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Triple coverings of algebraic surfaces according to the Cardano formula

By

Hiro-o TOKUNAGA

§ 0. Introduction

In this article, we consider a triple covering of an algebraic surface. In case of a cyclic covering, that is, its rational function field is obtained by a cyclic extension of degree 3, its structure is well-known. But in case of a non-Galois covering the structure is not well-known. In [6], R. Miranda obtained some results about a non-Galois triple covering by using a rank 2 vector bundle (called the "Tschirnhausen module"). T. Fujita and R. Lazarsfeld proved a beautiful theorem about a non-Galois triple covering over $\boldsymbol{P}^n$ ($n \geqq 4$) (see [3], [5]). In this paper, we study a non-Galois triple covering by using the Cardano formula. An outline of our method is as follows:

Let $p: X \to Y$ be a finite normal triple covering of a normal variety Y. First, we define the discriminant variety $D(X/Y)$ and the minimal splitting variety $\hat{X}$ associated to the triple covering $p: X \to Y$. For these varieties, we have a commutative diagram:

$$\begin{array}{ccccc} & & \hat{X} & & \\ & \swarrow^{\alpha} & & \searrow^{\beta_2} & \\ X & & \downarrow p_1 & & D(X/Y) \\ & \searrow_{p} & & \swarrow_{\beta_1} & \\ & & Y & & \end{array}$$

For details, see § 1 below. To study the triple covering $p: X \to Y$, we study structures of the morphisms $\beta_1: D(X/Y) \to Y$, $\beta_2: \hat{X} \to D(X, Y)$, and $\alpha: \hat{X} \to X$.

Our main results are as follows:

Proposition 3.1. *Let $p: X \to Y$ be a finite totally ramified triple covering of a smooth projective variety Y. Assume that*

(i) *X is smooth,*

(ii) *Y is simply connected.*

Then, p is cyclic, and the branch locus of p is smooth.

Proposition 3.4. *Let $p: S \to \Sigma$ be a finite triple covering where S and Σ are smooth surfaces. Assume that $\Delta(S/\Sigma)$ (the branch locus of p) is an irreducible divisor and has*

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singularities whose local euations are

$$x^2+y^{3k}=0\,,$$

where k is a natural number. (For two different singularities, corresponding k may be different.) Then the structures of $\beta_1: D(S/\Sigma)\to\Sigma$, $\beta_2: \hat{S}\to D(S/\Sigma)$ and $\alpha: \hat{S}\to S$ are as follows:

(i) *$D(S/\Sigma)$ is a normal double covering branched at $\Delta(S/\Sigma)$.*

(ii) *$\hat{S}$ is a nomal cyclic triple covering of $D(S/\Sigma)$ branched only at* $\mathrm{Sing}\,(D(S/\Sigma))$ *and singularities of $\hat{S}$ are of A_{k-1} type.*

(iii) *There exists an involution ι on $\hat{S}$, and we obtain S as quotient surface of $\hat{S}$ by ι.*

The above result is a slight generalization of the result of R. Miranda [6], Lemma 5.9.

Theorem 3.9. *Let $p: S\to\Sigma$ be a finite triple covering where S and Σ are smooth surfaces. Assume*

(i) *the surface $\hat{S}$ is smooth,*

(ii) *Σ is either a minimal rational surface or an abelian surface,*

(iii) *the Kodaira dimension $\kappa(S)$ of S is 2.*

Then, the structures of p, $\beta_1: D(S/\Sigma)\to\Sigma$, and $\beta_2: \hat{S}\to D(S/\Sigma)$ are one of the following:

(i) *$p: S\to\Sigma$ is a cyclic covering.*

(ii) *$p: S\to\Sigma$ is non-Galois and one of the following occurs:*

ii-a) *Σ is an abelian surface, $\boldsymbol{P}^2$ or $\boldsymbol{P}^1\times\boldsymbol{P}^1$.*

$\Delta(S/\Sigma)$ is an irreducible divisor with ordinary cusps (i.e. (2, 3)-cusp) and the structure of a triple covering at a small neighborhood of each cusp is isomorphic to Example 3, in § 2.

ii-b) *Σ is F_n $(n\geqq 2)$.*

If $\Delta(S/\Sigma)$ is irreducible, the structure of p is the same as case ii-a).

If $\Delta(S/\Sigma)$ is reducible, then, $\Delta(S/\Sigma)=s_0+D$ where $D\sim as_\infty$ for some $a\in\boldsymbol{N}$ and D is irreducible with some ordinary cusps.

(α) *$n=2k$ $(k\in\boldsymbol{N})$, $\beta_1: D(S/\Sigma)\to\Sigma$ is a double covering branched at $\Delta(S/\Sigma)$ and $\beta_2: \hat{S}\to D(S/\Sigma)$ is a cyclic triple covering branched at* $\mathrm{Sing}\,(D(S/\Sigma))$.

(β) *$n=3k$ $(k\in\boldsymbol{N})$, $\beta_j: D(S/\Sigma)\to\Sigma$ is a double covering branched at D and $\beta_2: \hat{S}\to D(S/\Sigma)$ is a cyclic triple covering branched at $\beta_1^{-1}(s_0)$ and* $\mathrm{Sing}\,(D(S/\Sigma))$.

Notations and Conventions. $\boldsymbol{N}$, $\boldsymbol{Z}$ and $\boldsymbol{C}$ mean natural numbers, integers, and the complex number field, respectively.

$k(X)$: the rational function field of X (k: the ground field).

$\mathrm{Sing}\,(X)$: the singular locus of X.

$k(X)$: the Kodaira dimension of X.

Let $f: X\to Y$ be a morphism between X and Y where both X and Y are normal varieties.

For $x\in X$, we say that "f is ramified at x", if f is not étale at x.

For $y \in Y$, we say that "f is branched at y", if f is not étale over y.

Therefore a ramification divisor is the divisor on X, and a branch divisor is a divisor on Y.

For a divisor D on Y, $f^{-1}(D)$ denotes a set theoretic inverse of D, and $f^*(D)$ denotes the ordinary pull back of the divisor D.

§ 1. The Cardano formula and preliminaries

In this section, we assume that the ground fields k is algebraically closed and its characteristic is neither equal to 2 nor 3. We review the classical "Cardano formula". Consider an equation

$$x^3 + ax + b = 0 \tag{1.1}$$

where a, b are elements of a field K ($\supset k$).

As is well-known, we can obtain solutions of the above equation as follows:

Put $x = u + v$. Then, $(u^3 + v^3 + b) + (u + v)(3uv + a) = 0$. Therefore, to obtain solutions of (1.1), it is sufficient to solve the equations

$$u^3 + v^3 = -b$$

$$uv = -\frac{a}{3}$$

So, we obtain solutions of (1.1) as follows:

$$x_1 = \sqrt[3]{-\frac{b}{2} + \sqrt{R}} + \sqrt[3]{-\frac{b}{2} - \sqrt{R}}$$

$$x_2 = \omega\sqrt[3]{-\frac{b}{2} + \sqrt{R}} + \omega^2\sqrt[3]{-\frac{b}{2} - \sqrt{R}}$$

$$x_3 = \omega^2\sqrt[3]{-\frac{b}{2} + \sqrt{R}} + \omega\sqrt[3]{-\frac{b}{2} - \sqrt{R}}$$

where $\omega^3 = 1$, $\omega \neq 1$ and $R = b^2/4 + a^3/27$.

Assume $R \notin K$. The above process consists of three parts.

Step 1. We have a quadratic extension $K_1 = K(\theta)$ with $\theta^2 = R$.

Step 2. We have a cyclic cubic extension $K_2 = K_1(\tilde{\theta})$ with $\tilde{\theta}^3 = -b/2 + R$. K_2 is the minimal splitting field for the equation (1.1). By the assumption on the characteristic of the ground field k, it is a Galois extension of K and its Galois group is isomorphic to $\mathfrak{S}_3$ (the symmetric group of degree 3).

Step 3. There exists a K-automorphism $\sigma \in \mathrm{Gal}(K_2/K)$ and the solution of (1.1) is contained in its invariant subfield K_2^σ.

In the case that R is contained in K, we put $K_1 = K$ in the Step 1, and omit the Step 3.

Let $p: X \to Y$ be a finite triple covering where X and Y are normal projective varieties. Let $k(X)$ and $k(Y)$ be their rational function fields, respectively. We apply the above argument to the fields $k(X)$, $k(Y)$. First, if R is not contained in $k(Y)$, take a quadratic extension of $k(Y)$ corresponding to K_1 in Step 1, and we also denote it K_1.

If R is contained in $k(Y)$, put $K_1=k(Y)$. Take the K_1-normalization of Y. (For the definition of the K_1-normalization, and its properties, see Iitaka [4], § 2.14.).

Definition 1.1. Let $p: X \to Y$ be a finite triple covering where X and Y are normal projective varieties. By the discriminant variety $D(X/Y)$ of Y, we mean the K_1-normalization of Y.

Remark. If p is a cyclic covering, $D(X/Y)$ is equal to Y.

Next, we consider a cubic cyclic extension of $k(D(X/Y))$ corresponding to K_2 in Step 3, and also denote it by K_2. Take the K_2-normalization of $D(X/Y)$, and denote it $\hat{X}$.

Definition 1.2. Let $p: X \to Y$ be the same as above. We call $\hat{X}$ obtained as above "the minimal splitting variety of X".

Remark. If p is a cyclic covering, $\hat{X}$ is isomorphic to X.

The following proposition is easy to prove, but important in our theory.

Proposition 1.3. *Let $p: X \to Y$ and $\hat{X}$ be the same as above, and $p_1: \hat{X} \to Y$ be the induced morphism. Then, the birational map over Y induced by an element of* Gal$(k(\hat{X})/k(Y))$ *is an automorphism of $\hat{X}$.*

Proof. Let σ be an element of Gal$(k(\hat{X})/k(Y))$. Then σ induces a birational map. $\tilde{\sigma}: \hat{X} \cdots \to \hat{X}$. Consider a commutative diagram

$$\begin{array}{ccc} \hat{X} & \overset{\tilde{\sigma}}{\dashrightarrow} & \hat{X} \\ & {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle p_1} & \\ & Y & \end{array}$$

Since $\hat{X}$, Y are projective and p_1 is finite, $\tilde{\sigma}$ is a morphism by Iitaka [4], Theorem 2.21, 2.22. Therefore, $\tilde{\sigma}$ is an isomorphism by Zariski's Main Theorem. Q.E.D.

By Proposition 1.3, if $p: X \to Y$ is not cyclic, we obtain X as a quotient variety of $\hat{X}$ for an automorphism $\tilde{\sigma}$ of order 2 where $\tilde{\sigma}$ is an isomorphism of $\hat{X}$ induced by an element $\sigma \in$ Gal$(k(\hat{X})/k(Y))$ of order 2. This corresponds to Step 3.

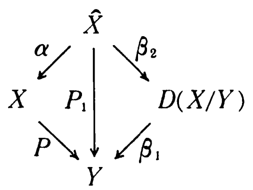
By the argument above, to study a triple dovering $p: X \to Y$, it is important to study $p_1: \hat{X} \to Y$, $D(X/Y)$, and the automorphism group induced by the Galois group Gal$(k(\hat{X})/k(Y))$. Moreover, in case Y is smooth, the following lemma plays an important role.

Lemma 1.4. *Let $\Delta(X/Y)$ and $\Delta(\hat{X}/Y)$ be the branch loci of p and p_1, respectively. (Both of them are divisors by the purity of the branch locus, Zariski* [9].) *Then, we have*

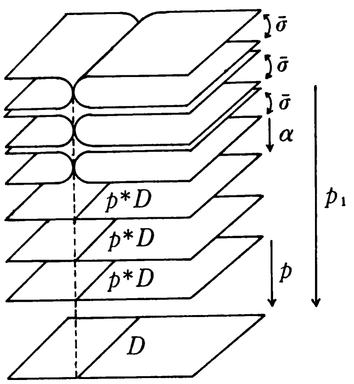
$$\Delta(X/Y)=\Delta(\hat{X}/Y).$$

Proof. Case I. $p: X\to Y$ is cyclic. In this case, $\hat{X}$ is equal to X. Therefore, our statement is obvious.

Case II. $p: X\to Y$ is non-Galois. Consider a commutative diagram



where $\alpha: \hat{X}\to X$ is a double covering, $\beta_1: D(X/Y)\to Y$ is a double covering, and $\beta_2: X\to D(X/Y)$ is a cyclic triple covering. Assume $\Delta(\hat{X}/Y)\supsetneqq\Delta(X/Y)$. (Note that $\Delta(\hat{X}/Y)\supset\Delta(X/Y)$.) Let D be an irreducible component of $\Delta(\hat{X}/Y)\setminus\Delta(X/Y)$. Since $p_1: \hat{X}\to Y$ is a Galois covering, p^*D is a part of the branch divisors of α. (Notice that p^*D is a reduced divisor.) Consider the action of automorphism group induced by Gal$(k(\hat{X})/k(Y))$ on a neighborhood of smooth parts of p_1^*D. Then, we know that the components of p_1^*D is fixed by the automorphism $\tilde{\sigma}$ of order 2 by which we have $X=\hat{X}/\langle\tilde{\sigma}\rangle$ (See Figure 1.) This means that $\sigma\in$Gal$(k(\hat{X})/k(Y))$ inducing $\tilde{\sigma}$ commutes with an element of order 3 of Gal$(k(\hat{X})/k(Y))$. This contradicts to the assumption that Gal$(k(\hat{X})/k(Y))$ is the third symmetric group. Q.E.D.



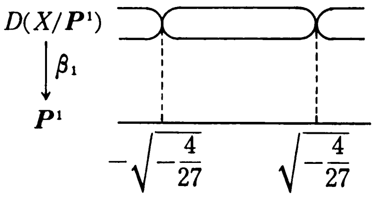
(Figure 1)

§ 2. Typical examples

In this section, we consider typical examples of triple coverings.

Examples 1. Put $Y=\boldsymbol{P}^1$. Let X be obtained by $\boldsymbol{C}(\boldsymbol{P}^1)(\theta)$-normalization of Y, where θ satisfies an equation $X^3+X+t=0$, and t is an inhomogeneous coordinate of $\boldsymbol{P}^1$. We will consider the structure of $\hat{X}$, $D(X/\boldsymbol{P}^1)$ and the action of an automorphism group induced by Gal$(\boldsymbol{C}(\hat{X})/\boldsymbol{C}(\boldsymbol{P}^1))$ for X and $\boldsymbol{P}^1$. Note that Gal$(\boldsymbol{C}(\hat{X})/\boldsymbol{C}(\boldsymbol{P}^1))$ is isomorphic to the symmetric group $\mathfrak{S}_3$ of degree 3. Note that we have $R=27t^2+4$.

Since $\boldsymbol{C}(D(X/\boldsymbol{P}^1))=\boldsymbol{C}(\boldsymbol{P}^1)(\sqrt{R})$, the double covering $D(X/\boldsymbol{P}^1)\to\boldsymbol{P}^1$ is illustrated os follows:



(Figure 2)

Therefore, $D(X/\boldsymbol{P}^1)\cong\boldsymbol{P}^1$, and $\beta_1: D(X/\boldsymbol{P}^1)\to\boldsymbol{P}^1$ is given by

$$\beta_1: z \longmapsto -\frac{2\sqrt{-1}}{3\sqrt{3}}\frac{z^2+1}{z^2-1} \quad (=t).$$

where z is a suitable inhomogeneous coordinate of $D(X/\boldsymbol{P}^1)$. Using the above coordinate z, we obtain

$$\begin{cases} \sqrt{\beta_1^* R}=\dfrac{2\sqrt{-1}}{3\sqrt{3}}\dfrac{z}{z^2-1} \\ \beta_1^* t \quad =\dfrac{2\sqrt{-1}}{3\sqrt{3}}\dfrac{z^2+1}{z^2-1} \end{cases}$$

and

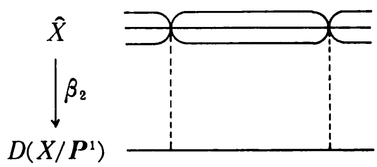
$$-\frac{1}{2}\beta_1^* t+\beta_1^* R=\frac{\sqrt{-1}}{3\sqrt{3}}\frac{z+1}{z-1}.$$

Since

$$\boldsymbol{C}(X)=\boldsymbol{C}(D(X/\boldsymbol{P}^1))\left(\sqrt[2]{-\frac{1}{2}\beta_1^* t+\sqrt{\beta_j^* R}}\right)$$

$$=\boldsymbol{C}(D(X/\boldsymbol{P}^1))\left(\sqrt[2]{\sqrt{-1}/3\sqrt{3}\frac{z+1}{z+1}}\right),$$

The cyclic triple convering $\hat{X}\to D(X/\boldsymbol{P}^1)$ is illustrated as follows:



(Figure 3)

Therefore, $\hat{X}\cong\boldsymbol{P}^1$, and the morphism $\beta_2: \hat{X}\to D(X/\boldsymbol{P}^1)$ is given by

$$\beta_2: w \longmapsto -\frac{w^3+1}{w^3-1} \quad (=z),$$

where w is a suitable inhomogeneous coordinate of $\hat{X}$. Next, let us consider the action of an automorphism group induced by $\mathrm{Gal}(\boldsymbol{C}(\hat{X})/\boldsymbol{C}(\boldsymbol{P}^1))$. On $D(X/\boldsymbol{P}^1)$, there is an involution σ which is induced by the non-trivial element of $\mathrm{Gal}(\boldsymbol{C}(D(X/\boldsymbol{P}^1)/\boldsymbol{C}(\boldsymbol{P}^1))$. By using the above coordinate z, this is represented by

$$\sigma : z \longmapsto -z .$$

This involution induces an involution $\tilde{\sigma}$ on $\hat{X}$. By using the above coordinate w, $\tilde{\sigma}$ is represented by

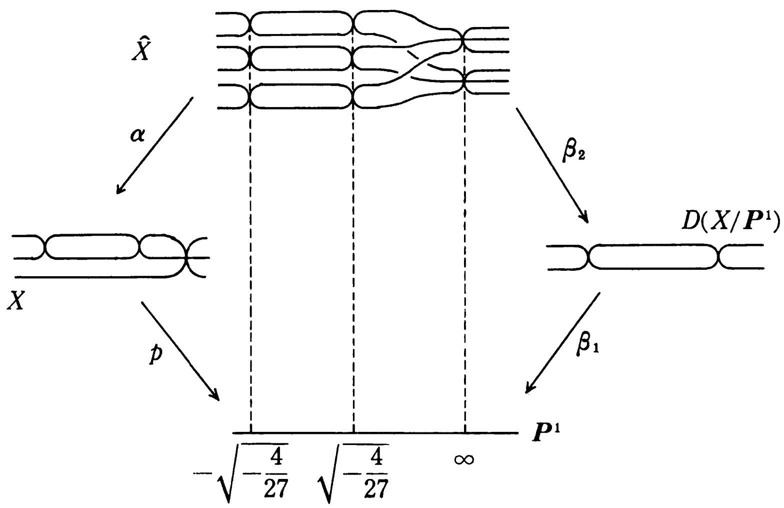
$$\tilde{\sigma} : w \longmapsto \frac{1}{w} .$$

Finally, let us consider the action of an automorphism τ of order 3 induced by an element of order 3 in $\mathrm{Gal}(\boldsymbol{C}(X)/\boldsymbol{C}(\boldsymbol{P}^1))$. Then, τ is represented by

$$\tau : w \longmapsto \varepsilon w ,$$

where $\varepsilon=\exp\left(\frac{2\pi\sqrt{-1}}{3}\right)$.

By the above argument, we obtain the structure of $D(X/\boldsymbol{P}^1)$, $\hat{X}$ and the action of the automorphism group induced by $\mathrm{Gal}(\boldsymbol{C}(\hat{X})/\boldsymbol{C}(\boldsymbol{P}^1))$. The following figure explains relations between $\boldsymbol{P}^1$, $D(X/\boldsymbol{P}^1)$, $\hat{X}$ and X.



(Figure 4)

Example 2 (Corollary to Example 1). Put $Y=\boldsymbol{P}^2$ and let $[z_0 : z_1 : z_2]$ be homogeneous coordinates of $\boldsymbol{P}^2$. Let X be a finite triple covering defined by the $\boldsymbol{C}(\boldsymbol{P}^2)(\theta)$-normalization of $\boldsymbol{P}^2$, where θ satisfies an equation $x^3+x+(z_1/z_0)=0$. Then, the minimal resolution of X is a rational ruled surface of degree 3, that is $\boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1}\oplus\mathcal{O}_{\boldsymbol{P}^1}(3))$. And X is obtained by contracting its negative section.

This fact is easily proved by blowing up at $[0:0:1]$ and we reducing the problem to Example 1.

Example 3. Put $X=\boldsymbol{C}^2$, $Y=\boldsymbol{C}^2$ and consider a covering

$$\pi : X \dashrightarrow Y$$
$$(x, y) \longmapsto (u, v)=(xy, x^3+y^3).$$

Clearly, X is a Galois covering of Y with $\mathrm{Gal}(X/Y)$ isomorphic to $\mathfrak{S}_3$. The Galois group $\mathfrak{S}_3$ aɔts on X by

$$\sigma : (x, y) \longmapsto (y, x)$$
$$\tau : (x, y) \longmapsto (\varepsilon x, \varepsilon^2 y)$$

where $\varepsilon=\exp\left(\frac{2\pi\sqrt{-1}}{3}\right)$, $\mathfrak{S}_3=\langle\sigma, \tau\rangle$, $\sigma^2=\tau^3=(\sigma\tau)^2=1$. Consider a diagram

$$\begin{array}{ccccc} & & X & & \\ & {\scriptstyle\varphi_{\langle\sigma\rangle}}\swarrow & & \searrow{\scriptstyle\varphi_{\langle\tau\rangle}} & \\ X/\langle\sigma\rangle & & \Big\downarrow{\scriptstyle\pi} & & X/\langle\tau\rangle \\ & {\scriptstyle\pi_{\langle\sigma\rangle}}\searrow & & \swarrow{\scriptstyle\pi_{\langle\tau\rangle}} & \\ & & Y & & \end{array}$$

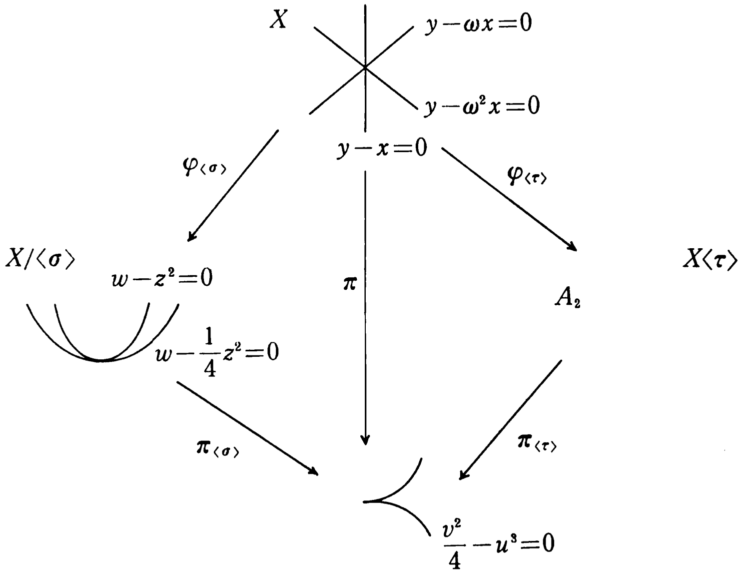
Let us analyse $X/\langle\sigma\rangle$, $X/\langle\tau\rangle$ and their ramification loci. The morphism $\varphi_{\langle\sigma\rangle}$, $\varphi_{\langle\tau\rangle}$, $\pi_{\langle\sigma\rangle}$, and $\pi_{\langle\tau\rangle}$ are written explicitly as follows:

$$\varphi_{\langle\sigma\rangle} : X \longrightarrow X/\langle\sigma\rangle \cong \boldsymbol{C}^2$$
$$(x, y) \longmapsto (z, w)=(x+y, xy)$$
$$\pi_{\langle\sigma\rangle} : X/\langle\sigma\rangle \dashrightarrow Y$$
$$(z, w) \longmapsto (u, v)=(w, z^3-3zw)$$
$$\varphi_{\langle\tau\rangle} : X \dashrightarrow X/\langle\tau\rangle \cong \mathrm{Spec}\,(\boldsymbol{C}[t_1, t_2, t_3]/(t_3^2-t_1t_2))$$
$$(x, y) \longmapsto (\bar{t}_1, \bar{t}_2, \bar{t}_3)=(x^3, y^3, xy)$$

Note that $X/\langle\sigma\rangle$ has a unique singularity and it is an A_2 singularity. The morphism $\pi_{\langle\tau\rangle}$ is given by

$$\pi_{\langle\tau\rangle} : X/\langle\tau\rangle \longrightarrow Y$$
$$(\bar{t}_1, \bar{t}_2, \bar{t}_3) \longmapsto (u, v)=(\bar{t}_3, \bar{t}_1+\bar{t}_2)$$

The ramification locus R_π of π is a divisor defined by an equation $(y-x)(y-\varepsilon x)\cdot(y-\varepsilon^2 x)=0$ where $\varepsilon=\exp(2\pi\sqrt{-1}/3)$. The support $\pi(R_\pi)$ is a divisor B_π on Y defined by an equation $(v^2/4)-u^3=0$. Let us consider the ramification loci of $\varphi_{\langle\sigma\rangle}$ and $\pi_{\langle\sigma\rangle}$. The ramification locus $\varphi_{\langle\sigma\rangle}$ is a divisor defined by an equation $y-x=0$. The support of its image of $\varphi_{\langle\sigma\rangle}$ is a divisor defined by an equation $w-(1/4)z^2=0$. Similary, we obtain the ramification locus of $\pi_{\langle\sigma\rangle}$, and it is a divisor on $X/\langle\sigma\rangle$ defined by an equation $w-z^2=0$. Note that images of $w-(1/4)z^2=0$ and $w-z^2=0$ are the same divisor on Y defined by an equation $(v^2/4)-u^3=0$. Finally, let us consider the ramification loci of $\varphi_{\langle\tau\rangle}$ and $\pi_{\langle\tau\rangle}$. It is clear that the ramification locus of $\varphi_{\langle\tau\rangle}$ is one point (0, 0). And its image of $\varphi_{\langle\tau\rangle}$ is the unique A_2 singularity of $X/\langle\tau\rangle$. The ramification locus of $\pi_{\langle\tau\rangle}$ is $(\pi^{-1}(B_\pi))_{\mathrm{red}}$. The following figure explains the above results.



(Figure 5)

Remark. In the above example, $\pi_{\langle\sigma\rangle}: X/\langle\sigma\rangle \to Y$ is a non-Galois triple covering. This is a typical example for the case of dimension 2. Locally, it is the same triple covering as the "generic triple covering of a surface" in the sense of Miranda [6].

§ 3. Applications

In this section, the ground field is always the complex number field $\boldsymbol{C}$.

(I) A totally ramified triple covering. Let $p: X \to Y$ be a finite triple covering of a smooth projective variety Y. We call p totally ramified, if for any irreducible component of the ramification divisors of p, its ramification index is equal to 3. For a totally ramified triple covering, we have the following:

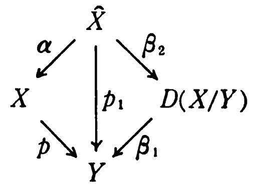
Proposition 3.1. *Let $p: X \to Y$ be a finite totally ramified triple covering of a smooth projective variety Y. Assume that*

(i) *X is smooth,*

(ii) *Y is simply connected.*

Then, p is cyclic, and the branch locus of p is smooth.

Proof. Assume that p is not cyclic. Then, from the arguments in § 1, there exists varieties $D(X/Y)$ and $\hat{X}$. For these two varieties, there exists the commutative diagram



Since Gal $(C(\hat{X})/C(Y))$ is isomorphic to $\mathfrak{S}_3$, there is no ramification point of p_1 whose ramification index is equal to 6. Hence, by lemma 1.4, α is étale. But this fact indicates that β_1 is étale. Since $D(X/Y)$ is irreducible and Y is simply connected, this is a contradiction. By Proposition 3.3, [8], it is easy to show that the branch locus of p is smooth. Q.E.D.

As is well-known, a trigonal curve is a curve which has a rational function of degree 3. Hence, we can regard C as a triple covering of $\boldsymbol{P}^1$. As an easy application of the above proposition, we have the following.

Corollary 3.2. *Let $p: C \to \boldsymbol{P}^1$ be a triple covering. We denote the branch points of p by $\mathfrak{p}_1, \cdots, \mathfrak{p}_r$ $(r \geqq 2)$. Assume that $p^{-1}(\mathfrak{p}_i)$ $(i=1, \cdots, r)$ consists of one point, that is, the ramification index of $p^{-1}(\mathfrak{p}_i)$ is 3. Then, $p: C \to \boldsymbol{P}^1$ is a cyclic triple covering.*

Remark 3.3. We can easily determine the cubic equation corresponding to the above triple covering $p: C \to \boldsymbol{P}^1$. There are three types.

(Type I) $$X^3 + \frac{(t-\mathfrak{p}_1) \cdots (t-\mathfrak{p}_r)}{t^r} = 0 \qquad r \equiv 0 \pmod 3$$

(Type II) $$X^3 + \frac{(t-\mathfrak{p}_1) \cdots (t-\mathfrak{p}_{r-2})(t-\mathfrak{p}_{r-1})^2(t-\mathfrak{p}_r)^2}{t^{r+1}} = 0 \qquad r \equiv 1 \pmod 3$$

(Type III) $$X^3 + \frac{(t-\mathfrak{p}_1) \cdots (t-\mathfrak{p}_{r-1})(t-\mathfrak{p}_r)^2}{t^{r+1}} = 0 \qquad r \equiv 2 \pmod 3$$

where t is an inhomogeneous coordinate of $\boldsymbol{P}^1$.

(II) Triple coverings of surfaces. In this part, we study a triple covering of a surface. Let $p: S \to \Sigma$ be a finite triple covering where both S and Σ are smooth surfaces. By $\hat{S}$ and $D(S/\Sigma)$, we denote the minimal splitting surface and the discriminant surface, respectively.

Proposition 3.4. *Let $p: S \to \Sigma$ be the same as above. Assume that $\Delta(S/\Sigma)$ (the branch locus of p) is an irreducible divisor and has singularities whose local equations are $x^2 + y^{3k} = 0$ where k is a natural number. (For two different singularieties, corresponding k may be different.) Then, the structures of $\beta_1: D(S/\Sigma) \to \Sigma$, $\beta_2: \hat{S} \to D(S/\Sigma)$ and $\alpha: \hat{S} \to S$ are as follows:*

(i) *$D(S/\Sigma)$ is a normal double covering of Σ branched along $\Delta(S/\Sigma)$.*

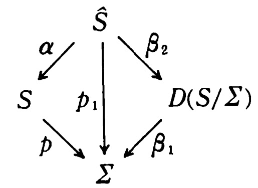
(ii) *$\hat{S}$ is a normal cyclic triple covering of $D(S/\Sigma)$ branched only at* Sing $(D(S/\Sigma))$ *and slngularities of $\hat{S}$ are of A_{k-1} type.*

(iii) *There exists an involution ι on $\hat{S}$ such that S is obtained the quotient surface of $\hat{S}$ by ι, and α is regarded as the quotient map.*

Proof. By the argument in § 1, the statement (iii) is clear. First we prove the following:

CLAIM 3.5. $p: S \to \Sigma$ is not a cyclic covering.

Proof of Claim 3.5. Assume that p is cyclic. Then, since $\Delta(S/\Sigma)$ is an irreducible divisor and $\deg p=3$, S is embedded in a total space of a line bundle over Σ. (See Tokunaga [8]. Proposition 3.3.) But in this case, S is singular. Therefore, p is not cyclic. When p is not cyclic, we have a diagram



where β_1 is a double covering, β_2 is a cyclic triple covering, and α is a double covering. Since $\Delta(S/\Sigma)$ is an irreducible divisor, there are three possibilities.

1) Both β_1 and β_2 are ramified at divisors, that is, β_1 is ramified to $\Delta(S/\Sigma)$ and β_2 is ramified at $\beta_1^{-1}\Delta(S/\Sigma)$.

2) β_1 is branched at $\Delta(S/\Sigma)$, but β_2 is not ramified at $\beta_1^{-1}(\Delta(S/\Sigma))$.

3) β_2 is branched at $\beta_1^{-1}(\Delta(S/\Sigma))$ and β_1 étale.

Case 1). In this case, the Galois covering $p_1: \hat{S} \to \Sigma$ is branched at $\Delta(S/\Sigma)$ and the ramification index of $p_1^{-1}(\Delta(S/\Sigma))$ is equal to 6. Consider the action of the Galois group at a smooth point of $p_1^{-1}(\Delta(S/\Sigma))$. Then, $\mathrm{Gal}(\boldsymbol{C}(\hat{S})/\boldsymbol{C}(\Sigma))$ have an element of order 6. This is a contradiction.

Case 2). In this case, $D(S/\Sigma)$ is a normal surface with A_{k-1} singularities. There are two possibilities

2—a) β_2 is étable, 2—b) β_2 is ramified.

Case 2—a). Let x be one of singularities on $D(S/\Sigma)$. Then, $\beta_2^{-1}(x)$ consists of 3 points which are A_{3k-1} singularities. Since S is smooth, the branch locus of α is a divisor on S by the purity of branch locus (see Zariski [9]). Moreover, $\alpha(\beta_2^{-1}(x))$ is contained in this divisor. This means that at least one of 3 points of $\beta_2^{-1}(x)$ has the stabilizer group $\mathfrak{S}_3$. This is a contradiction.

Case 2—b). By case 1), β_2 is branched at most some points. By the purity of a branch locus, they are singular points. Moreover, by the proof of 2—a), they consist of all singularites of $D(S/\Sigma)$. Let x be one of singularities and let U be its small neighborhood. Since singularities are all of type A_{3k-1}, we can take U in such a way that there is $V(\subset \boldsymbol{C}^2)$ a small neighborhood $V(\subset \boldsymbol{C}^2)$ of origin of $\boldsymbol{C}^2$ and that $\pi: V \to U$ is the quotient map by the group action of $\boldsymbol{Z}/3k\boldsymbol{Z}$. Moreover, $\pi|_{V\setminus(0,0)}: V\setminus(0, 0) \to U\setminus\{x\}$ is étale. Since the local fundamental group $\pi_1(U\setminus\{x\})$ is isomorphic to $\boldsymbol{Z}/3k\boldsymbol{Z}$ and $\beta_2|_{\beta_2^{-1}(U\setminus\{x\})}$ is cyclic and étale. $\beta_2^{-1}(U)\setminus\beta_2^{-1}(x)$ is isomorphic to a quotient space of $V\setminus(0, 0)$ by a subgroup Z/kZ. Moreover, since $\hat{S}$ is normal double covering of S, $\beta_2^{-1}(x)$ is an isolated hypersurface singularity. Therefore, $\beta_2^{-}(x)$ is an isolated hypersurface singularity. Therefore, $\beta_2^{-1}(x)$ is an A_{k-1} singularity.

Case 3). Clearly, $D(S/\Sigma)$ is smooth, and $\beta_2^{-1}(\Delta(S/\Sigma))$ has singularities. Therefore, $\hat{S}$ must be singular by Tokunaga [8]. Proposition 1.1. Hence α is not étale. Let x be smooth point of $\Delta(S/\Sigma)$, and let U be its small neighborhood. Consider a ramification index of $\beta_2^{-1}\beta_1^{-1}(\Delta(S/\Sigma))\cap p_1^{-1}(U)$ and $\alpha^{-1}p^{-1}(\Delta(S/\Sigma))\cap p_1^{-1}(U)$. They are equal to each other. But the ramification index of $\beta_2^{-1}\beta_1^{-1}(\Delta(S/\Sigma))\cap p_1^{-1}(U)$ is equal to 3 and the ramification index of $\alpha^{-1}p^{-1}(\Delta(S/\Sigma))\cap p_1^{-}(U)$ is even number because of α is a double cover. This is a contradiction.

By Cases 1), 2), and 3), only the possible case is Case 2-b). This proves proposition. Q.E.D.

Next, we consider the case that $\hat{S}$ is a smooth surface. In the following, α, β_1, β_2 p_0 mean the same morphisms which appear in the proof of Proposition 3.1, and $\hat{S}$ is always smooth.

First, we analyse the ramification divisor of $p_1: \hat{S}\to\Sigma$. Let $\hat{R}$, $\Delta(\hat{S}/\Sigma)(=\Delta(S/\Sigma))$ be the ramification locus and the branch locus of p_1, respectively. Let x be a point of $\hat{R}$. Then, a stabilizer at x (we denote it G_x) is a non-trivial subgroup of Gal$(\boldsymbol{C}(\hat{S})/\boldsymbol{C}(\Sigma))$. In the case that p is cyclic, $G_x\cong Z/3Z$ by Catanese [1], Proposition 1.1. In the case that p is not cyclic, there are three cases

1) $|G_x|=2$, 2) $|G_x|=3$, 3) $|G_x|=6$, *i.e.*, $G_x\cong\mathfrak{S}_3$ where $|G_x|$ is the order of the group G_x.

Case 1) By taking a suitable system of local coordinates, (u, v), the action of G_x is one of the following:

a) $\sigma:(u, v)\to(-u, -v)$

b) $\sigma:(u, v)\to(-u, v)$

where $G_x=\langle\sigma\rangle$, $\sigma^2=id$.

In case a), a quotient surface $\hat{S}/\langle\sigma\rangle$ has an A_1 singularity. On the other hand, there is an isomorphism over $\boldsymbol{C}(\Sigma)$ between $\boldsymbol{C}(S)$ and $\boldsymbol{C}(\hat{S}/\langle\sigma\rangle)$. Since $\hat{S}/\langle\sigma\rangle$ is normal and finite over Σ, $\hat{S}\langle\sigma\rangle$ is isomorphic to S by Iitaka [4], Theorem 2.21, 2.22. Since S is a contradiction. In case b), there exists a smooth divisor through x and for all points on it, the stabilizer group is isomorphic to $Z/2Z$.

Case 2) By taking a suitable system of local coordinate at x, the action of G_x is one of the following:

a) $\tau:(u, v)\mapsto(\varepsilon u, \varepsilon^2 v)$

b) $\tau:(u, v)\mapsto(\varepsilon u, \varepsilon v)$

c) $\tau:(u, v)\mapsto(\varepsilon u, v)$

$$G_x=\langle\tau\rangle, \quad \tau^3=id, \quad \text{and} \quad \varepsilon=\exp\left(\frac{2\pi\sqrt{-1}}{3}\right).$$

Since $\mathfrak{S}_3$ has a unique subgroup of order 3, the rational function field of the quotient surface $\hat{S}/G_x$ coincides with $\boldsymbol{C}(D(S/\Sigma))$. By the uniquness of $\boldsymbol{C}(D(S/\Sigma))$-normalition of Σ (see Iitaka [4], §2.14), $\hat{S}/G_x$ is equal to $D(S/\Sigma)$. Since $D(S/\Sigma)$ is a normal double covering, singularities of $D(S/\Sigma)$ must be hypersurface singularites. Therefore case b) does not occur, because in case b), $\hat{S}/\langle\tau\rangle$ has a rational triple point which can not be a hypersurface singularity. In case a), $\hat{S}/\langle\tau\rangle(=D(S/\Sigma))$ has an A_2 singularity. Since Σ is smooth, β_1 is not étale. Therefore, by the purity of branch loci, there exists a divisor on Σ which passes through $p_1(x)$, and $\beta_1: D(S/\Sigma)\rightarrow\Sigma$ is branched over its divisor. This show that the order of G_x is equal to 6. This is a contradiction. In case c), there exists a smooth divisor through x, and for all points on it, the stabilizer group is isomorphic to $\boldsymbol{Z}/3\boldsymbol{Z}$.

Case 3) By taking a suitable local coordinate system, the action of $G_x(\cong\mathfrak{S}_3)$ is represented as follows:

$$\sigma:(u, v)\longmapsto(v, u)$$

$$\tau:(u, v)\longrightarrow(\varepsilon u, \varepsilon^2 v)$$

$$G_x=\langle\sigma, \tau\rangle \quad \sigma^2=\tau^3=(\sigma\tau)^2=id, \quad \text{and} \quad \varepsilon=\exp\left(\frac{2\pi\sqrt{-1}}{3}\right).$$

Hence, in this case, the situation is the same as Example 3 in § 2. Thus, we obtain the following result.

Lemma 3.6. *Let $p: S\rightarrow\Sigma$ be a finite triple covering where both S and Σ are smooth surfaces. Assume that p is not étale and $\hat{S}$ is smooth. Then, if p is cyclic, the branch locus is a smooth divisor, while if p is not cyclic, there are two cases*

(a) *the branch divisor is a smooth divisor.*

(b) *the branch divisor has singular points and its singularities are all ordinary cups.* (*i.e.*, (2, 3)-*cusp*)

Lemma 3.7. *Let D be a divisor on $D(S/\Sigma)$ contained in the ramification locus of β_1. $D(S/\Sigma)$. Assume that D is smooth. Let D_1 be an irreducible component of D. Then, $\beta_2^{-1}(D_1)$ consists of* 3 *components which are isomorphic to each other.*

Proof. Since $p_1:\hat{S}\rightarrow\Sigma$ is Galois, $\beta_2^{-1}(D)$ is either irreducible or reducible with 3 components which are isomorphic to each other. Assume that $\beta_2^{-1}(D_1)$ is irreducible. Clearly, $\beta_2^{-1}(D_1)$ is a component of the ramification divisor of p_1. Therefore, there exists an automorphism σ such that $\sigma(x)=x$ for $x\in\beta_2^{-1}(D_1)$ and $\sigma^2=id$. Let τ be an automorphism with order 3. Then, by irreducibility of $\beta_2^{-1}(D_1)$, $\tau^*(\beta_2^{-1}(D_1))=\beta_2^{-1}(D_1)$. Let x be an arbitrary point of $\beta_2^{-1}(D_1)$. Consider a stabilizer at $\tau(x)$. Since $\tau(x)\in\beta_2^{-1}(D_1)$, we have $\sigma(\tau(x))=\tau(x)$. Moreover, we have $\tau\sigma\tau^{-1}(\tau(x))=\tau\sigma(x)=\tau(x)$. Therefore, $G_{\tau(x)}=\langle\sigma, \tau\sigma\tau^{-1}\rangle\cong\mathfrak{S}_3$. Hence, $\tau(x)=x$. Since x is an arbitrary point on $\beta_2^{-1}(D_1)$, this is a contradiction. Q.E.D.

Now we consider the case that Σ is a minimal rational surface or an abelian surface. We need the following lemma on connectedness of a divisor on a minimal rational surface and an abelian surface.

Lemma 3.8. *Let D be a divisor on a minimal rational surface or an abelian surface. Then, the divishr D is one of the following types:*

a) *Σ=an abelian surface*

a-1) *D is connected*

a-2) $D=E_1+\cdots+E_n$

E_i: an elliptic curve, $E_iE_j=0$, for all i, j.

b) $\Sigma=\boldsymbol{P}^2$

D is connected.

c) $\Sigma=F_n$ *(a rational ruled surface of degree n, $n\geqq 2$)*

c-1) *D: connected*

c-2) $D=f_1+\cdots+f_n$

f_i is a fibre of the fibration $F_n\to\boldsymbol{P}^1$.

c-3) $D=s_0+D$

s_0 is a negative section of F_n. (i.e., $s_0\cong\boldsymbol{P}^1$, $s_0^2=-n$)

D is a divisor linear equivalent to ks_∞ where k is a integer and s_∞ is a positive section of F_n. (i.e., $s_\infty\cong\boldsymbol{P}^1$, $s_\infty^2=n$)

d) $\Sigma=\boldsymbol{P}^1\times\boldsymbol{P}^1$

d-1) *D is connected*

d-2) $D=f_1+\cdots+f_n$

$f_i\cong\boldsymbol{P}^1$, $f_if_j=0$, for all i, j.

Proof. Case a), b) and d) is clear. We will prove case c). Let D be a divisor on F_n. Assume that $D=D_1+D_2$, $D_1D_2=0$, and $D_1\sim a_1s_0+b_1f$, $D_2\sim a_2s_0+b_2f$, where $\sim$ denotes linear equivalence, and f denotes a fibre. If one of D_i contains a fibre, then both D_1 and D_2 must be a finite sum of fibers. This is case c-2). From now on, we assume that neither D_1 nor D_2 are contained in a fibre. From $D_1D_2=0$, we obtain

$$-na_1a_2+a_1b_2+a_2b_1=0,\quad a_1a_2\neq 0,\qquad a_i>0,\ i=1, 2.$$

Put $e=\text{g.c.d.}(a_1, a_2)$. Then,

$$a_1'b_2=a_2'(na_1-b_1),\qquad a_2'b_1=a_1'(na_2-b_2),$$

where $a_1=a_1'e$, $a_2=a_2'e$.

Therefore we obtain

$$\begin{cases} D_1\sim a_1s_0+a_1'kf \\ D_2\sim a_2s_0+a_2'lf \end{cases}$$

where k and l are intergers satisfying $k+l=ne$. Without loss of generality, we may assume $k\leqq l$, *i.e.*, $k\leqq[ne/2]$ ([ ] denotes Gaussian symbol). Then,

$$D_1^2=-na_1^2+2a_1a_1'k=a_1a_1'(2k-ne)\leqq 0.$$

Hence D_1 contains at least one irreducible component whose self-intersection number $\leqq 0$. We denote it $\tilde{D}_1$.

Claim. $\tilde{D}_1=s_0$.

Proof of Claim. Assume $\tilde{D}_1\sim as_0+bf$, $a>0$. Then

$$\begin{cases} \tilde{D}_1^2 = a(2b-na) \\ \tilde{D}_1 K_{F_n} = na-2a-2b, \qquad (K_{F_n}\text{: a canonical divisor of } F) \end{cases}$$

Hence,

$$\tilde{D}_1^2 + \tilde{D}_1 K_{F_n} = (a-1)(2b-na) - 2a\ .$$

Since $\tilde{D}_1$ is irreducible, $\tilde{D}_1^2 + \tilde{D}_1 K_{F_n} \geqq -2$. So, from an inequality $\tilde{D}_1^2 = a(2b-na) \leqq 0$, $a>0$, we conclude $a=1$. Moreover, $s_0 \tilde{D}_1 = -n+b \leqq -n+2b \leqq 0$, and equalities can not hold simultaneously. Therefore, $s_0 = \tilde{D}_1$, and our claim is proved.

By the above claim, we obtain

$$\begin{cases} D = s_0 + D' \\ D' \sim a s_\infty \end{cases}$$

This is the case c-3). Q. E. D.

The rest of this section is devoted to prove the following.

Theorem 3.9. *Let $p: S \to \Sigma$ be a finite triple covering where both S and Σ are smooth surfaces. Assume the following:*

1) *$\hat{S}$ is smooth,*
2) *Σ is either a minimal rational surface or an abelian surface.*
3) *the Kodaira dimension $\kappa(S)$ of S is 2.*

Then the structures of p, $\beta: D(S/\Sigma) \to \Sigma$, and $\beta_2: \hat{S} \to D(S/\Sigma)$ are one of the following:

(i) *$p: S \to \Sigma$ is cyclic.*

(ii) *$p: S \to \Sigma$ is non-Galois and there are two possibilities*

ii-a) *Σ: $=$an abelian surface, $\boldsymbol{P}^2$, and $\boldsymbol{P}^1 \times \boldsymbol{P}^1$.*

$\Delta(S/\Sigma)$ is an irreducible divisor with ordinary cusps (e.e. (2, 3)-cusp) and a structure of a triple covering of a small neighborhood of each cusp is isomorphic to Example 3, §2.

ii-b) *$\Sigma = F_n$ $(n \geqq 2)$*

If $\Delta(S/\Sigma)$ is irreducible, the structure of p is the same as case ii-a).

If $\Delta(S/\Sigma)$ is reducible, then $\Delta(S/\Sigma) = s_0 + D$ where $D \sim as_\infty$ for some $a \in \boldsymbol{N}$ and D is irreducible and has ordinary cups.

(α) *$n=2k$ $(k \in \boldsymbol{N})$ $\beta_1: D(S/\Sigma) \to \Sigma$ is branched along $\Delta(S/\Sigma)$ and $\beta_2: \hat{S} \to D(S/\Sigma)$ is branched at* $\mathrm{Sing}\,(D(S/\Sigma))$.

(β) *$n=3k$ $(k \in \boldsymbol{N})$ $\beta_1: D(S/\Sigma) \to \Sigma$ is branched along D and $\beta_2: \hat{S} \to D(S/\Sigma)$ is branched at $\beta_1^{-1}(s_0)$ and* $\mathrm{Sing}\,(D(S/\Sigma))$.

Remark. 1) If $\kappa(S) < 2$, the above theorem dose not necesserally hold. See Example 2, in §2.

2) If Σ is a ruled surface whose base curve has a genus greater than 1, then the above theorem does not necessarily hold. For example, put $\Sigma = C \times \boldsymbol{P}^1$ where C is a curve with $g(C) \geqq 2$. Take a triple covering $\tilde{p}: C' \to \boldsymbol{P}^1$ where $g(C') \geqq 2$. Consider

$$p: S = C \times C' \longrightarrow C \times \boldsymbol{P}^1$$
$$(x,\ y) \longrightarrow (x,\ \tilde{p}(y))$$

This is a typical counter-example.

Proof of Theorem 3.9. We consider the case that p is non-Galois covering.

Case ii-a) If $\Sigma=\boldsymbol{P}^2$, then $\Delta(S/\Sigma)$ is always connected. Therefore, $D(S/\Sigma)$ is either smooth or one of the types in the statement in case ii-a). If $\Delta(S/\Sigma)$ is smooth, the fundamental group $\pi_1(\boldsymbol{P}^2\setminus\Delta(S\setminus\Sigma))$ is an abelian group. Therefore, p is cyclic. This is a contradiction. If $\Sigma=\boldsymbol{P}^1\times\boldsymbol{P}^1$ or an abelian surface, a disconnected divisor is one of the types stated in Lemma 3.5. Therefore, if $\Delta(S/\Sigma)$ is disconnected, then $\kappa(S)<2$. This is a contradiction. Hence $\Delta(S/\Sigma)$ is an irreducible divisor and it is smooth or one of the types in our statement. Assume that $\Delta(S/\Sigma)$ is smooth. In case $\Sigma=\boldsymbol{P}^1\times\boldsymbol{P}^1$, the fundamental group $\pi_1(\boldsymbol{P}^1\times\boldsymbol{P}^1\setminus\Delta(S/\Sigma)$ is abelian by Catanese [1], Theorem 1.6. Therefore, the situation is the same as the in case $\Sigma=\boldsymbol{P}^2$. In case Σ is an abelian surface, possible situations are as follows:

(1) $\beta_1: D(S/\Sigma)\to\Sigma$ is branched at $\Delta(S/\Sigma)$ and $\beta_2: \hat{S}\to D(S/\Sigma)$ is étale.

(2) $\beta_1: D(S/\Sigma)\to\Sigma$ is étale and $\beta_2: \hat{S}\to D(S/\Sigma)$ is branched at $\beta_1^{-1}(\Delta(S/\Sigma))$.

Case (1). Since $\Delta(S/\Sigma)$ is an ample divisor on Σ, $\beta_j^{-1}(\Delta(S/\Sigma))$ is also an ample divisor on $D(S/\Sigma)$. Hence $\beta_2^{-1}\beta_1^{-1}(\Delta(S/\Sigma))$ is ample, and smooth. So, it is an irreducible divisor. But by Lemma 3.7, this is a contradiction.

Case (2). By the same reason as in case (1), $\beta_1^*\Delta(S/\Sigma)$ is a smooth ample divisor on $D(S/\Sigma)$, and $D(S/\Sigma)$ is an abelian surface. By Tokunaga [8], $\beta_1^*\Delta(S/\Sigma)\sim 3L$ for a suitable $L\in\mathrm{Pic}\,(D(S/\Sigma))$, and $\hat{S}$ is embedded in the total space of L. Since $\deg\beta_1=2$. $\Delta(S/\Sigma)\sim 3\tilde{L}$ for a suitable $\tilde{L}\in\mathrm{Pic}\,(\Sigma)$. (Cf. Catanese [2] Lemma 4) Therefore, $L-\beta_1^*\tilde{L}\in\mathrm{Pic}^0(D(S/\Sigma))$. But since both $D(S/\Sigma)$ and Σ are abelian surfaces and β_1 is étale, $\mathrm{Pic}^0(\Sigma)\to\mathrm{Pic}^0(D(S/\Sigma))$ is surjective (see Mumford [7], p. 81). Therefore, $L=\beta_1^*(\tilde{L}+\tau)$ for a unique $\tau\in\mathrm{Pic}^0(\Sigma)$. Consider a diagram

$$\begin{array}{ccc} X\times_\Sigma D(S/\Sigma) & \xrightarrow[\tilde{f}]{} & D(S/\Sigma) \\ \downarrow g & & \downarrow \beta_1 \\ X & \xrightarrow[f]{} & \Sigma \end{array}$$

where X is a smooth cyclic triple convering branched at $\Delta(S/\Sigma)$ and it is embedded in the total space of the line bundle $\tilde{L}+\tau$. Note that $\tilde{f}$ is the same as β_2. Therefore, $X\times_\Sigma D(S/\Sigma)\cong\hat{S}$. But this is contradiction, since $\boldsymbol{C}(\hat{S})$ is a Galois extension of $\boldsymbol{C}(\Sigma)$ with Galois group $\mathfrak{S}_3$. From the above argument ii-a) follows.

Case ii-b). Assume that $\Delta(S/\Sigma)$ is a connected divisor. Then we obtain the same rəsult as in the case ii-a). In the following, we assume that $\Delta(S/\Sigma)$ is a disconnected divisor. Then by Lemma 3.5, $\Delta(S/\Sigma)=s_0+D$ where D is an effective divisor which is linearly equivalent to as_∞ for some $a\in\boldsymbol{Z}$. Possible cases are as follows:

Case (1) β_1 is branched at s_0+D, and β_2 is branched at $\mathrm{Sing}\,(D(S/\Sigma))$. In this case, $\mathrm{Sing}\,(D)$ is ɔonsists of (2, 3)-cusps.

Case (2) β_j is branched at s_0+D and β_2 is étale.

Case (3) β_1 is branched at D, and β_2 is branched at $\beta_1^*(s_0)\cup\mathrm{Sing}\,(D(S/\Sigma))$, ($\mathrm{Sing}\,(D(S/\Sigma))$ may be empty.)

Remrrk. The case that β_1 is branched at s_0 is impossible, since the class of s_0 in Pic (Σ) is not divisible by 2.

Case (1) Since the class of s_0+D in Pic (Σ) is divisible by 2, the integer n of F_n is even. This case is ii-b-(α).

Case (2) $\beta_2^*(\beta_1^{-1}(D))$ is a smooth irreducible divisor. By Lemma 3.4, this case does not occur.

Case (3) We can show that the integer n of F_n is divisible by 3. Moreover if D is non-singular, $\beta_2^*(\beta_1^{-1}(D))$ is a smooth irreducible divisor. Therefore, by Lemma 3.4, D must be singular. This case is ii-b-(β). Q.E.D.

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
KOCHI UNIVERSITY

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