1+A_2$ or $6A_1$. Since $m_1+m_4\geqq 3(\rho(\bar{V})-10)/5$, either $\rho(\bar{V})=15$ and $\operatorname{Sing}\bar{U}=3A_1$, or $\rho(\bar{V})=14$ and $\operatorname{Sing}\bar{U}=2A_1+A_4$, $3A_1+A_3$, $4A_1+A_2$ or $6A_1$. If $\rho(\bar{V})=15$ and $\operatorname{Sing}\bar{U}=3A_1$, then $(m_0, \cdots, m_6, \delta_4, \delta_6)=(12, 3, 0, \cdots, 0)$, which contradicts the above equality (1a). Thus $\rho(\bar{V})=14$. Then $(m_0, \cdots, m_6, \delta_4, \delta_6)=(12, 2, 0, 0, 1, 0, \cdots, 0)$, $(11, 3, 0, 1, 0, \cdots, 0)$, $(10, 4, 1, 0, \cdots, 0)$ or $(9, 6, 0, \cdots, 0)$. Actually, $(m_0, \cdots, m_6, \delta_4, \delta_6)=(9, 6, 0, \cdots, 0)$ by the above equality (1a), and $\rho(V)=\rho(\bar{V})+\#(D)=29$. This is the case given in the assertion (1).

(2) Assume $I=5$. Then $\rho(\bar{V})-c+2\leqq 21/I<5$ and $\rho(\bar{V})\leqq c+2$. Moreover, if $\rho(\bar{V})=c+2$, then $0<\rho(U)-\rho(\bar{U})=24-c-I(\rho(\bar{V})-c+2)=4-c\leqq 2$ and $\operatorname{Sing}\bar{U}=A_2$, $2A_1$ or A_1. On the other hand, if $c\geqq 16$ then $\rho(U)--\rho(\bar{U})=24-c-I(\rho(\bar{V})-c+2)\leqq 24-16-5=3$ and $\operatorname{Sing}\bar{U}=A_3$, A_1+A_2, $3A_1$, A_2, $2A_1$ or A_1. Therefore, in order to prove the assertion (2), we may assume that $\operatorname{Sing}\bar{U}=\sum_{i=1}^{3} m_iA_i$. Then D consists of c rods $B_d\ (1\leqq d\leqq n_1)$, $C_e\ (n_1+1\leqq e\leqq n_1+n_2=m_0=c-\sum_{i=1}^{3} m_i)$, $D_f\ (m_0+1\leqq f\leqq m_0+n_3)$, $E_g\ (m_0+n_3+1\leqq g\leqq m_0+n_3+n_4=m_0+m_1)$, $F_h\ (m_0+m_1+1\leqq h\leqq m_0+m_1+n_5)$, $G_i\ (m_0+m_1+n_5+1\leqq i\leqq m_0+m_1+n_5+n_6=m_0+m_1+m_2)$, $H_j\ (m_0+m_1+m_2+1\leqq j\leqq m_0+m_1+m_2+n_7)$ and $J_p\ (m_0+m_1+m_2+n_7+1\leqq p\leqq m_0+m_1+m_2+n_7+n_8=m_0+\cdots+m_3)$ which are defined as follows:

(i) B_d is a (-5)-curve,

(ii) C_e consists of one (-2)-curve C_{1e} and one (-3)-curve C_{2e},

(iii) D_f is a (-10)-curve,

(iv) E_g consists of two (-2)-curves E_{1g}, E_{2g} and one (-4)-curve E_{3g} with $(E_{bg}, E_{b+1,g})=1\ (b=1, 2)$,

(v) F_h consists of one (-2)-curve F_{1h} and one (-8)-curve F_{2h},

(vi) G_i consists of four (-2)-curves G_{1i}, G_{2i}, G_{4i}, G_{5i} and one (-3)-curve G_{3i} with $(G_{bi}, G_{b+1,i})=1\ (1\leqq b\leqq 4)$,

(vii) H_j consists of one (-3)-curve H_{1j} and one (-7)-curve H_{2j},

(viii) J_p consists of two (-2)-curves J_{1p}, J_{3p} and one (-6)-curve J_{2p} with $(J_{bp}, J_{b+1,p})=1$ $(b=1, 2)$.

Then $D^{\#}=(3/5)\sum B_d+(1/5)\sum(C_{1e}+2C_{2e})+(4/5)\sum D_f+(1/5)\sum(E_{1g}+2E_{2g}+3E_{3g})+(2/5)\sum(F_{1h}+2F_{2h})+(1/5)\sum(G_{1i}+2G_{2i}+3G_{3i}+2G_{4i}+G_{5i})+(1/5)\sum(3H_{1j}+4H_{2j})+(2/5)\sum(J_{1p}+2J_{2p}+J_{3p})$. Hence $-(9n_1+2n_2+32n_3+6n_4+24n_5+3n_6+23n_7+16n_8)/5=(D^{\#})^2=(K_V^2)=10-\rho(V)=10-\rho(\bar{V})-\#(D)=10-\rho(\bar{V})-(n_1+2n_2+n_3+3n_4+2n_5+5n_6+2n_7+3n_8)$. This implies:

(2a) $$5(\rho(\bar{V})-10)-4n_1+8n_2-27n_3+9n_4-14n_5+22n_6-13n_7-n_8=0.$$

On the other hand, by Proposition 1.6 we obtain $3n_1+n_2+8n_3+2n_4+6n_5+n_6+6n_7+4n_8=(D, K_V)<c-(K_V^2)=c+\rho(V)-10=c+\rho(\bar{V})-10+\#(D)=c+\rho(\bar{V})-10+(n_1+2n_2+n_3+3n_4+2n_5+5n_6+2n_7+3n_8)$. This implies:

(2b) $$c+\rho(\bar{V})-10>2n_1-n_2+7n_3-n_4+4n_5-4n_6+4n_7+n_8.$$

Assume $\rho(\bar{V})=c+2$. Then $\rho(U)-\rho(\bar{U})=4-c$ and $(c, \rho(U)-\rho(\bar{U}))=(2, 2)$ or $(3, 1)$. Consider the case $(c, \rho(U)-\rho(\bar{U}))=(2, 2)$. Then $\rho(\bar{V})=4$ and $\mathrm{Sing}\,\bar{U}=A_2$ or $2A_1$. Suppose $\mathrm{Sing}\,\bar{U}=2A_1$. Then $n_3+n_4=2$ and $n_i=0$ $(i\neq 3, 4)$. On the other hand, by the above equality (2a), we have $0=-30-27n_3+9n_4$. This leads to $9|30$, a contradiction. Hence $\mathrm{Sing}\,\bar{U}=A_2$. Then $n_1+n_2=n_5+n_6=1$ and $n_i=0$ $(i\neq 1, 2, 5, 6)$. By (2a), we have $0=-30-4n_1+8n_2-14n_5+22n_6=-48+12n_2+36n_6$, i. e., $n_2+3n_6=4$. Therefore, $n_2=n_6=1$, $(n_1, \cdots, n_8)=(0, 1, 0, 0, 0, 1, 0, 0)$ and $\rho(V)=\rho(\bar{V})+\#(D)=11$. This is one of the cases given in the assertion (2).

Consider the case $(c, \rho(U)-\rho(\bar{U}))=(3, 1)$. Then $\rho(\bar{V})=5$ and $\mathrm{Sing}\,\bar{U}=A_1$. Hence $n_1+n_2=2$, $n_3+n_4=1$ and $n_i=0$ $(i\geqq 5)$. By the above equality (2a), we have $0=-25-4n_1+8n_2-27n_3+9n_4=-24+12n_2-36n_3$, i. e., $n_2=2+3n_3\geqq 2$. So, $n_2=2$, $(n_1, \cdots, n_8)=(0, 2, 0, 1, 0, \cdots, 0)$ and $\rho(V)=\rho(\bar{V})+\#(D)=12$. This is one of the cases given in the assertion (2).

Now we shall prove $c\leqq 16$. Note that $c=24-(\rho(U)-\rho(\bar{U}))-I(\rho(\bar{V})-c+2)\leqq 24-1-5=18$.

Assume $c=18$. Then $1\leqq\rho(U)-\rho(\bar{U})=24-c-I(\rho(\bar{V})-c+2)=6-5(\rho(\bar{V})-16)\leqq 1$. Hence $\rho(\bar{V})=17$ and $\mathrm{Sing}\,\bar{U}=A_1$. So, $n_1+n_2=17$, $n_3+n_4=1$ and $n_i=0$ $(i\geqq 5)$. On the other hand, by the above equality (2a), we have $0=35-4n_1+8n_2-27n_3+9n_4=-24+12n_2-36n_3$, i. e., $n_2=2+3n_3$. Hence $(n_1, \cdots, n_8)=(12, 5, 1, 0, \cdots, 0)$ or $(15, 2, 0, 1, 0, \cdots, 0)$, either case contradicting the above inequality (2b).

Assume $c=17$. Then $0<\rho(U)-\rho(\bar{U})=24-c-I(\rho(\bar{V})-c+2)=7-5(\rho(\bar{V})-15)\leqq 2$. Hence $\rho(\bar{V})=16$ and $\mathrm{Sing}\,\bar{U}=A_2$ or $2A_1$. Consider the case where $\mathrm{Sing}\,\bar{U}=A_2$. Then $n_1+n_2=16$, $n_5+n_6=1$ and $n_i=0$ $(i\neq 1, 2, 5, 6)$. On the other hand, by the above equality (2a), we have $0=30-4n_1+8n_2-14n_5+22n_6=-48+12n_2+36n_6$, i. e., $n_2+3n_6=4$. Hence $(n_1, \cdots, n_8)=(12, 4, 0, 0, 1, 0, 0, 0)$ or $(15, 1, 0, 0, 0, 1, 0, 0)$, either case contradicting the above inequality (2b). Consider the case where $\mathrm{Sing}\,\bar{U}=2A_1$. Then $n_1+n_2=15$, $n_3+n_4=2$ and $n_i=0$ $(i\geqq 5)$. By (2a), we have $0=30-4n_1+8n_2-27n_3+9n_4=-12+12n_2-36n_3$, i. e., $n_2=1+3n_3$. Hence $(n_1, \cdots, n_8)=(8, 7, 2, 0, \cdots, 0)$, $(11, 4, 1, 1, 0, \cdots, 0)$ or $(14, 1, 0, 2, 0, \cdots, 0)$, either case contradicting the above inequality (2b).

So, we have proved $c\leqq 16$. We now assume $c=16$. Then $0<\rho(U)-\rho(\bar{U})=24-c-$

$I(\rho(\bar{V})-c+2)=8-5(\rho(\bar{V})-14)\leqq 3$. Hence $\rho(\bar{V})=15$ and Sing $\bar{U}=A_3$, A_1+A_2 or $3A_1$. Consider the case where Sing $\bar{U}=A_3$. Then $n_1+n_2=15$, $n_7+n_8=1$ and $n_i=0\ (i\neq 1, 2, 7, 8)$. By the above equality (2a), we have $0=25-4n_1+8n_2-13n_7-n_8=-36+12n_2-12n_7$, i. e., $n_2=3+n_7$. Hence $(n_1, \cdots, n_8)=(11, 4, 0, \cdots, 0, 1, 0)$ or $(12, 3, 0, \cdots, 0, 1)$, either case contradicting the above inequality (2b). Consider the case where Sing $\bar{U}=A_1+A_2$. Then $n_1+n_2=14$, $n_3+n_4=n_5+n_6=1$ and $n_i=0\ (i\geqq 7)$. By (2a), we have $0=25-4n_1+8n_2-27n_3+9n_4-14n_5+22n_6=-36+12n_2-36n_3+36n_6$, i. e., $n_2-3n_3+3n_6=3$. On the other hand, by (2b), we have $21>2n_1-n_2+7n_3-n_4+4n_5-4n_6=31-3n_2+8n_3-8n_6=31-3(n_2-3n_3+3n_6)-n_3+n_6=22-n_3+n_6$. Hence $n_3>1+n_6\geqq 1$. This contradicts $n_3+n_4=1$. Therefore we must have Sing $\bar{U}=3A_1$. Then $n_1+n_2=13$, $n_3+n_4=3$ and $n_i=0\ (i\geqq 5)$. By (2a), we have $0=25-4n_1+8n_2-27n_3+9n_4=12n_2-36n_3$, i. e., $n_2=3n_3$. Hence $(n_1, \cdots, n_8)=(4, 9, 3, 0, \cdots, 0)$, $(7, 6, 2, 1, 0, \cdots, 0)$, $(10, 3, 1, 2, 0, \cdots, 0)$ or $(13, 0, 0, 3, 0, \cdots, 0)$. By using the above inequality (2b), we must have $(n_1, \cdots, n_8)=(4, 9, 3, 0, \cdots, 0)$ and $\rho(V)=\rho(\bar{V})+\#(D)=40$. This is the case given in the assertion (2).

(3) Assume $I=7$. Then $\rho(\bar{V})-c+2\leqq 21/I=3$ and $\rho(\bar{V})\leqq c+1$. Moreover, if $\rho(\bar{V})=c+1$, then $1\leqq\rho(U)-\rho(\bar{U})=24-c-I(\rho(\bar{V})-c+2)=3-c\leqq 1$. Hence $c=2$, $\rho(\bar{V})=3$ and Sing $\bar{U}=A_1$. On the other hand, if $c\geqq 15$, then $0<\rho(U)-\rho(\bar{U})=24-c-I(\rho(\bar{V})-c+2)\leqq 24-15-7=2$ and Sing $\bar{U}=A_2$, $2A_1$ or A_1. Therefore, in order to prove the assertion (3), we may assume that Sing $\bar{U}=\sum_{i=1}^{2} m_iA_i$. Then D consists of c rods $B_d\ (1\leqq d\leqq n_1)$, $C_e\ (n_1+1\leqq e\leqq n_1+n_2)$, $D_f\ (n_1+n_2+1\leqq f\leqq n_1+n_2+n_3=m_0=c-m_1-m_2)$, $E_g\ (m_0+1\leqq g\leqq m_0+n_4)$, $F_h\ (m_0+n_4+1\leqq h\leqq m_0+n_4+n_5)$, $G_i\ (m_0+n_4+n_5+1\leqq i\leqq m_0+n_4+n_5+n_6=m_0+m_1)$, $H_j\ (m_0+m_1+1\leqq j\leqq m_0+m_1+n_7)$, $J_p\ (m_0+m_1+n_7+1\leqq p\leqq m_0+m_1+n_7+n_8)$ and $L_q\ (m_0+m_1+n_7+n_8+1\leqq q\leqq m_0+m_1+n_7+n_8+n_9=m_0+m_1+m_2)$ which are defined as follows:

(i) B_d is *a* (-7)-curve,

(ii) C_e consists of one (-2)-curve C_{1e} and one (-4)-curve C_{2e},

(iii) D_f consists of two (-2)-curves D_{1f}, D_{2f} and one (-3)-curve D_{3f} with $(D_{bf}, D_{b+1,f})=1\ (b=1, 2)$,

(iv) E_g is a (-14)-curve,

(v) F_h consists of one (-3)-curve F_{1h} and one (-5)-curve F_{2h},

(vi) G_i consists of four (-2)-curves G_{1i}, G_{2i}, G_{3i}, G_{5i} and one (-3)-curve G_{4i} with $(G_{bi}, G_{b+1,i})=1\ (1\leqq b\leqq 4)$,

(vii) H_j consists of one (-2)-curve H_{1j} and one (-11)-curve H_{2j},

(viii) J_p consists of four (-2)-curves $J_{1p}, \cdots, J_{4p}$ and one (-5)-curve J_{5p} with $(J_{bp}, J_{b+1,p})=1\ (1\leqq b\leqq 4)$,

(ix) L_q consists of three (-3)-curves L_{1q}, L_{2q}, L_{3q} with $(L_{bq}, L_{b+1,q})=1\ (b=1, 2)$.

Then $D^{\#}=(5/7)\sum B_d+(2/7)\sum(C_{1e}+2C_{2e})+(1/7)\sum(D_{1f}+2D_{2f}+3D_{3f})+(6/7)\sum E_g+(1/7)\sum(4F_{1h}+5F_{2h})+(1/7)\sum(G_{1i}+2G_{2i}+3G_{3i}+4G_{4i}+2G_{5i})+(3/7)\sum(H_{1j}+2H_{2j})+(1/7)\sum(J_{1p}+2J_{2p}+3J_{3p}+4J_{4p}+5J_{5p})+(1/7)\sum(4L_{1q}+5L_{2q}+4L_{3q})$. Hence $-(25n_1+8n_2+3n_3+72n_4+19n_5+4n_6+54n_7+15n_8+13n_9)/7=(D^{\#})^2=(K_V^2)=10-\rho(V)=10-\rho(\bar{V})-\#(D)=10-\rho(\bar{V})-(n_1+2n_2+3n_3+n_4+2n_5+5n_6+2n_7+5n_8+3n_9)$. This implies:

$$7(\rho(\bar{V})-10)-18n_1+6n_2+18n_3-65n_4-5n_5+31n_6-40n_7+20n_8+8n_9=0. \tag{3a}$$

On the other hand, by Proposition 1.6 we obtain $5n_1+2n_2+n_3+12n_4+4n_5+n_6+9n_7+$

$3n_8+3n_9=(D, K_V)<c-(K_{\bar{V}}^2)=c+\rho(V)-10=c+\rho(\bar{V})-10+\#(D)=c+\rho(\bar{V})-10+(n_1+2n_2+3n_3+n_4+2n_5+5n_6+2n_7+5n_8+3n_9)$. This implies:

$$\text{(3b)} \qquad c+\rho(\bar{V})-10>4n_1-2n_3+11n_4+2n_5-4n_6+7n_7-2n_8.$$

Assume $\rho(\bar{V})=c+1$. Then $c=2$, $\rho(\bar{V})=3$ and $\mathrm{Sing}\,\bar{U}=A_1$. Hence $n_1+n_2+n_3=n_4+n_5+n_6=1$ and $n_i=0\,(i\geqq 7)$. On the other hand, by the above equality (3a), we have $0=-49-18n_1+6n_2+18n_3-65n_4-5n_5+31n_6=-48-24n_1+12n_3-60n_4+36n_6$, i. e., $4\geqq n_3+3n_6=4+2n_1+5n_4\geqq 4$. Thus, we must have $(n_1, \cdots, n_9)=(0, 0, 1, 0, 0, 1, 0, 0, 0)$ and $\rho(V)=\rho(\bar{V})+\#(D)=11$. This is the case given in the assertion (3).

Now we shall prove $c\leqq 15$. Note that $c=24-(\rho(U)-\rho(\bar{U}))-I(\rho(\bar{V})-c+2)\leqq 24-1-7=16$.

Assume $c=16$. Then $1\leqq\rho(U)-\rho(\bar{U})=24-c-I(\rho(\bar{V})-c+2)=8-7(\rho(\bar{V})-14)\leqq 1$. Hence $\rho(\bar{V})=15$ and $\mathrm{Sing}\,\bar{U}=A_1$. So, $n_1+n_2+n_3=15$, $n_4+n_5+n_6=1$ and $n_i=0\,(i\geqq 7)$. Using the above equality (3a), we obtain $0=35-18n_1+6n_2+18n_3-65n_4-5n_5+31n_6=120-24n_1+12n_3-60n_4+36n_6$, i. e., $2n_1-n_3=10-5n_4+3n_6$. On the other hand, by the above inequality (3b), we have $21>4n_1-2n_3+11n_4+2n_5-4n_6=2(10-5n_4+3n_6)+11n_4+2n_5-4n_6=20+n_4+2n_5+2n_6=21+n_5+n_6\geqq 21$. This is absurd.

Assume $c=15$. Then $0<\rho(U)-\rho(\bar{U})=24-c-I(\rho(\bar{V})-c+2)=9-7(\rho(\bar{V})-13)\leqq 2$. Hence $\rho(\bar{V})=14$ and $\mathrm{Sing}\,\bar{U}=A_2$ or $2A_1$. Consider the case where $\mathrm{Sing}\,\bar{U}=A_2$. Then $n_1+n_2+n_3=14$, $n_7+n_8+n_9=1$ and $n_i=0\,(i\neq 1, 2, 3, 7, 8, 9)$. Using the above equality (3a), we obtain, $0=28-18n_1+6n_2+18n_3-40n_7+20n_8+8n_9=120-24n_1+12n_3-48n_7+12n_8$, i. e., $2n_1-n_3+4n_7-n_8=10$. On the other hand, by the above inequality (3b), we have $19>4n_1-2n_3+7n_7-2n_8=2(2n_1-n_3+4n_7-n_8)-n_7=20-n_7\geqq 19$. This is absurd. So, we must have $\mathrm{Sing}\,\bar{U}=2A_1$. Then $n_1+n_2+n_3=13$, $n_4+n_5+n_6=2$ and $n_i=0\,(i\geqq 7)$. By virtue of (3a), we obtain $0=28-18n_1+6n_2+18n_3-65n_4-5n_5+31n_6=96-24n_1+12n_3-60n_4+36n_6$, i. e., $2n_1-n_3+5n_4-3n_6=8$. On the other hand, by virtue of (3b), we have $19>4n_1-2n_3+11n_4+2n_5-4n_6=4+2\,(2n_1-n_3+5n_4-3n_6)-n_4=20-n_4$, i. e., $n_4>1$. Hence $n_4=2$, $n_5=n_6=0$ and $0=2n_1-n_3+5n_4-3n_6-8=2n_1-n_3+2$. So, $(n_1, \cdots, n_9)=(0, 11, 2, 2, 0, \cdots, 0)$, $(1, 8, 4, 2, 0, \cdots, 0)$, $(2, 5, 6, 2, 0, \cdots, 0)$ or $(3, 2, 8, 2, 0, \cdots, 0)$ and $\rho(V)\,(=\rho(\bar{V})+\#(D))=44$, 45, 46 or 47, respectively. They are the cases given in the assertion (3). The last assertion is now verified straightforwardly. Q. E. D.

Remark 6.7. (1) Let (V, D) be a log Enriques surface satisfying $I=3$, $\rho(\bar{V})=c+4=6$, $\mathrm{Sing}\,\bar{U}=D_4$ and $(m_0, m_1, m_2, m_3, \delta_4)=(1, 0, 0, 0, 1)$. Then $D=B_1+\sum_{r=0}^{3}S_{r1}$ with the notations of Proposition 6.6. Denote the intersection point $S_{01}\cap S_{i1}\,(1\leqq i\leqq 3)$ by P_i. Let $\tau\colon W\to V$ be the blowing-up of P_1 (resp. P_1 and P_2, or P_1, P_2 and P_3) and let $\Delta:=\tau'(D)$. Then (W, Δ) is a log Enriques surface satisfying $I=3$ and $\rho(\bar{W})=c+4$, where $\eta\colon W\to\bar{W}$ is the contraction of Δ. Moreover, $\mathrm{Sing}\,\bar{Z}=A_3$ (resp. A_2, or A_1), $c=3$ (resp. 4, or 5) and $(m_0, \cdots, m_3, \delta_4)=(2, 0, 0, 1, 0)$ (resp. $(3, 0, 1, 0, 0)$, or $(4, 1, 0, 0, 0)$), where $\bar{Z}$ is the canonical covering of $\bar{W}$. (See Example 6.8 below).

(2) Let (V, D) be a log Enriques surface satisfying $I=5$, $\rho(\bar{V})=c+2=4$, $\mathrm{Sing}\,\bar{U}=A_2$, $(m_0, m_1, m_2)=(1, 0, 1)$ and $(n_1, \cdots, n_6)=(0, 1, 0, 0, 0, 1)$. Then $D=C_{11}+C_{21}+G_{11}+\cdots+G_{51}$ with the notations of Proposition 6.6. Let $\tau\colon W\to V$ be the blowing-up of the

point $G_{21}\cap G_{31}$ and let $\varDelta:=\tau'(D)$. Then $(W, \varDelta)$ is a log Enriques surface satisfying $I=5$, $\rho(\overline{W})=c+2=5$, $\operatorname{Sing}\overline{Z}=A_1$, $(m_0, m_1, m_2)=(2, 1, 0)$ and $(n_1, \cdots, n_6)=(0, 2, 0, 1, 0, 0)$, where $\xi: W\to\overline{W}$ is the contraction of $\varDelta$ and $\overline{Z}$ is the canonical covering of $\overline{W}$. (See Example 6.9 below).

The following three examples show that the upper bounds of $\rho(\overline{V})-c$ in (1), (2) and (3) of Proposition 6.6 are the best possible ones.

Example 6.8 (for the case $(I, \rho(\overline{V})-c)=(3, 4)$). Let $\pi: \Sigma_0\to\boldsymbol{P}^1$ be a $\boldsymbol{P}^1$-fibration on the Hirzebruch surface Σ_0. Let M and L be a minimal section and a fiber of π, respectively. Take nonsingular members $A\in|2M+L|$ and $C\in|M+2L|$. Denote by $P_1, \cdots, P_5$ all five intersection points of $A\cap C$, where some points of them might be infinitely near to the other. Take a minimal section M_1 of π such that $P_6:=M_1\cap A\neq P_i$ $(1\leqq i\leqq 5)$ and M_1 meets C in two distinct points P_7 and P_8 other than P_i $(1\leqq i\leqq 6)$. Let L_1 and L_2 be the fibers of π containing P_7 and P_8, respectively. Let P_9 and P_{10} (resp. P_{11} and P_{12}) be all the intersection points of $A\cap L_1$ (resp. $A\cap L_2$), where the second point might be infinitely near to the first one. Let $\tau: V\to\Sigma_0$ be the blowing-up of nine points P_i's $(i\neq 5, 6, 12)$. Set $L_j:=\tau'(L_j)$, $M_1':=\tau'(M_1)$, $A':=\tau'(A)$, $C':=\tau'(C)$ and $D:=L_1'+L_2'+M_1'+A'+C'$. Noting that $L_1+L_2+M_1+C+2A\sim-3K_{\Sigma_0}$, we can check that $L_1'+L_2'+M_1'+C'+2A'\sim-3K_V$. Hence (V, D) is a log Enriques surface with $I=3$. Evidently, we have $c=2$, $\rho(V)=11$, $\rho(\overline{V})=6$, $\operatorname{Sing}\overline{U}=D_4$ and $(m_0, \cdots, m_3, \delta_4)=(1, 0, 0, 0, 1)$.

Example 6.9 (for the case $(I, \rho(\overline{V})-c)=(5, 2)$). Let $\pi: \Sigma_0\to\boldsymbol{P}^1$, M and L be the same as in Example 6.8. Take an irreducible rational curve A in $|2M+2L|$ such that the unique singular point P_1 of A is a node. Let P_2 $(\neq P_1)$ be a ramification point of $\pi_{|A}$. Denote by L_i $(i=1, 2)$ the fiber containing P_i. Take a minimal section M_1 of π so that M_1 meets A in two distinct points P_3 and P_4 other than P_i's $(i=1, 2)$. Then the point $P_{i+4}:=M\cap L_i$ $(i=1, 2)$ is different from P_j for each $1\leqq j\leqq 4$. Since $\dim|M+L|=3$, there is an irreducible member C in $|M+L|$ such that $P_2, P_5\in C$. Let P_i $(7\leqq i\leqq 9)$ be the intersection points of $A\cap C$ other than P_2, where some of P_i's might be infinitely near to the other. Let $\tau_1: V_1\to\Sigma_0$ be the blowing-up of seven points P_i's $(i\neq 6, 9)$ and set $E_j:=\tau_1^{-1}(P_j)$ $(j=1, 2)$. Let $\tau_2: V\to V_1$ be the blowing-up of the point $\tau_1'(A)\cap E_2$ and one of two points $\tau_1'(A)\cap E_1$. Set $\tau:=\tau_1\circ\tau_2$, $E_i':=\tau_2'(E_i)$, $L_i':=\tau'(L_i)$, $M_1':=\tau'(M_1)$, $A':=\tau'(A)$, $C':=\tau'(C)$ and $D:=L_2'+M_1'+L_1'+E_1'+A'+C'+E_2'$. Noting that $L_2+2M_1+L_1+3A+2C\sim-5K_{\Sigma_0}$, we can check that $L_2'+2M_1'+L_1'+2E_1'+3A'+2C'+E_2'\sim-5K_V$. Hence (V, D) is a log Enriques surface with $I=5$. Evidently, we have $c=2$, $\rho(V)=11$, $\rho(\overline{V})=4$, $\operatorname{Sing}\overline{U}=A_2$ and $(n_1, \cdots, n_6)=(0, 1, 0, 0, 0, 1)$.

Example 6.10 (for the case $(I, \rho(\overline{V})-c)=(7, 1)$). Let (V, D) be the log Enriques surface given in Example 5.5. Then $\operatorname{Index}(K_{\overline{V}})=7$ and the canonical covering $\overline{U}$ of $\overline{V}$ is a $K3$-surface. Let $\sigma: V\to W$ be the blowing-down of the (-1)-curve $\tau^{-1}(P_6)$ of V where P_6 is defined in Example 5.5. Set $\varDelta:=\sigma(D)$. Then $(W, \varDelta)$ is a log Enriques surface satisfying $I=7$, $c=2$, $\rho(W)=11$, $\rho(\overline{W})=3$, $\operatorname{Sing}\overline{Z}=A_1$ and $(n_1, \cdots, n_6)=(0, 0, 1, 0, 0, 1)$, where $\eta: W\to\overline{W}$ is the contraction of $\varDelta$ and $\overline{Z}$ is the canonical covering of $\overline{W}$.

In view of the following three examples, the upper bounds of c in (1), (2) and (3) of Proposition 6.6 are the best possible ones. In the case of Example 6.11, there is a reduced effective divisor G on U with only simple normal crossings such that G consists of (-2)-curves and its dual graph Dual (G) is as given in Figure (7). Several subgraphs of Dual (G) of Dynkin type $A_r+D_s+E_t$ with $r+s+t=19$ are obtainable. In particular, there is a subgraph Γ of Dynkin type D_{19}. Hence U is a $K3$-surface with $\rho(U)=20$.

We shall use the same notations $\pi: \Sigma_2 \to \boldsymbol{P}^1$, M, A, P_1, P_2, L_1 and L_2 as defined before Example 5.4.

Example 6.11 (for the case $(c, I)=(15, 3)$). Let P_3 $(\neq P_1, P_2)$ be a ramification point of $\pi_{|A}$ and let L_3 be the fiber of π containing P_3. Let $\tau_1: V_1 \to \Sigma_2$ be the blowing-up of three points P_i's $(1 \leqq i \leqq 3)$ and let $E_i := \tau_1^{-1}(P_i)$. Let $\tau_2: V_2 \to V_1$ be the blowing-up of three points $P_4 :=$ one of two intersection points $\tau_1'(A) \cap E_1$, $P_5 := \tau_1'(A) \cap E_2$ and $P_6 := \tau_1'(A) \cap E_3$, and set $F_1 := \tau_1^{-1}(P_4)$, $E_{i-1} := \tau_2^{-1}(P_i)$ $(i=5, 6)$. Let $\tau_3: V_3 \to V_2$ be the blowing-up of two points $P_7 := \tau_2'\tau_1'(A) \cap E_4$ and $P_8 := \tau_2'(E_3) \cap E_5$, and set $E_6 := \tau_3^{-1}(P_7)$ and $F_2 := \tau_3^{-1}(P_8)$. Let $\tau_4: V' \to V_3$ be the blowing-up of the point $P_9 := \tau_3'\tau_2'\tau_1'(A) \cap E_6$, and set $F_3 := \tau_4^{-1}(P_9)$. Denote by E_i', F_j', $L_k'(k=2, 3)$, M' and A' the proper transforms on V' of E_i, F_j, L_k, M and A, respectively. Set $\tau := \tau_1 \circ \tau_2 \circ \tau_3 \circ \tau_4$, $F_4' := \tau'(L_1)$ and $D' := \sum E_i' + \sum L_k' + M' + A'$. Note that F_p' $(1 \leqq p \leqq 4)$ is a (-1)-curve of V'. Noting that $2L_2+2L_3+2M+2A \sim -3K_{\Sigma_2}$, we can check that $E_3'+E_1'+2(A'+E_5'+L_3'+M'+L_2'+E_4')+E_2'+E_6' \sim -3K_{V'}$. Hence (V', D') is a log Enriques surface with $(c, I)=(2, 3)$. $D'+\sum F_p'$ has only simple normal crossings and has the dual graph as shown in Figure (5), where the self-intersection number of each irreducible component of D' is attached. Here recall the Remark to Proposition 6.1 and note that $F_1'+E_1'+F_4'=\tau^{-1}(L_1)$.

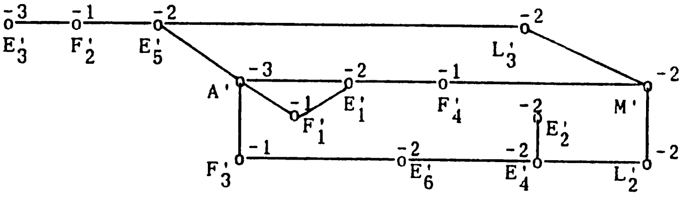


Figure (5)

We can find a blowing-up $\sigma: V \to V'$ of several singular points of $\Delta' := E_1'+A'+E_5'+L_3'+M'+L_2'+E_4'+E_2'+E_6'$ in such a way that the dual graph of $\sigma^{-1}(\Delta')$ is given in Figure (6), where $\tilde{E}_i := \sigma'(E_i')$, $\tilde{L}_k := \sigma'(L_k')$, $\tilde{M} := \sigma'(M')$ and $\tilde{A} := \sigma'(A')$.

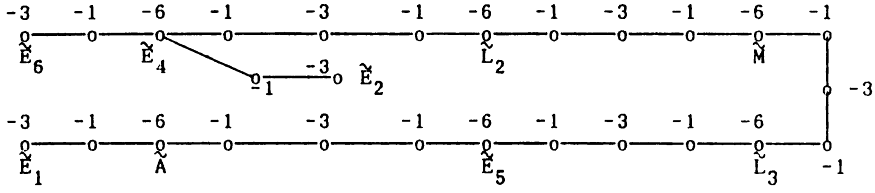


Figure (6)

Denote by $D := \sigma^{-1}(D') - \{(-1)\text{-curve of } V \text{ contained in } \sigma^{-1}(D')\}$. Then (V, D) is a log Enriques surface satisfying $I=3$, $c=15$, $\rho(V)=29$, $\rho(\bar{V})=14$, $\mathrm{Sing}\,\bar{U}=6A_1$ and $(m_0, m_1)=(9, 6)$. Since $20 \geqq \rho(U)=\rho(\bar{U})+\#\{\text{irreducible component of } g^{-1}(\mathrm{Sing}\,\bar{U})\}=\rho(\bar{U})+6 \geqq \rho(\bar{V})+6=20$, we have $\rho(U)=20$ and $\rho(\bar{U})=14$. We use the notation $\hat{\pi}: \hat{U} \to V$ defined at the beginning of §2. Let $\eta: \tilde{U} \to \hat{U}$ be a minimal desingularization. Then there is a birational morphism $\xi: \tilde{U} \to U$ whose exceptional curves are contained in $(\hat{\pi} \circ \eta)^{-1}(D)$. Denote by $\tilde{F}_p$ and Γ the reduced total transforms on U of $\sigma'(F_p')$ and $\sigma^{-1}(D')$, respectively. Then $\tilde{F}_p$ is a (-2)-curve and Γ is a (-2)-fork of Dynkin type D_{19}. Set $H_1 := \tilde{F}_1$. Then we can write $\Gamma = H_2 + \sum_{i=2}^{19} C_i$ so that $G_k := \Gamma + \sum_{j=1}^{4} \tilde{F}_p - H_k$ $(k=1, 2)$ has only simple normal crossings and has the dual graph as shown in Figure (7). Moreover, $(H_1, H_2)=1$ and H_1 passes the intersection point $H_2 \cap \tilde{F}_4$.

Let $\varphi: U \to \bar{U}'$ be the contraction of Γ. Then $\bar{U}'$ is the canonical covering of $\bar{V}'$ and $\mathrm{Sing}\,\bar{U}'=D_{19}$, where $\bar{V}'$ is obtained from V' by the contraction of D'.

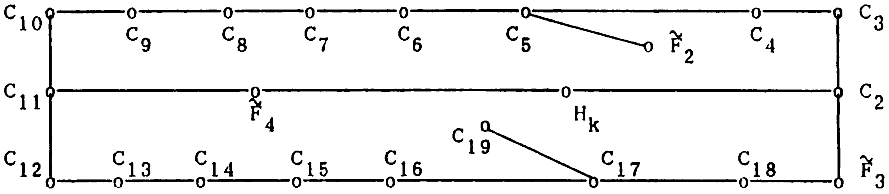


Figure (7)

Example 6.12 (for the case $(c, I)=(16, 5)$). Let P_3 $(\neq P_1, P_2)$ be a ramification point of $\pi_{|A}$ and let L_3 be the fiber of π containing P_3. Denote by P_4 the intersection point $M \cap L_1$. Let $\tau_1: V_1 \to \Sigma_2$ be the blowing-up of four points P_i's and set $E_i := \tau_1^{-1}(P_i)$ $(1 \leqq i \leqq 3)$ and $F_1 := \tau_1^{-1}(P_4)$. Let $\tau_2: V_2 \to V_1$ be the blowing-up of three points $P_5 :=$ one of two intersection points $\tau_1'(A) \cap E_1$, $P_6 := \tau_1'(A) \cap E_2$ and $P_7 := \tau_1'(A) \cap E_3$ and set $E_{j-2} := \tau_2^{-1}(P_j)$ $(j=6, 7)$. Let $\tau_3: V_3 \to V_2$ be the blowing-up of two points $P_8 := \tau_2'\tau_1'(A) \cap E_4$ and $P_9 := \tau_2'(E_3) \cap E_5$, and set $E_6 := \tau_3^{-1}(P_9)$. Let $\tau_4: V' \to V_3$ be the blowing-up of the point $P_{10} := \tau_3'(E_5) \cap E_6$ and set $F_2 := \tau_4^{-1}(P_{10})$. Denote by E_i', F_j', L_k', A' and M' the proper transforms on V' of E_i, F_j, L_k, A and M, respectively. Set $\tau := \tau_1 \circ \tau_2 \circ \tau_3 \circ \tau_4$ and $D' := \sum E_i' + \sum L_k' + A' + M'$. Noting that $L_1 + 3A + 4L_3 + 4M + 3L_2 \sim -5K_{\Sigma_2}$, we can check that $L_1' + 2E_1' + 3A' + 4E_5' + 4L_3' + 4M' + 3L_2' + 2E_4' + E_2' + E_6' + 2E_3' \sim -5K_{V'}$. Hence (V', D') is a log Enriques surface with $(c, I)=(2, 5)$.

Since $\dim|M+2L|=3$, we can find an irreducible member F_3 in $|M+2L|$ such that $P_1, P_5, P_2 \in F_3$, where P_5 is an infinitely near point of P_1 as defined above. Then $F_3' := \tau'(F_3)$ is a (-1)-curve satisfying $(F_3', \tau_4'\tau_3'\tau_2^{-1}(P_5))=(F_3', E_2')=(F_3', L_3')=1$. Then

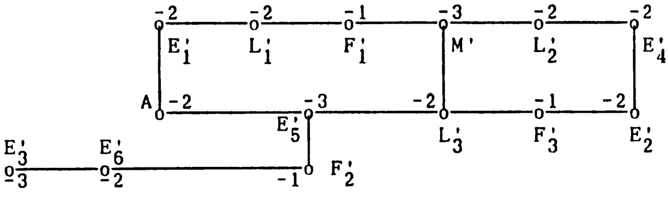


Figure (8)

$D'+\sum_{p=1}^{3}F_p'$ has only simple normal crossings and has the dual graph as shown in Figure (8), where the self-intersection number of each irreducible component of D' is attached and where $E_3'+E_6'+F_2'+E_5'+L_3'=\tau^{-1}(L_3)$.

We can find a blowing-up σ: $V\to V'$ of several singular points of $\Delta':=L_1'+E_1'+A'+E_5'+L_3'+M'+L_2'+E_4'+E_2'$ such that the dual graph of $\sigma^{-1}(\Delta')$ is as given in Figure (9), where the proper transforms of E_i', L_k', A' and M' on V are denoted by $\tilde{E}_i$, $\tilde{L}_k$, $\tilde{A}$ and $\tilde{M}$, respectively.

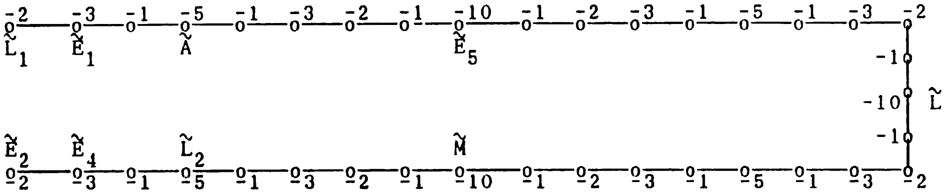


Figure (9)

Let $D:=\sigma^{-1}(D')-\{(-1)\text{-curve of } V \text{ contained in } \sigma^{-1}(D')\}$. Then (V, D) is a log Enriques surface satisfying $I=5$, $c=16$, $\rho(V)=40$, $\rho(\bar{V})=15$, $\mathrm{Sing}\,\bar{U}=3A_1$ and $(n_1, \cdots, n_4)=(4, 9, 3, 0)$. We use the same notations $\hat{\pi}$: $\hat{U}\to V$, η: $\tilde{U}\to\hat{U}$ and ξ: $\tilde{U}\to U$ as in Example 6.11. Denote by $\tilde{F}_p$ and Γ the reduced total transforms on U of $\sigma'(F_p')$ and $\sigma^{-1}(D')$, respectively. Then $\tilde{F}_p$ is a (-2)-curve and Γ is a (-2)-rod of Dynkin type A_{17}. The canonical covering $\bar{U}'$ of (V', D') is obtained from U by contracting Γ. Moreover, $\Gamma+\sum\tilde{F}_p$ has only simple normal crossings and has the dual graph as shown in Figure (10), where $\Gamma=\sum_{i=1}^{17}C_i$ and $C_{17+i}:=\tilde{F}_i$ $(1\leqq i\leqq 3)$.

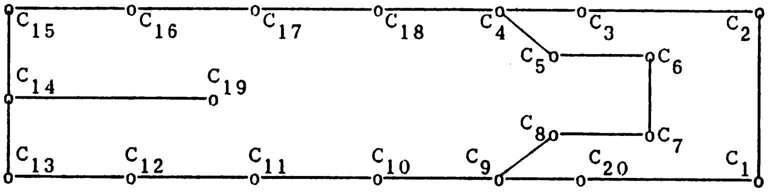


Figure (10)

Example 6.13 (for the case $(c, I)=(15, 7)$). Let τ_1: $V_1\to\Sigma_2$ be the blowing-up of two points P_i's $(i=1, 2)$ and set $E_i:=\tau_1^{-1}(P_i)$. Let τ_2: $V_2\to V_1$ be the blowing-up of two points $P_3:=$ one of two intersection points of $\tau_1'(A)\cap E_1$ and $P_4:=\tau_1'(A)\cap E_2$, and set $E_i:=\tau_2^{-1}(P_i)$. Let τ_3: $V_3\to V_2$ be the blowing-up of two points $P_5:=\tau_2'\tau_1'(A)\cap E_3$ and $P_6:=\tau_2'(E_2)\cap E_4$ and set $E_i:=\tau_3^{-1}(P_i)$. Let τ_4: $V_4\to V_3$ be the blowing-up of two points $P_7:=\tau_3'\tau_2'\tau_1'(A)\cap E_5$ and $P_8:=\tau_3'(E_4)\cap E_6$ and set $E_i:=\tau_4^{-1}(P_i)$. Let τ_5: $V_5\to V_4$ be the blowing-up of two points $P_9:=(\tau_1\cdots\tau_4)'(A)\cap E_7$ and $P_{10}:=\tau_4'\tau_3'(E_4)\cap E_8$ and set $E_9:=\tau_5^{-1}(P_9)$ and $F_1:=\tau_5^{-1}(P_{10})$. Let τ_6: $V'\to V_5$ be the blowing-up of the point $P_{11}:=(\tau_1\cdots\tau_5)'(A)\cap E_9$ and set $F_2:=\tau_6^{-1}(P_{11})$. Denote the proper transforms on V' of E_i, F_j, M, L_2 and A by E_i', F_j', M', L_2' and A', respectively. Set $\tau:=\tau_1\cdots\tau_6$ and $D':=\sum E_i'+M'+L_2'+A'$. Noting that $2M+4L_2+6A\sim-7K_{\Sigma_2}$, we can check that $2M'+4L_2'+$

$6E_4'+6A'+5E_1'+4E_3'+3E_5'+2E_7'+E_9'+E_8'+2E_6'+3E_2'\sim -7K_{V'}$. Hence (V', D') is a log Enriques surface with $(c, I)=(2, 7)$. The dual graph of $D'+F_1'+F_2'$ is as given in Figure (11), where the self-intersection number of each irreducible component of $D'+F_1'+F_2'$ is attached and where $E_2'+E_6'+E_8'+F_1'+E_4'+L_2'=\tau^{-1}(L_2)$.

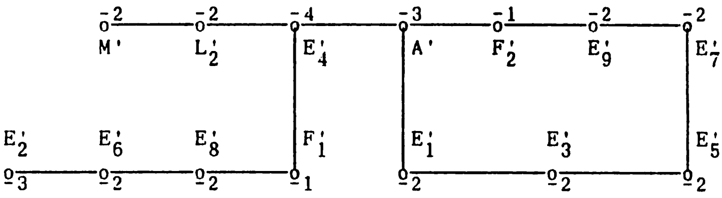


Figure (11)

We can find a blowing-up $\sigma: V\to V'$ of several singular points of $\Delta':=M'+L_2'+E_4'+A'+E_1'+E_3'+E_5'+E_7'+E_9'$ such that the dual graph of $\sigma^{-1}(\Delta')$ is as given in Figure (12), where the proper transforms of E_i', M', L_2' and A' are denoted by $\tilde{E}_i$, $\tilde{M}$, $\tilde{L}_2$ and $\tilde{A}$, respectively.

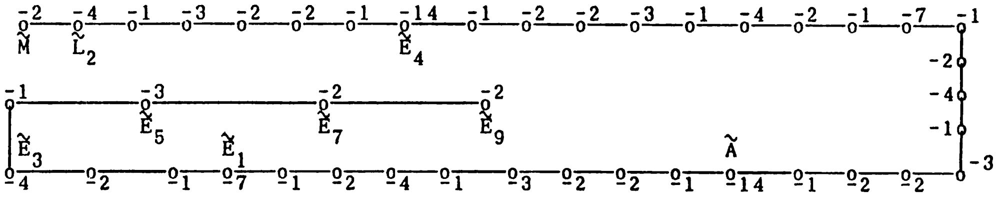


Figure (12)

Let $D:=\sigma^{-1}(D')-\{(-1)\text{-curve of } V \text{ contained in } \sigma^{-1}(D')\}$. Then (V, D) is a log Enriques surface satisfying $I=7$, $c=15$, $\rho(V)=46$, $\rho(\bar{V})=14$, $\mathrm{Sing}\,\bar{U}=2A_1$ and $(n_1, \cdots, n_6)=(2, 5, 6, 2, 0, 0)$.

Let $\tilde{F}_j$ and Γ be the reduced total transforms on U of $\sigma'(F_j')$ and $\sigma^{-1}(D')$, respectively. Then $\tilde{F}_j$ is a (-2)-curve and Γ is a (-2)-rod of Dynkin type A_{15}. The canonical covering $\bar{U}'$ of (V', D') is obtained from U by contracting Γ. Moreover, $\Gamma+\sum\tilde{F}_j$ has only simple normal crossings and has the dual graph as shown in Figure (13).

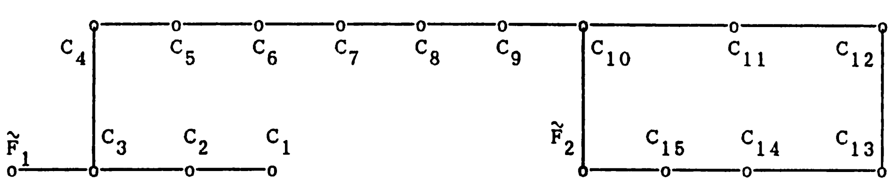


Figure (13)

Let (V', D') be one of the log Enriques surfaces given in Examples 5.7, 5.8, 6.11, 6.12 and 6.13. Let $f': V'\to\bar{V}'$ be the contraction of D'. Then we see that $\#(\mathrm{Sing}\,\bar{V}')=2$ and $\rho(\bar{V}')=1$. Hence the lower bound -1 for $\rho(\bar{V})-c$ in Proposition 6.6 is the best possible one.

The following lemma gives an upper bound for $\#(\mathrm{Sing}\,\bar{U})$.

Lemma 6.14. *Let $\overline{V}$ be a log Enriques surface. Then* $\#(\operatorname{Sing}\overline{U})\leqq\operatorname{Min}\{10, (24-p)/2\}$ *for every prime divisor p of I.*

Proof. It suffices to consider the case where $\operatorname{Sing}\overline{U}\neq\emptyset$. In this case, if $g: U\rightarrow\overline{U}$ is a minimal desingularization then U is a $K3$-surface. In view of Lemma 2.2, we may assume that $I=p$ which is a prime number. For each $x\in\operatorname{Sing}\overline{U}$, we have $\pi(x)\in\operatorname{Sing}\overline{V}$ and $\pi^{-1}\pi(x)=x$. Hence, $\#(\operatorname{Sing}\overline{U})\leqq c$. Note that $\rho(U)-\rho(\overline{U})$ is the number of all irreducible components of exceptional divisors of g, which is apparently not less than $\#(\operatorname{Sing}\overline{U})$. So, we have $\#(\operatorname{Sing}\overline{U})\leqq\operatorname{Min}\{c, \rho(U)-\rho(\overline{U})\}\leqq[c+\rho(U)-\rho(\overline{U})]/2=[24-I(\rho(\overline{V})-c+2)]/2\leqq(24-I)/2$ by Lemma 2.4. This, together with Lemma 3.1, implies Lemma 6.14. Q. E. D.

In the forthcoming article [14], we shall reduce the general cases $\overline{V}$ of log Enriques surfaces to the case $\overline{W}$ with at worst singularities of Dynkin type A_1.

Added in proof:

In the proof of Lemma 2.3, actually, we do not need to use the fact that one of ξ_i's is a primitive I-th root of the unity. We have another elementary proof for Lemma 2.3 as follows. Note that ξ_i is a primitive n_i-th root of the unity for some $n_i\geqq 1$. We may assume that $n_1<\cdots<n_r$ and each $n_j(1\leqq j\leqq h)$ is equal to one of $n_1, \cdots, n_r$. Note that l. c. m. $\{n_1, \cdots, n_r\}=I$. Let $f(T)$ (resp. $g_i(T)$) be the minimal polynomial of A (resp. ξ_i) over $\boldsymbol{Q}$. Then $\deg g_i(T)=\phi(n_i)$. We have also $f(T)=$l. c. m. $\{g_1(T), \cdots, g_h(T)\}=g_1(T)\cdots g_r(T)$. Hence $\phi(I)\leqq\phi(n_1)\cdots\phi(n_r)=\deg f(T)\leqq\dim H=b_2(U)-\rho(U)$.

DEPARTMENT OF MATHEMATICS
OSAKA UNIVERSITY

References

[1] M. Artin, Some numerical criteria for contractibility of curves on algebraic surfaces, Amer. J. Math., **84** (1962), 485-496.

[2] E. Brieskorn, Rationale Singularitäten komploxer Flächen, Invent. Math., **4** (1968), 336-358.

[3] F. Hirzebruch, Über vierdimensionale Riemannsche Flächen mehrdeutiger analytischer Funktionen von zwei komplexen Veränderlichen, Math. Ann., **126** (1953), 1-22.

[4] M. Kato and I. Naruki, On rational double points on quartic surfaces, Preprint of Research Institute for Mathematical Science, Kyoto University, RIMS-390 (1982), 1-85.

[5] Y. Kawamata, Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces, Ann. Math., **127** (1988), 93-163.

[6] M. Miyanishi, Projective degenerations of surfaces according to S. Tsunoda, In: Oda, T. (ed.) Proceedings of Algebraic Geometry, Sendai 1985, Advanced Studies in Pure Mathematics 10 (1987) 415-427, Kinokuniya, Tokyo and North-Holland, Amsterdam.

[7] M. Miyanishi and P. Russell, Purely inseparable coverings of exponent one of the affine plane, J. Pure App. Alg., **28** (1983), 279-317.

[8] M. Miyanishi and S. Tsunoda, Non-complete algebraic surfaces with logarithmic Kodaira dimension $-\infty$ and with non-connected boundaries at infinity, Japan J. Math., **10** (1984), 195-242.

[9] M. Miyanishi and S. Tsunoda, Open algebraic surfaces with Kodaira dimension $-\infty$, Proceedings of Symposia in Pure Mathematics, **46** (1987), 435-450.

[10] S. Tsunoda, Structure of open algebraic surfaces, I, J. Math. Kyoto Univ., **23** (1983), 95-125.

[11] S. Tsunoda, Degeneration of surfaces, In: Oda, T. (ed.) Proceedings of Algebraic Geometry, Sendai 1985, Advanced Studies in Pure Mathematics 10 (1987), 755-764, Kinokuniya, Tokyo and North-Holland, Amsterdam.

[12] D.-Q. Zhang, Logarithmic del Pezzo surfaces of rank one with contractible boundaries, Osaka J. Math., **25** (1988), 461-497.

[13] D.-Q. Zhang, Logarithmic del Pezzo surfaces with rational double and triple singular points, Tohoku Math. J., **41** (1989), 399-452.

[14] D.-Q. Zhang, Logarithmic Enriques surfaces, II, forthcoming.