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Homology planes with quotient singularities

By

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Introduction

We are interested in a nonsingular affine surface X defined over the complex field $\mathbf{C}$ whose homology groups $H_i(X; \mathbf{Z})$ vanish for all $i > 0$. We then call X a *homology plane*. Such a surface X is a *contractible surface* if it is furthermore simply-connected. In trying to solve the cancellation problem in dimension 2, C.P. Ramanujam [13] found a contractible surface which is not isomorphic to the affine plane $\mathbf{A}^2$. Since then, it has become an interesting problem to find other examples of contractible surfaces or possibly give structure theorems on contractible surfaces or homology planes.

In a recent trend of birational classification of algebraic varieties, we have the fundamental invariant called the Kodaira dimension, and it is also useful in classifying contractible surfaces or homology planes. We refer to Miyanishi [7] for the definition of Kodaira dimension and relevant results. Let X be either a contractible surface or a homology plane and let $\kappa(X)$ be its Kodaira dimension which takes values $-\infty, 0, 1, 2$. By Gurjar-Miyanishi [3], we know that

(1) If $\kappa(X) = -\infty$ any homology plane X is isomorphic to $\mathbf{A}^2$;
(2) If $\kappa(X) = 0$ there are no homology planes; (the result is essentially due to Fujita [2]);
(3) If $\kappa(X) = 1$ all homology planes and contractible surfaces are completely classified and homology planes are not necessarily contractible surfaces.

By this result, any homology plane of $\kappa \leq 1$ is rational. In this direction, Gurjar-Shastri [4] proved in full generality that any homology plane is rational. As for the structure theorems, the case of Kodaira dimension 2 is left open, though there are now abundant examples; we found in [3] a contractible surface similar to but not isomorphic to Ramanujam's example, and in Miyanishi-Sugie [9] infinitely many contractible surfaces which are obtained by the blowing-up method from the configurations of two curves on the projective plane $\mathbf{P}^2$. Independently, Petrie and tom Dieck [15, 16, 17] constructed infinitely many examples by the blowing-up method from the arrangements of lines and curves of low degree on $\mathbf{P}^2$. See furthermore tom Dieck [18].

Besides the classification of such surfaces, it seems interesting to consider the automorphism groups of homology planes. Petrie [12] posed the following

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conjecture and proved it for all contractible surfaces known by then.

Homology Plane Conjecture. *Let X be a homology plane admitting a non-trivial automorphism of finite order. Then X is isomorphic to* $\mathbf{A}^2$.

The conjecture itself was negated by examples of tom Dieck [18] and the authors [19]. We include one of examples in this article (see §4).

In the present article, we consider a $\mathbf{Q}$-homology plane. A nonsingular affine surface X is a $\mathbf{Q}$-*homology plane* if $H_i(X;\mathbf{Q})=(0)$ for all $i>0$. We also define a *logarithmic homology plane* (resp. *a logarithmic* $\mathbf{Q}$-*homology plane*) as a normal affine surface X with at worst quotient singularities for which $H_i(X;\mathbf{Z})=(0)$ (resp. $H_i(X;\mathbf{Q})=(0)$) for all $i>0$. Surfaces of these kinds are more abundant and have more chances to admit non-trivial automorphisms. These surfaces are, moreover, interrelated via finite automorphisms. Indeed, we shall show that if X is a homology plane admitting an effective action of a finite group of prime order such that the fixed point locus is isolated then the quotient surface X/G is a logarithmic homology plane. On the other hand, let Y be a logarithmic $\mathbf{Q}$-homology plane and let $Y^\circ = Y - \operatorname{Sing} Y$. Let Z° be a finite unramified Galois covering of Y° and let Z be the normalization of Y in the function field $\mathbf{C}(Z^\circ)$. Then Z has only quotient singularities and the Galois group G of $\mathbf{C}(Z^\circ)/\mathbf{C}(Y^\circ)$ acts on Z so that $Y=Z/G$. Such a covering as Z°/Y° does occur as the one associated with $H_1(Y^\circ;\mathbf{Z})$. In most cases, Y has positive second Betti number. So, it would be interesting to ask when Y becomes a logarithmic $\mathbf{Q}$-homology plane.

In §1, we classify $\mathbf{Q}$-homology planes with Kodaira dimension less than 2. We again encounter a difficulty in the case of Kodaira dimension 2. We do not even know whether or not a $\mathbf{Q}$-homology plane of Kodaira dimension 2 is rational.

In §2, we consider a logarithmic homology plane and give a structure theorem in the case of Kodaira dimension $-\infty$ and the case where it has a $\mathbf{C}^*$-fibration. In the first case, there appear two more types other than the affine plane, one of which is a quotient space $\mathbf{C}^2/G$ by a small finite subgroup G of $GL(2,\mathbf{C})$. Moreover, they are all contractible surfaces. In the second case, we can determine which of logarithmic homology planes are contractible.

In §3, we give an example of homology plane X with Kodaira dimension 2 which admits an involution. This example is given independently by tom Dieck [18].

In §4, we give a remark on the boundary divisor of a $\mathbf{Q}$-homology plane with Kodaira dimension 2 and a result on the automorphism group $Aut(X)$ of such a surface X.

§1. Q-homology planes which are not homology planes

1.1. Let X be a nonsingular affine algebraic surface defined over the complex field $\mathbf{C}$. It a $\mathbf{Q}$-*homology plane* if $H_i(X;\mathbf{Q})=0$ for $i>0$. More precisely, for a positive integer n, we define an n-primary homology plane to be a nonsingular affine surface X defined over $\mathbf{C}$ such that $H_i(X; \mathbf{Z}/n\mathbf{Z}) = 0$ for all $i > 0$. An n-primary homology plane is clearly a $\mathbf{Q}$-homology plane, but not *vice versa*. Given such a surface X, we can find a nonsingular projective surface V and an effective reduced divisor D with simple normal crossings on V such that $V - D = X$. We always consider the following cohomology exact sequence for a pair (V, D) with coefficients in K, K being $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{Z}/n\mathbf{Z}$ etc.:

$$\begin{aligned} 0 &\longrightarrow H^0(V, D) \longrightarrow H^0(V) \longrightarrow H^0(D) \longrightarrow H^1(V, D) \longrightarrow H^1(V) \longrightarrow H^1(D) \\ &\longrightarrow H^2(V, D) \longrightarrow H^2(V) \longrightarrow H^2(D) \longrightarrow H^3(V, D) \longrightarrow H^3(V) \longrightarrow 0 \\ &\longrightarrow H^4(V, D) \longrightarrow H^4(V) \longrightarrow 0, \end{aligned}$$

where

$$H^i(V, D) \cong H_{4-i}(X) \text{ and } H^i(V) \cong H_{4-i}(V) \qquad \text{for all } i$$

by Lefschetz and Poincaré dualities. From this exact sequence, we deduce the next

Lemma 1.1. (1) *Let X be a $\mathbf{Q}$-homology plane. Then the irregularity $q(V)$ and the geometric genus $p_g(V)$ vanish, and D is simply connected, i.e., each irreducible component of D is isomorphic to $\mathbf{P}^1$ and its dual graph is a tree. Moreover, $Pic(X)$ is a finite group and $\Gamma(\mathcal{O}_X)^* = \mathbf{C}^*$. If X is rational, $H_i(X; \mathbf{Z}) = 0$ for $i \geq 2$, $H_0(X; \mathbf{Z}) = \mathbf{Z}$ and $H_1(X; \mathbf{Z}) = Coker(H^2(V; \mathbf{Z}) \to H^2(D; \mathbf{Z}))$.*

(2) *If X is rational then we have the isomorphisms*

$$Pic(X) \cong H_1(X; \mathbf{Z}) \cong H^2(X; \mathbf{Z}) \cong Coker(H_2(D; \mathbf{Z}) \longrightarrow H^2(V; \mathbf{Z}))$$

(3) *If X is an n-primary homology plane then $H_1(X; \mathbf{Z}/n\mathbf{Z})$ is an n-tosion group.*

Proof. Since $H^3(V, D; \mathbf{C}) = 0$, we have $H^3(V; \mathbf{C}) \cong H_1(V; \mathbf{C}) \cong H^1(V; \mathbf{C}) = 0$. Then we obtain $q(V) = dim\, H^1(V, O_V) = 0$ by the Hodge decomposition. Since $H^2(V, D; \mathbf{C}) = 0$, we have $H^2(V; \mathbf{C}) = H^1(V, \Omega_V) = H^2(D; \mathbf{C})$, whence $p_g(V) = dim\, H^2(V, O_V) = 0$. We then have $H^1(D; \mathbf{C}) = 0$, which implies that D is simply connected. If X is rational, $H^2(V; \mathbf{Z})$ and $H^2(D; \mathbf{Z})$ are free abelian groups of the same rank. Hence the homomorphism $H^2(V; \mathbf{Z}) \to H^2(D; \mathbf{Z})$ is injective. Thence we can verify the assertions (1) and (3) above.

In order to prove the assertion (2), observe the following sequence of integral homology groups;

$$\begin{aligned} \cdots &\longrightarrow H_2(D; \mathbf{Z}) \longrightarrow H_2(V; \mathbf{Z}) \longrightarrow H_2(V, D; \mathbf{Z}) \\ &\longrightarrow H_1(D; \mathbf{Z}) \longrightarrow H_1(V; \mathbf{Z}) \longrightarrow \cdots, \end{aligned}$$

where $H_2(V, D; \mathbf{Z}) \cong H^2(X; \mathbf{Z}) \cong Pic(X)$. Since $H_1(D; \mathbf{Z}) = 0$ and $H_2(V; \mathbf{Z}) \cong H^2(V; \mathbf{Z})$, we have

$$H^2(X; \mathbf{Z}) \cong Coker(H_2(D; \mathbf{Z}) \longrightarrow H^2(V; \mathbf{Z})).$$

On the other hand, by the universal coefficient theorem we have

$$H^2(X\,;\,\mathbf{Z}) \cong Hom(H_2(X\,;\,\mathbf{Z}), \mathbf{Z}) \oplus Ext^1(H_1(X\,;\,\mathbf{Z}), \mathbf{Z}),$$

where $H_2(X\,;\,\mathbf{Z}) = (0)$ by the assertion (1) above and $Ext^1(H_1(X\,;\,\mathbf{Z}), \mathbf{Z}) \cong H_1(X\,;\,\mathbf{Z})$ because $H_1(X\,;\,\mathbf{Z})$ is a finite abelian group. Thus we have an isomorphism $H^2(X\,;\,\mathbf{Z}) \cong H_1(X\,;\,\mathbf{Z})$. ■

1.2. Let X be a $\mathbf{Q}$-homology plane and let $\kappa(X)$ be the Kodaira dimension of X, which takes value $-\infty$, 0, 1 or 2. If $\kappa(X) = -\infty$, we can describe the surface X as follows:

Theorem 1.2. *Let X be a $\mathbf{Q}$-homology plane of Kodaira dimension $-\infty$. Then X has an $\mathbf{A}^1$-fibration $f\colon X \to \mathbf{A}^1$ such that for every point P of $\mathbf{A}^1$, $f^{-1}(P)$ is irreducible and $f^{-1}(P)_{red}$ is isomorphic to $\mathbf{A}^1$. The homology group $H_1(X\,;\,\mathbf{Z})$ is a product of cyclic components $\mathbf{Z}/\mu_i\mathbf{Z}$, where μ_i is the multiplicity of a multiple fiber F_i of f and F_i runs through all multiple fibers of f.*

Proof. Since $\kappa(X) = -\infty$, X has an $\mathbf{A}^1$-fibration $f\colon X \to C$, where C is a smooth curve and $f(X) = C$. Since the boundary D consists of nonsingular rational curves, C must be rational. Since $H^2(X\,;\,\mathbf{Q}) = 0$, it is not difficult to show that C is isomorphic to $\mathbf{A}^1$ and every fiber of f is irreducible. The rest of the assertion is easy to verify. ■

Next we consider the case $\kappa(X) = 0$. According to Fujita [2, (8.64)] we have the following:

Theorem 1.3. *Let X be a $\mathbf{Q}$-homology plane with $\kappa(X) = 0$. Then X is isomorphic to one of the surfaces listed below:*

(1) *$H[k, -k]$ with $k \geq 1$. Let Σ_k be the Hirzebruch surface of degree k, let M be the minimal section and let ℓ be a fiber of the $\mathbf{P}^1$-fibration $f\colon \Sigma_k \to \mathbf{P}^1$ of Σ_k. Let M_1 be a cross-section of f such that $M_1 \sim M + (k+1)\ell$, let ℓ_1, ℓ_2 be two fibers of f not passing the point $P := M \cap M_1$, let $P_{i1} := \ell_i \cap M_1$ and let P_{i2} be the infinitely near point to P_{i1} of the first order lying on the fiber ℓ_i, where $i = 1, 2$. Let $\sigma\colon V \to \Sigma_k$ be the blowing-up of the points P_{ij} with $i, j = 1, 2$ and let E_i be the exceptional curve obtained by blowing up the point P_{i2} with $i = 1, 2$. Then $H[k, -k]$ is the surface $X := V - (\sigma^{-1}(M + M_1 + \ell_1 + \ell_2) - E_1 - E_2)$. We have $H_1(X\,;\,\mathbf{Z}) \cong \mathbf{Z}/4k\mathbf{Z}$.*

(2) *$Y\{3, 3, 3\}$. Let ℓ_i $(i = 1, 2, 3)$ be three non-concurrent lines on $\mathbf{P}^2$, let $P_{ij} := \ell_i \cap \ell_j$, let P'_{ij} be the infinitely near point to P_{ij} of the first order lying on the line ℓ_i, where $(ij) = (12), (23), (31)$ and let ℓ_4 be a fourth line not passing any of the points P'_{ij}s. Let $\sigma\colon V \to \mathbf{P}^2$ be the blowing-up of six points P_{ij} and P'_{ij} and let E'_i be the exceptional curve obtained by blowing up the point P'_{ij}, where $i = 1, 2, 3$. Then $Y\{3, 3, 3\}$ is the surface $X := V - (\sigma^{-1}(\ell_1 + \ell_2 + \ell_3 + \ell_4) - E'_1 - E'_2 - E'_3)$. Moreover, $H_1(X\,;\,\mathbf{Z}) \cong \mathbf{Z}/9\mathbf{Z}$.*

(3) *$Y\{2, 4, 4\}$. Let M_i and ℓ_i be respectively cross-sections and fibers of the $\mathbf{P}^1$-fibration f on the Hirzebruch surface $\Sigma_0 = \mathbf{P}^1 \times \mathbf{P}^1$ with $i = 1, 2, 3$, let $P_{ij} := M_i$*

$\cap \ell_j$ and let P'_{23} and P'_{32} be the infinitely near points to P_{23} and P_{32} of the first order lying on the cross-sections M_2 and M_3, respectively. Let $\sigma: V \to \Sigma_0$ be the blowing-up of six points $P_{12}, P_{13}, P_{23}, P_{32}, P'_{23}, P'_{32}$ and let $E_{12}, E_{13}, E'_{23}, E'_{32}$ be the exceptional curves obtained from the blowing-up of the points $P_{12}, P_{13}, P'_{23}, P'_{32}$. Then $Y\{2, 4, 4\}$ is the surface $X := V - (\sigma^{-1}(M_1 + M_2 + M_3 + \ell_1 + \ell_2 + \ell_3) - E_{12} - E_{13} - E'_{23} - E'_{32})$. Moreover, $H_1(X;\mathbf{Z}) \cong \mathbf{Z}/8\mathbf{Z}$.

(4) *$Y\{2, 3, 6\}$. We do not specify the construction of this surface. See* [2, (8.59) and (8.61)]. *We have $H_1(X;\mathbf{Z}) \cong \mathbf{Z}/6\mathbf{Z}$.*

1.3. Now let X be a $\mathbf{Q}$-homology plane with $\kappa(X) = 1$. Our argument in this case will basically follow the one in [3]. It is known [7] that there is a $\mathbf{C}^*$-fibration $\pi: X \to C$, where C is isomorphic to either $\mathbf{P}^1$ or $\mathbf{A}^1$ because $\Gamma(\mathcal{O}_X)^* = \mathbf{C}^*$ (cf. Lemma 1.1). The fibration $\pi: X \to C$ is extended to a $\mathbf{P}^1$-fibration $p: V \to B$, where V is a nonsingular rational surface such that X is an open subset of V and $D := V - X$ is an effective divisor with simple normal crossings, B is a nonsingular complete curve containing C as an open set and the restriction of p onto X is π. The boundary divisor D contains one or two horizontal components according as the $\mathbf{C}^*$-fibration π is twisted or not. If π is twisted, we denote the unique horizontal component by H, and if π is untwisted, we denote two horizontal components by H_1 and H_2. Except for the horizontal components, all the other components of D are contained in singular fibers of the $\mathbf{P}^1$-fibration p. Since X is affine, any singular fiber F of the $\mathbf{C}^*$-fibration π has the following form. Namely, $F = \Delta + \Gamma$, where $\Gamma_{red} = \phi$, $\mathbf{C}^*$ or $\mathbf{A}^1 + \mathbf{A}^1$, and Δ_{red} is a disjoint union of $\mathbf{A}^1$'s. We have the following:

Lemma 1.4. (1) *If C is isomorphic to $\mathbf{P}^1$ then π is untwisted, every fiber of π is irreducible and there is exactly one fiber F such that $F_{red} \cong \mathbf{A}^1$.*

(2) *If C is isomorphic to $\mathbf{A}^1$ and π is untwisted, then all the fibers of π are irreducible except for one singular fiber which consists of two irreducible components. If C is isomorphic to $\mathbf{A}^1$ and π is twisted, all fibers are irreducible and there is exactly one fiber which is isomorphic to $\mathbf{A}^1$.*

Proof. We follow the proof of Lemma 3.2 in [3]. Let $F_1, \ldots, F_n$ exhaust all the singular fibers of π. Let $\tilde{F}_i$ be the fiber of p containing F_i. Write

$$\tilde{F}_i = \sum_{j=1}^{r_i} \mu_{ij} C_{ij} + \sum_{j=r_i+1}^{s_i} \delta_{ij} D_{ij},$$

where $C'_{ij}s$ and $D'_{ij}s$ are irreducible components such that $C_{ij} \cap X \neq \phi$ and $D_{ij} \cap X = \phi$.

Suppose π is twisted and $C \cong \mathbf{P}^1$. Then

$$\mathit{rank}\, \mathit{Pic}(X) = 2 + \sum_{i=1}^{n} (s_i - 1) - (1 + \sum_{i=1}^{n} (s_i - r_i)) = 1 + \sum_{i=1}^{n} (r_i - 1) \geq 1,$$

which is not the case because $\mathit{rank}\, \mathit{Pic}(X) = 0$ by Lemma 1.1.

Suppose π is twisted and $C \cong \mathbf{A}^1$. Then H has two ramification points

P_1, P_2 of $\pi|_H \colon H \to B$. Since D is a tree, the unique fiber of π contained in D must pass through P_1 or P_2, say P_1. Then the fiber $\pi^{-1}(p(P_2))$ is isomorphic to $\mathbf{A}^1$ and other fibers are isomorphic to $\mathbf{A}^1_*$. For otherwise, D would not be a tree.

Suppose $C \cong \mathbf{P}^1$, whence π is untwisted. Then we have a relation

$$H_1 - H_2 \sim \sum_{i,j} \alpha_{ij} C_{ij} + \begin{pmatrix}\text{a linear combination} \\ \text{of fiber components of } D\end{pmatrix}.$$

Therefore $Pic(X)$ is the abelian group defined by the following generators and relations

$$Pic(X) = \left\langle [C_{ij}] \begin{array}{l} i = 1,\dots,n \\ \\ j = 1,\dots,r_i \end{array} \middle| \begin{array}{l} \sum_{i,j} \alpha_{ij}[C_{ij}] = 0 \\ \sum_{j=1}^{r_i} \mu_{1j}[C_{1j}] = \cdots = \sum_{j=1}^{r_n} \mu_{nj}[C_{nj}] \end{array} \right\rangle,$$

if $C \cong \mathbf{P}^1$ and

$$Pic(X) = \left\langle [C_{ij}] \begin{array}{l} i = 1,\dots,n \\ \\ j = 1,\dots,r_i \end{array} \middle| \begin{array}{l} \sum_{i,j} \alpha_{ij}[C_{ij}] = 0 \\ \sum_{j=1}^{r_1} \mu_{1j}[C_{1j}] = \cdots = \sum_{j=1}^{r_n} \mu_{nj}[C_{nj}] = 0 \end{array} \right\rangle,$$

if $C \cong \mathbf{A}^1$. Hence $rank(Pic(X)) \geq \sum_{i=1}^n r_i - n$ if $C \cong \mathbf{P}^1$ and $rank(Pic(X)) \geq \sum_{i=1}^n r_i - (n+1)$ if $C \cong \mathbf{A}^1$. Since $Pic(X)$ is a finite group, we have $r_i = 1$ for all i if $C \cong \mathbf{P}^1$, and $r_i = 1$ for all $i \geq 2$ and $r_1 = 2$ if $C \cong \mathbf{A}^1$.

If $C \cong \mathbf{P}^1$ one of the fibers $F_1, \dots, F_n$ must be isomorphic to $\mathbf{A}^1$, for otherwise D would be disconnected. However there are no two such fibers, for otherwise D would not be a tree. ■

1.4. In the subsequent paragraphs, we construct examples of a **Q**-homology planes with an automorphism of finite order whose fixed point locus consists of several disconnected components, one of them being isomorphic to the complex line **C** and the others being isomorphic to the complex line with one point deleted off **C***. These surfaces are not homology planes by virtue of the following

Lemma 1.5. *Let X be a homology plane with an effective action of a cyclic group G of prime order p. Then we have:*

(1) *The fixed point locus X^G is connected.*
(2) *If $\dim X^G = 1$ then X^G is the complex line* **C**.

Proof. Recall the following result from the theory of transformation groups.

Let G be a cyclic group of finite order p, let X be a finite-dimensional G-space and let X^G be the fixed point locus. Then, for cohomology groups with $\mathbf{Z}/p\mathbf{Z}$-coefficients, we have

$$\sum_{i=n}^{\infty} rank\, H^i(X^G; \mathbf{Z}/p\mathbf{Z}) \leq \sum_{i=n}^{\infty} rank\, H^i(X; \mathbf{Z}/p\mathbf{Z}) \qquad \text{for } \forall n \in \mathbf{Z}.$$

We apply this result to our homology plane X with $n = 0$. Since $H^i(X; \mathbf{Z}/p\mathbf{Z}) = 0$ for all $n > 0$ and $H^0(X^G; \mathbf{Z}/p\mathbf{Z}) \neq 0$, we must have $H^i(X^G; \mathbf{Z}/p\mathbf{Z}) = 0$ for all $n > 0$ and $rank\, H^0(X^G; \mathbf{Z}/p\mathbf{Z}) = 1$. This implies that the fixed point locus X^G is connected and the Euler characteristic $e(X^G)$ equals 1. We shall show that X^G is nonsingular. If $dim\, X^G = 0$ we have nothing to prove. So, suppose $dim\, X^G = 1$. Suppose that X^G has two irreducible components, say C_1 and C_2, meeting at a point P. If C_1 and C_2 have different tangents at the point P, then the action of G on X is trivial near the point P, hence everywhere on X. This is a contradiction. Suppose that C_1 and C_2 have the same tangent at P. Then blow up G-equivariantly the point P. The proper transforms C_1' and C_2' of C_1 and C_2 either get separated over the point P or still have a common point P'. Then repeat the above argument again with C_1', C_2' and P'. After repeating several G-equivariant blowing-ups, the proper transforms of C_1 and C_2 have a common point and different tangents there. This is a contradiction. So, we conclude that X^G is irreducible. We shall show that the curve X^G is nonsingular. If X^G has a singular point P, the induced action of G on its Zariski cotangent space is trivial and the Zariski cotangent space is identified with the Zariski cotangent space of X at P. So, the action of G on X is trivial near the point P, hence everywhere on X, which is a contradiction. Hence X^G is nonsingular. If $dim\, X^G = 1$, X^G is homeomorphic to the complex line. Therefore, X^G is algebracally isomorphic to the complex line. ■

Similar results hold for $\mathbf{Q}$-homology planes. Namely we have

Lemma 1.6. *Let X be a $\mathbf{Q}$-homology plane with an effective action of a cyclic group G of prime order p. Then X^G is nonsingular. Namely, if X^G contains an irreducible curve, it is nonsingular and every connected component of X^G is irreducible. Moreover, X^G has the Euler number $e(X^G) = 1$. If X is an n-primary rational homology plane and n is prime to p then either X^G is a single point or isomorphic to the complex line. If X is p-primary and rational, we have*

$$N - 1 + b_1^p(X^G) \leq b_1^p(X),$$

where N is the number of irreducible components of X^G and $b_1^p(X) = dim_{\mathbf{Z}/p\mathbf{Z}} H_1(X; \mathbf{Z}/p\mathbf{Z})$ and $b_1^p(X^G) = dim_{\mathbf{Z}/p\mathbf{Z}} H_1(X^G; \mathbf{Z}/p\mathbf{Z})$.

Proof. Similar arguments as in Lemma 1.5 apply to this case, as well. The Euler number $e(X^G) = 1$ is obtained from the Lefschetz fixed point theorem. The last inequality is an easy consequence of the inequality cited in the proof of Lemma 1.6. ■

Example 1. Let ℓ_1, ℓ_2, ℓ_3 be concurrent lines on P^2 with the common point P and let ℓ_4 be a line not passing through P. Let $\sigma\colon Y \to P^2$ be a composite of the following blowing-ups: First blow up the point P and let E be the resulting (-1) curve. For $1 \leq i \leq 3$, let P_i (resp. Q_i) be the intersection point of E (resp. ℓ_4) with the proper transform of ℓ_i. Next blow up these six point P_i's and Q_i's to obtain the surface Y and let A_i's and B_i's be the resulting (-1)-curves on Y as the inverse images of P_i's and Q_i's, respectively. Name the proper transform of ℓ_i on X as L_i for $1 \leq i \leq 4$ and the proper transform of E as C. Then we have $(C^2) = -4, (L_i^2) = -2$ for $1 \leq i \leq 3$ and $(L_4^2) = -2$.

Let $\rho: V \to Y$ be the double covering with branch locus C, L_1, L_2, L_3 and L_4. Then V is a rational surface with rank 8. Let $M = \rho^{-1}(C)$, $E_i = \rho^{-1}(A_i)$, $F_i = \rho^{-1}(B_i)$, $H_i = \rho^{-1}(L_i)$ for $1 \leq i \leq 3$ and let $N = \rho^{-1}(L_4)$. Then $\rho^*(C) = 2M$, $\rho^*(A_i) = E_i$, $\rho^*(B_i) = F_i$, $\rho^*(L_i) = 2H_i$ for $1 \leq i \leq 3$ and $\rho^*(L_4) = 2N$. Hence $(M^2) = -2$, $(N^2) = -1$, $(E_i^2) = (F_i^2) = -2$ and $(H_i^2) = -1$ for $1 \leq i \leq 3$. The covering involution ι acts on V in such a way that the fixed point locus V^ι consists of M, N, H_1, H_2 and H_3. Now obtain an affine surface X as an open set of V by removing the curves $M, N, E_2, E_3, H_3, F_3, F_2$ and F_1. Then there exists a C^*-fibration on X obtained as the restriction of the P^1-fibration $|E_3 + H_3 + F_3|$ onto P^1. The involution ι acts along this C^*-fibration and there is only one reducible fiber $(E_1 + H_1) \cap X$. It is now straightforward to verify that $H^i(X; \mathbf{Q}) = 0$ for all $i > 0$. So, X is a $\mathbf{Q}$-homology plane. Furthermore X^ι consists of $(H_1 + H_2) \cap X$ which consists of two disjoint curves, one being isomorphic to $\mathbf{C}$ and the other being isomorphic to $\mathbf{C}^*$.

Example 2. Let C be an irreducible rational curve of degree $n > 1$ in $\mathbf{P}^2$ with a cusp P of order $n-1$ and let ℓ_i $(1 \leq i \leq n)$ be lines passing through the cusp P which are not tangent to the curve C at P, thus ℓ_i meeting C in a single point different from P. Let W be a cyclic covering of $\mathbf{P}^2$ of order n branched over $C \cup \ell_1 \cup \cdots \cup \ell_n$ and let V be the minimal resolution of singularities of W. Then V has a $\mathbf{P}^1$-fibration $\rho: V \to \mathbf{P}^1$ and the proper transform of ℓ_i $(1 \leq i \leq n)$ on V is a (-1)-curve E_i contained in a reducible fiber Γ_i of ρ with a linear chain of length $n+1$ as its dual graph which is described as follows:

$$\Gamma_i = A^{(i)} + nE_i + (n-1)B_{n-1}^{(i)} + (n-2)B_{n-2}^{(i)} \cdots + B_1^{(i)},$$

where $(A^{(i)})^2 = -n$, $(B_{n-1}^{(i)})^2 = (B_{n-2}^{(i)})^2 = \cdots = (B_1^{(i)})^2 = -2$, $(A^{(i)} \cdot E_i) = (E_i \cdot B_{n-1}^{(i)}) = (B_{n-1}^{(i)} \cdot B_{n-2}^{(i)}) = \cdots = (B_2^{(i)} \cdot B_1^{(i)}) = 1$. The fibration ρ has two sections M and N, where M is a (-2)-curve mapped to the point P by the composite π of the resolution morphism and the covering morphism and N is a (-1)-curve mapped to the curve C and where M meets $A^{(i)}$ and N meets $B_1^{(i)}$. Except for fibers $\Gamma_1, \ldots, \Gamma_n$, ρ has one more singular fiber Δ, which is contained in the inverse image by π of the tangent line of C at P and can be contracted to a smooth fiber without changing the self-intersection numbers of M and N. Let X be an affine open set obtained from V by removing the curves M, N, Δ, all components of Γ_1 other than E_1 and $B_{n-1}^{(1)}$ and all components of Γ_i other than E_i for every $2 \leq i \leq n$. Then it is straightforward to show that:

(1) X is an n-primary rational homology plane with $H_1(X; \mathbf{Z}/n\mathbf{Z}) \cong (\mathbf{Z}/n\mathbf{Z})^{n-1}$.

(2) X has an automorphism of order N inheriting the covering transformation of $\pi: V \to \mathbf{P}^2$.

(3) X^G consists of $X \cap E_1$ which is isomorphic to $\mathbf{C}$ and $X \cap E_i$ for $2 \leq i \leq n$ which is isomorphic to $\mathbf{C}^*$.

1.5. We shall construct further examples of **Q**-homology planes of Kodaira dimension 0 admitting non-trivial automorphisms.

Example 3. Take X to be $H[k, -k]$ in Theorem 1.3. Choose an inhomogeneous coordinate t on the base curve $\mathbf{P}^1$ so that $t = 0, 1, -1$ at the points $f(P), f(P_{11}), f(P_{21})$, respectively. Let $\iota: \mathbf{P}^1 \to \mathbf{P}^1$ be the involution such that $\iota^*(t) = -t$. Suppose $k \equiv 1 \pmod 2$. Then ι induces an involution on Σ_k, which we denote by the same letter ι. This involution ι fixes two fibers including the one passing the point P and exchanges the fibers passing the points P_{11} and P_{21}. So, ι induces an involution on X such that the fixed point locus is the fiber $t = \infty$ plus a point on the fiber $t = 0$.

Example 4. Take X to be the surface $Y\{3, 3, 3\}$ in Theorem 1.3. We choose the points P_{23}, P_{31} and P_{12} to be $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, respectively. Choose the line ℓ_4 to be defined by the equation $x_0 + \omega x_1 + \omega^2 x_2 = 0$. So, ℓ_4 meets the lines ℓ_1, ℓ_2 and ℓ_3 respectively at the points $(0, -\omega, 1)$, $(-\omega^2, 0, 1)$ and $(-\omega, 1, 0)$. Let ρ be an automorphism of order 3 on $\mathbf{P}^2$ defined by

$$\rho\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}.$$

Then the line ℓ_4 and the set $\{P_{ij}, P'_{ij} | (ij) = (12), (23), (31)\}$ are ρ-stable. Hence ρ induces an automorphism of order 3 on X such that the fixed point locus consists of three points $(1, 1, 1)$, $(1, \omega, \omega^2)$ and $(1, \omega^2, \omega)$.

Example 5. Take X to be the surface $Y\{2, 4, 4\}$ in Theorem 1.3. Choose inhomogeneous coordinates (t, u) on $\mathbf{P}^1 \times \mathbf{P}^1$ in such a way that the cross-sections (resp. fibers) M_1, M_2, M_3 (resp. ℓ_1, ℓ_2, ℓ_3) are defined by $u = 0, 1, -1$ (resp. $t = 0, 1, -1$). Define an involution ι on $\mathbf{P}^1 \times \mathbf{P}^1$ by $\iota^*(t, u) = (-t, u)$. Then ι induces an involution on X such that the fixed point locus consists of a single point $(t^{-1}, u^{-1}) = (0, 0)$.

1.6. As we have observed above, **Q**-homology planes with Kodaira dimension ≤ 1 are rational and usually have non-trivial automorphisms of finite order unless they are not homology planes. So, we can raise the following questions:

Question 1. Is a **Q**-homology plane of Kodaira dimension 2 rational?

If such a **Q**-homology plane X exists, a smooth completion V of X must satisfy $q(V) = p_g(V) = 0$ in view of Lemma 1.1.

Question 2. Can one evaluate the order of the automorphism group $Aut(X)$ when X is a **Q**-homology plane of Kodaira dimension 2 ?

§2. Logarithmic Q-homology planes

2.1. Let (X, x) be a germ of normal surface singularity. It is called *logterminal* if the following two conditions are satisfied:

(1) K_X is $\mathbf{Q}$-Cartier,

(2) Let $f\colon Y \to X$ be a minimal resolution of singularity and let $F_1, \dots, F_n$ exhaust irreducible exceptional curves of f. If we write

$$K_Y = f^*(K_X) + \sum_{i=1}^{n} a_i F_i \qquad a_i \in \mathbf{Q},$$

then we have $0 \geq a_i > -1$ for every i.

It is known (cf. Tsunoda [14]) that a log-terminal singularity (X, x) is a quotient singularity. Namely, (X, x) is isomorphic to $(\mathbf{C}^2/G, (0))$, where G is a small finite subgroup of $GL(2, \mathbf{C})$. We call a normal algebraic surface X *logarithmic* if it has only log-terminal singularities. A logarithmic affine surface X is called a *logarithmic homology plane* (resp. *logarithmic* $\mathbf{Q}$*-homology plane*) if $H_i(X; \mathbf{Z}) = (0)$ (resp. $H_i(X; \mathbf{Q}) = (0)$) for all $i > 0$. In this section, we shall look into structures of logarithmic $\mathbf{Q}$-homology planes. We shall begin with

Lemma 2.1. *Let X be a $\mathbf{Q}$-homology plane with a finite group G acting on it. Let $S = X/G$ be the algebraic quotient of X by G. Then S is a logarithmic $\mathbf{Q}$-homology plane.*

Proof. Let $\pi\colon X \to S$ be the quotient morphism. Then π induces an isomorphism of the cohomology rings

$$\pi^*\colon H^*(S; \mathbf{Q}) \longrightarrow H^*(X; \mathbf{Q})^G.$$

Since $H^*(X; \mathbf{Q}) \cong H^0(X; \mathbf{Q}) \cong \mathbf{Q}$ by the hypothesis, we conclude that $H^i(S; \mathbf{Q}) = (0)$ for $i > 0$. Hence S is a $\mathbf{Q}$-homology plane. ■

Let X be a logarithmic affine algebraic surface, let $\Sigma = \{x_1, \dots, x_N\}$ be the singular locus and let $\varphi\colon Y \to X$ be the minimal resolution of singularities. Let V be a normal projective surface such that X is an open subset of V and the complement $D := V - X$ is an effective divisor with simple normal crossings and let $f\colon W \to V$ be the minimal resolution of singularities of V. Then Y is an open set of W and $f|_Y = \varphi$. We identify the divisor D on V with the divisor $f^{-1}(D)$ on W. Set $\Delta = f^{-1}(\Sigma)$ and $X^\circ = X - \Sigma$.

Lemma 2.2. *With these notations we have the following:*

(1) *We have isomorphisms and exact sequences*

$$H^1(V, D; \mathbf{Z}) \cong H_3(Y; \mathbf{Z}), \quad H^2(V, D; \mathbf{Z}) \cong H_2(X^\circ; \mathbf{Z})$$

$$H^i(V, D; \mathbf{Z}) \cong H_{4-i}(X^\circ; \mathbf{Z}) \cong H_{4-i}(Y; \mathbf{Z}) \qquad \textit{for } i \neq 1, 2$$

$$0 \longrightarrow \mathbf{Z}^{\oplus N} \longrightarrow H_3(X^\circ; \mathbf{Z}) \longrightarrow H^1(V, D; \mathbf{Z}) \longrightarrow 0$$

$$0 \longrightarrow \mathbf{Z}^{\oplus N} \longrightarrow H_3(X^\circ; \mathbf{Z}) \longrightarrow H_3(Y; \mathbf{Z}) \longrightarrow 0$$
$$0 \longrightarrow H_2(X^\circ; \mathbf{Z}) \longrightarrow H_2(Y; \mathbf{Z}) \longrightarrow \bigoplus_{i=1}^{N} \mathbf{Z}^{\oplus r_i} \longrightarrow 0,$$

where N is the number of singular points and r_i is the number of irreducible components of $f^{-1}(x_i)$.

(2) *We have isomorphisms and an exact sequence*

$$H^i(V, D; \mathbf{Z}) \cong H_{4-i}(X; \mathbf{Z}) \qquad \text{for } i = 0, 1, 4,$$
$$0 \longrightarrow H^2(V, D; \mathbf{Z}) \longrightarrow H_2(X; \mathbf{Z}) \longrightarrow H_1(\partial T; \mathbf{Z}) \longrightarrow H^3(V, D; \mathbf{Z}) \longrightarrow H_1(X; \mathbf{Z}) \longrightarrow 0.$$

(3) *If X is a logarithmic $\mathbf{Q}$-homology plane, then D is simply connected and $p_g(W) = q(W) = 0$. Moreover, $\Gamma(X, \mathcal{O}_X)^* = \mathbf{C}^*$.*

Proof. (1) Write a long exact sequence of integral cohomologies attached to the inclusion of pairs($(D \cup \Sigma, D) \subset (V, D)$;

$$\begin{aligned}
0 &\longrightarrow H^0(V, D \cup \Sigma; \mathbf{Z}) \longrightarrow H^0(V, D; \mathbf{Z}) \longrightarrow H^0(D \cup \Sigma, D; \mathbf{Z}) \\
&\longrightarrow H^1(V, D \cup \Sigma; \mathbf{Z}) \longrightarrow H^1(V, D; \mathbf{Z}) \longrightarrow H^1(D \cup \Sigma, D; \mathbf{Z}) \\
&\longrightarrow H^2(V, D \cup \Sigma; \mathbf{Z}) \longrightarrow H^2(V, D; \mathbf{Z}) \longrightarrow H^2(D \cup \Sigma, D; \mathbf{Z}) \\
&\longrightarrow H^3(V, D \cup \Sigma; \mathbf{Z}) \longrightarrow H^3(V, D; \mathbf{Z}) \longrightarrow 0 \\
&\longrightarrow H^4(V, D \cup \Sigma; \mathbf{Z}) \longrightarrow H^4(V, D; \mathbf{Z}) \longrightarrow 0,
\end{aligned}$$

where

$$H^0(D \cup \Sigma, D; \mathbf{Z}) \cong \mathbf{Z}^{\oplus N} \quad \text{and} \quad H^i(D \cup \Sigma, D; \mathbf{Z}) = (0) \qquad \text{for } i \neq 0, 2.$$

Furthermore, we have isomorphisms

$$H^i(V, D \cup \Sigma; \mathbf{Z}) \cong H_{4-i}(X^\circ : \mathbf{Z}) \qquad \text{for } \forall i > 0.$$

Hence we obtain the isomorphisms and exact sequence involving (V, D) and X°. In a similar way, write down a long exact sequence of integral cohomologies attached to the inclusion $(D \cup \Delta, D) \subset (W, D)$. Noting that $W - (D \cup \Delta) \cong X^\circ$ and that each connected component of Δ is a tree of nonsingular rational curves, we obtain the isomorphisms and exact sequences,

$$H_i(X^\circ; \mathbf{Z}) \cong H_i(Y; \mathbf{Z}) \qquad \text{for } i \neq 2, 3$$
$$0 \longrightarrow \mathbf{Z}^{\oplus N} \longrightarrow H_3(X^\circ; \mathbf{Z}) \longrightarrow H_3(Y; \mathbf{Z}) \longrightarrow 0$$
$$0 \longrightarrow H_2(X^\circ; \mathbf{Z}) \longrightarrow H_2(Y; \mathbf{Z}) \longrightarrow \bigoplus_{i=1}^{N} \mathbf{Z}^{\oplus r_i} \longrightarrow 0.$$

Combining these two results, we obtain the stated isomorphisms and exact sequences.

(2) Write $\Sigma = \{x_1, \dots, x_N\}$. For all i, choose a closed neighbourhood T_i of x_i so that $T_i \cap T_j = \emptyset$ whenever $i \neq j$. Let ∂T_i be the boundary of T_i. Write $T := T_1 \cup \cdots \cup T_N$ and $\partial T = \partial T_1 \cup \cdots \cup \partial T_N$. Consider a long exact sequence of integral

homologies:

$$0 \longrightarrow H_4(X; \mathbf{Z}) \longrightarrow H_4(X, T; \mathbf{Z}) \longrightarrow H_3(T; \mathbf{Z}) \longrightarrow H_3(X; \mathbf{Z}) \longrightarrow$$
$$H_3(X, T; \mathbf{Z}) \longrightarrow H_2(T; \mathbf{Z}) \longrightarrow H_2(X; \mathbf{Z}) \longrightarrow H_2(X, T; \mathbf{Z}) \longrightarrow H_1(T; \mathbf{Z}) \longrightarrow$$
$$H_1(X; \mathbf{Z}) \longrightarrow H_1(X, T; \mathbf{Z}) \longrightarrow H_0(T; \mathbf{Z}) \longrightarrow H_0(X; \mathbf{Z}) \longrightarrow H_0(X, T; \mathbf{Z})$$
$$\longrightarrow 0,$$

where $H_i(X, T; \mathbf{Z}) \cong H_i(X^\circ, \partial T; \mathbf{Z})$ for every i by the excision theorem. Since every x_i is a quotient singular point, we may assume that T_i is a quotient B_i/G_i of the closed ball $B_i := \{(z_1, z_2) \mid |z_1|^2 + |z_2|^2 \leq \varepsilon\}$ in $\mathbf{C}^2$ which is stable under the induced linear action of a small finite subgroup G_i of $GL(2, \mathbf{C})$. Since B_i is contractible, T_i is contractible by Conner's theorem (cf. Kraft-Petrie-Randall [5]). Hence we have

$$H_0(T; \mathbf{Z}) = \mathbf{Z}^{\oplus N} \quad \text{and} \quad H_i(T; \mathbf{Z}) = (0) \qquad \text{for } \forall i > 0.$$

Thence we have isomorphisms and an exact sequence

$$H_i(X; \mathbf{Z}) \cong H_i(X, T; \mathbf{Z}) \qquad \text{for } i \geq 2,$$
$$0 \longrightarrow H_1(X; \mathbf{Z}) \longrightarrow H_1(X, T; \mathbf{Z}) \longrightarrow H_0(T; \mathbf{Z}) \longrightarrow H_0(X; \mathbf{Z}) \longrightarrow 0.$$

On the other hand, we have a long exact sequence of homologies attached to a pair $(X^\circ, \partial T)$,

$$0 \longrightarrow H_4(X^\circ; \mathbf{Z}) \longrightarrow H_4(X^\circ, \partial T; \mathbf{Z}) \longrightarrow H_3(\partial T; \mathbf{Z}) \longrightarrow H_3(X^\circ; \mathbf{Z})$$
$$\longrightarrow H_3(X^\circ, \partial T; \mathbf{Z}) \longrightarrow H_2(\partial T; \mathbf{Z}) \longrightarrow H_2(X^\circ; \mathbf{Z}) \longrightarrow H_2(X^\circ, \partial T; \mathbf{Z})$$
$$\longrightarrow H_1(\partial T; \mathbf{Z}) \longrightarrow H_1(X^\circ; \mathbf{Z}) \longrightarrow H_1(X^\circ, \partial T; \mathbf{Z}) \longrightarrow H_0(\partial T; \mathbf{Z})$$
$$\longrightarrow H_0(X^\circ; \mathbf{Z}) \longrightarrow 0.$$

Since the action of G_i on $\partial B_i \cong S^3$ is orientation preserving, we have

$$H_0(\partial T_i; \mathbf{Z}) \cong H_3(\partial T_i; \mathbf{Z}) \cong \mathbf{Z}.$$

It is known that $H_1(\partial T_i; \mathbf{Z}) \cong \pi_1(\partial T_i)/[\pi_1(\partial T_i), \pi_1(\partial T_i)]$, which is a finite group. Then, by Poincaré duality and the universal coefficient theorem, we know that $H_2(\partial T_i; \mathbf{Z}) = (0)$. Therefore we have an isomorphism and exact sequences,

$$H_4(X^\circ; \mathbf{Z}) \cong H_4(X^\circ, \partial T; \mathbf{Z})$$
$$0 \longrightarrow H_3(\partial T; \mathbf{Z}) \longrightarrow H_3(X^\circ; \mathbf{Z}) \longrightarrow H_3(X^\circ, \partial T; \mathbf{Z}) \longrightarrow 0$$
$$0 \longrightarrow H_2(X^\circ; \mathbf{Z}) \longrightarrow H_2(X^\circ, \partial T; \mathbf{Z}) \longrightarrow H_1(\partial T; \mathbf{Z}) \longrightarrow H_1(X^\circ; \mathbf{Z})$$
$$\longrightarrow H_1(X^\circ, \partial T; \mathbf{Z}) \longrightarrow H_0(\partial T; \mathbf{Z}) \longrightarrow H_0(X^\circ; \mathbf{Z}) \longrightarrow 0.$$

Now comparing the above isomorphisms and exact sequences, we obtain

$$H_4(X^\circ; \mathbf{Z}) \cong H_4(X^\circ, \partial T; \mathbf{Z}) \cong H_4(X, T; \mathbf{Z}) \cong H_4(X; \mathbf{Z}),$$
$$H^1(V, D; \mathbf{Z}) \cong H_3(X^\circ, \partial T; \mathbf{Z}) \cong H_3(X, T; \mathbf{Z}) \cong H_3(X; \mathbf{Z}),$$

$$0 \longrightarrow H_2(X^\circ; \mathbf{Z}) \longrightarrow H_2(X; \mathbf{Z}) \longrightarrow H_1(\partial T; \mathbf{Z})$$
$$\longrightarrow H_1(X^\circ; \mathbf{Z}) \longrightarrow H_1(X; \mathbf{Z}) \longrightarrow 0.$$

Now the stated results are easily verified in view of the assertion (1) above.

(3) Since $H_1(\partial T; \mathbf{Z})$ is a torsion group, we have isomorphisms

$$H^2(V, D; \mathbf{Q}) \cong H_2(X; \mathbf{Q}) \quad \text{and} \quad H^3(V, D; \mathbf{Q}) \cong H_1(X; \mathbf{Q}).$$

Suppose X is a logarithmic $\mathbf{Q}$-homology plane. Then, by virtue of the assertions (1) and (2) above, we have $H_i(Y; \mathbf{Q}) = (0)$ for $i \neq 0, 2$ and $H_2(Y; \mathbf{Q}) \cong \bigoplus_{i=1}^N \mathbf{Q}^{\oplus r_i}$. Look at a long exact sequence of rational cohomologies attached to a pair (W, D),

$$0 \longrightarrow H^0(W, D) \longrightarrow H^0(W) \longrightarrow H^0(D) \longrightarrow H^1(W, D) \longrightarrow H^1(W) \longrightarrow H^1(D)$$
$$\longrightarrow H^2(W, D) \longrightarrow H^2(W) \longrightarrow H^2(D) \longrightarrow H^3(W, D) \longrightarrow H^3(W) \longrightarrow 0$$
$$\longrightarrow H^4(W, D) \longrightarrow H^4(W) \longrightarrow 0,$$

where

$$H^i(W, D) \cong H_{4-i}(Y) \qquad \text{for } \forall i.$$

Taking the above remark into account, we know from this exact sequence that

$$H_3(W) \cong H^1(W) = (0) \quad \text{and} \quad H^1(D) = (0),$$

whence we conclude the assertion (3) as in Lemma 1.1. ■

Lemma 2.1 can be strengthened to the following effect:

Theorem 2.3. *Let X be a homology plane with an effective action of a cyclic group G of prime order p. Suppose that X^G is isolated. Let $Z = X/G$. Then Z is a logarithmic homology plane.*

Proof. Let P be the unique G-fixed point of X (cf. Lemma 1.5). Let $q: X \to Z$ be the quotient morphism and let $Q = q(P)$. Then Q has a cyclic quotient singularity, and Z is a logarithmic $\mathbf{Q}$-homology plane by Lemma 2.1. Hence $H_*(Z; \mathbf{Q}) \cong \mathbf{Q}$. Suppose $H_1(Z; \mathbf{Z}) \neq (0)$. Then $H_1(Z^\circ; \mathbf{Z}) \neq (0)$ and is a finite group, where $Z^\circ = Z - \{Q\}$ (cf. Lemma 2.2). Since $q: X^\circ \to Z^\circ$ is an unramified covering, $H_1(\mathbf{Z}^\circ; \mathbf{Z}) \supseteq \mathbf{Z}/p\mathbf{Z}$. On the other hand, $|H_1(Z^\circ; \mathbf{Z})/Im(H_1(X^\circ: \mathbf{Z})| \leq p$ by a general theory. Hence $H_1(Z^\circ; \mathbf{Z}) = \mathbf{Z}/p\mathbf{Z}$. Since $H_1(\partial T; \mathbf{Z}) = \mathbf{Z}/p\mathbf{Z}$ and since $H_2(Z; \mathbf{Z}) = (0)$ (cf. the proof of Lemma 2.6), we obtain $H_1(Z; \mathbf{Z}) = (0)$ by Lemma 2.2 (or an exact sequence in its proof). This is a contradiction. Thus Z is a logarithmic homology plane. ■

2.2. Let X be a normal affine surface with a $\mathbf{A}^1$-fibration $\rho: X \to C$. Let $\varphi: Y \to X$ be a minimal resolution of singularities. Then we have the following result by Miyanishi [6]:

Lemma 2.4. *Let X be as above. Then we have*:

(1) *Every singular point P of X is a cyclic quotient singular point.*

(2) *Every fiber of ρ is a disjoint union of curves which are isomorphic to $\mathbf{A}^1$. Each irreducible component has at most one singular point of X.*

(3) *Let P be a singular point lying on an irreducible component T of $\rho^{-1}(\rho(P))$ and let $C_1,\dots,C_r$ exhaust all irreducible components of $\varphi^{-1}(P)$, where C_r meets the proper transform $\varphi'(T)$, $(C_i\cdot C_{i+1})=1$ for $1\le i<r$ and $(C_i^2)=-\alpha_i$ with $\alpha_i\ge 2$. (We say that P is of type $(\alpha_1,\dots,\alpha_r)$.) Then the multiplicity of P is equal to*

$$\mu(P)=\sum_{i=1}^{r}\alpha_i-2(r-1).$$

(4) *Let $[\alpha_1,\dots,\alpha_r]$ be the continued fraction*

$$[\alpha_1,\dots,\alpha_r]=\alpha_1-\cfrac{1}{\alpha_2-\cfrac{1}{\alpha_3-\cfrac{1}{\ddots-\alpha_{r-1}-\cfrac{1}{\alpha_r}}}},$$

and write $d/e=[\alpha_1,\dots,\alpha_r]$ with $e<d$ and e and d relatively prime. If P is a singular point of type $[\alpha_1,\dots,\alpha_r]$ then (X, P) has the same singularity as $(\mathbf{C}^2/G, (0))$, where $G=\langle\zeta\rangle$ is a cyclic group of order d, ζ being a primitive d-th root of unity, and G acts on $\mathbf{C}^2$ by $(x, y)\to(\zeta x, \zeta^e x)$.

In order to construct a normal affine surface X with an $\mathbf{A}^1$-fibration carrying a singular point of type $(\alpha_1,\dots,\alpha_r)$, we start with a normal projective surface V with a $\mathbf{P}^1$-fibration $p\colon V\to B$, where B is a nonsingular complete curve. We assume furthermore that there is a cross-section M of p. For example, the surface X with the $\mathbf{A}^1$-fibration $\rho\colon X\to C$ considered in the above lemma has an open immersion into a normal projective surface V endowed with a $\mathbf{P}^1$-fibration $p\colon V\to C$ such that $p|_X=\rho$ and that the complement $D:=V-X$ is an effective divisor with simple normal crossings and that D contains an irreducible component M which is a cross-section of p; *we may assume without loss of generality that there are no (-1)-curves contained in D possibly except for the component M.* Let F be a smooth fiber of p and let P_0 be a point on F. Let $\sigma_0\colon V_1\to V$ be the blowing-up of P_0 and let $E_0=\sigma_0^{-1}(P_0)$. Let $P_1=\sigma_0'(F)\cap E_0$, $\sigma_0'(F)$ signifying the proper transform of F via σ_0, let $\sigma_1\colon V_2\to V_1$ be the blowing-up of P_1 and let $E_2=\sigma_1^{-1}(P_1)$. Now let P_2 be either $E_2\cap(\sigma_0\sigma_1)'(F)$ or $E_2\cap\sigma_1'(E_0)$, let $\sigma_2\colon V_2\to V_1$ be the blowing-up of P_2 and let $E_3=\sigma_2^{-1}(P_2)$. Next, let P_4 be one of the intersection points of E_3 with two adjacent components. We continue this process altogether $(n+1)$-times to obtain a normal projective surface W by the blowing-up of points $P_0,\dots,P_n$, say $f\colon W\to V$, where the i-th point P_i is one of the intersection points of the exceptional curve $E_i=\sigma_{i-1}^{-1}(P_{i-1})$ with two adjacent components. We denote the proper transform of E_i on W also by the same letter E_i. The surface W has a $\mathbf{P}^1$-fibration given by a morphism $p\cdot f\colon W\to B$ and the fiber $(p\cdot f)^{-1}(p(F))$ cosists of irreducible components $f'(F)$, $E_0,\dots,E_n$, which form a chain of rational curves. Call this fiber $\tilde{F}$. Then we can write $\tilde{F}_{red} = \Sigma_0 + E_n + \Sigma_1$, where Σ_0 is the one of two connected components of $\tilde{F}_{red} - E_n$ which meets $f'(M)$ and Σ_1 is the other one. We note that E_n is the unique (-1)-curve in the fiber $\tilde{F}$ provided $n \geq 1$. Then Σ_1 can be contracted to a cyclic quotient singular point of type $(\alpha_1,\dots,\alpha_r)$, which is uniquely determined by the above blowing-up process. We denote by $\bar{V}$ the normal algebraic surface obtained from W by contracting Σ_1 to a point. Then $\bar{V}$ will inherit the $\mathbf{P}^1$-fibration from W, which we denote by $\bar{p}\colon \bar{V} \to B$. Denote the fiber $\bar{p}^{-1}(p(F))$ by $\bar{F}$. It is proved in [6] that a cyclic quotient singular point with arbitrarily given r-tuple $(\alpha_1,\dots,\alpha_r)$ can be produced by this process. When we start with a normal affine surface X with an $\mathbf{A}^1$-fibration $\rho\colon X \to C$, we consider its completion V with a $\mathbf{P}^1$-fibration $p\colon V \to B$. Then choose a smooth fiber F, which corresponds to a reduced and irreducible fiber of ρ and apply the above-described process to F to produce a cyclic quotient singular point on the replaced fiber $\bar{F}$. It is straightforward to show that the open subset of $\bar{V}$ with the image of $\sigma^{-1}(D - M) \cup f'(M) \cup \Sigma_0$ removed off is affine. We denote this affine surface by $\bar{X}$ and say that $\bar{X}$ is obtained from X by attaching a cyclic singular point of type $(\alpha_1,\dots,\alpha_r)$ (or equivalently, of type d/e if $d/e = [\alpha_1,\dots,\alpha_r]$ as in Lemma 2.4) to the fiber $F^0 = F \cap X$. Indeed, $\bar{X}$ has an $\mathbf{A}^1$-fibration $\bar{\rho}\colon \bar{X} \to C$ which is the same as the $\mathbf{A}^1$-fibration ρ on X except that the fiber F° is replaced by a multiple fiber $\bar{F}^\circ = \bar{F} \cap \bar{X}$. Its multiplicity is equal to d in Lemma 2.4 (cf. Miyanishi [7, pp. 76–80]).

2.3. Let X be a logarithmic $\mathbf{Q}$-homology plane. With the notations of 2.1, we can consider the Kodaira dimension $\kappa(X^\circ)$ and call it the Kodaira dimension of X. We write $\kappa(X)$ instead of $\kappa(X^\circ)$. We shall now look into the structure of a logarithmic homology plane X of Kodaira dimension $-\infty$. We employ here the notations Σ, X°, V, Y, W, etc. introduced in 2.1. We apply the results [7; Theorem 2.11] and [8; Lemma 3] to obtain the next result.

Lemma 2.5. *Let X be a logarithmic $\mathbf{Q}$-homology plane of Kodaira dimension $-\infty$. Then one of the following holds:*

(I) *X has an $\mathbf{A}^1$-fibration $\rho\colon X \to C$;*

(II) *There exists a Zariski open set U of X° and a proper birational morphism $\phi\colon U \to T'$ onto a nonsingular algebraic surface T' defined over $\mathbf{C}$ such that:*

(1) *Either $U = X^\circ$ or $X^\circ - U$ has pure dimension one;*

(2) *T' is an open set of a Platonic $\mathbf{C}^*$-fiber space T with $dim(T - T') \leq 0$. For the definition of Platonic $\mathbf{C}^*$-fiber space, see [8].*

Consider the case (II). Since U is quasi-affine, it has no complete curves. Hence ϕ is an isomorphism. This implies that $\Gamma(U, \mathcal{O}_X)^* = \Gamma(T', \mathcal{O}_T)^* = \Gamma(T, \mathcal{O}_T)^* = \mathbf{C}^*$ (cf. [8]). Suppose $X^\circ - U$ has pure dimension one. Let $G_1,\dots,G_s$ be the closures of irreducible components of $X - U$ in V. Since $H^2(V; \mathbf{Q}) \cong H^2(D; \mathbf{Q})$ and V has at worst quotient singularities, some positive multiple of $(G_1 + \cdots + G_s)$ is linearly equivalent to a divisor supported by $V - X$. Hence there exists a non-constant function h on V which is invertible on

U. This is a contradiction. So, $X^\circ = U$. Since $dim(T - T') \leq 0$, X is affine and T is isomorphic to $\mathbf{C}^2/G - (0)$ for some small finite subgroup G of $GL(2, \mathbf{C})$, we conclude that

$$\Gamma(X, \mathcal{O}_X) \cong \Gamma(T', \mathcal{O}_T) \cong \Gamma(T, \mathcal{O}_T) \cong \Gamma(\mathbf{C}^2/G, \mathcal{O}_{\mathbf{C}^2/G}),$$

whence $X \cong \mathbf{C}^2/G$. Since $\mathbf{C}^2/G$ is contractible by Conner's theorem, X is a logarithmic homology plane.

Next, consider the case (I) above. Since $\Gamma(X, \mathcal{O}_X)^* = \mathbf{C}^*$, $q(W) = 0$ and $H^2(V; \mathbf{Q}) \cong H^2(D; \mathbf{Q})$, the base curve C is isomorphic to $\mathbf{A}^1$ and every fiber of ρ is irreducible; indeed, it is some multiple of $\mathbf{A}^1$. Moreover, every singular point of X is a cyclic quotient singular point and each fiber of ρ has at most one singular point lying on it. We consider a completion $p: V \to B$ of $\rho: X \to C$ as in 2.1; in particular, we assume that $D := V - X$ contains no (-1)-curves possibly except for the cross-section M. Furthermore, consider a $\mathbf{P}^1$-firation $q := p \cdot f: W \to B$ on the surface W obtained by the minimal resolution of singularities $f: W \to V$.

Lemma 2.6. (1) *Let $H = mL$ be a fiber of ρ, where $L \cong \mathbf{A}^1$ and $m \geq 1$, let $\tilde{F} = q^{-1}(\rho(H))$, and let E be the closure of $\varphi^{-1}(L)$ in W. If $m > 1$, then $\tilde{F}$ is a reducible fiber. Suppose that H has a cyclic quotient singular point P of type d/e with $(d, e) = 1$. Write $\tilde{F}_{red} = \Sigma_0 + E + \Sigma_1$, where $f^{-1}(P) = \Sigma_1$. Then E is a unique (-1)-curve in the fiber $\tilde{F}$, m is divisible by d, and $\tilde{F}_{red}$ is a linear chain if and only if $m = d$.*

(2) *Let $H_1, \dots, H_{N'}$ be all multiple fibers of ρ. Then $H_1(X^\circ; \mathbf{Z}) \cong \bigoplus_{i=1}^{N'} \mathbf{Z}/m_i\mathbf{Z}$, where m_i is the multiplicity of H_i. Let $P_1, \dots, P_N$ exhaust all singular points of X; we may assume $P_i \in H_i$ for $1 \leq i \leq N$. Let d_i/e_i be the type of P_i. Then $H_1(\partial T; \mathbf{Z}) \cong \bigoplus_{i=1}^{N} \mathbf{Z}/d_i\mathbf{Z}$. Moreover, we have an exact sequence*

$$0 \longrightarrow \bigoplus_{i=1}^{N} \mathbf{Z}/d_i\mathbf{Z} \longrightarrow \bigoplus_{i=1}^{N'} \mathbf{Z}/m_i\mathbf{Z} \longrightarrow H_1(X; \mathbf{Z}) \longrightarrow 0.$$

Proof. (1) Since E is a unique (-1)-curve in the fiber, we can contract E. Then, after the contraction of E, one of the adjacent components of E becomes a (-1)-curve if the contracted fiber is still reducible. Then contract again a (-1)-curve. Thus, starting with E, we can contract successively all irreducible components but one and finally obtain a nonsingular rational curve with multiplicity 1 and self-intersection number 0. In this process, there appear no (-1)-curves which meet three other components. Hence, when the last component of Σ_1 is contracted, all the components contracted already in the preceding contraction process form a linear chain, say Z. Call A the component of $\tilde{F}$ which then becomes a (-1)-curve or a curve with self-intersection number 0 and let a be its multiplicity. Then m is divisible by a. Think of A as a smooth fiber of a $\mathbf{P}^1$-fibration and reverse the above contraction process to regain the chain Z back. We then readily show that $d = m/a$.

If $m = d$ then A has multiplicity 1. If A becomes a (-1)-curve when the last component of Σ_1 is contracted, then $\tilde{F}$ contains another (-1)-curve. This contradicts the hypothesis. So, A becomes a curve with self-intersection number 0, which must be a smooth fiber of a $\mathbf{P}^1$-fibration. Thus $\tilde{F}_{red}$ is a linear chain if $m = d$. The converse is already verified.

(2) It is straightforward to show that $H_1(X^\circ; \mathbf{Z}) \cong \bigoplus_{i=1}^{N'} \mathbf{Z}/m_i\mathbf{Z}$ via an isomorphism $H_1(X^\circ; \mathbf{Z}) \cong H^3(W, D \cup \Sigma; \mathbf{Z})$. Meanwhile, $H_1(\partial T; \mathbf{Z}) \cong \bigoplus_{i=1}^{N} H_1(\partial T_i; \mathbf{Z})$ and $H_1(\partial T_i; \mathbf{Z}) \cong \pi_1(\partial T_i) \cong \mathbf{Z}/d_i\mathbf{Z}$. Thence we obtain $H_1(\partial T; \mathbf{Z}) \cong \bigoplus_{i=1}^{N} \mathbf{Z}/d_i\mathbf{Z}$. The exact sequence in the assertion is obtained from the sequence in Lemma 2.2, (2) if we show that $H_2(X; \mathbf{Z}) = (0)$. In order to show this, note that $H^3(X; \mathbf{Z}) = (0)$ because X is an affine surface. By the universal coefficient theorem, we have

$$H^3(X; \mathbf{Z}) \cong Hom(H_3(X; \mathbf{Z}), \mathbf{Z}) \oplus Ext^1(H_2(X; \mathbf{Z}), \mathbf{Z})$$

and $H_2(X; \mathbf{Z})$ is a torsion group. Hence the above isomorphism entails $H_2(X; \mathbf{Z}) = (0)$. ■

Now we have proved a structure theorem on logarithmic homology planes of Kodaira dimension $-\infty$.

Theorem 2.7. *Let X be a logarithmic homology plane of Kodaira dimension $-\infty$. Then X is isomorphic to one of the following three surfaces:*

(1) $\mathbf{A}^2$;

(2) $\mathbf{C}^2/G$, *where G is a small finite subgroup of $GL(2, \mathbf{C})$;*

(3) *A surface X with an $\mathbf{A}^1$-fibration $\rho: X \to \mathbf{A}^1$ such that every fiber is irreducible and that there are exactly N multiple fibers $H_1, \dots, H_N$ with respective multiplicities $d_1, \dots, d_N$, each of them carrying a cyclic quotient singular point P_i of type d_i/e_i, where N is an arbitrary positive integer.*

These surfaces are all contractible surfaces and dominated by $\mathbf{A}^2$. In particular, a surface of type (3) *is birationally dominated by $\mathbf{A}^2$.*

Proof. The last two assertions need proofs. Note that $\mathbf{A}^2$ and $\mathbf{C}^2/G$ are contractible and dominated by $\mathbf{A}^2$. Consider the case (3). Let $\{Q_1, \dots, Q_N, Q_\infty\}$ be the points of $\mathbf{P}^1$ such that the multiple fiber H_i lies over Q_i for $1 \le i \le N$ and Q_∞ is the point at infinity. Let $U = \mathbf{P}^1 - \{Q_1, \dots, Q_N, Q_\infty\}$. Then $\rho^{-1}(U) \cong \mathbf{A}^1 \times U$ and $\pi_1(\rho^{-1}(U)) \to \pi_1(X^\circ)$ is surjective. Set

$$\Gamma = \langle \gamma_1, \dots, \gamma_N, \gamma_\infty | \gamma_1^{d_1} = \cdots = \gamma_N^{d_N} = \gamma_1 \cdots \gamma_N \gamma_\infty = 1 \rangle.$$

Then, as in [3; Lemma 3.2], one knows that $\pi_1(X^\circ) \cong \Gamma$. Hence $|\pi_1(X^\circ)| < \infty$ if and only if $N = 1$, and $\pi_1(X^\circ) \cong \mathbf{Z}/\gamma_1\mathbf{Z}$ if $N = 1$. By this observation, we know that any unramified covering space of X° is obtained as the normalization of a cartesian product of X° over $\mathbf{A}^1$ with a covering $T \to \mathbf{A}^1$ ramified over $\{Q_1, \dots, Q_N\}$. In particular, any covering space of X° has an $\mathbf{A}^1$-fibration which comes from the $\mathbf{A}^1$-fibration $\rho: X \to \mathbf{A}^1$. Now, consider an unramified covering space $\sigma: Y \to X$. For any P_i, $1 \le i \le N$, $\sigma^{-1}(P_i)$ consists of singular points on Y with the same singularity as P_i. Hence the exceptional locus of the minimal resolution of any point of $\sigma^{-1}(P_i)$ is the same as that of P_i. Therefore, the surface $\tilde{Y}$ obtained from Y by resolving minimally all points of $\sigma^{-1}(P_i)$ is an unramified covering space of the surface $\tilde{X}$ obtained from X by resolving minimally the singular point P_i. If the exceptional locus of the minimal resolution of P_i is added to the fiber H_i, it is considered as a fiber of multiplicity 1. This implies the following result:

Let $\alpha = \{P_{i_1}, \dots, P_{i_r}\}$ be a subset of $\{P_1, \dots, P_N\}$ and let $X_\alpha = X^\circ \cup \{P_{i_1}, \dots, P_{i_r}\}$. Then $\pi_1(X_\alpha) = \Gamma / \langle \gamma_{i_1} = \cdots = \gamma_{i_r} = 1 \rangle$. In particular, $\pi_1(X) = (1)$.

Therefore, all surfaces in Theorem 2.7 are contractible. It is easy to see that X in the case (3) with $N = 1$ is written as $\mathbf{C}^2/\pi_1(X^\circ)$. If $N \geq 2$, X is not isomorphic to $\mathbf{A}^2$ nor $\mathbf{C}^2/G$.

We shall show that a surface X in the case (3) is birationally dominated by $\mathbf{A}^2$. We can embed X into a projective normal surface V so that $\rho: X \to \mathbf{A}^1$ extends to a $\mathbf{P}^1$-fibration $p: V \to \mathbf{P}^1$. Let $f: W \to V$ be the minimal resolution of singularities. Then $q = p \cdot f: W \to \mathbf{P}^1$ is still a $\mathbf{P}^1$-fibration. We can choose the projective surface V so that the fiber of q containing H_i, for each i, has a linear chain of rational curves $\Sigma_0 + \bar{H}_i + \Sigma_1$ as in 2.2, where Σ_1 is the exceptional locus $f^{-1}(P_i)$. Let K_i be the terminal component of the graph contained in Σ_1. Noting that K_i has multiplicity 1 in the fiber, it is easy to show that $X' = (X - \bigcup_{i=1}^N H_i) \cup (\bigcup_{i=1}^N K_i) \cong \mathbf{A}^2$. Thus we obtain a birational morphism $\mathbf{A}^2 \cong X' \to X$. ■

Theorem 2.8. *Let X be a logarithmic $\mathbf{Q}$-homology plane of Kodaira dimension $-\infty$. If it is not isomorphic to one of three kinds listed in Theorem* 3.6, *it is isomorphic to a surface of the following kind*:

(4) *A surface X with an $\mathbf{A}^1$-fibration $\rho: X \to \mathbf{A}^1$ such that every fiber is irreducible and $H_1(X; \mathbf{Z}) \neq (0)$.*

Let $H_1, \dots, H_{N'}$ be all multiple fibers with respective multiplicities $m_1, \dots, m_{N'}$, where $H_1, \dots, H_N$ carry cyclic quotient singular points $P_1, \dots, P_N$ of type $d_1/e_1, \dots, d_N/e_N$, respectively while $H_{N+1}, \dots, H_{N'}$ have none of singular points. Then $H_1(X; \mathbf{Z}) \cong \bigoplus_{i=1}^N \mathbf{Z}/(m_i/d_i)\mathbf{Z} \oplus \bigoplus_{i=N+1}^{N'} \mathbf{Z}/m_i\mathbf{Z}$ and $H_i(X; \mathbf{Z}) = (0)$ for $\forall i \neq 1$.

Example 1. Let (u, t) be a coordinate system on $\mathbf{C}^* \times \mathbf{C}$ and let σ be an involution on $\mathbf{C}^* \times \mathbf{C}$ defined by $\sigma^*(u, t) = (u^{-1}, -u^2 t)$. The fixed point locus consists of two points $Q_1: (u, t) = (1, 0)$ and $Q_2: (u, t) = (-1, 0)$. Let $X = \mathbf{C}^* \times \mathbf{C}/\langle \sigma \rangle$. Then $X = Spec(A)$, where $A = \mathbf{C}[u + u^{-1}, u^2 t^2, t(1 - u^2)]$. Let $C = Spec\,\mathbf{C}[v]$ with $v = u + u^{-1}$. Then the morphism $\rho: X \to C$ induced by the inclusion $\mathbf{C}[v] \subset A$ is an $\mathbf{A}^1$-fibration which has two multiple fibers H_1 and H_2 of multiplicities 2 carrying double points P_1 and P_2 of type 2/1, i.e., of type A_1. Indeed, H_1 and H_2 are defined by $v = 2$ and $v = -2$, respectively.

2.4. Next, we consider a logarithmic affine surface X with a $\mathbf{C}^*$-fibration $\pi: X \to C$. We employ the notations of 1.3 and 2.2 mixed up. As in 1.3, the fibration π is extended to a $\mathbf{P}^1$-fibration $p: V \to B$, where V is a normal projective surface which is smooth along $D := V - X$. Let $f: W \to V$ be the minimal resolution of singularities. Then $q = p \cdot f: W \to B$ is a $\mathbf{P}^1$-fibration on a smooth surface W. We call a (-1) curve E a (-1) component if E is a fiber component, i.e., E is contained in a fiber of p or q. We assume that D contains no (-1) components possibly except for a (-1) component meeting either two cross-sections or a 2-section contained in D.

Lemma 2.9. *Let F be a fiber of π. Then we have:*

(1) *$F = \Gamma + \Delta$, where $\Gamma_{red} = \emptyset$, $\mathbf{A}^1_*$ or $C_1 + C_2$ with $C_1 \cong C_2 \cong \mathbf{A}^1$ meeting transversally in one point, and Δ_{red} is a disjoint union of $\mathbf{A}^1$'s.*

(2) *If $\Gamma_{red} = \mathbf{A}^1_*$, there are no singular points lying on Γ.*

(3) *If $\Gamma_{red} = C_1 + C_2$, then $P := C_1 \cap C_2$ can be a cyclic quotient singular point.*

(4) *Each component of Δ_{red} has at most one cyclic quotient singular point except in the case:*

(4–1) *Suppose a smooth fiber L touches a 2-section H in a point Q. Blowing up Q and its infinitely near point of the first order lying on H, we have a degenerate fiber $E_1 + E_2 + 2E$ such that $(E_1^2) = (E_2^2) = -2$, $(E^2) = -1$ and E meets the proper transform H' of H. Starting with a point $P_0 = E \cap H'$, apply a process of blowing-ups to produce a chain of curves $\Sigma_0 + E_n + \Sigma_1$ connecting E and H', where $E \cap \Sigma_0 \neq \emptyset$, $\Sigma_1 \cap H' \neq \emptyset$ and $(E_n^2) = -1$, and contract the curves $E_1 + E_2 + E + \Sigma_0$ to produce a quotient singular point $\bar{P}$ of Dynkin type D_r. Then the image of E_n gives a curve in X isomorphic to $\mathbf{A}^1$ with a singular point $\bar{P}$.*

(4–2) *In the situation of* (4–1), contract *E_1 and E_2 to rational double points of type A_1. Then E gives a fiber isomorphic to $\mathbf{A}^1$ with these two singular points.*

(5) *Let P be a cyclic quotient singular point of type d/e lying on a component A of Δ. Let m be the multiplicity of A in F. Then d divides m.*

Proof. Let $\tilde{F} = q^{-1}(\pi(F))$. All (-1) components of $\tilde{F}$ then come from the components of F except for a (-1) component meeting either two cross-sections or a 2-section in D. Furthermore, since X is affine, D is connected and every connected component of $\tilde{F} - F$ which is disjoint from D is contracted to a singular point. Hence every irreducible component of F gives rise to a (-1) component of $\tilde{F}$ except for C_1 or C_2 of $\Gamma_{red} = C_1 + C_2$. On the other hand, it is well-known that any (-1) component of a degenerate fiber of a $\mathbf{P}^1$-fibration does not meet three other fiber components and that if a (-1) component has multiplicity 1 there is another (-1) component. All the stated assertions can be verified without difficulty by virtue of these observations. ∎

Lemma 2.10. *Let X be a logarithmic $\mathbf{Q}$-homology plane with a $\mathbf{C}^*$-fibration $\pi: X \to C$. Then we have:*

(1) *C is either $\mathbf{P}^1$ or $\mathbf{A}^1$. If the fibration is twisted then $C \cong \mathbf{A}^1$.*

(2) *If $C \cong \mathbf{P}^1$ then every fiber of π is irreducible, and there is exactly one fiber isomorphic to $\mathbf{A}^1$.*

(3) *If $C \cong \mathbf{A}^1$ and π is untwisted, all fibers are irreducible except one which consists of two irreducible components. If $C \cong \mathbf{A}^1$ and π is twisted, all fibers are irreducible and there is exactly one fiber isomorphic to $\mathbf{A}^1$.*

Proof. This is the restatement of Lemma 1.4 except for the assumption that X has possibly quotient singularities. ■

Lemma 2.11. *Let X be as above. Suppose π is untwisted and $C \cong \mathbf{P}^1$. Let $F_0,\dots,F_n$ be all singular fibers with respective multiplicities $m_0,\dots,m_n$, where $F_{0,red} \cong \mathbf{A}^1$. As in [3], let H_1 and H_2 be cross-sections of p contained in D such that $H_1 \cap H_2 = \emptyset$ and $(H_1^2) = -a$ with $a \geq 0$; so, by the blowing-downs of (-1) components in W and its images, H_1 is mapped to the minimal section $\bar{H}_1$ of a Hirzebruch surface Σ_a, let δ_i be the multiplicity of $(\bar{F}_i)_{red}$ (= the closure of the proper transform of $(F_i)_{red}$ on W) in the total transform of $\bar{H}_2$ on W. Then we have:*

(1) *$\kappa(X) = 1, 0$ or $-\infty$ if and only if*

$$(n-1) - \sum_{i=1}^{n} \frac{1}{m_i} > 0, \ = 0 \ \textit{or} \ < 0, \ \textit{respectively}.$$

(2) *Let d/e be the type of a cyclic quotient singular point lying on F_0 if it exists at all. Then $H_1(X;\mathbf{Z})$ is a torsion group of order equal to*

$$\frac{1}{d}\cdot|m_0\cdots m_n a - \sum_{i=0}^{n} m_0\cdots \hat{m}_i\cdots m_n\delta_i|,$$

where d divides m_0 and δ_0. Moreover, $H_i(X;\mathbf{Z}) = 0$ if $i \neq 0, 1$.

(3) *The surface X is contractible if and only if $a = 1$, $m_0 = d$, $n = 2$ and $m_1 m_2 - m_1\delta_2 - m_2\delta_1 = \pm 1$. Moreover, if X is contractible, there is a birational morphism $\rho\colon X_1 \to X$, where X_1 is a nonsingular contractible surface with $\kappa(X_1) = 1$.*

Proof. (1) Consider the peeling of the bark of $K_V + D$,

$$K_V + D = (K_V + D^{\#}) + Bk(D)$$

(cf. [10]). Let E be a unique (-1) component of the fiber $q^{-1}(\pi(F_0))$ which is the closure of $(F_0)_{red}$ on W. Then it is not hard to show that $(K_V + D^{\#}\cdot E) < 0$. Hence, by the theory of peeling (cf. [ibid]), we may include E in the boundary divisor without changing $\kappa(X)$. The rest of the proof of the assertion (1) is just a repetition of the proof of Lemma 3.8 in [3].

(2) It is straightforward to show that d divides m_0 and δ_0 (cf. the proof of Lemma 2.6). Let $X^\circ = X - Sing(X)$. By Lemma 2.2 (or its proof), we have an exact sequence

$$\begin{aligned} 0 \longrightarrow H_2(X^\circ;\mathbf{Z}) \longrightarrow H_2(X;\mathbf{Z}) &\longrightarrow H_1(\partial T;\mathbf{Z}) \\ &\longrightarrow H_1(X^\circ;\mathbf{Z}) \longrightarrow H_1(X;\mathbf{Z}) \longrightarrow 0, \end{aligned}$$

where $H_2(X^\circ;\mathbf{Z}) = 0$, $H_1(\partial T;\mathbf{Z}) \cong \mathbf{Z}/d\mathbf{Z}$ and

$$H_1(X^\circ;\mathbf{Z}) \cong Coker(H_2(D \cup f^{-1}(Sing\ X);\mathbf{Z}) \longrightarrow H_2(W;\mathbf{Z})) \cong Pic(X^\circ).$$

Thence $H_2(X;\mathbf{Z})$ is given as in the proof of Lemma 2.6. The order of $Pic(X^\circ)$ is computed in the same fashion as in [3; pp. 110–111], and is equal to

$$|m_0 \cdots m_n a - \sum_{i=0}^{n} m_0 \cdots \hat{m}_i \cdots m_n \delta_i|.$$

Hence we obtain the order of $H_1(X ; \mathbf{Z})$ as it is given above.

(3) The proof for the first assertion is almost the same as for [3; Theorem 3]. So, we do not reproduce it. To find a smooth contractible surface X_1 birationally mapping to X, consider the minimal resolution of the cyclic singular point P_0 lying on the fiber F_0. Contain the terminal component of the exceptional locus, not meeting the proper transform of F_0, into the interior and throw away, instead, the proper transform of F_0 as a component of the boundary divisor. Thus, we obtain a nonsingular affine surface X_1, which is a contractible surface with $\kappa(X_1) = 1$ by [3; Theorem 3]. This construction shows also the existence of a birational morphism $\rho: X_1 \to X$. ■

Remark 2.12. By Lemma 2.11, (1), $\kappa(X) = 0$ if and only if $n = 2 = m_1 = m_2$. Then the order of $H_1(X ; \mathbf{Z})$ is equal to $\frac{1}{d}|4m_0 a - 4\delta_0 - 2m_0\delta_1 - 2m_0\delta_2|$, which is an even integer. So, X is not a homology plane.

Example 2. Let $a = 1$, $(m_0, \delta_0)^+ = (2, 1)$, $(m_1, \delta_1)^* = (3, 1)$ and $(m_2, \delta_2)^* = (5, 1)$; for the notation $(m_i, \delta_i)^+$ or $(m_i, \delta_i)^*$, see [3; pp. 112]. Then X is a nonsingular homology plane with $\kappa(X) = 1$, but X is not contractible. One can show that X has an involution σ and the quotient surface $Y = X/\langle\sigma\rangle$ is a contractible surface of Kodaira dimension 1 which has a rational double point of Dynkin type A_1.

Example 3. Let $a = 1$, $(m_1, \delta_1)^* = (3, 1)$ and $(m_2, \delta_2)^* = (5, 1)$. Take the fiber F_0 so that $m_0 = 4$, $\delta_0 = 2$ and $d/e = 2/1$. Then X is a homology plane with one rational double point of type A_1 and $\kappa(X) = 1$.

In the subsequent arguments, we need a notion of oscilating sequence of blowing-ups. Let Z be a nonsingular projective surface and let P_0 be a point of Z. A birational morphism $\sigma: Y \to Z$ from a nonsingular projective surface Y onto Z is called *an oscilating sequence of blowing-ups with initial point* P_0 if σ induces an isomorphism between $Y - \sigma^{-1}(P_0)$ and $Z - \{P_0\}$, $\sigma^{-1}(P_0)$ is a linear chain of nonsingular rational curves and $\sigma^{-1}(P_0)$ contains a unique (-1) curve.

Lemma 2.13. (1) *Let Z be as above. Let H and ℓ be irreducible curves such that $(\ell^2) = 0$ and $(\ell \cdot H) = 1$, and let $P_0 = \ell \cap H$. Let $\sigma: Y \to Z$ be an oscilating sequence of blowing-ups with initial point P_0 such that $(\sigma'(\ell))^2 \leq -2$. Write the linear chain $\sigma'(H) \cup \sigma^{-1}(P_0) \cup \sigma'(\ell)$ as a graph*

$$\sigma'(H) - \{-c_1, \ldots, -c_t\} - (-1) - \{-b_s, \ldots, -b_2\} - \sigma'(\ell),$$

where $(-b_i)$ or $(-c_j)$ represents a nonsingular rational curve whose self-intersection number is $(-b_i)$ or $(-c_j)$. Let $(\sigma'(\ell))^2 = -b_1$ and let E be the unique (-1) curve. Let m and δ be the multiplicities of E in $\sigma^*(\ell)$ and $\sigma^*(H)$. Then we have

$$\frac{m}{\delta} = [b_1, b_2, \dots, b_s] \quad and \quad \frac{m}{m-\delta} = [c_1, c_2, \dots, c_t],$$

where $(m, \delta) = 1$ (*cf. Lemma* 2.4 *for the notation*).

(2) *Let* Z *be as above, let* H *be an irreducible curve on* Z *and let* P_0 *be a smooth point of* H. *Let* $\sigma: Y \to Z$ *be an oscilating sequence of blowing-ups with initial point* P_0. *Write* $\sigma'(H) \cup \sigma^{-1}(P_0)$ *as a linear graph*

$$\sigma'(H) - \{-c_1, \dots, -c_t\} - (-1) - \{-b_s, \dots, -b_1\}$$

and denote by E *the unique* (-1) *curve. Let* δ *be the multiplicity of* E *in* $\sigma^*(H)$. *Then we have*

$$\frac{\delta}{q} = [b_1, \dots, b_s],$$

where $(\delta, q) = 1$.

(3) *Let the situation be as in* (1) *above. Rewrite the chain* $\sigma'(H) \cup \sigma^{-1}(P_0) \cup \sigma'(\ell)$ *as follows*:

$$\sigma'(H) - \{-c_1, \dots, c_u\} - (c_u + 1) - \{-c_{u+2}, \dots, c_t\} - (-1)$$
$$- \{-b_s, \dots, -b_{v+2}\} - (-b_{v+1}) - \{-b_v, \dots, -b_1\}.$$

Let C_1 *and* C_2 *be components expressed by* $(-c_{u+1})$ *and* $(-b_{v+1})$, *respectively. Let* n_i *and* γ_i *be the multiplicities of* C_i *in* $\sigma^*\ell$ *and* σ^*H, *respectively, where* $i = 1, 2$. *Write*

$$\frac{N_1}{q_1} = [c_{u+2}, \dots, c_t] \quad and \quad \frac{N_2}{q_2} = [b_{v+2}, \dots, b_s],$$

where $(N_1, q_1) = (N_2, q_2) = 1$. *Then we have*

$$m\gamma_1 - n_1\delta = N_1 \quad and \quad m\gamma_2 - n_2\delta = N_2.$$

(4) *In the situation of* (3) *above, consider an oscilating sequence of blowing-ups with initial point* $(-c_u) \cap (-c_{u+1})$ *and let* G *be the resulting, unique* (-1) *curve. Let* n *and* γ *be the multiplicities of* G *in the total transform of* ℓ *and* H. *As the graph of the total transform of* ℓ *is linear, let* $\{-d_1, \dots, -d_r\}$ *be the sub-chain in between* G *and* E *and let* N *be the numerator of* $[d_1, \dots, d_N]$. *Then* $|m\gamma - n\delta| = N$.

Proof. (1) As in [7; p. 78], we rewrite the chain as

$$\sigma'(H) - \{(-2)^{m_1-1}, -(m_2+2), (-2)^{m_3-1}, \dots, (-2)^{m_{a-1}-1}, -(m_a+1)\}$$
$$-(-1) - \{(-2)^{m_a-1}, -(m_{a-1}+2), \dots, (-2)^{m_2-1}, -(m_1+1)\}$$

if a is even and

$$\sigma'(H) - \{(-2)^{m_1-1}, -(m_2+2), \dots, -(m_{a-1}+2), (-2)^{m_a-1}\}$$
$$-(-1) - \{-(m_a+1), (-2)^{m_{a-1}-1}, -(m_{a-2}+2), \dots,$$
$$(-2)^{m_2-1}, -(m_1+1)\}$$

if a is odd, where $(-2)^m$ represents a sub-chain $\{-2, -2, \ldots, -2\}$ with -2 iterated m times and $-(m_1+1)$ represents $\sigma'(\ell)$. Then the argument in [7] shows that

$$\frac{m}{\delta} = \begin{cases} [m_1+1, (2)^{m_2-1}, \ldots, (2)^{m_a-1}] & \text{if } a \equiv 0 \pmod 2 \\ [m_1+1, (2)^{m_2-1}, \ldots, (2)^{m_{a-1}+1}, m_a+1] & \text{if } a \not\equiv 0 \pmod 2. \end{cases}$$

It is straightforward to show that the right-hand side of the above equality is equal to

$$[[m_1, m_2, \ldots, m_a]] = m_1 + \cfrac{1}{m_2 + \cfrac{1}{\ddots + \cfrac{1}{m_a}}}.$$

If $a \equiv 0 \pmod 2$, write

$$\frac{c}{d} = [m_2+1, (2)^{m_2-1}, \ldots, m_a+1] = [[m_2, \ldots, m_a]],$$

where $(c, d) = 1$. Then

$$\frac{c+d}{d} = [m_2+2, (2)^{m_3-1}, \ldots, m_a+1],$$

and

$$\frac{m_1 c + d}{(m_1-1)c+d} = [(2)^{m_1-1}, m_2+2, \ldots, m_a+1].$$

On the other hand, we have $m = m_1 c + d$ and $\delta = c$ because

$$[[m_1, \ldots, m_a]] = m_1 + \frac{d}{c} = \frac{m_1 c + d}{c}.$$

Hence we have $(m_1-1)c+d = m-\delta$. This shows that

$$\frac{m}{m-\delta} = [c_1, \ldots, c_t].$$

The case $a \not\equiv 0 \pmod 2$ can be treated similarly.

(2) First of all, we work in the situation of (1) above. Let $\sigma_0\colon Z_1 \to Z_0$ be the blowing-up of P_0 and let $E_0 = \sigma_0^{-1}(P_0)$. Then the next point to be blown up in the oscilating sequence σ of blowing-ups is $P_1 := E_0 \cap \sigma_0'(\ell)$, for otherwise the proper transform of E_0 or $\sigma_0'(\ell)$ would remain as a (-1) curve in $\sigma^{-1}(P_0)$. Hence P_1 is the intersection point of two components with multiplicity 1 in $\sigma_0^*(\ell)$, where the component leading to a branch $\{-b_s, \ldots, -b_1\}$ is a (-1) curve.

Now in the present situation, let $\sigma_0\colon Z_1 \to Z_0$ be also the blowing-up of P_0 and let $E_0 = \sigma_0^{-1}(P_0)$. If the point P_1 to be blown up next is $\sigma_0'(H) \cap E_0$, we are in a situation to the one as observed above. Hence the multiplicity δ of E in the present situation corresponds to m in the case (1). If $P_1 \neq E_0 \cap \sigma_0'(H)$, let $\sigma_1: Z_2 \to Z_1$ be the blowing-up of P_1 and let $E_1 = \sigma_1^{-1}(P_1)$. If the point P_2 to be blown up next is $\sigma_1'(E_0) \cap E_1$, we are in the situation as observed above. Otherwise, blow up P_2. Continuing this way, either we are in the situation as observed above or the last exceptional curve E appears in $\sigma^* H$ as a terminal component with multiplicity 1. In this case we understand $1 = [\emptyset]$.

(3) Define sequences of integers $\{p_i\}_{i=0}^t$ and $\{q_i\}_{i=0}^t$ successively as follows:

$$\text{(i)} \qquad p_0 = 1,\ p_1 = c_1, \dots, p_i = c_i p_{i-1} - p_{i-2} \qquad \text{for } 2 \le i \le t,$$

$$\text{(ii)} \qquad q_0 = 0,\ q_1 = 1, \dots, q_i = c_i q_{i-1} - q_{i-2} \qquad \text{for } 2 \le i \le t.$$

Then we have

$$\frac{p_i}{q_i} = [c_1, \dots, c_i],\ \frac{p_i}{p_{i-1}} = [c_i, \dots, c_1] \quad \text{and} \quad p_i q_{i-1} - q_i p_{i-1} = -1$$

for $0 < i \le t$ (cf. Orlik-Wagreich [10]). By (1) above, we have

$$\frac{p_t}{q_t} = \frac{m}{m - \delta} \quad \text{and} \quad \frac{p_u}{q_u} = \frac{n_1}{n_1 - \gamma_1}.$$

Hence we have

$$m\gamma_1 - n_1\delta = p_t(p_u - q_u) - p_u(p_t - q_t) = p_u q_t - p_t q_u.$$

Note that

$$\gamma_0 := p_u q_{u+1} - q_u p_{u+1} = 1$$

$$\gamma_1 := p_u q_{u+2} - q_u p_{u+2} = p_u(c_{u+2} q_{u+1} - q_u) - q_u(c_{u+2} p_{u-1} - p_u)$$

$$= c_{u+2}(p_u q_{u+1} - q_u p_{u+1}) = c_{u+2}$$

$$\gamma_j := p_u q_{u+j+1} - q_u p_{u+j+1} = c_{u+j+1}\gamma_{j-1} - \gamma_{j-2}$$

for $2 \le j \le t - u - 1$. Then $\gamma_{t-u-1} = N_1 = m\gamma_1 - n_1\delta$. The other case can be proved in a similar way.

(4) An oscilating sequence of blowing-ups is a composition of blowing-ups of points. After each blowing-up of as point, let n', γ' be the multiplicities of the new exceptional curve in the total transforms of ℓ and H, respectively. Then one can show that $|m\gamma' - n'\delta|$ is equal to the numerator of $[d_1', \dots, d_{r'}']$, where $\{-d_1', \dots, -d_{r'}'\}$ is the sub-chain in the middle between the new exceptional curve and E in the total transform of ℓ. We omit the details of the proof. ■

Lemma 2.14. *Let X be as above. Suppose π is twisted and $C \cong \mathbf{A}^1$. Let H be a 2-section of p contained in D. Then we have:*

(1) *$p|_H: H \to B$ has two ramifying points P_0, P_∞, and we may assume that the fiber $\ell_\infty := p^{-1}(Q_\infty)$, which is necessarily reducible, is contained in D, where $Q_\infty = p(P_\infty)$. The fiber $F_0 := \pi^*(Q_0)$, $Q_0 := \pi(P_0)$, is written as $F_0 = m_0 C_0$, where*

$C_0 \cong \mathbf{A}^1$ and C_0 has

(i) *one cyclic quotient singular point,*

(ii) *one quotient singular point of Dynkin type D_r (cf. Lemma 2.9, (4–1)), or*

(iii) *two rational double points of type A_1 (cf. ibid., (4–2))*

provided X is not smooth.

(2) *Let $F_i := m_i C_i$ $(1 \le i \le r)$ be all other singular fibers of π. Then $C_i \cong \mathbf{A}^1_*$.*

(3) *$H_i(X\,;\mathbf{Z}) = (0)$ if $i \neq 0, 1$, and $H_1(X\,;\mathbf{Z}) \cong \prod_{i=1}^r \mathbf{Z}/m_i\mathbf{Z}$ in the cases* (ii) *and* (iii) *above and $H_1(X\,;\mathbf{Z})$ is an extension of $\mathbf{Z}/m'_0\mathbf{Z} \times \prod_{i=1}^r \mathbf{Z}/m_i\mathbf{Z}$ by $\mathbf{Z}/2\mathbf{Z}$ if X is smooth or in the case* (i) *above, where m'_0 divides m_0.*

(4) *$\kappa(X) = 1, 0$ or $-\infty$ if and only if*

$$(r-1) - \sum_{i=1}^{r} \frac{1}{m_i} > 0, \ = 0 \quad \textit{or} \quad < 0, \ \textit{respectively.}$$

(5) *If $H_1(X\,;\mathbf{Z}) = (0)$ then $\kappa(X) = -\infty$.*

Proof. (1) Since D is a tree, it is clear that one fiber of p totally contained in D meets H in P_0 or P_∞. So, we assume that it passes through P_∞. It is also easy to show that $(F_0)_{red} \cong \mathbf{A}^1$.

(2) If $(F_i)_{red} \cong \mathbf{A}^1$ for some $1 \le i \le r$, then D would not be a tree. So, $(F_i)_{red} \cong \mathbf{A}^1_*$ for all $1 \le i \le r$.

(3) We shall show that $H_1(X\,;\mathbf{Z})$ is a finite abelian group. As in the proof of Lemma 2.6 this implies $H_i(X\,;\mathbf{Z}) = (0)$ if $i \neq 0, 1$. Let $X^\circ = X - Sing(X)$. Then we have $H_1(X^\circ\,;\mathbf{Z}) \cong Pic(W)/L(D \cup \Sigma)$, where $\Sigma = f^{-1}(Sing(X))$ and $L(D \cup \Sigma)$ is the subgroup of $Pic(W)$ generated by all irreducible components of D and Σ. Indeed, we have an exact sequence

$$H_3(W;\mathbf{Z}) \longrightarrow H_3(W, D\cup\Sigma;\mathbf{Z}) \longrightarrow H_2(D\cup\Sigma;\mathbf{Z}) \xrightarrow{\theta} H_2(W;\mathbf{Z})$$
$$\longrightarrow H_2(W, D\cup\Sigma;\mathbf{Z}) \longrightarrow H_1(D\cup\Sigma;\mathbf{Z}),$$

where

(i) $H_1(D\cup\Sigma;\mathbf{Z}) = (0)$ because $D\cup\Sigma$ is a tree of nonsingular rational curves,

(ii) $H_2(W;\mathbf{Z}) \cong H^2(W;\mathbf{Z}) \cong Pic(W)$ and $H_3(W;\mathbf{Z}) = (0)$ because W is rational, and

(iii) $H_3(W, D\cup\Sigma;\mathbf{Z}) = (0)$ because the homomorphism θ followed by the Poincare duality isomorphism $\delta\colon H_2(W;\mathbf{Z}) \to H^2(W;\mathbf{Z})$ is an injection.

On the other hand, by the universal coefficient theorem, we have

$$H^3(W, D\cup\Sigma;\mathbf{Z}) \cong Hom(H_3(W, D\cup\Sigma;\mathbf{Z}), \mathbf{Z})$$
$$\oplus\, Ext^1(H_2(W, D\cup\Sigma;\mathbf{Z}), \mathbf{Z}) \cong H_2(W, D\cup\Sigma;\mathbf{Z}).$$

Hence, by the Lefschetz duality, we have

$$H_1(X^\circ;\mathbf{Z}) \cong H_2(W, D\cup\Sigma;\mathbf{Z}).$$

In order to compute $H_1(X^\circ;\mathbf{Z})$, note that W with the $\mathbf{P}^1$-fibration $q\colon W \to B$ is obtained by blowing up points from a relatively minimal ruled surface $\bar{q}: \Sigma_a \to B$ with a minimal section M. Thus H is obtained as the proper transform of a 2-section $\bar{H}$ of $\bar{q}$. Since $\bar{H} \sim 2M + s\ell$ on Σ_a, we have a relation

$$H \sim 2M' + \sum_{i=0}^{r} \delta_i C_i + \begin{pmatrix} \text{fiber components of} \\ D \cup \Sigma \end{pmatrix},$$

where M' is the proper transform of M. Furthermore, we have a relation

$$\ell_\infty \sim m_i C_i + \begin{pmatrix} \text{fiber components of} \\ D \cup \Sigma \end{pmatrix} \quad \text{for } 0 \leq i \leq r.$$

Thus $H_1(X^\circ; \mathbf{Z})$ has generators $\xi_i := [C_i]$ and $\eta := [-M']$ and relations

$$m_0 \xi_0 = \cdots = m_r \xi_r = 0 \quad \text{and} \quad 2\eta = \delta_0 \xi_0 + \cdots + \delta_r \xi_r.$$

Therefore we have an exact sequence

$$0 \longrightarrow \prod_{i=0}^{r} \mathbf{Z}/m_i\mathbf{Z} \longrightarrow H_1(X^\circ; \mathbf{Z}) \longrightarrow \mathbf{Z}/2\mathbf{Z} \longrightarrow 0.$$

Suppose $Sing(X) = \emptyset$. Then $H_1(X; \mathbf{Z}) \cong H_1(X^\circ; \mathbf{Z})$ and it is an extension of $\prod_{i=0}^{r} \mathbf{Z}/m_i\mathbf{Z}$ by $\mathbf{Z}/2\mathbf{Z}$. Suppose $Sing(X) \neq \emptyset$. As in the proof of Lemma 2.11, we have an exact sequence

$$0 \longrightarrow H_1(\partial T; \mathbf{Z}) \longrightarrow H_1(X^\circ; \mathbf{Z}) \longrightarrow H_1(X; \mathbf{Z}) \longrightarrow 0.$$

To compute $H_1(\partial T; \mathbf{Z})$, we employ the same notation as used in Lemma 2.13: A graph expressed by a string of integers

$$\{-b_1, \ldots, -b_s\} - (-1) - \{-c_t, \ldots, -c_1\} \text{ with } \forall b_i \geq 2 \quad \text{and} \quad \forall c_j \geq 2$$

is a linear chain of nonsingular rational curves consisting of components expressed by their self-intersection numbers $(-b_i)$,$(-c_j)$, and (-1).

If X has a unique cyclic singular point, it is produced as follows:

Choose a point P_0 on E_1 or E_2, say E_1 (cf. Lemma 2.9, case (4–1)), which can be $E \cap E_1$ or $E \cap E_2$. Apply an oscilating sequence of blowing-ups with initial point P_0. We obtain thus a linear chain of the form as indicated above

$$\{-b_1, \ldots, -b_s\} - (-1) - \{-c_t, \ldots, -c_1\},$$

where the component $(-c_1)$ is connected to the proper transform of E if $P_0 = E \cap E_1$ (or E_1 if $P_0 \neq E \cap E_1$). The unique (-1) component will give rise to a component C_0 in the fiber F_0. Then contract the sub-chain $\{-b_1, \ldots, -b_s\}$ to a cyclic quotient singular point of type $(b_1, \ldots, b_s)$ (cf. the paragraph 2.2 above). Lemma 2.13 or its variation then shows that the multiplicity m_0 is divisible by $|H_1(\partial T; \mathbf{Z})|$, which is the numerator of a continued fraction $[b_1, \ldots, b_s]$.

Suppose X has a quotient singular point of Dynkin type D_r. Then it is produced as follows:

Produce a tree below by an oscilating sequence of blowing-ups with initial point $H \cap E$,

$$\{-c_1, \ldots, -c_t\} - (-1) - \{-b_s, \ldots, -b_1\} - [E] \begin{array}{l} \nearrow [E_1] \\ \\ \searrow [E_2] \end{array},$$

where $[E]$, $[E_1]$ and $[E_2]$ are the proper transforms of E, E_1 and E_2. Then contract the sub-tree on the right-hand side of a unique (-1) component to produce a singular point of Dynkin type D_r. Note that the multiplicity of $[E]$ in $q^*(Q_0)$ is 2 and $|H_1(\partial T; \mathbf{Z})| \cong 4m$, where

$$m = (b-1)n - q, \, b = -([E])^2 \quad \text{and} \quad \frac{n}{q} = [b_1, \ldots, b_s]$$

(cf. Brieskorn [1]). Lemma 2.13 shows that $m_0/2$ is equal to the multiplicity of a (-1) curve in a linear chain

$$\{-c_1, \ldots, -c_t\} - (-1) - \{-b_s, \ldots, -b_1, -(b-1)\}$$

which is obtained by an oscilating sequence of blowing-ups from a smooth fiber of a $\mathbf{P}^1$-fibration. Hence $m_0/2$ is the numerator of $[b-1, b_1, \ldots, b_s]$ which is equal to m. So, $m_0 = 2m$ and $H_1(X; \mathbf{Z}) \cong \prod_{i=1}^r \mathbf{Z}/m_i\mathbf{Z}$.

The case X has two rational double points of type A_1 can be treated in the same fashion.

(4) A straightforward computation shows that

$$D + \Sigma + K_W \sim \sum_{i=1}^r \tilde{F}_i - \sum_{i=1}^r \bar{C}_i + G_0 + G_\infty - \ell_\infty,$$

where G_0 and G_∞ are effective divisors supported by $p^{-1}(Q_0)$ and $p^{-1}(Q_\infty)$, respectively such that the components appearing in G_0 and G_∞ have negative-definite intersection matrices. Hence $\kappa(X) = 1, 0$ or $-\infty$ if and only if

$$(r-1) - \sum_{i=1}^r \frac{1}{m_i} > 0, \ = 0 \quad \text{or} \quad < 0, \text{ respectively.}$$

(5) If $H_1(X; \mathbf{Z}) = (0)$ then there are no singular fibers of type mC with $m \geq 2$ and $C \cong \mathbf{A}^1_*$. Hence $\kappa(X) = -\infty$. ■

We consider next the case where π is untwisted and $C \cong \mathbf{A}^1$. Let H_1 and H_2 be two cross-sections of p contained in D, and let ℓ_∞ be the unique fiber of p totally contained in D. If H_1 and H_2 meet each other, then ℓ_∞ must pass through the point $H_1 \cap H_2$, for otherwise D would contain a loop. Then, by repeating elementary transformations with center $H_1 \cap H_2$ (i.e., blowing up a point on a fiber and contracting the proper transform of the fiber), we may assume that $H_1 \cap H_2 = \emptyset$. Then we may also assume that ℓ_∞ is a smooth fiber. By Lemma 2.9, all fibers of π are irreducible except for one fiber F_0 which consists of two conponents. We have two cases to consider:

(I) $F_0 = n_1 G_1 + n_2 G_2$, where $G_1 \cong G_2 \cong \mathbf{A}^1$ and G_1 and G_2 meet in one point which might be a cyclic quotient singular point;

(II) $F_0 = m_0 C_0 + nG$, where $C_0 \cong \mathbf{A}^1_*$, $G \cong \mathbf{A}^1$, $C_0 \cap G = \emptyset$ and there might be a single cyclic quotient singular point on G.

We denote by $F_i = m_i C_i$ $(1 \leq i \leq r)$ all the irreducible singular fibers of π. Then $C_i \cong \mathbf{A}^1_*$.

Let $\tau: W \to \Sigma_a$ be a birational morphism onto a Hirzebruch surface Σ_a which preserves the $\mathbf{P}^1$-fibration and $\bar{H}_1 \cap \bar{H}_2 = \emptyset$, where $\bar{H}_i = \tau(H_i)$, $i = 1, 2$. We may assume that $(H_1^2) = (\bar{H}_1^2) = -a$. So, the fibers $\bar{F}_i = q^{-1}(\pi(F_i))$ $(0 \leq i \leq r)$ are obtained from smooth fibers by blowing up first the intersection points of the fibers with $\bar{H}_2$. We write

$$\tau^* \bar{H}_2 = \sum_{i=1}^{r} \delta_i \bar{C}_i + R + (\text{other components}),$$

where $R = \gamma_1 \bar{G}_1 + \gamma_2 \bar{G}_2$ in the case (I) and $R = \delta_0 \bar{C}_0 + \gamma \bar{G}$ in the case (II), and $\bar{C}_i$'s, $\bar{G}_j$'s and $\bar{G}$ are the closure of C_i's, G_j's and G.

Then we have the following result.

Lemma 2.15. *Let the assumptions and notations be as above. Suppose the case* (I) *occurs. Then we have*:

(1) $H_i(X; \mathbf{Z}) = (0)$ *if* $i \neq 0, 1$ *and* $H_1(X; \mathbf{Z}) \cong \prod_{i=1}^{r} \mathbf{Z}/m_i\mathbf{Z}$.

(2) $\kappa(X) = 1, 0$ *or* $-\infty$ *if and only if*

$$(r-1) - \sum_{i=1}^{r} \frac{1}{m_i} > 0, \quad = 0 \quad \text{or} \quad < 0, \text{ respectively.}$$

(3) *If* $H_1(X; \mathbf{Z}) = (0)$ *in particular, then* $\kappa(X) = -\infty$.

Proof. (1) Let $\tilde{F}_0$ be the fiber of $q: W \to B$ containing F_0. Then $\tilde{F}_0$ is a linear chain, and one of $\bar{G}_1$ and $\bar{G}_2$ is a (-1) curve. If the other is not a (-1) curve, the graph of $\tilde{F}_0$ looks like the one in Lemma 2.13, (3), where we take $H = H_2$ and $\ell = \tau(\tilde{F}_0)$ on Σ_a, and either $\bar{G}_1 = (-c_{u+1})$ and $\bar{G}_2 = (-1)$, or $\bar{G}_1 = (-1)$ and $\bar{G}_2 = (-b_{v+1})$. With the notations in Lemma 2.13, the sub-chain $\{-c_{u+2}, \dots, -c_t\}$ or $\{-b_s, \dots, -b_{v+2}\}$ is contracted to a cyclic quotient singular point. If both $\bar{G}_1$ and $\bar{G}_2$ are (-1) curves, the graph of $\tilde{F}_0$ looks like the one in Lemma 2.13, (4).

On the other hand, $H_1(X^\circ; \mathbf{Z})$ is generated by $\xi_1, \dots, \xi_r, \eta_1$ and η_2 with relations

$$n_1\eta_1 + n_2\eta_2 = m_1\xi_1 = \cdots = m_r\xi_r = 0$$

$$\gamma_1\eta_1 + \gamma_2\eta_2 + \delta_1\xi_1 + \cdots + \delta_r\xi_r = 0,$$

where ξ_i, η_1 and η_2 are respectively the classes of C_i, G_1 and G_2 in $H_1(X^\circ; \mathbf{Z})$. Therefore we have an exact sequence

$$0 \longrightarrow \prod_{i=1}^{r} \mathbf{Z}/m_i\mathbf{Z} \longrightarrow H_1(X^\circ; \mathbf{Z}) \longrightarrow \Gamma \longrightarrow 0,$$

where $\Gamma = (\mathbf{Z}\eta_1 + \mathbf{Z}\eta_2)/\langle n_1\eta_1 + n_2\eta_2, \gamma_1\eta_1 + \gamma_2\eta_2\rangle$ and $|\Gamma| = |n_1\gamma_2 - n_2\gamma_1|$. So, by Lemma 2.13, $|\Gamma| = |H_1(\partial T; \mathbf{Z})|$. Then, invoking the exact sequence

$$0 \longrightarrow H_1(\partial T; \mathbf{Z}) \longrightarrow H_1(X^\circ; \mathbf{Z}) \longrightarrow H_1(X; \mathbf{Z}) \longrightarrow 0,$$

we know that $H_1(X; \mathbf{Z}) \cong \prod_{i=1}^r \mathbf{Z}/m_i\mathbf{Z}$.

(2) This can be proved straightforwardly as in Lemma 2.14.

(3) If $H_1(X; \mathbf{Z}) = (0)$ then $m_1 = \cdots = m_r = 1$. So, $\tilde{F}_i = q^{-1}(\pi(F_i))$ is a smooth fiber for $1 \leq i \leq r$, and the boundary $D \cup \Sigma$ of X° in W contains a connected linear chain

$$\ell_\infty + H_2 + (\text{a connected maximal subchain of } \tilde{F}_0 \cap D),$$

where $(\ell_\infty^2) = 0$. Then, by [7; Cor 2.4.3], X° contains an $\mathbf{A}^1$-fibration. Hence $\kappa(X) - \infty$. ■

Lemma 2.16. *Let the assumptions and notations be as above. Suppose the case* (II) *occurs. Then we have:*

(1) $H_i(X; \mathbf{Z}) = (0)$ *if* $i \neq 0, 1$ *and* $\prod_{i=1}^r \mathbf{Z}/m_i\mathbf{Z}$ *is a subgroup of* $H_1(X; \mathbf{Z})$.

(2) $\kappa(X) = 1, 0$ *or* $-\infty$ *if and only if*

$$(r-1) - \sum_{i=1}^r \frac{1}{m_i} > 0, \quad = 0 \quad \textit{or} \quad < 0, \textit{ respectively.}$$

(3) *If* $H_1(X; \mathbf{Z}) - (0)$ *in particular, then* $\kappa(X) = -\infty$.

Proof. (1) As in the previous lemma, let $\tilde{F}_0$ be the fiber of q containing F_0. Its graph is obtained from the graph in Lemma 2.13, (3) by applying an oscilating sequence of blowing-ups with initial point on a component $(-c_{u+1})$ or $(-b_{v+1})$. Moreover, $H_1(X^\circ; \mathbf{Z})$ has generators and relations as indicated in Lemma 2.15, where η_1 and η_2 are replaced by the classes of C_0 and G in $H_1(X^\circ; \mathbf{Z})$. We can show by Lemma 2.13 that $|H_1(\partial T; \mathbf{Z})|$ divides $|m_0\gamma - n\delta_0|$ but does not necessarily equal $|m_0\gamma - n\delta_0|$. The assertion (2) is straightforward. The proof of the assertion (3) is similar to Lemma 2.15. ■

Summarizing Lemmas 2.11, 2.14, 2.15 and 2.16, we obtain the following result.

Theorem 2.17. *Let X be a logarithmic homology plane with a $\mathbf{C}^*$-fibration $\pi: X \to C$. Suppose $\kappa(X) \geq 0$. Then $\kappa(X) = 1$ and π* is an untwisted *$\mathbf{C}^*$-fibration with $C \cong \mathbf{P}^1$. Moreover, X has at most one singular point which has necessarily a cyclic quotient singularity. The structure of X and the condition for X to be contractible are described in Lemma* 2.11.

§3. A homology plane with non-trivial involution

3.1. We work on the projective plane $\mathbf{P}^2$ with homogeneous coordinates (x_0, x_1, x_2). Let C be a conic defined by $x_0^2 = x_1^2 + x_2^2$ and let ℓ_1, ℓ_2 and ℓ_3 be three concurrent lines defined by $x_1 = -x_0$, $x_1 = 0$ and $x_1 = x_0$, respectively. So,

ℓ_1 and ℓ_3 touch C in points $P_1 := (1, -1, 0)$ and $P_3 := (1, 1, 0)$, respectively, while ℓ_2 meets C in two points P_{21} and P_{22}. Let $P_\infty := (0, 0, 1)$ and $P_0 := (0, 1, 0)$. This configuration of curves admits an involution σ defined by $\sigma: (x_0, x_1, x_2) \mapsto (x_0, -x_1, x_2)$. The action σ fixes the line ℓ_2 pointwise and stabilizes the line ℓ_0 defined by $x_0 = 0$; the points P_0 and P_∞ are fixed points on ℓ_0.

Let $\sigma_1 : \Sigma_1 \to \mathbf{P}^2$ be the blowing-up of the point P_∞ and let $M_\infty = \sigma_1^{-1}(P_\infty)$, where Σ_1 is the Hirzebruch surface of degree 1. Let $Q_1^{(1)} = \sigma_1'(\ell_1) \cap M_\infty$ and $Q_3^{(1)} = \sigma_1'(\ell_3) \cap M_\infty$, where $\sigma_1'(\ell_1)$ and $\sigma_1'(\ell_3)$ signify, as usual, the proper transforms of ℓ_1 and ℓ_3 by σ_1. Next, let $\sigma_2 : V \to \Sigma_1$ be a composite of blowing-ups with centers $Q_1^{(1)}, Q_3^{(1)}, P_1^{(i)}, P_3^{(i)}$ $(i = 0, 1)$ and $P_{21}^{(j)} (0 \le j \le 4)$, where, for $s = 1$ and 3, $P_s^{(0)} = P_s$ and $P_s^{(1)}$ is an infinitely near point of P_s of order one lying on the conic $\sigma_1'(C)$, and where $P_{21}^{(0)} = P_{21}$ and, for $1 \le j \le 4$, $P_{21}^{(j)}$ is an infinitely near point of P_{21} of order j lying on the conic $\sigma_1'(C)$. The surface V inherits the $\mathbf{P}^1$-fibration $p: V \to \mathbf{P}^1$ from the Hirzebruch surface Σ_1. The lines ℓ_1, ℓ_2, ℓ_3 give rise to the fibers $T^{(1)}, T^{(2)}, T^{(3)}$ of p, which have the graphs indicated below:

$$\begin{tikzcd}[column sep=small] T^{(1)}: S_\infty \arrow[r, dash] & A^{(1)} \arrow[r, dash] & (-3) \arrow[r, dash] & (-1) \arrow[r, dash] \arrow[d, dash] & (-2) \\ & & & S_0 & \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] T^{(2)}: S_\infty \arrow[r, dash] & A^{(2)} \arrow[r, dash] \arrow[drr, dash] & (-2) \arrow[r, dash] & (-2) \arrow[r, dash] & (-2) \arrow[r, dash] & (-2) \arrow[r, dash] & (-1) = \tilde{A}^{(2)} \arrow[dlll, dash] \\ & & & S_0 & & & \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] T^{(3)}: S_\infty \arrow[r, dash] & A^{(3)} \arrow[r, dash] & (-3) \arrow[r, dash] & (-1) \arrow[r, dash] \arrow[d, dash] & (-2), \\ & & & S_0 & \end{tikzcd}$$

where $S_0, S_\infty, A^{(2)}, A^{(1)}$ and $A^{(3)}$ are respectively the proper transforms of $\sigma_1'(C)$, M_∞ and $\sigma_1'(\ell_i)$ by σ_2 and $\sigma_2^{-1}(Q_1^{(1)})$ and $\sigma_2^{-1}(Q_3^{(1)})$, and where $(S_\infty^2) = -1$, $(S_0^2) = -5$ and $(A^{(i)})^2 = -1$. Let $D = (\sigma_2 \cdot \sigma_1)^{-1}(C \cup \ell_1 \cup \ell_2 \cup \ell_3) - A^{(1)} - \tilde{A}^{(2)} - A^{(3)}$ and let $X = V - D$. Then we have the following result:

Theorem 3.1. (1) *X has an involution σ induced by the involution σ on $\mathbf{P}^2$ as defined as above.*

(2) *X is a homology plane of Kodaira dimension 2.*

(3) *The involution σ on X has a unique fixed point $(\sigma_2\sigma_1)^{-1}(P_0)$.*

(4) *Let Y be the quotient space of X by σ. Then Y is a logarithmic homology plane of Kodaira dimension 2 with a cyclic quotient singular point of Dynkin type A_1.*

Proof. (1) Note that the centers of the above blowing-ups are fixed points under the action of σ or the induced action of σ. So, σ preserves the boundary divisor D, and hence induces an involution on X. The assertion (3) is straightforward.

(2) As divisors on V, we can write

$$T^{(i)} = u_i A^{(i)} + (\text{other fiber components}),\ i = 1, 3$$
$$T^{(2)} = u_2 \tilde{A}^{(2)} + (\text{other fiber components})$$
$$\sigma_2^*(M_\infty) = u_1 A^{(1)} + v_3 A^{(3)} + (\text{other boundary components})$$
$$\sigma_2^* \sigma_1'(C) = w\tilde{A}^{(2)} + (\text{other boundary components}),$$

where $u_1 = u_2 = u_3 = v_1 = v_2 = 1$ and $w = 5$. Since we have relations $T^{(1)} \sim T^{(2)} \sim T^{(3)}$ and $\sigma_2^* \sigma_1'(C) \sim 2\sigma_2^*(M_\infty) + 2\sigma_2^*(\sigma_1'(\ell_2))$ in $Pic(V)$, we compute

$$|H_1(X\,;\mathbf{Z})| = |\Delta| \quad \text{with} \quad \Delta = det \begin{pmatrix} u_1 & -u_2 & 0 \\ 0 & u_2 & -u_3 \\ 2v_1 & 2v_3 & 2u_2 - w \end{pmatrix}.$$

Hence $H_1(X\,;\mathbf{Z}) = (0)$. This implies that X is a homology plane.

(4) The involution σ on V interchanges the fibers $T^{(1)}$ and $T^{(3)}$, has the fixed curves $A^{(2)}$ and two (-2) curves, all appearing in the fiber $T^{(2)}$ and disjoint from each other, has the fixed points $Q_0 := (\sigma_2\sigma_1)^{-1}(P_0)$, $Q_\infty := S_\infty \cap (\sigma_2\sigma_1)'(\ell_0)$ and $Q_2 := S_0 \cap \tilde{A}^{(2)}$ and stabilizes the curves S_∞ and S_0. Let $\sigma_3 \colon \tilde{V} \to V$ be the blowing-ups of the points Q_0, Q_∞ and Q_2. Let $W = \tilde{V}/\langle\sigma\rangle$ and let $\rho \colon \tilde{V} \to W$ be the quotient morphism. Then the quotient space W is a nonsingular projective surface with a $\mathbf{P}^1$-fibration $q \colon W \to \mathbf{P}^1$ such that

(1) $R_\infty := \rho(S_\infty)$ is a cross-section of q and $R_0 := \rho(S_0)$ is a 2-section of q;

(2) $U_1 := \rho_*(T^{(1)}) = \rho_*(T^{(3)})$ and it is a fiber of q with the graph:

$$\begin{array}{ccccccccccc} R_\infty & - & (-1)^* & - & (-3) & - & (-1) & - & (-2)\,; \\ & & & & & & | & & \\ & & & & & & R_0 & & \end{array}$$

(3) $U_2 := \rho_*(T^{(2)})$ is a fiber of q with the graph:

$$\begin{tikzcd}[column sep=tiny] R_\infty \arrow[r, dash] & (-2) \arrow[r, dash] \arrow[drrr, dash] & (-1) \arrow[r, dash] & (-4) \arrow[r, dash] & (-1) \arrow[r, dash] & (-4) \arrow[r, dash] & (-1)^* \arrow[r, dash] & (-2)\,; \arrow[dllll, dash] \\ & & & & R_0 & & & \end{tikzcd}$$

(4) $U_0 := \rho_*(\sigma_3^*((\sigma_2\sigma_1)'(\ell_0)))$ is a fiber of q with the graph:

$$\begin{array}{ccccccc} R_\infty & - & (-2) & - & (-1)^* & - & (-2)^{**}\,. \\ & & & & | & & \\ & & & & R_0 & & \end{array}$$

Let $\tilde{Y}$ be the open set of W obtained by removing all components of $R_\infty \cup R_0 \cup U_0 \cup U_1 \cup U_2$ but the components marked with (*). Contract the component in U_0 marked with (**) to obtain an affine surface Y with a unique cyclic quotient singular point of Dynkin type A_1. Then $Y \cong X/\langle\sigma\rangle$. It is straightforward to show that Y is a logarithmic homology plane of Kodaira dimension 2. ∎

§4. The automorphism group of a homology plane

4.1. In this section we shall prove the following

Theorem 4.1. *Let X be a $\mathbf{Q}$-homology plane defined over $\mathbf{C}$ admitting an effective action of a cyclic group G, let (V, D) be a G-normal completion, i.e., V is a nonsingular projective surface containing X as an open set such that the G-action on X extends to a G-action on V and $D = V - X$ is a divisor with simple normal crossings, and let $T = T(V, D)$ be the weighted dual graph of D. Suppose G acts trivially on T and $\kappa(X) = 2$. Then D has no irreducible components with positive self-intersection number.*

Proof. Our proof proceeds by *reduction absurdum.*

(1) If T is a linear chain, X must contain a cylinderlike open set (cf. [7]). Hence $\kappa(X) = -\infty$, a contradiction. So, T has a branch point. Let C be an irreducible component with $(C^2) = n > 0$. If the vertex $v(C)$ corresponding to C is not a branch point, one can show easily that either X contains a cylinderlike open set or X admits a $\mathbf{C}^*$-fibration. In either case, $\kappa(X) \neq 2$, a contradiction. So, $v(C)$ is a branch point. Let $B_1, \dots, B_r$ be all irreducible components of D with $(B_i \cdot C) > 0$, where $r \geq 3$. On the other hand, every irreducible component of D is a nonsingular rational curve and stable under the G-action by the assumption. In particular, C is a component of $V^G =$ the G-fixed point locus.

(2) Choose $(n-1)$ points $P_1, \dots, P_{n-1}$ on C which are distinct from $B_j \cap C$ $(1 \leq j \leq r)$. Let $\sigma\colon V' \to V$ be the blowing-up of points $P_1, \dots, P_{n-1}$ and let $C' = \sigma'(C)$, $B_j' = \sigma'(B_j)$ $(1 \leq j \leq r)$ and $E_i = \sigma^{-1}(P_i)$ $(1 \leq i \leq n-1)$. Then $(C')^2 = 1$, whence the linear system $|C'|$ has dimension 2. Write the parameter space of $|C'|$ by W; so, $W \cong \mathbf{P}^2 = \mathbf{P}(H^0(V', \mathcal{O}_{V'}(C')))$. The group G acts on $H^0(V', \mathcal{O}_{V'}(C'))$ linearly and hence on $\mathbf{P}^2$.

Claim. *Let t_0 be the point of W corresponding to C'. We then have a line $L \subseteq W^G$ such that $W^G = L \cup \{t_0\}$ and $t_0 \notin L$.*

Proof. Choose a system of homogeneous coordinates $\{x_0, x_1, x_2\}$ on W such that t_0 is given by $(1, 0, 0)$ and

$$\zeta\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta^\alpha & 0 \\ 0 & 0 & \zeta^\beta \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix},$$

where ζ is a primitive root of unity of order $|G|$ and $\alpha, \beta \geq 0$. This is possible because G is a cyclic group. Now, let P be a general point of C' (other than P_i's and $B_j' \cap C$'s) and let $\tau_P\colon V'' \to V'$ be the blowing-up of P. Let $C'' = \tau_P'(C')$, $E = \tau_P^{-1}(P)$. Then $(C'')^2 = 0$ and $dim|C''| = 1$. Moreover, G acts on the linear pencil $|C''|$. Hence there is a member $D(P) \in |C'|$, $D(P) \neq C'$, which is G-stable. The linear pencil $(\tau_P)_*|C''|$ is a sublinear system of $|C'|$ corresponding to the line $\ell(P)$ through t_0 and the point $[D(P)]$. The G-action acts along the line $\ell(P)$. Hence we must have $\alpha = \beta$. Then the line L: $x_0 = 0$ is contained in W^G, and $t_0 \notin L$. If $\alpha > 0$ then we are done. If $\alpha = 0$ then $W = W^G$. Namely, every member of $|C'|$ is G-stable. By the next claim, each member of $|C'|$ then contains a pointwise G-fixed component and, in fact, there are too many (infinitely many, in fact) such components. A contradiction. ■

(3) We shall verify

Claim. *With the above notations, $D(P)$ is reducible and contains a G-fixed component.*

Proof. The curve $D(P)$ is reduced to a single G-stable rational curve by G-equivariant blowing downs. After these G-equivariant blowing downs, the last component is G-stable and meet the images of $B_1, \ldots, B_r$, $E_1, \ldots, E_{n-1}$ transversally. So, the last curve must be pointwise fixed by the G-action. So, back on the surface V', the proper transform of this last curve is pointwise G-fixed. This component, after contracting E back on V', is still pointwise G-fixed and hence cannot meet C'. If so, the G-action on V' would be trivial. Namely, $D(P)$ is reducible. This argument can apply to any G-stable member of $|C'|$. ■

Let $H(P)$ be the irreducible component of $D(P)$ which meets C' at the point P. Then $H(P)$ is G-stable and not contained in D. Let $K(P) = H(P)$ with $D \cap H(P)$, $E_i \cap H(P)$'s (if not empty) and $H(P) \cap$(other components of $D(P)$) removed off. Note that these removed points on $H(P)$ are G-fixed points. This is clear for $D \cap H(P)$ and $E_i \cap H(P)$'s. If there are several non G-fixed points contained in $H(P) \cap$(other components of $D(P)$), one can readily show that $D(P)$ contains a loop, which is not the case. Therefore we have shown that $\{K(P)\}_{P \in C' - S} \subset X$ possibly with a finite set S, and the G-action on X is along $K(P)$'s. Since $K(P) \subset H(P) \cong P^1$, $K(P)$ must be either C or C^* for general P's. So, $\kappa(X) \leq 1$, a contradiction. ■

4.2. Let X be a **Q**-homology plane of Kodaira dimension 2. Then it is well known that the automorphism group $Aut(X)$ is a finite group. Let (V, D) be an $Aut(X)$-normal completion of X, whose existence is guaranteed by Sumihiro's theorem. Let T be the weighted dual graph of D. Then $Aut(X)$ acts on the graph T in a natural way. Let N be the normal subgroup of $Aut(X)$ consisting of elements which acts trivially on T. As in the proof of Theorem 4.1, D contains an irreducible component, say C, which corresponds to a branch point of T. Let $B_1, \ldots, B_r$ be all irreducible components of D which meet the component C. Then C is pointwise fixed by N and N acts effectively on each of the components $B_1, \ldots, B_r$. Then one can easily show that N is a subgroup of the multiplicative group G_m. Hence N is a cyclic group. We have proved the following:

Theorem 4.2. *Let X be a **Q**-homology plane with $\kappa(X) = 2$. Then $Aut(X)$*

contains a cyclic normal subgroup N such that the quotient group $Aut(X)/N$ is a subgroup of $Aut(T)$, which is the symmetric automorphism group of the boundary weighted graph T of X.

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