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Relations on pfaffians II: a counterexample

By

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1. Introduction

Let R be a commutative ring with unity and fix an integer $n \geq 1$. We denote by S the polynomial ring $R[\{x_{ij} | 1 \leq i < j \leq n\}]$. $S = \bigoplus_{i \geq 0} S_i$ has a structure of graded ring with $\deg(x_{ij}) = 1$ for $1 \leq i < j \leq n$. For a positive integer t such that $1 < 2t \leq n$, Pf_{2t} denotes the ideal generated by all $2t$-order pfaffians of the n by n generic antisymmetric matrix $X = (x_{ij})$, where $x_{ij} = -x_{ji}$ for $i > j$, $x_{ii} = 0$ for $i = 1, \ldots, n$.

$X = (x_{ij})$ has $\binom{n}{2t}$ distinct $2t$-order pfaffians and they are linearly independent over R. Hence there exists a graded exact sequence

$$S(-t)^{\binom{n}{2t}} \xrightarrow{\partial_1} S \longrightarrow S/Pf_{2t} \longrightarrow 0$$

where ∂_1 sends basis elements of the graded free module $S(-t)^{\binom{n}{2t}}$ to distinct $2t$-order pfaffians of X. Since ∂_1 is homogeneous, $\operatorname{Ker}(\partial_1)$ is decomposed into homogeneous components. By the linear independence of pfaffians, $\operatorname{Ker}(\partial_1)$ is described in the form $\bigoplus_{a>0} \operatorname{Ker}(\partial_1)_{t+a}$. We call each element of $\operatorname{Ker}(\partial_1)_{t+a}$ *a relation (on $2t$-order pfaffians) of degree a*.

When R contains the rationals $\mathbf{Q}$, $\operatorname{Ker}(\partial_1)$ is generated as an S-module by relations of degree 1 ([4]). Furthermore over an arbitrary commutative ring R, it is true when $t = 1$, $n = 2t$, $n = 2t + 1$ ([2]) or $n = 2t + 2$ ([7]).

In [6] we have already shown that:

1) $\operatorname{Ker}(\partial_1)$ is generated as an S-module by $\bigoplus_{a=1}^{t} \operatorname{Ker}(\partial_1)_{t+a}$. (By the general theory on Gröbner bases ([1]) we can show this fact immediately, because the set of all $2t$-order pfaffians forms a Gröbner basis of the ideal Pf_{2t}.)

2) Suppose that R is the prime field of characteristic $p > 0$. If $2p > n - 2t$, then $\operatorname{Ker}(\partial_1)$ is generated as an S-module by relations of degree 1.

We can prove 2) in the same way as in the case of generic matrices ([5]). By 2) we know that $\operatorname{Ker}(\partial_1)$ is generated by relations of degree 1 over an arbitrary commutative ring R when $n \leq 2t + 3$. (In this paper we will not use these results in [6].)

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In this article we show that $\mathrm{Ker}(\partial_1)$ is not generated as an S-module by $\mathrm{Ker}(\partial_1)_{t+1}$ in general. In fact there exists a relation of degree 2 which is not contained in $S_1 \cdot \mathrm{Ker}(\partial_1)_{t+1}$ when $n = 8$, $t = 2$ and R is a field of characteristic 2. Consequently we know that the Betti numbers of S/Pf_{2t} depend on the characteristic of the coefficient field in this case. Therefore S/Pf_{2t} does not have generic minimal free resolutions in general by Proposition 2 of Section 4 in [8]. (In the case of determinantal ideals of generic matrices, the second syzygies of the ideals are not generated by degree 1 relations on the first syzygies in general ([3]). Hence there do not exist generic minimal free resolutions in this case, either.)

Section 2 is devoted to introducing the main theorem. In Section 3 we reduce its proof to Lemma 3.3 which will be proved in Section 4.

The author would like to thank Professor J. Nishimura for his valuable advice and encouragement.

2. Notation and main theorem

Throughout this article let $\mathbf{F}_2$ be the prime field of characteristic 2 and E the $\mathbf{F}_2$-vector space of dimension 8 with basis $\{e_1, \ldots, e_8\}$. Then $(e_i \wedge e_j)$ is a generic 8 by 8 antisymmetric matrix with entries in $\mathrm{S}(\wedge^2 E) = \bigoplus_{r \geq 0} \mathrm{S}_r(\wedge^2 E)$, where $\mathrm{S}_r(*)$ or $\wedge^r(*)$ stands for the rth symmetric or exterior module, respectively. We sometimes denote $\mathrm{S}(\wedge^2 E)$ or $\mathrm{S}_r(\wedge^2 E)$ simply by S or S_r and call S_r the homogeneous component of degree r.

Pf_4 denotes the ideal generated by all 4-order pfaffians of $(e_i \wedge e_j)$. This antisymmetric matrix has $\binom{8}{4} = 70$ distinct 4-order pfaffians and they are linearly independent over $\mathbf{F}_2$.

Since S is a polynomial ring over $\mathbf{F}_2$, S/Pf_4 has a graded minimal free resolution

$$\cdots \longrightarrow \bigoplus_{i>0} S(-2-i)^{\beta_i} \longrightarrow S(-2)^{70} \xrightarrow{\partial_1} S \longrightarrow S/Pf_4 \longrightarrow 0,$$

where $S(a)$ is a graded free module with grading $[S(a)]_c = S_{a+c}$ and ∂_1 sends every generators of $S(-2)^{70}$ to distinct 4-order pfaffians of $(e_i \wedge e_j)$. Then we have:

Theorem 2.1. *With notation as above, β_2 is not 0, i.e., Pf_4 has the non-linear first syzygy.*

Our purpose in this article is to prove this theorem.

3. How to calculate β_2

Throughout this article, for a graded module N, N_a stands for the homogeneous component of degree a.

Let $\mathcal{M}$ be the homogeneous maximal ideal of S. Then β_2 is equal to $\dim_{\mathbf{F}_2}([\mathrm{Tor}_2^S(S/Pf_4, S/\mathcal{M})]_4)$. Since both S/Pf_4 and $S/\mathcal{M}$ have the graded S-

module structures, $\mathrm{Tor}_2^S(S/Pf_4, S/\mathscr{M})$ is also graded.) Furthermore $[\mathrm{Tor}_2^S(S/Pf_4, S/\mathscr{M})]_4$ is isomorphic to $[\mathrm{Tor}_1^S(Pf_4, S/\mathscr{M})]_4$. ($Pf_4$ is a graded submodule of S.) In order to compute $\dim_{\mathbf{F}_2}([\mathrm{Tor}_1^S(Pf_4, S/\mathscr{M})]_4)$, consider the Koszul complex

$$\mathbf{C}.: \cdots \longrightarrow \wedge^i(\wedge^2 E) \otimes_{\mathbf{F}_2} S(-i) \longrightarrow \cdots \longrightarrow \wedge^2 E \otimes_{\mathbf{F}_2} S(-1) \longrightarrow S \longrightarrow 0,$$

which is a graded free resolution of $S/\mathscr{M}$. (Boundary maps are defined to be the following composition;

$$\wedge^i(\wedge^2 E) \otimes_{\mathbf{F}_2} S(-i) \xrightarrow{\Delta \otimes 1} \wedge^{i-1}(\wedge^2 E) \otimes_{\mathbf{F}_2} \wedge^2 E \otimes_{\mathbf{F}_2} S(-i)$$

$$= \wedge^{i-1}(\wedge^2 E) \otimes_{\mathbf{F}_2} S_1 \otimes_{\mathbf{F}_2} S(-i) \xrightarrow{1 \otimes m} \wedge^{i-1}(\wedge^2 E) \otimes_{\mathbf{F}_2} S(-i+1),$$

where Δ is the comultiplication and m is the multiplication.) Since $(\wedge^i \wedge^2 E) \otimes_{\mathbf{F}_2} S(-i)) \otimes_S Pf_4 = \wedge^i(\wedge^2 E) \otimes_{\mathbf{F}_2} Pf_4(-i)$, the degree 4 component of the graded complex $\mathbf{C}. \otimes_S (Pf_4)$ is written in the form

$$0 \longrightarrow \wedge^2(\wedge^2 E) \otimes_{\mathbf{F}_2} (Pf_4)_2 \xrightarrow{\phi} \wedge^2 E \otimes_{\mathbf{F}_2} (Pf_4)_3 \xrightarrow{\psi} (Pf_4)_4 \longrightarrow 0.$$

Since $[\mathrm{Tor}_1^S(Pf_4, S/\mathscr{M})]_4 = \mathrm{Ker}(\psi)/\mathrm{Im}(\phi)$, β_2 is equal to $\dim_{\mathbf{F}_2}(\mathrm{Ker}(\psi)/\mathrm{Im}(\phi))$. Hence in order to prove Theorem 2.1, we have only to show the following lemma.

Lemma 3.1. *$\wedge^2(\wedge^2 E) \otimes_{\mathbf{F}_2} (Pf_4)_2 \xrightarrow{\phi} \wedge^2 E \otimes_{\mathbf{F}_2} (Pf_4)_3 \xrightarrow{\psi} (Pf_4)_4 \longrightarrow 0$ is not exact.*

Since modules in Lemma 3.1 are so big to calculate, we deal with only some direct summand as follows.

Definition 3.2. Let $(\wedge^2(\wedge^2 E) \otimes_{\mathbf{F}_2} (Pf_4)_2)^*$ be the subspace of $\wedge^2(\wedge^2 E) \otimes_{\mathbf{F}_2} (Pf_4)_2$ where each basis element e_i appears exactly once and define $(\wedge^2 E \otimes_{\mathbf{F}_2} (Pf_4)_3)^*$ and $((Pf_4)_4)^*$ similarly. Then it is clear that

$$(\wedge^2(\wedge^2 E) \otimes_{\mathbf{F}_2} (Pf_4)_2)^* \xrightarrow{\phi^*} (\wedge^2 E \otimes_{\mathbf{F}_2} (Pf_4)_3)^* \xrightarrow{\psi^*} ((Pf_4)_4)^* \longrightarrow 0 \qquad (1)$$

is a direct summand of the complex in Lemma 3.1. (ϕ^*(resp. ψ^*) is the restriction of ϕ (resp. ψ).)

We will show that the sequence (1) is not exact.

By direct computations (using plethysm formulas in [6]) it is easy to check that

$$\dim_{\mathbf{F}_2}(\wedge^2(\wedge^2 E) \otimes_{\mathbf{F}_2} (Pf_4)_2)^* = 210,$$
$$\dim_{\mathbf{F}_2}(\wedge^2 E \otimes_{\mathbf{F}_2} (Pf_4)_3)^* = 280,$$
$$\dim_{\mathbf{F}_2}((Pf_4)_4)^* = 91.$$

Therefore in order to prove Lemma 3.1, it is sufficient to show the following lemma;

Lemma 3.3. $\dim_{\mathbf{F}_2} \mathrm{Ker}(\phi^*) \geq 22$.

This will be proved in the next section.

Remark 3.4. By using the same mothod as in [6] it is proved that $\beta_2 \leq 1$. So, by Lemma 3.3, we have $\beta_2 = 1$. (From Theorem 5.3 in [6], $\beta_2 = 0$ in other characteristic. In fact, if we determine appropriate signatures, a set consists of 21 elements indexed by $2 \leq i \leq j \leq 8$ in Definition 4.1 forms a free basis of $\mathrm{Ker}(\phi^*)$ in the case of characteristic 0.) For instance, one of the Koszul relations on 4-order pfaffians

$$(e_1 \wedge e_2 \wedge e_3 \wedge e_4) \otimes pf_4(e_5 \wedge e_6 \wedge e_7 \wedge e_8) - (e_4 \wedge e_5 \wedge e_6 \wedge e_7) \otimes pf_4(e_1 \wedge e_2 \wedge e_3 \wedge e_4) \in \mathrm{Ker}(M_{2,2})$$

is not generated by relations of degree 1. ($M_{2,2}$ is a map defined in Definition 5.1 in [6].)

4. Proof of Lemma 3.3

This section is devoted to proving Lemma 3.3.

Throughout this section we denote e_i simply by $\mathbf{i}$ for each i.

Definition 4.1. For i and j such that $2 \leq i < j \leq 8$, define K_{ij} in $(\wedge^2(\wedge^2 E) \otimes_{\mathbf{F}_2} (Pf_4)_2)^*$ to be

$$\sum_{\sigma \in \mathbf{S}_3} \sum_{\substack{\tau \in \mathbf{S}_5 \\ \tau(1)<\tau(2) \\ \tau(3)<\tau(4)<\tau(5)}} (\mathbf{a}_{\sigma(1)} \wedge \mathbf{b}_{\tau(1)}) \wedge (\mathbf{a}_{\sigma(2)} \wedge \mathbf{b}_{\tau(2)}) \otimes pf_4(\mathbf{a}_{\sigma(3)} \wedge \mathbf{b}_{\tau(3)} \wedge \mathbf{b}_{\tau(4)} \wedge \mathbf{b}_{\tau(5)}),$$

where $a_1 = 1$, $a_2 = i$, $a_3 = j$, $2 \leq b_1 < \cdots < b_5 \leq 8$, $\{a_1, a_2, a_3, b_1, \ldots, b_5\} = \{1, \ldots, 8\}$, $pf_4(\mathbf{r} \wedge \mathbf{s} \wedge \mathbf{t} \wedge \mathbf{u}) = (\mathbf{r} \wedge \mathbf{s}) \cdot (\mathbf{t} \wedge \mathbf{u}) + (\mathbf{r} \wedge \mathbf{t}) \cdot (\mathbf{s} \wedge \mathbf{u}) + (\mathbf{r} \wedge \mathbf{u}) \cdot (\mathbf{s} \wedge \mathbf{t})$ and $\mathbf{S}_k$ is the symmetric group on $\{1, 2, \ldots, k\}$. ($\mathbf{a}_{\sigma(r)}$ or $\mathbf{b}_{r(s)}$ stands for $e_{a_{\sigma(r)}}$ or $e_{b_{\tau(s)}}$, respectively. Note that we are doing on $\mathbf{F}_2$. Moreover summations above run over appropriate permutations.)

Lemma 4.2. *For any i and j such that $2 \leq i < j \leq 8$, K_{ij} is contained in* $\mathrm{Ker}(\phi^*)$.

Proof. It is easily checked by direct computations.

Definition 4.3. We denote by V the $\mathbf{F}_2$-subspace of $(\wedge^2(\wedge^2 E) \otimes_{\mathbf{F}_2} (Pf_4)_2)^*$ spanned by $\{K_{ij} | 2 \leq i < j \leq 8\}$.

Remark 4.4. A symmetric group $\mathbf{S}_8$ acts E with permutations on $\{e_1, \ldots, e_8\}$, i.e., σ contained in $\mathbf{S}_8$ maps e_i to $e_{\sigma(i)}$ for $i = 1, \ldots, 8$. So, $\mathrm{GL}(E)$ has a subgroup isomorphic to $\mathbf{S}_8$.

It is easy to see that both $(\wedge^2(\wedge^2 E)\otimes_{\mathbf{F}_2}(Pf_4)_2)^*$ and $(\wedge^2 E\otimes_{\mathbf{F}_2}(Pf_4)_3)^*$ have the structures of $\mathbf{S}_8$-modules and ϕ^* is an $\mathbf{S}_8$-homomorphism. Therefore for any σ in $\mathbf{S}_8$, $\sigma(\mathrm{Ker}(\phi^*))=\mathrm{Ker}(\phi^*)$. Let $\mathbf{S}_7$ be the subgroup of $\mathbf{S}_8$ consists of permutations fixing 1.

Lemma 4.5. *V is an $\mathbf{S}_8$-submodule of* $(\wedge^2(\wedge^2 E)\otimes_{\mathbf{F}_2}(Pf_4)_2)^*$.

Proof. For $s>t$, we put $K_{st}=K_{ts}$. Then for any μ in $\mathbf{S}_7$, we have $\mu(K_{ij}) = K_{\mu(i)\mu(j)}$. Therefore V is an $\mathbf{S}_7$-submodule of $(\wedge^2(\wedge^2 E)\otimes_{\mathbf{F}_2}(Pf_4)_2)^*$.

For distinct positive integers p, q and r which satisfy $1\le p, q, r\le 8$, N_{pqr} is defined to be

$$\sum_{\sigma\in\mathbf{S}_3}\sum_{\substack{\tau\in\mathbf{S}_5\\ \tau(1)<\tau(2)\\ \tau(3)<\tau(4)<\tau(5)}}(\mathbf{a}_{\sigma(1)}\wedge\mathbf{b}_{\tau(1)})\wedge(\mathbf{a}_{\sigma(2)}\wedge\mathbf{b}_{\tau(2)})\otimes pf_4(\mathbf{a}_{\sigma(3)}\wedge\mathbf{b}_{\tau(3)}\wedge\mathbf{b}_{\tau(4)}\wedge\mathbf{b}_{\tau(5)}),$$

where $a_1=p$, $a_2=q$, $a_3=r$, $b_1<\cdots<b_5$ and $\{a_1, a_2, a_3, b_1,\dots,b_5\} = \{1,\dots,8\}$. By this definition $N_{1ij}=K_{ij}$, and $\mu(N_{pqr})=N_{\mu(p)\mu(q)\mu(r)}$ for any permutation μ.

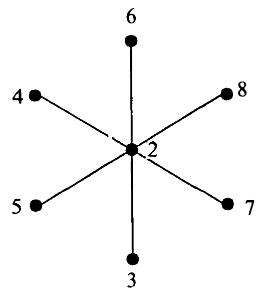
By direct computations we have $N_{1pq}+N_{1pr}+N_{1qr}+N_{pqr}=0$. Therefore N_{pqr} is contained in V. Consequently V is an $\mathbf{S}_8$-submodule of $(\wedge^2(\wedge^2 E)\otimes_{\mathbf{F}_2}(Pf_4)_2)^*$. Q.E.D.

Definition 4.6. For an element $\sum_{2\le i<j\le 8}a_{ij}K_{ij}$ in V, consider the following graph;

1) the vertex set is $\{2, 3,\dots, 8\}$,
2) draw a line between i and j $(i<j)$ if and only if $a_{ij}\neq 0$.

(This graph is not determined uniquely, because $\{K_{ij}|2\le i<j\le 8\}$ is not linearly independent.)

Example. The following graph is one corresponding to $\sum_{j=3}^{8}K_{2j}$.



Lemma 4.7. 1) $\sum_{2\le i<j\le 8}K_{ij}=0$.
2) $\sum_{j=3}^{8}K_{2j}=0$.

Proof. Before proving this lemma, consider the following set.

$$T = \left\{ (\mathbf{c}_1 \wedge \mathbf{c}_2) \wedge (\mathbf{c}_3 \wedge \mathbf{c}_4) \otimes pf_4(\mathbf{c}_5 \wedge \mathbf{c}_6 \wedge \mathbf{c}_7 \wedge \mathbf{c}_8) \;\middle|\; \begin{array}{l} \{c_1, \ldots, c_8\} = \{1, \ldots, 8\} \\ c_1 < c_2,\ c_1 < c_3 < c_4 \\ c_5 < c_6 < c_7 < c_8 \end{array} \right\}$$

T is a free basis of $(\wedge^2(\wedge^2 E) \otimes_{\mathbf{F}_2} (Pf_4)_2)^*$ and each K_{ij} is a sum of distinct 60 elements in T.

Now we start to prove 1). Obviously $\sum_{2 \le i < j \le 8} K_{ij}$ is $\mathbf{S}_7$-invariant. Since each element in $\mathbf{S}_7$ acts $(\wedge^2(\wedge^2 E) \otimes_{\mathbf{F}_2} (Pf_4)_2)^*$ as a certain permutation on T, we have only to compute the coefficients of $(\mathbf{1} \wedge \mathbf{2}) \wedge (\mathbf{3} \wedge \mathbf{4}) \otimes pf_4(\mathbf{5} \wedge \mathbf{6} \wedge \mathbf{7} \wedge \mathbf{8})$ and $(\mathbf{2} \wedge \mathbf{3}) \wedge (\mathbf{4} \wedge \mathbf{5}) \otimes pf_4(\mathbf{1} \wedge \mathbf{6} \wedge \mathbf{7} \wedge \mathbf{8})$. It is easy to check that both of them are zero.

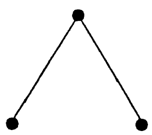
Next we show 2). Since $\sum_{j=3}^{8} K_{2j}$ is invariant under permutations fixing 1 and 2, it suffices to compute the coefficients of $(\mathbf{1} \wedge \mathbf{2}) \wedge (\mathbf{3} \wedge \mathbf{4}) \otimes pf_4(\mathbf{5} \wedge \mathbf{6} \wedge \mathbf{7} \wedge \mathbf{8})$, $(\mathbf{1} \wedge \mathbf{3}) \wedge (\mathbf{2} \wedge \mathbf{4}) \otimes pf_4(\mathbf{5} \wedge \mathbf{6} \wedge \mathbf{7} \wedge \mathbf{8})$, $(\mathbf{1} \wedge \mathbf{3}) \wedge (\mathbf{4} \wedge \mathbf{5}) \otimes pf_4(\mathbf{2} \wedge \mathbf{6} \wedge \mathbf{7} \wedge \mathbf{8})$, $(\mathbf{2} \wedge \mathbf{3}) \wedge (\mathbf{4} \wedge \mathbf{5}) \otimes pf_4(\mathbf{1} \wedge \mathbf{6} \wedge \mathbf{7} \wedge \mathbf{8})$, and $(\mathbf{3} \wedge \mathbf{4}) \wedge (\mathbf{5} \wedge \mathbf{6}) \otimes pf_4(\mathbf{1} \wedge \mathbf{2} \wedge \mathbf{7} \wedge \mathbf{8})$. It will be easily checked. Q.E.D.

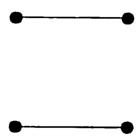
Lemma 4.8. *Any four elements in* $\{K_{ij} | 2 \le i < j \le 8\}$ *are linearly independent over* $\mathbf{F}_2$.

Proof. Suppose that there exists a relation $\sum_{2 \le i < j \le 8} a_{ij} K_{ij} = 0$ such that ${}^{\#}\{(i, j) | i < j,\ a_{ij} \neq 0\}$ is less than or equal to 4, where a_{ij}'s are elements in $\mathbf{F}_2$ and ${}^{\#}\{\ \}$ stands for the number of elements satisfying a given condition.

Since $K_{ij} \neq 0$ for $2 \le i < j \le 8$, we may assume ${}^{\#}\{(i, j) | i < j,\ a_{ij} \neq 0\} = 2$, 3 or 4.

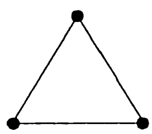
First suppose ${}^{\#}\{(i, j) | i < j,\ a_{ij} \neq 0\} = 2$. Consider the graph corresponding to the given element $\sum_{2 \le i < j \le 8} a_{ij} K_{ij}$. It is either Graph A or Graph B.

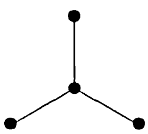

Graph A

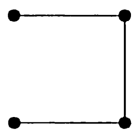

Graph B

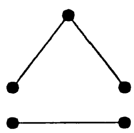
(In these graphs the vertices with no lines have been omitted.) For Graph A (resp. Graph B), we may assume that the given relation is $K_{23} + K_{24} = 0$ (resp. $K_{23} + K_{45} = 0$) by an appropriate permutation. It is easy to check that both $K_{23} + K_{24}$ and $K_{23} + K_{45}$ are not zero. (Compute the coefficients of $(\mathbf{2} \wedge \mathbf{5}) \wedge (\mathbf{3} \wedge \mathbf{6}) \otimes pf_4(\mathbf{1} \wedge \mathbf{4} \wedge \mathbf{7} \wedge \mathbf{8})$.)

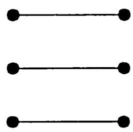
Next suppose ${}^{\#}\{(i, j) | i < j,\ a_{ij} \neq 0\} = 3$. The set of graphs with three lines consists of the following five elements:


Graph C


Graph D


Graph E


Graph F


Graph G

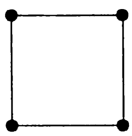
For Graph G, we may assume that the given relation is $K_{23}+K_{45}+K_{67}=0$. Since $\mathrm{Ker}(\phi^*)$ is an S_8-submodule of $(\wedge^2(\wedge^2 E)\otimes_{F_2}(Pf_4)_2)^*$, we obtain

$$(78)(K_{23}+K_{45}+K_{67})=K_{23}+K_{45}+K_{68}=0,$$

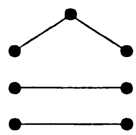
where (78) is the element in S_8 exchanging 7 and 8. Then we have $K_{67}+K_{68}=0$. Contradiction. For Graph D, Graph E and Graph F, we can show the linear independence similarly.

For Graph C, we may assume that the given relation is $K_{23}+K_{34}+K_{24}$. But it is not zero because $(\mathbf{2}\wedge\mathbf{5})\wedge(\mathbf{3}\wedge\mathbf{6})\otimes pf_4(\mathbf{1}\wedge\mathbf{4}\wedge\mathbf{7}\wedge\mathbf{8})$ appears in $K_{23}+K_{34}+K_{24}$ with non-zero coefficient.

Lastly assume $^\#\{(i,j)|i<j,\ a_{ij}\neq 0\}=4$. We have only to compute relations whose graphs is Graph H or Graph I.



Graph H



Graph I

(For other graphs we can use the same technique as in the case of Graph G.)

For Graph H the given relation is $K_{23}+K_{34}+K_{45}+K_{25}$. But $(\mathbf{2}\wedge\mathbf{6})\wedge(\mathbf{3}\wedge\mathbf{7})\otimes pf_4(\mathbf{1}\wedge\mathbf{4}\wedge\mathbf{5}\wedge\mathbf{8})$ has non-zero coefficient.

For Graph I we may assume that the given relation is $K_{23}+K_{34}+K_{56}+K_{78}$. Then we have

$$(67)(K_{23}+K_{34}+K_{56}+K_{78})=K_{23}+K_{34}+K_{57}+K_{68}=0,$$

In this case, we obtain another relation $K_{56}+K_{78}+K_{57}+K_{68}=0$ whose graph is Graph H. Contradiction.

We have completed the proof of Lemma 4.8. Q.E.D.

Lemma 4.9. *Any fourteen elements in* $\{K_{ij}|3\leq i<j\leq 8\}$ *are linearly independent. In particular,* $\dim_{F_2}V=14$.

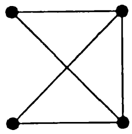
Proof. From 2) of Lemma 4.7, V is spanned by $\{K_{ij}|3\leq i<j\leq 8\}$. Furthermore we have

$$\sum_{3\leq i<j\leq 8}K_{ij}=\sum_{2\leq i<j\leq 8}K_{ij}-\sum_{j=3}^{8}K_{2j}=0$$

by Lemma 4.7. Therefore $\dim_{F_2}V$ is less that or equal to 14.

Assume that there exists a relation $\sum_{3\leq i<j\leq 8}a_{ij}K_{ij}$ such that $^\#\{(i,j)|i<j,\ a_{ij}\neq 0\}\leq 14$. It is easy to see that we may assume that $^\#\{(i,j)|i<j,\ a_{ij}\neq 0\}=5$, 6 or 7. (See Lemma 4.8 and remember that $\sum_{3\leq i<j\leq 8}K_{ij}=0$.)

First assume that the graph corresponding to the given relation is Graph J.



Graph J

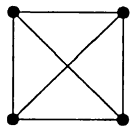
We may assume that the given relation is equal to $K_{34}+K_{35}+K_{36}+K_{45}+K_{56}=0$. Then we have

$$(24)(K_{34}+K_{35}+K_{36}+K_{45}+K_{56})=K_{23}+K_{35}+K_{36}+K_{25}+K_{56}=0,$$

Then $K_{23}+K_{25}+K_{34}+K_{45}=0$. Contradiction to Lemma 4.8.

More generally when ${}^{\#}\{(i,j)|i<j,\ a_{ij}\neq 0\}=5$ or 7, we can show the linear independence by the same technique as in the cases of Graph G or Graph J.

Suppose ${}^{\#}\{(i,j)|i<j,\ a_{ij}\neq 0\}=6$. We may assume that the graph corresponding to the given relation is the following one:



Graph K

(For other graphs, we can use the same technique as in the cases of ${}^{\#}\{(i,j)|i<j,\ a_{ij}\neq 0\}=5$ or 7.) In this case the given relation is equal to $K_{34}+K_{35}+K_{36}+K_{45}+K_{46}+K_{56}$. But $(\mathbf{3}\wedge\mathbf{7})\wedge(\mathbf{4}\wedge\mathbf{8})\otimes pf_4(\mathbf{1}\wedge\mathbf{2}\wedge\mathbf{5}\wedge\mathbf{6})$ appears in it with non-zero coefficient.

We have completed the proof of Lemma 4.9. Q.E.D.

Definition 4.10. Let h be a positive integer such that $1\leq h\leq 8$. Define A_h to be

$$\sum(\mathbf{c}_1\wedge\mathbf{c}_2)\wedge(\mathbf{c}_3\wedge\mathbf{c}_4)\otimes pf_4(\mathbf{c}_5\wedge\mathbf{c}_6\wedge\mathbf{c}_7\wedge\mathbf{h}),$$

where the above sum runs over the set satisfying the following conditions.

- $\{c_1,\dots,c_7\}=\{1,\dots,h-1,\ h+1,\dots,8\}$.
- $c_1<c_2$, $c_1<c_3<c_4$ and $c_5<c_6<c_7$.

It is easy to see that A_h is a sum of distinct 105 elements in T.

Lemma 4.11. *For a positive integer h such that $1\leq h\leq 8$, A_h is contained in* Ker(ϕ^*).

Proof. It is easy to compute.

Lemma 4.12. $\sum_{h=1}^{8}A_h=0$.

Proof. For any σ in $\mathbf{S}_8$, we have $\sigma(A_h) = A_{\sigma(h)}$. Therefore $\sigma(\sum_{h=1}^{8} A_h) = \sum_{h=1}^{8} A_h$. Hence we have only to compute the coefficient of $(\mathbf{1} \wedge \mathbf{2}) \wedge (\mathbf{3} \wedge \mathbf{4}) \otimes pf_4(\mathbf{5} \wedge \mathbf{6} \wedge \mathbf{7} \wedge \mathbf{8})$. It is obviously zero. Q.E.D.

We put $W = V + \sum_{h=1}^{8} \mathbf{F}_2 \cdot A_h$, which is an $\mathbf{S}_8$-submodule of $\mathrm{Ker}(\phi^*)$.

Lemma 4.13. $\dim_{\mathbf{F}_2} W = 21$, *i.e., any seven elements in* $\{A_h | 1 \leq h \leq 8\}$ *are linearly independent in* W/V.

Proof. Suppose that $\sum_{h=1}^{8} b_h A_h$ is contained in V, where b_h's are elements in $\mathbf{F}_2$ and ${}^{\#}\{h | b_h \neq 0\}$ is neither 0 nor 8.

We may assume $1 \leq {}^{\#}\{h | b_h \neq 0\} \leq 4$. Furthermore by computing the number of elements in T, we know that ${}^{\#}\{h | b_h \neq 0\}$ must be even.

First suppose ${}^{\#}\{h | b_h \neq 0\} = 2$. Since V is $\mathbf{S}_8$-invariant by Lemma 4.5, we may assume that $A_1 + A_2$ is contained in V. Put $A_1 + A_2 = \sum_{3 \leq i < j \leq 8} c_{ij} K_{ij}$ such that ${}^{\#}\{(i, j) | c_{ij} \neq 0\} \leq 7$. Then for any permutation μ fixing 1 and 2, we have

$$A_1 + A_2 = \mu(A_1 + A_2) = \sum_{3 \leq i < j \leq 8} c_{ij} K_{\mu(i)\mu(j)}.$$

Therefore $\sum_{3 \leq i < j \leq 8} c_{ij} K_{ij} - \sum_{3 \leq i < j \leq 8} c_{ij} K_{\mu(i)\mu(j)} = 0$. By Lemma 4.9, $\sum_{3 \leq i < j \leq 8} c_{ij} K_{ij}$ must be 0. Hence $A_1 + A_2 = 0$. But $(\mathbf{1} \wedge \mathbf{3}) \wedge (\mathbf{4} \wedge \mathbf{5}) \otimes pf_4(\mathbf{2} \wedge \mathbf{6} \wedge \mathbf{7} \wedge \mathbf{8})$ appears in $A_1 + A_2$ with non-zero coefficient. Contradiction.

Next suppose ${}^{\#}\{h | b_h \neq 0\} = 4$. We may assume $A_1 + A_2 + A_3 + A_4$ is in V. Then

$$(45)(A_1 + A_2 + A_3 + A_4) = A_1 + A_2 + A_3 + A_5$$

is contained in V. Therefore $A_4 + A_5$ is also in V. Contradiction. Q.E.D.

Definition 4.14. Define a subset Z of T (see the proof of Lemma 4.7) to be

$$Z = \left\{ \begin{array}{l|l} (\mathbf{c}_1 \wedge \mathbf{c}_2) \wedge (\mathbf{c}_3 \wedge \mathbf{c}_4) & \text{If } c_5 = 1, \text{ then } c_1 < c_3 < c_2 < c_4. \\ \quad \otimes pf_4(\mathbf{c}_5 \wedge \mathbf{c}_6 \wedge \mathbf{c}_7 \wedge \mathbf{c}_8) \in T & \text{If } c_1 = 1 \text{ and } c_2 < c_3 < c_4, \text{ then } c_3 + c_4 \text{ is even.} \\ & \text{If } c_1 = 1 \text{ and } c_3 < c_2 < c_4, \text{ then } c_3 + c_4 \text{ is odd.} \\ & \text{If } c_1 = 1 \text{ and } c_3 < c_4 < c_2, \text{ then } c_3 + c_4 \text{ is even.} \end{array} \right\}$$

Then put $B = \sum_{q \in Z} q$.

It is easy to check that ${}^{\#}Z$ is odd.

Lemma 4.15. $(2345678)(B) = B$.

Proof. It is easy to compute.

Lemma 4.16. *B is contained in* $\mathrm{Ker}(\phi^*)$.

Proof. By direct computations it will be easily checked (use Lemma 4.15).

Proposition 4.17. *B is not contained in W, i.e.,* $\dim_{\mathbf{F}_2}(W+\mathbf{F}_2\cdot B)=22$.

Proof. Suppose that $B+\sum_{h=1}^{8} b_h A_h$ is contained in V, where b_h's are elements in $\mathbf{F}_2$. We may assume $^{\#}\{h|b_h\neq 0\}=1$ or 3, because $^{\#}Z$ is odd.

First suppose $^{\#}\{h|b_h\neq 0\}=1$. If $B+A_h$ is contained in V for $h\geq 2$, so is $B+A_{\sigma(h)}$ where $\sigma=(2345678)$. Then $A_h+A_{\sigma(h)}$ is in V, too. Contradiction. Assume $B+A_1$ is contained in V. Put

$$B+A_1=\sum_{\substack{2\leq i<j\leq 8\\ i\neq 5,\, j\neq 5}} g_{ij}K_{ij}.$$

(Obviously V is spanned by $\{K_{ij}|2\leq i<j\leq 8,\ i\neq 5,\ j\neq 5\}$ and any fourteen elements of them are linearly independent.) Computing the coefficients of $(\mathbf{2}\wedge\mathbf{5})\wedge(\mathbf{3}\wedge\mathbf{6})\otimes pf_4(\mathbf{1}\wedge\mathbf{4}\wedge\mathbf{7}\wedge\mathbf{8})$, $(\mathbf{2}\wedge\mathbf{5})\wedge(\mathbf{4}\wedge\mathbf{6})\otimes pf_4(\mathbf{1}\wedge\mathbf{3}\wedge\mathbf{7}\wedge\mathbf{8})$ and $(\mathbf{2}\wedge\mathbf{5})\wedge(\mathbf{3}\wedge\mathbf{4})\otimes pf_4(\mathbf{1}\wedge\mathbf{6}\wedge\mathbf{7}\wedge\mathbf{8})$, we obtain $g_{23}+g_{26}=0$, $g_{24}+g_{26}=0$ and $g_{23}+g_{24}=1$. Contradiction.

Next suppose $^{\#}\{h|b_h\neq 0\}=3$. By using $\sigma=(2345678)$, we can check this case easily. Q.E.D.

We have completed the proof of Lemma 3.3.

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References

[1] B. Buchberger, Gröbner bases: An algorithmic method in polynomial ideal theory, in Multidimensional System Theory, Ed. N. K. Bose, D. Reidel Publ. Comp. (1985), Chapter 6.

[2] D. A. Buchsbaum and D. Eisenbud, Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3, Amer. J. of Math., **99** (1977), 447–485.

[3] M. Hashimoto, Determinantal ideals without generic minimal free resolutions, Nagoya J. Math., to appear.

[4] T. Józefiak, P. Pragacz, and J. Weyman, Resolutions of determinantal varieties and tensor complexes associated with symmetric and antisymmetric matrices, Astérisque, **87-88**, 109–189.

[5] K. Kurano, The first syzygies of syzygies of determinantal ideals, J. Alg., **124** (1989), 414–436.

[6] K. Kurano, Relations on pfaffians I: Plethysm formulas, preprint.

[7] P. Pragacz, Characteristic free resolution of $(n-2)$-order pfaffians of $n\times n$-antisymmetric matrix, J. Alg., **78** (1982), 386–396.

[8] P. Roberts, Homological invariants of modules over commutative rings, Presses Univ. Montreal, Montreal, 1980.