J. Math. Kyoto Univ. (JMKYAZ)
32-4 (1992) 701-713

The Bers projection and the λ-lemma

By

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§ 0. Introduction

In order to discuss the connection between univalent functions and Teichmüller spaces, it is important to consider the class S of Schwarzians of all schlicht functions on the exterior Δ^* of the unit disk, i.e.,

$$S=\left\{S_f=(f''/f')'-\frac{1}{2}(f''/f')^2 : f\in\Sigma_0\right\},$$

where Σ_0 is the class of all univalent meromorphic functions f on Δ^* having an expansion

$$f(z)=z+\sum_{n=1}^{\infty} b_n z^{-n}.$$

It should be noted that the correspondence $f\mapsto S_f$ is a bijection from Σ_0 to S.

The class S inherits a topology by the hyperbolic sup-norm of weight -2 (so-called the Nehari norm) of the space of holomorphic quadratic differentials. The space S has been studied by many authors (Bers [6], Gehring [15], [16], Žuravlev [28], Flinn [13], Shiga [24], Overholt [20], Sugawa [25], etc.). In particular, the first remarkable result by Gehring [15] states that

$$\operatorname{Int} S=T(=\text{the universal Teichmüller space}).$$

As the Bers projection plays a very important role in the Teichmüller theory, the (generalized) Bers projection is thought to do so in the investigation of the space S, too. § 1 is devoted to study the (generalized) Bers projection mainly in the case that the domain has no exterior (Theorem 1). As a corollary of Theorem 1, we give a simple proof of a theorem of Overholt [19].

In § 2 and succesive sections, we shall consider $\operatorname{Int} S(\Gamma)$, where Γ is an arbitrary Fuchsian group. The "λ-lemma" and the "improved λ-lemma" first introduced by Mañé-Sad-Sullivan [17] and Sullivan-Thurston [26] are greatly powerful tools to study the structure of holomorphic families and, indeed, have many applications in various aspects (for example, see [7], [9], [11], [21], [24]).

As a new application of the "extended λ-lemma" (Bers-Royden [9]), in § 2, we give another proof of a theorem of Žuravlev: $T(\Gamma)$ is the zero component of $\operatorname{Int} S(\Gamma)$. Our proof is based only on the openness of the universal Teichmüller space due to Ahlfors and the λ-lemma while Žuravlev's one is essentially relies upon the Grunsky's in-

Received January 16, 1991

equality.

In §3 we extend the λ-lemma to the Banach space version and as an application of this extension, in §4, we shall give a proof of the submersivity of the Bers projection corresponding to the element of $\operatorname{Int} S(\Gamma)$ under certain additional hypothesis.

In Appendix, we shall show the holomorphy of the (generalized) Bers projection (Proposition 1) for the convenience of the reader.

§1. The Bers projection

Let D be a hyperbolic open subset of the extended plane $\hat{\boldsymbol{C}}$, that is, the complement $E=\hat{\boldsymbol{C}}\setminus D$ contains at least three points. We shall consider the complex Banach space $B_2(D)$ consisting of all holomorphic functions φ on D with norm

$$\|\varphi\|_D=\sup_{z\in D}\rho_D(z)^{-2}|\varphi(z)|<\infty ,$$

where $\rho_D(z)|dz|$ is the Poincaré metric of each component of D which is of constant negative curvature -4. We note that the finiteness of the norm $\|\varphi\|_D$ implies that $\varphi(z)=O(|z|^{-4})$ as $z\to\infty$ if $\infty\in D$.

Now let $M(E)$ be the open unit ball with center 0 of the Banach space $L^\infty(E)=\{\mu\in L^\infty(\boldsymbol{C}): \mu=0 \text{ on } \boldsymbol{C}\setminus E\}$, where E is any measurable set of $\hat{\boldsymbol{C}}$. For each $\mu\in M(\boldsymbol{C})$, we denote by w^μ the normalized μ-conformal map, precisely, the quasiconformal self-map of $\hat{\boldsymbol{C}}$ fixing 0, 1 and ∞, which satisfies the Beltrami equation

$$(w^\mu)_{\bar{z}}=\mu\cdot(w^\mu)_z$$

on $\boldsymbol{C}$ (for the details, see [2]). Set $E=\hat{\boldsymbol{C}}\setminus D$ for a hyperbolic open set D. For $\mu\in M(E)$, $w^\mu|_D$ is a univalent meromorphic function on D, therefore the Beardon-Gehring theorem [4] implies that $\Phi(\mu)=S_{w^\mu|D}$ belongs to $B_2(D)$, where S_f denotes the Schwarzian derivative of a locally univalent meromorphic function f:

$$S_f=(f''/f')'-\frac{1}{2}(f''/f')^2.$$

The map $\Phi=\Phi_D: M(E)\to B_2(D)$ is called the (generalized) Bers projection.

The following proposition (essentially due to Bers) is of basic importance in the sequel.

Proposition 1. *Let D be any hyperbolic open subset of $\hat{\boldsymbol{C}}$ and E be its complement. Then the Bers projection $\Phi: M(E)\to B_2(D)$ is holomorphic and its differential map $d_\mu\Phi: L^\infty(E)\to B_2(D)$ at $\mu\in M(E)$ admits an estimate of operator norm:*

$$\|d_\mu\Phi\|\leqq\frac{12}{1-\|\mu\|_\infty}.$$

In particular, the derivative at 0 is explicitly described as follows:

$$d_0\Phi[\nu](z)=-\frac{6}{\pi}\iint_E\frac{\nu(\zeta)}{(\zeta-z)^4}d\xi d\eta \qquad (\zeta=\xi+i\eta)$$

for every $\nu\in L^\infty(E)$.

The proof of this proposition is well-known (at least, in the case that D is a quasidisk). For convenience, we shall give a proof of this proposition in Appendix.

For a while, let us impose the case that E is nowhere dense. The following theorem, at a glance, may seem slightly curious.

Theorem 1. *Suppose that D is a dense hyperbolic open set in $\hat{\boldsymbol{C}}$ consisting of finitely many components D_j $(j=1, \cdots, n)$. Then the Bers projection $\Phi: M(E)\to B_2(D)$ is injective, where $E=\hat{\boldsymbol{C}}\setminus D$. Further suppose that D is connected, then the differntial map of Φ is injective at each point of $M(E)$. In particular, $d_0\Phi: L^\infty(E)\to B_2(D)$ is a (bounded) embedding.*

Remark. Observe that the class of nowhere dense compact sets E with positive area is sufficiently large. In particular, when E is connected, it is worth to note that $B_2(D)$ is isomorphic to $B_2(\Delta)$ where $\Delta=\{|z|<1\}$ is the unit disk. Further, we remark that for any $\mu\in M(E)\setminus\{0\}$, w^μ is holomorphic on an open dense set and self-homeomorphism of $\boldsymbol{C}$ with a finite modulus of the Hölder continuity of order $1-\varepsilon$, where ε is small if $\|\mu\|_\infty$ is sufficiently small (cf. [2]), but not globally holomorphic.

We shall prepare a simple lemma.

Lemma. *Under the hypothesis of Theorem 1, consider a homeomorphism A: $\boldsymbol{C}\to\boldsymbol{C}$. If each restriction $A|_{D_j}$ is a restriction of Möbius transformation A_j $(j=1, \cdots, n)$, then A is a Möbius transformation in itself.*

Proof. We can deduce the desired conclusion from a simple observation that if $\bar{D}_j\cap\bar{D}_k$ contains at least three points then $A_j=A_k$. ■

Proof of Theorem 1. The injectivity of Φ is almost trivial. In fact, if $\Phi(\mu_1)=\Phi(\mu_2)$ for $\mu_1, \mu_2\in M(E)$, then $A=w^{\mu_2}\circ(w^{\mu_1})^{-1}$ satisfies the hypotheses of the lemma above, thus $A\in$Möb. From the normalization, we conclude that $A=$id, i.e., $w^{\mu_1}=w^{\mu_1}$ on $\hat{\boldsymbol{C}}$, hence $\mu_1=\mu_2$.

Secondly, we prove the injectivity of the differential map:

$$d_0\Phi[\nu](z)=-\frac{6}{\pi}\iint_E\frac{\nu(\zeta)}{(\zeta-z)^4}d\xi d\eta\,.$$

Without loss of generality, we may assume that $\infty\in D$. Then $d_0\Phi[\nu]$ has the expansion

$$d_0\Phi[\nu]=-\frac{1}{\pi}\sum_{n=0}^{\infty}(n+1)(n+2)(n+3)\iint_E\zeta^n\nu(\zeta)d\xi d\eta\cdot z^{-n-4}\,,$$

near ∞. Suppose that $\nu\in$Ker $d_0\Phi$, then $\iint_E\zeta^n\nu(\zeta)d\xi d\eta=0$ for each non-negative integer n, hence

$$(*)\qquad \iint_E f(\zeta)\nu(\zeta)d\xi d\eta=0$$

for all (holomorphic) polynomials $f(\zeta)$ of ζ. Since E has no interior points, famous

Mergelyan's theorem (cf. [14], [22]) says that any continuous function on E is uniformly approximated by polynomials on E, therefore $(*)$ holds for all continuous functions f on E. Because the space of continuous functions $C(E)$ is dense in $L^1(E)$, further $(*)$ holds for all integrable functions f on E, therefore ν must vanish a.e. on E.

Finally, the injectivity of $d_\mu\Phi$ for general $\mu\in M(E)$ easily follows by the argument of the change of base points. We shall illustrate this argument, here. Let $R^\mu(\nu)$ denote the Beltrami coefficient (or the complex dilatation) of the quasiconformal map $w^\nu\circ w^\mu$ for $\nu\in M(\boldsymbol{C})$. Direct computation shows that $R^\mu: M(\boldsymbol{C})\to M(\boldsymbol{C})$ is biholomorphic and R^μ sends $M(E^\mu)$ onto $M(E)$, where $E^\mu=w^\mu(E)$. On the other hand, since $w^\mu|_D$ is conformal, induced map $(w^\mu)^*: B_2(D^\mu)\to B_2(D)$ is an isometric (Banach space) isomorphism where $(w^\mu)^*\varphi=\varphi\circ(w^\mu|_D)\cdot(dw^\mu/dz)^2$ and $D^\mu=w^\mu(D)$. Then a formal calculation shows that the following diagram commutes:

$$\begin{tikzcd} M(E^\mu) \arrow[r, "R^\mu"] \arrow[d, "d_0\Phi_{D^\mu}"'] & M(E) \arrow[d, "d_\mu\Phi_D"'] \\ B_2(D^\mu) \arrow[r, "(w^\mu)^*"] & B_2(D)\,. \end{tikzcd}$$

By the former step, $d_0\Phi_{D^\mu}$ is injective, therefore $d_\mu\Phi_D$ is injective, too. ■

We should note that differential map $d_\mu\Phi$ does not has necessarily closed range, therefore above theorem does *not* state that $\Phi: M(E)\to B_2(D)$ is immersive.

Example. In Theorem 1 and the previous lemma, necessary is the hypothesis that D consists of finitely many components. Here, we exhibit a simple counterexample.

Let I be the interval $[0, 1]$ and U be a dense open subset of I such that $m(U)<1$, where m denotes the 1-dimensional Lebesgue measure. We can divide the complement of U into measurable two parts, say C_1 and C_2, with positive linear measure. Choose $k_1, k_2\in(0, 1)$ such that $k_1m(C_1)=k_2m(C_2)$, and define a function u by

$$u(x)=\int_0^x(1+k_1\chi_{C_1}(t)-k_2\chi_{C_2}(t))dt\,,$$

weere χ_{C_j} denotes the characteristic function of C_j. Then u is an absolutely continuous strictly increasing function from I onto I and $u'=1$ on U.

Now we set

$$F(x+iy)=[x]+u(x-[x])+iy \qquad \text{for } x+iy\in\boldsymbol{C}$$

and $F(\infty)=\infty$, where $[x]$ denotes the largest integer not exceeding x. Simple calculations show that $F: \hat{\boldsymbol{C}}\to\hat{\boldsymbol{C}}$ is $(1+k/1-k)$-quasiconformal map, where $k=\max\{k_1/2+k_1, k_2/2-k_2\}$. Furthermore, F is a translation on each component of D, where $D=\{x+iy\in\boldsymbol{C}: x-[x]\in U\}$, but not globally a translation. Let μ be the Beltrami coefficient of F and $E=\hat{\boldsymbol{C}}\setminus D$, then $\mu\in M(E)\setminus\{0\}$ but $\Phi_D(\mu)=0$, therefore $\Phi_D: M(E)\to B_2(D)$ is not injective whereas D is open dense in $\hat{\boldsymbol{C}}$.

As a corollary of the first part of above theorem, we can give a simple proof of a result of Overholt. Before stating this result, we shall explain a needed terminology. A hyperbolic domain D is called (conformally) rigid if any univalent function $f: D \to \hat{C}$ whose Schwarzian derivative has norm smaller than a positive constant depending only on D must be a Möbius transformation. In particular, when D is simply connected, D is rigid if and only if the Schwarzian derivative of a Riemann mapping of D corresponds to an isolated point of S. Concerning with the existence of rigid domains, we refer articles Thurston [27] and Astala [3].

Corollary (Overholt [19]). *The complement of any (conformally) rigid domain is of zero area.*

Proof. Let D be a rigid domain and E be its complement. By definition, the Bers projection $\Phi: M(E) \to B_2(D)$ must be the constant map 0. If D would have exterior points, clearly Φ would be non-constant, therefore E must have no interior points. Now Theorem 1 yields that $M(E)$ is a singleton, hence E has the Lebesgue measure zero. ■

Note that, in the above proof, we have used only the fact that $\Phi: M(E) \to B_2(D)$ is a constant map, therefore this corollary also holds under a weaker hypothesis, for example, that the connected component of $S(D)=\{S_f \in B_2(D): f$ is univalent meromorphic function on $D\}$ containing 0 has topological dimension at most 1.

Remark. The above corollary means that any rigid domain has a complement of 2-dimensional Hausdorff measure zero. On the other hand, for any given value $t \in (1, 2)$, Astala [3] constructed the rigid domain whose Hausdorff dimension equals to t.

§2. The λ-lemma and another proof of Žuravlev's theorem

Throughout this section, we assume that D is a simply connected domain of hyperbolic type. For every $\varphi \in B_2(D)$ there exists a locally univalent meromorphic function f on D whose Schwarzian derivative equals to φ. Once a normalization assigned, say for example,

$$f(z)=(z-a)+\sum_{n=2}^{\infty} c_n(z-a)^n$$

near a fixed point $a \in D \setminus \{\infty\}$, f satisfying the equation $S_f=\varphi$ is uniquely determined, denoted by f^{φ}, and it turns out that $\varphi \mapsto f^{\varphi}(z)$ is a holomorphic map from $B_2(D)$ to $\hat{C}$ for each fixed point $z \in D$.

Now let G be a Kleinian group discontinuously acting on D and set

$$B_2(D, G)=\{\varphi \in B_2(D): (\varphi \circ g) \cdot (g')^2=\varphi \text{ for all } g \in G\}$$

$$S(D, G)=\{\varphi \in B_2(D, G): f^{\varphi} \text{ is univalent in } D\}.$$

As is easily seen, $S(D, G)$ does not depend on the normalization.

For $\varphi \in B_2(D, G)$, there exists a unique homomorphism $\chi^{\varphi}: G \to \mathrm{PSL}_2(\boldsymbol{C})=\text{Möb}$ so that $f^{\varphi} \circ g=\chi^{\varphi}(g) \circ f^{\varphi}$ for all $g \in G$, called the monodromy homomorphism of φ. Note that the holomorphy of the map $\varphi \mapsto f^{\varphi}(z)$ forces the holomorphy of the map $\varphi \mapsto \chi^{\varphi}(g)$ for each fixed $g \in G$. Observe that χ^{φ} is a monomorphism and that $\chi^{\varphi}(G)$ is also a Kleinian group for every $\varphi \in S(D, G)$. Let $L^{\infty}(E, G)$ be the complex Banach space of the Beltrami differentials $\nu \in L^{\infty}(E)$ which is $(-1, 1)$-form for G, precisely,

$$(\nu \circ g) \cdot \overline{g'}/g' = \nu$$

for every $g \in G$ and denote by $M(E, G)$ the unit ball of $L^{\infty}(E, G)$, that is, $M(E, G) = L^{\infty}(E, G) \cap M(E)$, where E is any G-invariant measurable set in $\boldsymbol{C}$.

It is known that for every $\mu \in M(E, G)$, $\Phi_D(\mu)$ belongs to $B_2(D, G)$, where $E = \hat{\boldsymbol{C}} \setminus D$. We now set

$$T(D, G) = \Phi_D(M(E, G)).$$

Note that $T(D, G) \subset S(D, G)$ and that $T(D, G)$ is connected.

In case that D is the exterior of the unit disk Δ, denoted by Δ^*, and that Γ is a Fuchsian group acting on Δ^*, $T(\Gamma) = T(\Delta^*, \Gamma)$ is called the (Bers model of) Teichmüller space of Fuchsian group Γ. For abbriviation, let $S(\Gamma)$ denote $S(\Delta^*, \Gamma)$. In particular, when Γ is the trivial group 1, we call $T = T(1)$ the universal Teichmüller space and $S = S(1)$ the quasi-Teichmüller space. It should be mentioned that Nehari's theorem implies that for any univalent function $f : \Delta^* \to \hat{\boldsymbol{C}}$, its Schwarzian derivative S_f has norm not greater than 6, thus $S_f \in S$.

The followmg theorem is a basic fact for our argument in the present section.

Theorem A (Ahlfors [1]). *The universal Teichmüller space is a bounded domain in* $B_2(\Delta^*)$.

We can characterize the Teichmüller space of Γ as the set of all holomorphic quadratic differentials $\varphi \in B_2(\Delta^*, \Gamma)$ such that f^{φ} can be extended to a Γ-compatible quasiconformal self-map of $\hat{\boldsymbol{C}}$, where a quasiconformal map $w : \hat{\boldsymbol{C}} \to \hat{\boldsymbol{C}}$ is Γ-compatible if and only if for every $\gamma \in \Gamma$, $w \circ \gamma = A_{\gamma} \circ w$ on $\hat{\boldsymbol{C}}$ for some $A_{\gamma} \in$ Möb.

Now we state the Bers-Royden version of the λ-lemma in the form which is convenient only for our present aim and which has not necessarily full generality.

Theorem B (extended λ-lemma; [9] and [8]). *Let E be a subset of $\hat{\boldsymbol{C}}$ containing at least four points and G be a subgroup of* Möb *acting on E. Suppose that maps $f : \Delta_r \times E \to \hat{\boldsymbol{C}}$ and $\chi : \Delta_r \to$* Hom$(G$, Möb$)$ *have the following four properties:*

(i) $f(0, \cdot) = id_E$,

(ii) $f_{\lambda} = f(\lambda, \cdot) : E \to \hat{\boldsymbol{C}}$ *is an injection for every fixed* $\lambda \in \Delta$,

(iii) $f(\cdot, z) : \Delta_r \to \hat{\boldsymbol{C}}$ *is a holomorphic map for every fixed* $z \in E$,

(iv) $f_{\lambda} \circ g = \chi_{\lambda}(g) \circ f_{\lambda}$ *for all* $\lambda \in \Delta_r$ *and* $g \in G$,

where $\Delta_r = \{z \in \boldsymbol{C} : |z| < r\}$.

Then the restriction $f|\Delta_{r/3} \times E$ has a canonical extension $\hat{f} : \Delta_{r/3} \times \hat{\boldsymbol{C}} \to \hat{\boldsymbol{C}}$ with the following properties:

(a) $\hat{f}$ *has also above four properties* (i)-(iv) *in which Δ_r, E is just replaced by $\Delta_{r/3}$, $\hat{\boldsymbol{C}}$ respectively,*

(b) $\hat{f}_{\lambda} = \hat{f}(\lambda, \cdot) : \hat{\boldsymbol{C}} \to \hat{\boldsymbol{C}}$ *is a quasiconformal map,*

(c) *the Beltrami coefficient* $\mu(\lambda)$ *of* $\hat{f}_\lambda$ *is a holomorphic map from* $\Delta_{r/3}$ *to* $L^\infty(\boldsymbol{C}, G)$,

(d) *for each* $\lambda \in \Delta_{r/3}$, $\mu(\lambda)$ *is harmonic on* $D=\hat{\boldsymbol{C}} \setminus \bar{E}$, *precisely*, $\mu(\lambda)=\rho_D^{-2} \cdot \overline{\psi(\lambda)}$ *on* D *for a* $\psi(\lambda) \in B_2(D, G)$.

The extension $\hat{f}: \Delta_{r/3} \times \hat{\boldsymbol{C}} \to \hat{\boldsymbol{C}}$ *with the properties* (a) *and* (d) *is uniquely determined.*

The map $f: \Delta_r \times E \to \hat{\boldsymbol{C}}$ with the above properties (i)-(iv) is called admissible.

As an application of the λ-lemma (in the above form), we shall give another proof of Žuravlev's theorem.

Theorem C ([28]). *For any Fuchsian group* Γ, $T(\Gamma)$ *is the connected component of* $\operatorname{Int} S(\Gamma)$ *which contains the origin.*

Here, note that we shall not use the fact that $T(\Gamma)$ is open in $B_2(\Delta^*, \Gamma)$ except Ahlfors' theorem (Theorem A).

Theorem A implies that $T(\Gamma) \subset T \cap B_2(\Delta^*, \Gamma) \subset \operatorname{Int} S(\Gamma)$. Since $T(\Gamma)$ is connected, Theorem C is obtained by the special case $L=B_2(\Delta^*, \Gamma)$ of the following theorem which is a slight generalization of Theorem 2 of Žuravlev [28].

Theorem 2. *Suppose that* L *is a (complex Banach) submanifold of* $B_2(\Delta^*, \Gamma)$ *and* V *is a connected component of* $\operatorname{Int}_L(L \cap \operatorname{Int} S(\Gamma))$, *where* $\operatorname{Int}_L X$ *is the interior of* X *in* L. *Then the condition* $V \cap T(\Gamma) \neq \emptyset$ *implies that* $V \subset T(\Gamma)$.

In order to prove this theorem, we shall prepare the following proposition which, in some sense, is a weak version of Theorem 1 of Žuravlev [28].

Proposition 2. *Suppose that a holomorphic function* $F: \Delta \to B_2(\Delta^*, \Gamma)$ *satisfies that* $F(\Delta) \subset S(\Gamma)$. *If* $F(\Delta) \cap T(\Gamma) \neq \emptyset$, *then* $F(\Delta) \subset T(\Gamma)$.

Proof. Set $\Omega=F^{-1}(T(\Gamma)) \subset \Delta$. We know that Ω is open and closed, hence $\Omega=\Delta$ or $\emptyset$ thus the proof completes, if we show the following claim: *for each* $\lambda_0 \in \Omega$, *the Poincaré disk with center* λ_0 *and radius* $\log 2$ *is contained in* Ω.

Now we shall show the above claim. Set $\varphi_0=F(\lambda_0) \in T(\Gamma)$ and $G=\chi^{\varphi_0}(\Gamma)$. We may assume that $\lambda_0=0$ because the unit disk Δ is analytically homogeneous and the Poincaré distance is invariant under the $\operatorname{Aut} \Delta$. Now we define the mapping $g: \Delta \times \Delta^* \to \hat{\boldsymbol{C}}$ by $g(\lambda, z)=f^{F(\lambda)}(z)$ and $h: \Delta \times g_0(\Delta_0) \to \hat{\boldsymbol{C}}$ by $h(\lambda, z)=g(\lambda, g_0^{-1}(z))=g_\lambda \circ g_0^{-1}(z)$ where $g_\lambda=g(\lambda, \cdot)$. Since $F(\Delta) \subset S(\Gamma)$, h has properties (i)-(iv) listed in Theorem B, therefore for each $\lambda \in \Delta_{1/3}$, $h(\lambda, \cdot)$ can be canonically extended to the G-compatible quasiconformal self-map $\hat{h}_\lambda$ of $\hat{\boldsymbol{C}}$. On the other hand, $g_0=f^{\varphi_0}$ can be extended to a Γ-compatible quasiconformal self-map $\hat{g}_0$ of $\hat{\boldsymbol{C}}$ because $\varphi \in T(\Gamma)$. Hence, for every $\lambda \in \Delta_{1/3}$, g_λ is extended to a Γ-compatible quasiconformal self-map $\hat{h}_\lambda \circ \hat{g}_0$ of $\hat{\boldsymbol{C}}$, therefore $F(\lambda) \in T(\Gamma)$. ■

Proof of Theorem 2. Suppose that L is a complex Banach submanifold of $B_2(\Delta^*, \Gamma)$ modeled on a Banach space A with norm $\|\cdot\|$. Let $\theta: U \to A$ be a holomorphic chart of the componnent V such that U is a subdomain of V and thus $W=\theta(U)$ is a subdomain of A. Now for every $a\in\theta(U\cap T(\Gamma))$, set $\delta(a)=\inf_{b\in\partial W}\|a-b\|$. We take $b\in A$ with $\|b\|=\delta(a)$ and define a holomorphic mapping $F: \Delta\to B_2(\Delta^*, \Gamma)$ by the formula $F(\lambda)=\theta^{-1}(a+\lambda b)$. Since $F(\Delta)\subset S(\Gamma)$ and $F(\Delta)\cap T(\Gamma)\neq\emptyset$, Proposition 1 yields that $F(\Delta)\subset T(\Gamma)$. Consequently the following assertion holds: if $a\in\theta(U\cap T(\Gamma))$ then the ball in A with center a and radius $\delta(a)$ is also contained in $\theta(U\cap T(\Gamma))$. By this assertion, it is known that $\theta(U\cap T(\Gamma))$ is open and closed in W thus $\theta(U\cap T(\Gamma))=W$ or $\emptyset$. This conclusion means that if a coodinate neiborhood U of V has non-empty intersection with $T(\Gamma)$, then $U\subset T(\Gamma)$. Hence, it is easy to see that if $V\cap T(\Gamma)\neq\emptyset$, then $V\subset T(\Gamma)$. ■

§ 3. An extension of the λ-lemma

In preceding section, we applied the λ-lemma by making the functions one complex variable, but sometimes it is essential to treat functions with several complex variables. From this reason, we shall make a generalization of the λ-lemma in this direction and apply this generalized λ-lemma to the investigation of Int $S(\Gamma)$.

Theorem 3. *Let A be a complex Banach space with norm $\|\cdot\|$ and A_r be the ball $\{x\in A: \|x\|<r\}$. Let E be a subset of $\hat{\boldsymbol{C}}$ containing at least four points and G be a subgroup of* Möb *acting on E. Suppose that maps $f: A_r\times E\to\hat{\boldsymbol{C}}$ and $\chi: A_r\to$*Hom$(G,$ Möb$)$ *have the following four properties:*

(i) $f(0, \cdot)=id_E$,

(ii) $f_x=f(x, \cdot): E\to\hat{\boldsymbol{C}}$ *is an injection for every fixed* $x\in A_r$,

(iii) $f(\cdot, z): A_r\to\hat{\boldsymbol{C}}$ *is a holomorphic map for every fixed* $z\in E$,

(iv) $f_x\circ g=\chi_x(g)\circ f_x$ *for all* $x\in A_r$ *and* $g\in G$.

Then the restriction $f|A_{r/3}\times E$ has a canonical extension $\hat{f}: A_{r/3}\times\hat{\boldsymbol{C}}\to\hat{\boldsymbol{C}}$ with the following properties:

(a) $\hat{f}$ *has also above four properties* (i)-(iv) *in which A_r, E is just replaced by $A_{r/3}$, $\hat{\boldsymbol{C}}$ respectively,*

(b) $\hat{f}_x=\hat{f}(x, \cdot): \hat{\boldsymbol{C}}\to\hat{\boldsymbol{C}}$ *is a quasiconformal map,*

(c) *the Beltrami coefficient $\mu(x)$ of $\hat{f}_x$ is a holomorphic map from $A_{r/3}$ to $L^\infty(\boldsymbol{C}, G)$.*

(d) *for each $x\in A_{r/3}$, $\mu(x)$ is harmonic on $D=\hat{\boldsymbol{C}}\setminus\bar{E}$, precisely, $\mu(x)=\rho_D^{-2}\cdot\overline{\phi(x)}$ on D for a $\phi(x)\in B_2(D, G)$.*

The extension $\hat{f}: A_{r/3}\times\hat{\boldsymbol{C}}\to\boldsymbol{C}$ with the properties (a) *and* (d) *is uniquely determined.*

Sketch of Proof. First, normalizing by Möbius transformations, we may assume that $0, 1, \infty\in E$ and that $f(x, \cdot)$ fixes 0, 1 and ∞ for each $x\in A_r$ (cf. [9, § 1]). For every $x\in A$ with $\|x\|=1$, we can define an admissible map $g_x: \Delta_r\times E\to\hat{\boldsymbol{C}}$ by the formula $g_x(\lambda, z)=f(\lambda x, z)$. By Theorem B, $g_x|\Delta_{r/3}\times E$ can be canonically extended to $\hat{g}_x: \Delta_{r/3}\times\hat{\boldsymbol{C}}\to\hat{\boldsymbol{C}}$ with properties (a)-(d) in Theorem B. Set $\hat{f}(\lambda x, z)=\hat{g}_x(\lambda, z)$ for every $x\in\partial A_1$, $\lambda\in\Delta_{r/3}$, $z\in\hat{\boldsymbol{C}}$. By the uniqueness part of Theorem B, $\hat{f}: A_{r/3}\times\hat{\boldsymbol{C}}\to\hat{\boldsymbol{C}}$ is a well-defined map, and this should be the canonical extension of $f|A_{r/3}\times E$.

The nontrivial part of Theorem 3 are only (a) and (c). (The part (b) is a consequence of Theorem B.) We now outline the proof of the property (a). It suffices to prove the assertion that $\hat{f}(\cdot, z): A_{r/3}\to \boldsymbol{C}$ is holomorphic for fixed $z\in\hat{\boldsymbol{C}}\setminus\{0, 1, \infty\}$. Since the map $\hat{f}(\cdot, z)$ excludes three points 0, 1 and ∞, Schottky theorem yields the local boundedness of $\hat{f}(\cdot, z)$. The holomorphy of locally bounded functions on Banach spaces follows from the existence of the Gateaux derivative at every point (see, for example [10]), therefore the proof of the holomorphy of $\hat{f}(\cdot, z)$ is reduced in the case $A=\boldsymbol{C}^2$. In this case, we can prove the above assertion by the quite same method as in [9], thus we shall leave the proof for the reader as an exercise.

Once the admissibility of $\hat{f}$ established, the proof of the property (c) can be proceeded in the same way as in [9, §4], which we shall omit.

§4. Investigation of $\mathrm{Int}\, S(\Gamma)$

Gehring showed that $\mathrm{Int}\, S=T$ in [15] and recently Shiga [24] proved that $\mathrm{Int}\, S(\Gamma)=T(\Gamma)$ for any finitely generated Fuchsian group of the first kind. Generally, it is conjectured that $\mathrm{Int}\, S(\Gamma)=T(\Gamma)$ for any Fuchsian group Γ. By virture of Žuravlev's theorem (Theorem C), this conjecture is equivalent to the claim that $\mathrm{Int}\, S(\Gamma)$ is connected.

Through this section, let Γ denote an arbitrary Fuchsian group acting on Δ^*.

We begin with a proof of the following proposition which is a consequence of the extended λ-lemma (Theorem B).

Proposition 2. *Let V be a connected component of* $\mathrm{Int}\, S(\Gamma)$. *For any $\varphi_1, \varphi_2\in V$, there exists a quasiconformal map F from $\hat{\boldsymbol{C}}$ onto itself with the following properties:*

(1) $f^{\varphi_2}=F\circ f^{\varphi_1}$ *on* Δ_*,

(2) $\chi^{\varphi_2}(\gamma)=F\circ\chi^{\varphi_2}(\gamma)\circ F^{-1}$ *on* $\hat{\boldsymbol{C}}$ *for all* $\gamma\in\Gamma$.

Remark. It should be mentioned that the Beltrami coefficient $\mu=F_{\bar{z}}/F_z$ of the above F automatically belongs to $M(\hat{\boldsymbol{C}}\setminus f^{\varphi_1}(\Delta), \chi^{\varphi_1}(\Gamma))$.

Proof. For $\varphi_1, \varphi_2\in V$, we define an equivalence relation $\sim$ by the rule $\varphi_1\sim\varphi_2$ if and only if the claim in the above proposition is valid for φ_1 and φ_2. Since V is connected, it suffices to show that each equivalence class is open in V. Let V_1 be any equivalence class. Then the following is true. For arbitrary $\varphi_1\in V_1$, if $\varphi_2\in B_2(\Delta^*, \Gamma)$ satisfies $\|\varphi_2-\varphi_1\|_{\Delta^*}<a/3$, then $\varphi_2\in V_1$, where

$$a=\mathrm{dist}(\varphi_1, \partial S(\Gamma))=\inf_{\varphi\in\partial S(\Gamma)}\|\varphi-\varphi_1\|_{\Delta^*}(>0).$$

Indeed, for any $\psi\in B_2(\Delta^*, \Gamma)$ with $\|\psi\|_{\Delta^*}=a$, we have $\varphi_1+\lambda\psi\in S(\Gamma)$ for each $\lambda\in\Delta$, therefore we can define an admissible map $f: \Delta\times f^{\varphi_1}(\Delta^*)\to\hat{\boldsymbol{C}}$ by the formula:

$$f(\lambda, z)=f^{\varphi_1+\lambda\psi}\circ(f^{\varphi_1})^{-1}(z).$$

Now Theorem B is applicable. Let $\hat{f}: \Delta_{1/3}\times\hat{\boldsymbol{C}}\to\hat{\boldsymbol{C}}$ be the canonical extension of $f: \Delta_{1/3}\times f^{\varphi_1}(\Delta^*)\to\hat{\boldsymbol{C}}$. Then, for $\varphi_2=\varphi_1+\lambda\psi$ $(\lambda\in\Delta_{1/3})$, $F=\hat{f}(\lambda, \cdot)$ works. ∎

For $g\in$ Möb $=\mathrm{PSL}_2(\boldsymbol{C})$ such that $g\neq\mathrm{id}$, g is called elliptic, parabolic, loxodromic if $\mathrm{tr}^2 g\in[0,4)$, $\mathrm{tr}^2 g=4$, $\mathrm{tr}^2 g\notin[0,4]$ respectively, where $\mathrm{tr}^2 g=(a+d)^2$ if g is represented by a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\mathrm{SL}_2(\boldsymbol{C})$. We note that an element $g\neq\mathrm{id}$ of Kleinian group is elliptic if and only if g has finite order.

Corollary. *For each* $\varphi\in\mathrm{Int}\,S(\Gamma)$, *the monodromy homomorphism* $\chi^{\varphi}:\Gamma\to$ Möb *is type-preserving, precisely,* χ^{φ} *sends elliptic, parabolic and loxodromic elements to elliptic, parabolic and loxodromic elements, respectively, and* $\mathrm{tr}^2\gamma=\mathrm{tr}^2\chi^{\varphi}(\gamma)$ *for any elliptic element* $\gamma\in\Gamma$.

Proof. At first, observe that the map $\tau_\gamma:\varphi\mapsto\mathrm{tr}^2\chi^{\varphi}(\gamma)$ is holomorphic for fixed $\gamma\in\Gamma$. Since the isomorphism induced by a quasiconformal self-map of $\hat{\boldsymbol{C}}$ is type-preserving, Proposition 2 implies that if $\chi^{\varphi_0}(\gamma)$ is parabolic or elliptic for a $\varphi_0\in\mathrm{Int}\,S(\Gamma)$, then τ_γ is constant on the component of $\mathrm{Int}\,S(\Gamma)$ containing φ_0, thus τ_γ is constant on the whole $B_2(\Delta^*,\Gamma)$. From the above facts, the desired conclusion immediately follows. ■

Let $\varphi_0\in\mathrm{Int}\,S(\Gamma)$ and set $D=f^{\varphi_0}(\Delta^*)$, $G=\chi^{\varphi_0}(\Gamma)$ and $E=\hat{\boldsymbol{C}}\setminus D$. The Bers projection $\Phi: M(E,G)\to B_2(D,G)$ is expected to reflect the properties of domain D, therefore the author thinks that it is important to study the Bers projection $\Phi=\Phi_D$.

As an application of Theorem 3, we shall show submersivity of the Bers projection Φ_D.

Theorem 4. *Let* $\varphi_0\in\mathrm{Int}\,S(\Gamma)$ *and set* $D=f^{\varphi_0}(\Delta^*)$, $G=\chi^{\varphi_0}(\Gamma)$ *and* $E=\hat{\boldsymbol{C}}\setminus D$. *Let* V *be the zero-component of* $\mathrm{Int}\,S(D,G)$. *Concerning with the Bers projection* $\Phi: M(E,G)\to B_2(D,G)$, *the following is true.* $V\subset T(D,G)$ *and* $\Phi:\Phi^{-1}(V)\to V$ *is a (split) submersion.*

Proof. Define a biholomorphic isometry $\mathcal{A}: B_2(D,G)\to B_2(\Delta^*,\Gamma)$ by the formula

$$\mathcal{A}(\psi)=S_{f^{\varphi}\circ f^{\varphi_0}}=(\psi\circ f^{\varphi_0})\cdot\left(\frac{df^{\varphi_0}}{dz}\right)^2+\varphi_0\,.$$

Noting that $\mathcal{A}(S(D,G))=S(\Gamma)$, we set $V_0=\mathcal{A}(V)$. Let any $\psi_1\in V$ be fixed. First let us show that $\psi_1\in T(D,G)$. Set $\varphi_1=\mathcal{A}(\psi_1)\in V_0$. Then Proposition 2 produces a quasiconformal map $F:\hat{\boldsymbol{C}}\to\hat{\boldsymbol{C}}$ with the properties:

(1) $f^{\varphi_1}=F\circ f^{\varphi_0}$ on Δ^*,

(2) $\chi^{\varphi_1}(\gamma)=F\circ\chi^{\varphi_0}(\gamma)\circ F^{-1}$ on $\hat{\boldsymbol{C}}$ for all $\gamma\in\Gamma$.

We denote by ν the Beltrami coefficient of F. By Remark below Proposition 2, $\nu\in M(E,G)$, and by (1), we can conclude that

$$\mathcal{A}(\psi_1)=\varphi_1=S_{F\circ f^{\varphi_0}}=\mathcal{A}(S_F)\,,$$

hence $\psi_1=S_F=\Phi(\nu)\in\Phi(M(E,G))=T(D,G)$.

Next, we shall show the submersivity of Φ at 0. (The submersivity of Φ at general points follows also from the one at 0 and the typical argument of the change of base points.) To this end, it is sufficient to prove the existence of the local holomorphic section of Φ which sends 0 to 0. Set $a=\mathrm{dist}(0, \partial S(D, G))=\mathrm{dist}(\varphi_0, \partial S(\Gamma))$ (>0). Now we define admissible map $f: B_2(D, G)_a \times D \to \hat{C}$ by the formula $f(\psi, z)= f^{\psi}(z)$, then, by virture of Theorem 3, $f \mid B_2(D, G)_{a/3} \times D$ can be canonically extended to $\hat{f}: B_2(D, G)_{a/3} \times \hat{C} \to \hat{C}$ with the properties (a)-(d) listed in Theorem 3. Denote by $\alpha(\psi)$ the Beltrami coefficient of $\hat{f}(\psi, \cdot)$ for every $\psi \in B_2(D, G)_{a/3}$, then by (b) in Theorem 3, $\alpha: B_2(D, G)_{a/3} \to M(E, G)$ is a desired holomorphic section of Φ. ∎

Remark 1. In the case $\varphi_0=0$, the above theorem provides a proof of the submersivity of the (original) Bers projection $\Phi: M(\Delta, \Gamma) \to T(\Gamma)$ cf. Bers [5], Earle-Nag [12]). The uniqueness part of Theorem B shows that the local holomorphic section α constructed above is, in this case, nothing but the Ahlfors-Weil section (cf. [1], [7]). More generally, in the case $\varphi_0 \in T(\Gamma)$ (i.e. D is a quasidisk and G is a quasi-Fuchsian group), $V=T(D, G)$ and the submersivity of $\Phi_D: M(E, G) \to T(D, G)$ is well-known, in fact, which follows from the case $\varphi_0=0$ by the argument of the change of base points.

Remark 2. The author thinks the above theorem to be meaningful in the sense that it may provide informations as to the question whether Int $S(\Gamma)=T(\Gamma)$. By this theorem, for example, it seems to the author that E would have interior points. In fact, if E has no interior points, then both Theorem 1 and Theorem 4 implies that $\Phi: M(E, G) \to B_2(D, G)$ is a local embedding at 0, in particular, the differential map $d_0\Phi$ is a Banach space isomorphism $L^\infty(E, G) \to B_2(D, G) \cong B_2(\Delta^*, \Gamma)$, which seems to him impossible.

Appendix: The holomorphy of the Bers projection

The proof of Proposition 1 can be performed in the quite same way as in Nag [18]. Theorem of Beardon-Gehring [4] yields that

$$\|\Phi(\mu)\|_D \leqq 12 \qquad \text{for any } \mu \in M(E).$$

Since, as is seen above, Φ is (globally) bounded, it suffices to prove the existence of the Gateaux derivative $d_\mu \Phi$ at each point $\mu \in M(E)$ (see, for example, [10]). Without loss of generality, we may assume that $\infty \notin D$.

Let any ε, $a \in (0, \infty)$ be fixed. We choose a $K \in (0, \infty)$ satisfying

$$(*) \qquad \|\mu\|_\infty + (1+a)\varepsilon K < 1,$$

and pick $\tilde{\varepsilon} \in (0, \infty)$ so that $(1+a)\varepsilon < \tilde{\varepsilon}$ and $\|\mu\|_\infty + \tilde{\varepsilon} K < 1$. Let $\nu \in L^\infty(E)$ with $\|\nu\|_\infty < K$. Since $\|\mu + t\nu\|_\infty < 1$ for $t \in \Delta_{\tilde{\varepsilon}} = \{|t| < \tilde{\varepsilon}\}$, the map

$$(t, z) \longmapsto w^{\mu+t\nu}(z) \in C \qquad \text{for } (t, z) \in \Delta_{\tilde{\varepsilon}} \times D$$

is holomorphic (cf. [2]) and $(d/dz) w^{\mu+t\nu}(z) \neq 0$ for $z \in D$. Therefore $(t, z) \mapsto S_{w^{\mu+t\nu}|D}(z) = \Phi(\mu+t\nu)(z)$ is a holomorphic function of $(t, z) \in \Delta_{\tilde{\varepsilon}} \times D$. Now we set $\theta(z) = (d/dt)\Phi(\mu+t\nu)(z)|_{t=0}$. In particular, when $\mu=0$, we can obtain the concrete form of θ:

$$\theta(z) = -\frac{6}{\pi} \iint_E \frac{\nu(\zeta)}{(\zeta - z)^4} d\xi d\eta$$

by the formal calculation and the formula:

$$\frac{d}{dt}w^{t\nu}(z)\Big|_{t=0}=-\frac{z(z-1)}{\pi}\iint_E\frac{\nu(\zeta)d\xi d\eta}{\zeta(\zeta-1)(\zeta-z)}$$

(cf. [1], [2]). Fix any $z_0\in D$ and $\varphi(t)=\rho_D(z_0)^{-2}\cdot\Phi(\mu+t\nu)(z_0)$, then we obtain that $|\varphi(t)|\leqq 12$. On the other hand, for $t\in\Delta_\varepsilon$

$$\varphi(t)-\varphi(0)-t\varphi'(0)=\frac{1}{2\pi i}\int_{|\zeta|=(1+a)\varepsilon}\left(\frac{1}{\zeta-t}-\frac{1}{\zeta}-\frac{t}{\zeta^2}\right)\varphi(\zeta)d\zeta$$
$$=\frac{1}{2\pi i}\int_{|\zeta|=(1+a)\varepsilon}\frac{t^2}{\zeta^2(\zeta-t)}\varphi(\zeta)d\zeta\,,$$

hence we have an estimate

$$|\varphi(t)-\varphi(0)-t\varphi'(0)|\leqq\frac{12|t|^2}{(1+a)a\varepsilon^2}.$$

Since $z_0\in D$ is arbitrary, we conclude that

$$\|\Phi(\mu+t\nu)-\Phi(\mu)-t\theta\|_D\leqq\frac{12|t|^2}{(1+a)a\varepsilon^2}=o(|t|)\qquad\text{as } t\to 0\,.$$

In this way, we can show the existence of the Gateaux derivative $d_\mu\Phi=\theta\in B_2(D)$.

Finally, let us estimate the operator norm $\|d_\mu\Phi\|$. By using same notations as above, we obtain

$$\varphi(t)-\varphi(0)=\frac{1}{2\pi i}\int_{|\zeta|=(1+a)\varepsilon}\frac{t}{\zeta(\zeta-t)}\varphi(\zeta)d\zeta\,,$$

thus $|\varphi(t)-\varphi(0)|\leqq 12|t|/a\varepsilon$. This estimate implies that

$$\left\|\frac{\Phi(\mu+t\nu)-\Phi(\mu)}{t}\right\|_D\leqq\frac{12}{a\varepsilon}\,,$$

and by taking the limit for $t\to 0$,

$$\|d_\mu\Phi[\nu]\|_D\leqq\frac{12}{a\varepsilon}$$

for all $\nu\in L^\infty(E)_K$, hence $\|d_\mu\Phi\|\leqq 12/a\varepsilon K$. Since ε and K is arbitrary as long as (*) holds, now we have

$$\|d_\mu\Phi\|\leqq\frac{1+a}{a}\cdot\frac{12}{1-\|\mu\|_\infty}\,,$$

finally, by taking the limit for $a\to\infty$, the desired estimate follows. ■

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