J. Math. Kyoto Univ. (JMKYAZ)
32-3 (1992) 451–459

On homotopy associative mod 2 H-spaces[1]

By

John McCleary

In [1], [3], and [6] the following question is considered:

If Y is a mod 2 H-space, when does $Y \times S^7$ admit the structure of a homotopy associative mod 2 H-space?

Among the simple Lie groups, the results in [1] reveal that the only possible examples are the following:

$$Spin(7)_{(2)} \simeq (G_2 \times S^7)_{(2)}$$

$$Spin(8)_{(2)} \simeq (Spin(7) \times S^7)_{(2)}$$

$$\text{and } SO(8)_{(2)} \simeq (SO(7) \times S^7)_{(2)} .$$

The focus of [6] is on generalizing the results of [1] to finite H-spaces. Here the Hopf algebra over the mod 2 Steenrod algebra, $\mathscr{A}_2$, given by

$$A = F_2[x_3]/x_3^4 \otimes \Lambda(Sq^2 x_3) \cong H^*(G_2; F_2)$$

plays a crucial role. The main results of [6] are summarized in the following

Theorem (Lin-Williams). *Let Y be a finite simply-connected CW-complex and suppose $H^*(Y; F_2)$ contains no subalgebras isomorphic to A. Then $Y \times S^7$ cannot be a homotopy associative H-space. Suppose $H^*(Y; F_2)$ contains at most one subalgebra isomorphic to A. Then $Y \times (S^7)^k$ cannot be a homotopy associative H-space for $k \geq 3$.*

The method of proof of this theorem suggests an extension that is the principal result of this paper.

Main Theorem. *Let Y be a finite simply-connected CW-complex and suppose that $H^*(Y; F_2)$ contains $A^{\otimes \ell}$ for some $\ell \geq 0$. Then $Y \times (S^7)^k$ cannot be a homotopy associative H-space for $k \geq 2\ell + 1$.*

This shows that, at the prime 2, from ℓ copies of G_2 and k copies of S^7, the examples of $Spin(7)$ and $Spin(8)$ above are the only homotopy associative

Communicated by Prof. K. Ueno, June 8, 1990

[1] This paper was written while I was visiting the Mathematical Sciences Research Institute, Berkeley, CA.

factors that can be constructed out of $(G_2)^\ell \times (S^7)^k$. John Harper has told me that Daciberg Goncalves had considered the question treated in the main theorem and that he had successfully handled cases of it. This paper gives a uniform treatment of all cases.

The paper is organized as follows. In § 1 we recall the methods used in [9], [1], and [6] for recognition of obstructions to homotopy associative H-space structures on spaces of the type above. This leads to a careful accounting of the Stasheff spectral sequence associated to a 4-connected cover of such a product $Y \times (S^7)^k$. In § 2 we introduce an auxiliary spectral sequence to aid in the computation of the Stasheff spectral sequence and determine its properties. From this construction we reduce the proof of the main theorem to an algebraic proposition about polynomial algebras over $\boldsymbol{F}_2$ which is proved in § 3.

My thanks to Jim Lin for suggesting the problem and for making a copy of [6] available to me. Conversations with Jim Lin, Joe Lipman, and Lou van den Dries were valuable in this work. My thanks to Chun-Nip Lee for suggesting the tool of commutative algebra in § 3. The hospitable and lively atmosphere at MSRI was instrumental in bringing about such interactions.

§ 1. Background

The space S^7 admits no associative H-space structure. This is a simple consequence of the unstable axioms for the action of the mod odd Steenrod algebra and the structure that the cohomology algebra $H^*(BS^7; \boldsymbol{F}_p)$ would have if BS^7 existed. Localizing at the prime 2, it is known further that $S^7_{(2)}$ admits no homotopy associative H-space structures. The proof in [1] is based on secondary cohomology operations, which provide a factorization of u_8^3 for an element u_8 in the cohomology of a space such that $Sq^1 u_8 = Sq^2 u_8 = Sq^4 u_8 = 0$. The existence of a homotopy associative H-structure on $S^7_{(2)}$ implies the existence of the projective 3-space of $S^7_{(2)}$, $B_3 S^7_{(2)}$ (see [9]), which may be thought of as the third stage of a filtration of the classifying space for $S^7_{(2)}$, if it were an associative H-space. The existence of $B_3 S^7_{(2)}$ implies the existence of a non-trivial class $u_8^3 \in H^{24}(B_3 S^7_{(2)}; \boldsymbol{F}_2)$, and the factorization of u_8^3 through secondary operations leads to a contradiction.

The construction of the relevant secondary cohomology operations is given in [6]. The universal example is a 3-stage Postnikov system

$$\begin{array}{ccc} U_2 & & \\ \downarrow{\scriptstyle q_2} & & \\ U_1 & \xrightarrow[Bk_1]{} & K(\boldsymbol{Z}/2\boldsymbol{Z}, 16, 17, 12, 15) \\ \downarrow{\scriptstyle q_1} & & \\ K(\boldsymbol{Z}, 8) & \xrightarrow[Bk_0]{} & K(\boldsymbol{Z}/2\boldsymbol{Z}, 10, 12)\,. \end{array}$$

Applying the based loop space functor, we get a tower

$$\begin{array}{ccc}
\Omega U_2 & & \\
\Big\downarrow{\scriptstyle \Omega q_2} & & \\
\Omega U_1 & \xrightarrow[k_1]{} & K(\boldsymbol{Z}/2\boldsymbol{Z}, 15, 16, 11, 14) \\
\Big\downarrow{\scriptstyle \Omega q_1} & & \\
K(\boldsymbol{Z}, 7) & \xrightarrow[k_0]{} & K(\boldsymbol{Z}/2\boldsymbol{Z}, 9, 11).
\end{array}$$

In the Eilenberg-Moore spectral sequence with E_2-term given by $\mathrm{Ext}_{H_*(\Omega U_2; \boldsymbol{F}_2)}(\boldsymbol{F}_2, \boldsymbol{F}_2)$, converging to $H^*(U_2; \boldsymbol{F}_2)$, there is a primitive class $v \in H^{22}(\Omega U_2; \boldsymbol{F}_2)$ with

$$d_2([v]) = [\kappa_7 | \kappa_7 | \kappa_7] ,$$

where $\kappa_7 = (q_1 q_2)^*(\iota_7)$.

Now suppose $X \simeq_{(2)} Y \times (S^7)^k$, that X is homotopy associative, and that $H^*(X; \boldsymbol{F}_2)$ contains $A^{\otimes \ell}$ where

$$A = \boldsymbol{F}_2[x_3]/x_3^4 \otimes \Lambda(Sq^2 x_3) .$$

We consider the following diagram from [6]:

$$\begin{array}{ccccc}
X_2 & & & & (\Omega U_2)^k \\
\Big\downarrow{\scriptstyle p_2} & & & {\scriptstyle h_2'}\nearrow & \Big\downarrow \\
X_1 & & & & (\Omega U_1)^k \\
\Big\downarrow{\scriptstyle p_1} & & & {\scriptstyle h_1'}\nearrow & \Big\downarrow \\
X & \xrightarrow[\pi]{} & (S^7)^k & \xrightarrow[h']{} & K(\boldsymbol{Z}, 7)^k
\end{array}$$

where $h' \colon (S^7)^k \to K(\boldsymbol{Z}, 7)^k$ is the product of the integral generating classes in $H^7(S^7; \boldsymbol{Z})$. The spaces X_1 and X_2 are homotopy associative if X is, and they are constructed as connective covers of X. The multiplication on $(\Omega U_2)^k$ may be chosen so that $h_2 \colon X_2 \to (\Omega U_2)^k$ given by $h_2 = h_2' \circ \pi \circ p_1 \circ p_2$ is an H-map and, in fact, an A_3-map ([9]), and so it induces a map of projective 3-spaces

$$B_3 h_2 \colon B_3 X_2 \to B_3(\Omega U_2)^k .$$

Since the cubes of the images of the fundamental classes, $[\kappa_7^i | \kappa_7^i | \kappa_7^i]$, $1 \le i \le k$, vanish in $H^*(B_3(\Omega U_2)^k; \boldsymbol{F}_2)$, and $h_2^*([\kappa_7^i]) = a_i$, a class representing the suspension of the i^{th} S^7 in $X \simeq Y \times (S^7)^k$, then $a_i^3 = 0$ in $H^*(B_3 X_2; \boldsymbol{F}_2)$. This provides an obstruction to the homotopy associativity of X.

The difficult step in carrying this analysis further lies in the choice of connective covers X_1, X_2, and the computation of the relevant parts of their cohomology. Lin's work [5] on connections by Steenrod operations among generators of the cohomology of H-spaces guides these choices and computations. In [6], when $H^*(X; \boldsymbol{F}_2)$ contains $A^{\otimes \ell}$ as a sub-Hopf algebra over $\mathscr{A}_2$, Lin and Williams show that for $X_1 = X\langle 3 \rangle$, the 3-connective cover of X, and X_2, the fibre of a suitable k-invariant, $X_1 \underset{k}{\to} K(\boldsymbol{Z}/2\boldsymbol{Z}, 4, 5, 7, 8)$, the mod 2 cohomology of X_2 has elements w_i in degree 14, and y_i in degree 22, $1 \le i \le \ell$, and no other classes in dimensions that could contribute to the vanishing of any of the $[\kappa_7^i | \kappa_7^i | \kappa_7^i]$.

In the next section, we describe the Stasheff spectral sequence going from part of the bar construction on $H^*(X_2; \boldsymbol{F}_2)$ and converging to $H^*(B_3 X_2; \boldsymbol{F}_2)$.

§ 2. An auxiliary spectral sequence

Following the strategy of the previous section, we study the Stasheff spectral sequence [9], [4], which is built of the first three filtration degrees of the bar construction on $H^*(X_2; \boldsymbol{F}_2)$ and converges to $H^*(B_3 X_2; \boldsymbol{F}_2)$. The results of [1] and [6] on the cohomology of X_2 determine the following picture for the E_1-term of the spectral sequence in the relevant degrees. We write $[S_i^7]$ for the generator coming from a copy of S^7 in $Y \times (S^7)^k$ (note $S_i^7 S_i^7 = 0$):

	1	2	3
22	$\{[y_j], 1 \le j \le \ell\}$		
21	$\{[S_i^7 S_j^7 S_k^7]\}\{[w_i S_j^7], [S_i^7 w_j]\}$	$\{[S_i^7 \vert S_j^7 S_k^7], [S_i^7 S_j^7 \vert S_k^7], [w_i \vert S_j^7], [S_i^7 \vert w_j]\}$	$\{[S_i^7 \vert S_j^7 \vert S_k^7]\}$
20			
19			
⋮			
16			
15			
14	$\{[w_j], 1 \le j \le \ell\}\{[S_i^7 S_j^7], i \ne j\}$	$\{[S_i^7 \vert S_j^7]\}$	
13			
12			
⋮			
9			
8			
7	$\{[S_i^7], 1 \le i \le k\}$		

The classes which obstruct homotopy associativity are the 'cubes', $\{[S_i^7 | S_i^7 | S_i^7], 1 \le i \le k\}$, lying in bidegree (3, 21). Since the bidegree of d_1 is (1, 0), $d_1(y_i) = 0$ for all i. The differential d_2 carries the classes y_i to a quotient of the span of $\{[S_i^7 | S_j^7 | S_k^7]\}$ where they can eliminate some cubes. The dimension of the space of y_i's is ℓ, so they can eliminate at most ℓ of the cubes.

The situation regarding cubes is more complicated for d_1. We simplify the discussion considerably by introducing an auxiliary spectral sequence associated to (E_1, d_1). Form the total complex, Tot E_1, and filter it as follows: consider the copies of A in $A^{\otimes \ell}$ as ordered so that the pairs $\{w_i, y_i\}$ are indexed by the ordered set $1 \le i \le \ell$. The filtration of $\operatorname{Tot} E_1$ is given by

$$\begin{aligned} \mathscr{F}_1 = \{0\} \subset \mathscr{F}_0 &= \text{the subalgebra determined by } (S^7)^k \\ \subset \mathscr{F}_{-1} &= \text{the subalgebra determined by } \mathscr{F}_0 \text{ and } \{w_1, y_1\} \\ \subset \mathscr{F}_{-2} &= \text{the subalgebra determined by } \mathscr{F}_{-1} \text{ and } \{w_2, y_2\} \\ &\vdots \end{aligned}$$

This gives a filtered complex

$$\{0\} \subset \mathscr{F}_0 \subset \mathscr{F}_{-1} \subset \cdots \subset \mathscr{F}_{-\ell+1} \subset \mathscr{F}_{-\ell} = \operatorname{Tot} E_1 .$$

From the original spectral sequence it is evident that $d_1(\mathscr{F}_{-j}) \subset \mathscr{F}_{-j}$. Apply the usual construction [8] and obtain a spectral sequence with

$${}_0E^{-i,*} \cong \mathscr{F}_{-i}/\mathscr{F}_{-i+1}, \qquad {}_0d \text{ induced by } d_1 .$$

Restricting d_1 to the filtration degree 0, we are computing the homology of the bar construction applied to $\Lambda(S_1^7, \ldots, S_k^7)$, an exterior algebra on k generators of degree 7. This gives ${}_1E^{0,*} \cong \boldsymbol{F}_2[a_1, \ldots, a_k]$ where $a_i = [S_i^7]$ is a class of degree 8. From the original spectral sequence we also see that $d_1(\mathscr{F}_{-j})$ lies in $\mathscr{F}_0$ and so ${}_0d \equiv 0$ on ${}_0E^{-i,*}$ for $i \ge 1$. This leads to the following picture of ${}_1E$:

$\ddots$					27
	$\{y_3, a_iw_3, w_3a_i\}$				26
	$\{[w_3S_j^7], [S_i^7w_3]\}$	$\{y_2, a_iw_2, w_2a_i\}$			25
		$\{[w_2S_j^7], [S_i^7w_2]\}$	$\{y_1, a_iw_1, w_1a_i\}$	$\{a_ra_sa_t, 1 \le r \le s \le t\}$	24
			$\{[w_1S_j^7], [S_i^7w_1]\}$		23
					22
					21
					20
$\ddots$					19
	w_3				18
		w_2			17
			w_1	$\{a_ra_s, 1 \le r \le s \le k\}$	16
					15
					14
					$\vdots$
					10
					9
				$\{a_r, 1 \le r \le k\}$	8
					7
-4	-3	-2	-1	0	

By construction ${}_md\colon {}_mE^{-i,j} \to {}_mE^{-i+m,j-m+1}$, ${}_md(y_i) = 0$ for all i and m, and ${}_\infty E^{-i,j}$ is an associated graded vector space for $(\operatorname{Tot} E_2)^{i+j}$. Since d_1 is a derivation,

$\vdots \quad \vdots \quad \cdots \quad \cdots$

so is ${}_m d$ for all $m \geq 0$. Finally, ${}_\infty E^{*,*} = {}_{\ell+1}E^{*,*}$ by the finite length of the filtration.

It is useful to observe that ${}_m d(a_i w_j) = {}_m d(w_j a_i)$ since the target of the differential is commutative, and ${}_m d$ is a derivation. Thus we can concentrate our attention on classes $\{a_i w_j\}$. To determine the cubes which remain in E_2, we consider how cubes vanish in this auxiliary spectral sequence. By construction, ${}_m d(w_i) = 0$ unless $m = i$ and ${}_i d(w_i)$ is an element of ${}_i E^{0,16}$, the quotient of the space of homogeneous quadratic polynomials in $F_2[a_1, \ldots, a_k]$ by the images of the previous differentials. The cubes represent classes in ${}_i E^{0,24}$ and so, if we write

$$ {}_i d(w_i) \equiv p_i = \sum_{1 \leq m \leq n \leq k} x_{mn} a_m a_n \qquad \text{in} \quad {}_i E^{0,16}, $$

for some $x_{mn} \in F_2$, then we are seeking solutions to equations of the form

$$ {}_i d\left(\sum_{n=1}^{k} \beta_n^r a_n p_i \right) \equiv a_r^3 \qquad \text{in} \quad {}_i E^{0,24}. $$

In the next section we address this algebraic problem.

§ 3. Elementary algebra and the proof of the Main Theorem

The quotients by previous differentials and the associated indeterminacy in ${}_i E^{0,*}$ can be expressed more conveniently by observing that

$$ {}_{i+1}E^{0,*} \cong F_2[a_1, \ldots, a_k]/(p_1, \ldots, p_i), $$

where ${}_j d(w_j) \equiv p_j \bmod (p_1, \ldots, p_{j-1})$, and $(u_1, \ldots, u_s)$ is the ideal generated by u_1, $\ldots$, u_s. Let $V = \{\alpha_1 a_1^3 + \cdots \alpha_k a_k^3 | \alpha_i \in F_2\}$ be the k-dimensional span of the cubes in $F_2[a_1, \ldots, a_k]$. We introduce the following subspace of V associated to the choice of quadratic homogeneous polynomials, p_1, $\ldots$, p_j:

$$ \begin{aligned} V(p_1, \ldots, p_j) = \Big\{ & \alpha_1 a_1^3 + \cdots \alpha_k a_k^3 | \exists (\beta_1^r, \ldots, \beta_k^r) \in F_2^{\times k}, \qquad 1 \leq r \leq j, \\ & \text{and } \alpha_1 a_1^3 + \cdots \alpha_k a_k^3 = \sum_{r=1}^{j} (\beta_1^r a_1 + \cdots + \beta_k^r a_k) p_r \Big\} \\ = {} & V \cap (p_1, \ldots, p_j). \end{aligned} $$

Thus $V(p_1, \ldots, p_j)$ is the subspace of V that vanishes in $F_2[a_1, \ldots, a_k]/(p_1, \ldots, p_j)$. The key to the main theorem, as reduced in the discussion above, is the following innocent algebraic assertion.

Proposition 3.1. $\dim_{F_2} V(p_1, \ldots, p_j) \leq j$.

Proof. We proceed by induction on j, the number of quadratic relations, and k the number of free variables. When $j \geq k$, the assertion is trivial since

$\dim V(p_1, \ldots, p_j) \le \dim V = k \le j$. For arbitrary $k \ge 1$ and $j = 1$, we now show that $V(p_1)$ has dimension less than or equal to 1.

Write $p_1 = \sum_{1 \le m \le n \le k} x_{mn} a_m a_n$ with $x_{mn} \in F_2$. The equation

$$\alpha_1 a_1^3 + \cdots + \alpha_k a_k^3 = (\beta_1 a_1 + \cdots + \beta_k a_k) p_1$$

implies the following relations among coefficients

$$\alpha_m = \beta_m x_{mm}, \qquad 1 \le m \le k,$$

$$0 = \begin{cases} \beta_n x_{mm} + \beta_m x_{mn} \\ \beta_m x_{nn} + \beta_n x_{mn} & \qquad 1 \le l < m < n \le k. \\ \beta_l x_{mn} + \beta_m x_{ln} + \beta_n x_{lm} \end{cases}$$

Without loss of generality, $\alpha_1 = 1$. Elementary manipulations in F_2 lead to the following consequences of $\alpha_1 = 1$:

$$\beta_1 = 1, \qquad x_{11} = 1, \qquad \alpha_m = \beta_m = x_{1m} = x_{mm}, \qquad m > 1,$$

$$x_{mn} = 0, \qquad \beta_m x_{nn} = \beta_m \beta_n = 0, \qquad m \ne n, \qquad m, n > 1.$$

It follows immediately that only one other coefficient α_m can be non-zero. The two cases—'all other $\alpha_m = 0$' and 'some $\alpha_m = 1$'—lead to the only possible factorizations which are

$$a_1^3 = a_1 \cdot a_1^2,$$

$$a_1^3 + a_m^3 = (a_1 + a_m)(a_1^2 + a_1 a_m + a_m^2).$$

If p_1 is not one of the quadratic polynomials above, $V(p_1) = \{0\}$; otherwise, $\dim V(p_1) = 1$. (Notice that this implies that $a^3 + b^3 + c^3$ cannot be factored as a linear polynomial times a quadratic polynomial in $F_2[a, b, c]$.) We now make the following

INDUCTIVE HYPOTHESIS. *For all $J \le K < k$ and $q_1, \ldots, q_J$, homogeneous quadratic polynomials in $F_2[b_1, \ldots, b_K]$, $\dim_{F_2} V(q_1, \ldots, q_J) \le J$.*

This is the inductive hypothesis based on the lexicographic well-ordering of the relevant pairs, $(1, 1) < (2, 1) < (2, 2) < (3, 1) < \cdots$.

For k variables, we know already that $\dim V(p_1) \le 1$. For some $j < k$, consider the subspace $V(p_1, \ldots, p_j)$ for a choice of p_i's. If $V(p_1, \ldots, p_j) \ne V$, then there is some $a_m^3 \notin V(p_1, \ldots, p_j)$. Consider the mapping of algebras

$$\varepsilon_m : F_2[a_1, \ldots, a_k] \to F_2[a_1, \ldots, \widehat{a_m}, \ldots, a_k],$$

gotten by setting $a_m = 0$. Define

$$q_i = \varepsilon_m(p_i) = p_i(a_1, \ldots, a_{m-1}, 0, a_{m+1}, \ldots, a_k).$$

Then $\varepsilon_m(V(p_1, \ldots, p_j)) \subset V(q_1, \ldots, q_j)$, and

$$\dim V(p_1, \ldots, p_j) = \dim(\ker \varepsilon_m \cap V(p_1, \ldots, p_j)) + \dim \varepsilon_m(V(p_1, \ldots, p_j)).$$

From the definition of ε_m, $\ker \varepsilon_m \cap V = \langle a_m^3 \rangle$, the vector space spanned by a_m^3. By the above discussion then, $\dim \ker \varepsilon_m \cap V(p_1, \dots, p_j) = 0$. Applying the Inductive Hypothesis, we obtain

$$\dim V(p_1, \dots, p_j) = \dim \varepsilon_m(V(p_1, \dots, p_j)) \leq \dim V(q_1, \dots, q_j) \leq j .$$

There is only one case left to consider—for some $j < k$, and for some choice of p_i's, $V(p_1, \dots, p_j) = V$. We show that this cannot happen by applying some commutative algebra. Recall the notion of the **height** of an ideal in a Noetherian ring R, like $F_2[a_1, \dots, a_k]$. For a prime ideal $\mathfrak{p}$, $\operatorname{ht}(\mathfrak{p}) = n$, if there is at least one strictly ascending chain of prime ideals in the ring,

$$(0) \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_{n-1} \subset \mathfrak{p}_n = \mathfrak{p} ,$$

and there are no longer such chains. For an arbitrary ideal I,

$$\operatorname{ht}(I) = \inf_{\text{prime } \mathfrak{p} \supset I} \{\operatorname{ht}(\mathfrak{p})\} .$$

The key lemma for our purposes is the following ([7], p. 77)

Lemma 3.2. *If R is a Noetherian ring, $I = (p_1, \dots, p_j) \subset \mathfrak{p}$, a minimal prime over-ideal of I, then* $\operatorname{ht}(\mathfrak{p}) \leq j$, *and so* $\operatorname{ht}(I) \leq j$.

Also recall the equivalent definitions of the **radical** of an ideal I:

$$\sqrt{I} = \bigcap_{\text{prime } \mathfrak{p} \supset I} \mathfrak{p} ,$$

$$= \{x \in R \mid \text{for some } n,\ x^n \in I\} .$$

From the assumption $V(p_1, \dots, p_j) = V$, we see that $a_m^3 \in V(p_1, \dots, p_j)$ for all m, and so $a_m \in \sqrt{(p_1, \dots, p_j)}$ for all m. This implies that

$$\sqrt{(p_1, \dots, p_j)} = \mathfrak{m} ,$$

where $\mathfrak{m}$ is the maximal ideal in $F_2[a_1, \dots, a_k]$ given by the non-constant polynomials. By the definition of the radical, it is clear that $\mathfrak{m}$ is the minimal prime over-ideal containing $(p_1, \dots, p_j)$, and so, by the lemma,

$$\operatorname{ht}(\mathfrak{m}) \leq j .$$

However, the chain of prime ideals

$$(0) \subset (a_1) \subset (a_1, a_2) \subset \cdots \subset (a_1, \dots, a_k) = \mathfrak{m}$$

shows that

$$\operatorname{ht}(\mathfrak{m}) \geq k .$$

Since we have assumed that $j < k$, this gives a contradiction and proves the Proposition.

Proof of the Main Theorem. In §2 we showed that the homotopy associativity of the H-space $X \simeq Y \times (S^7)^k$ depended on the elimination in the

Stasheff spectral sequence of the k cubes that would be found in $H^*(B_3X_2; F_2)$. The classes y_i, $1 \leq i \leq \ell$, can eliminate only ℓ cubes and the Proposition above shows that the classes w_i, $1 \leq i \leq \ell$, can eliminate only ℓ other cubes. Thus, if $k \geq 2\ell + 1$, X cannot be a homotopy associative H-space.

DEPARTMENT OF MATHEMATICS
VASSAR COLLEGE

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