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Spectral flow and intersection number

By

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Dedicated to Professor Nobuhiko Tatsuuma on his 60-th birthday

§ 1. Introduction

Let $\mathscr{F} = \mathscr{F}(H)$ be the set of bounded Fredholm operators on a separable (complex) Hilbert space H of infinite dimension. $\mathscr{F}$ is a classifying space for the complex K-group. A subset $\hat{\mathscr{F}}$ of $\mathscr{F}$ consisting of selfadjoint operators has three components:

$$\hat{\mathscr{F}} = \hat{\mathscr{F}}_+ \cup \hat{\mathscr{F}}_- \cup \hat{\mathscr{F}}_* . \tag{1-1}$$

$\hat{\mathscr{F}}_+$ ($\hat{\mathscr{F}}_-$) consists of essentially positive (negative) operators and $\hat{\mathscr{F}}_*$ consists of others. $\hat{\mathscr{F}}_\pm$ are contractible and $\hat{\mathscr{F}}_*$ is a classifying space for K^{-1}-group ([AS]). Especially we have

$$\pi_1(\hat{\mathscr{F}}_*) \cong \mathbf{Z} . \tag{1-2}$$

An isomorphism of (1-2) is given by, so called, the spectral flow. It is defined as the number of eigenvalues (with directions) that change signs when the parameter of a loop in $\hat{\mathscr{F}}_*$ goes around ([APS1, 2]). This definition is more clarified by considering a subspace $\hat{F}(\infty)$ of $\hat{\mathscr{F}}_*$, which has the same homotopy type with the whole space $\hat{\mathscr{F}}_*$ and has a spectrally nice property in a sense ([BW1]):

$$\begin{aligned}\hat{F}(\infty) = \{A \in \hat{\mathscr{F}}_*\colon \|A\| = 1, \ &\text{the essential spectra } \sigma_{ess}(A)\\ &\text{of } A \text{ are just } \{-1, 1\} \text{ and other spectra } \sigma(A)\backslash\sigma_{ess}(A)\\ &\text{are the finite number of eigenvalues}\} .\end{aligned} \tag{1-3}$$

Let $l\colon [0, 1] \to \hat{F}(\infty)$ be a continuous loop, then the graph of the spectrum of l can be parametrized through a finite monotone sequence of continuous functions:

$$\lambda_j\colon [0, 1] \to [-1, 1] \qquad j = 1, \ldots, N , \tag{1-4}$$

(N is the maximal number of the eigenvalues $\in \sigma(l(t))\backslash\sigma_{ess}(l(t))$ with multiplicities of the operator $l(t)$ $(0 \le t \le 1)$)
$-1 \le \lambda_1(t) \le \cdots \le \lambda_N(t) \le 1$

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and we regard as

$$-1 = \cdots = \lambda_{-1}(t) = \lambda_0(t)\,, \qquad \lambda_{N+1}(t) = \lambda_{N+2}(t) = \cdots = 1\,,$$

because ± 1 are eigenvalues with infinite multiplicities.

Since $\{\lambda_k(0)\}_{k=1}^N = \{\lambda_k(1)\}_{k=1}^N$, there exists a unique integer $s \in \mathbf{Z}$ ($|s| \leq N$) such that

$$\lambda_{k+s}(0) = \lambda_k(1) \qquad \text{for any } k \in \mathbf{Z}\,. \tag{1-5}$$

This integer s is the spectral flow of the given loop l, and is invariant for the homotopy class of l. Spectral flows appear in various contexts ([At], [B], [BW1], [BW3], [DW1], [DW2], [F], [OF], [T], [VW], [Y] etc.).

The Bott periodicity theorem for complex K-groups says that the spectral flow and the Fredholm index are equivalent as topological invariants through the suspension or the desuspension. However it seems for us that there are analytic difficulties in dealing with the spectral flows arising from the family of differential operators with different domains of definitions, especially, since we need continuous loops of bounded operators in a fixed Hilbert space to define the spectral flow.

Our purpose here is to give a non-trivial example of the spectral flow arising from a variation of non-local elliptic boundary conditions imposed on a fixed elliptic differential operator and its spectral flow formula in terms of intersection numbers between certain singular cycles. Our situation was already suggested in [B] and [BW3].

Now we can regard that the isomorphism $sf: \pi_1(\hat{\mathscr{F}}_*) \cong \mathbf{Z}$ defined by the spectral flow gives us a cohomology class $[sf] \in H^1(\hat{\mathscr{F}}_*, \mathbf{Z})$. On the other hand, let $\mathscr{Z}$ be the subset of $\hat{\mathscr{F}}_*$ consisting of the operators with non-zero kernels. Then according to the definition of the spectral flow, it may be regarded as the intersection number of the set $\mathscr{Z}$ with the loop in the space $\hat{\mathscr{F}}_*$. Obviously this interpretation may not be rigorous. However,

(1) if the loop is contained in a certain closed oriented manifold M (of finite dimension), continuously embedded in $\hat{\mathscr{F}}_*$ with an embedding

$$\gamma\colon M \to \hat{\mathscr{F}}_*\,, \tag{1-6}$$

and moreover

(2) if the set $\gamma^{-1}(\mathscr{Z})$ represents the Poincaré dual of the cohomology class $\gamma^*([sf]) \in H^1(M, \mathbf{Z})$, then the spectral flow for the loop coming from M might be equal to the intersection number of the cycle $\gamma^{-1}(\mathscr{Z})$ with the loop.

Here we say that a subset S of a manifold M represents a homology class, if there exists a closed manifold Y and a smooth map $\tau\colon Y \to M$ satisfying,

(a) $\tau(Y) = S$

(b) τ is one to one on an open dense subset Y_r of Y, with the open dense image $\tau(Y_r)$ in S.

(c) rank $(d\tau)_p = \dim Y$ for $p \in Y_r$.

Then with a smooth triangulation of Y, S can be seen as a singular cycle whose homology class is the image of the fundamental class $[Y]$ under the map τ.

Remark 1.1. Even if we have a map $\gamma: M \to \hat{\mathscr{F}}_*$, it is not so clear to be able to find such a manifold Y and a map τ for $\gamma^{-1}(\mathscr{Z})$.

In this paper we will give an embedding of the unitary group $\gamma: U(N) \to \hat{\mathscr{F}}_*$ for each N satisfying all the above conditions (1), (2) and give a spectral flow formula in terms of intersection numbers (Theorem 4.2). For that purpose we will identify the space of unitary matrices with a certain space of non-local elliptic boundary conditions. The loops of Fredholm operators, then, are obtained as realizations of an elliptic differential operator by varying such boundary conditions.

In §2, we introduce the space of boundary conditions (see [BW3]) and construct the map $\gamma: U(N) \to \hat{\mathscr{F}}_*$. In §3, we prove the continuity of the map γ and give a spectral flow formula, which is analogous to the formula in [BW3]. In §4, we show that the set $\gamma^{-1}(\mathscr{Z})$ represents the Poincaré dual of $\gamma^*([sf]) \in H^1(U(N), \mathbf{Z})$ and consequently we have a spectral flow formula in terms of the intersection number (Theorem 4.2). Also we note a relation of the spectral flow with the Maslov index as a special case of the theorem 4.2 (Theorem 4.5). Finally in §5, we remark how we can obtain a loop in $\hat{\mathscr{F}}_*$ from two invertible differential operators with a same principal symbol and show a spectral flow formula, for the case of ordinary differential operators, in terms of intersection numbers through monodromy matrices.

§2. A space of elliptic boundary conditions

In this section we construct a continuous map $\gamma: \varinjlim U(N) = U(\infty) \to \hat{\mathscr{F}}_*$. For this purpose, we introduce a certain space of non-local elliptic boundary conditions (so called, generalized Atiyah-Patodi-Singer boundary conditions ([BW3], [DW1])) and identify it with the unitary group $U(N)$. Then the map γ is defined as the realizations of an elliptic differential operator by imposing such boundary conditions.

Let A be a symmetric, first order elliptic differential operator on a closed manifold Y. A is acting on a smooth Hermitian vector bundle E. We denote by $L_2(Y, E)$ the Hilbert space of L_2-sections of E with the inner product $(\cdot, \cdot)_Y$. We assume that there exists a unitary bundle isomorphism G of E satisfying following conditons (2-1) ~ (2-3):

$$G^2 = -Id \tag{2-1}$$

$$G \circ A = -A \circ G \tag{2-2}$$

$$\dim \operatorname{Ker} A \cap \operatorname{Ker}(G - \sqrt{-1}) = \dim \operatorname{Ker} A \cap \operatorname{Ker}(G + \sqrt{-1}) . \tag{2-3}$$

Here G is regarded as a unitary operator on $L_2(Y, E)$, and is denoted with the same symbol.

Remark 2.1. On any $4k$-dim manifold with the signature zero we have always such an operator. See also [BW3] and [Y] for such examples of the pair A and G. If $\operatorname{Ker} A = 0$, then the condition (2-3) is automatically satisfied.

Let $\hat{E}$ be the pull back of the vector bundle E to the product manifold $X = [0, 1] \times Y$ via the projection $X \to Y$, and by $(\phi, \psi)_X$ the inner product for sections $\phi, \psi \in \Gamma(X, \hat{E})$:

$$(\phi, \psi)_X = \int_0^1 (\phi(t, \cdot), \psi(t, \cdot))_Y \, dt . \tag{2-4}$$

Now let B be the elliptic differential operator

$$B = G\left(\frac{\partial}{\partial t} + A\right) \tag{2-5}$$

acting on the vector bundle $\hat{E}$. We would like to define a map $\gamma = \gamma_N \colon U(N) \to \hat{\mathscr{F}}_*(L_2(X, \hat{E}))$ for each N by selfadjoint realizations of B. Note here that the actions of G and A on $\Gamma(X, \hat{E})$ are naturally defined, so we denote them with the same symbols for the sake of simplicity.

By the conditions (2-1) and (2-2), the operator B is formally symmetric, namely, for any smooth $\phi, \psi \in \Gamma(X \backslash \partial X, \hat{E})$ with compact supports,

$$(B\phi, \psi)_X = (\phi, B\psi)_X . \tag{2-6}$$

Again owing to the conditions (2-1) $\sim$ (2-3) we can choose an orthonormal basis $\{\phi_n\}_{n=\pm 1, \pm 2, \pm 3, \dots}$ of $L_2(Y, E)$ such that

(2-7) ϕ_n is a smooth eigensection of A with the eigenvalue λ_n,

$n = \pm 1, \pm 2, \pm 3, \dots,$

(2-8) $0 \leq \lambda_1 \leq \cdots \leq \lambda_n \leq \cdots, \qquad \lambda_n = -\lambda_{-n}, \qquad n = 1, 2, 3, \dots,$

(2-9) $G\phi_n = \phi_{-n}, \qquad G\phi_{-n} = -\phi_n, \qquad n = 1, 2, 3, \dots.$

Let $V(N)$ be a subspace of $L_2(Y, E)$ spanned by $\{\phi_n\}_{n=\pm 1, \pm 2, \dots, \pm N}$ with complex coefficients, and $\mathscr{B}_N$ the set of N-dim subspaces L of $V(N)$ satisfying

$$L \perp GL \quad \text{(orthogonal)}. \tag{2-10}$$

For each $L \in \mathscr{B}_N$, we denote by π_L the orthogonal projection operator in $L_2(Y, E)$ onto the subspace:

$$L + \overline{\sum_{n \geq N+1} \mathbf{C}\phi_{-n}} , \tag{2-11}$$

and also denote by π_+ the orthogonal projection operator in $L_2(Y, E)$ onto the subspace

$$\overline{\sum_{k \geq 1} \mathbf{C}\phi_k} . \tag{2-12}$$

These π_L and π_+ are pseudo-differential operators of order zero.

Now let γ_i ($i = 0, 1$) be the restriction operator from the first order L_2-Sobolev space $W^1(X, \hat{E})$ on X to $L_2(Y_i, \hat{E}_{|Y_i})$, where $Y_i \equiv \{i\} \times Y \cong Y$, $\hat{E}_{|Y_i} \cong E$, $i = 0, 1$, and let D_L be the subspace of $W^1(X, \hat{E})$ such that

$$D_L = \{\phi \in W^1 : \pi_+ \circ \gamma_0(\phi) = 0, \pi_L \circ \gamma_1(\phi) = 0\} . \tag{2-13}$$

Let B_L denote the operator B with the domain D_L, then we have

Proposition 2.1. *For each $L \in \mathscr{B}_N$, B_L is selfadjoint and has compact resolvent operators.*

This proposition is proved by constructing a parametrix for B_L. Such a parametrix is obtained by patching parametrices near boundaries Y_i ($i = 0, 1$) and an interior one. A parametrix near boundary $\{0\} \times Y$ is constructed in the same way as [APS1, Proposition 2.5], and an interior one is obtained by the standard way. To construct a parametrix near the boundary $\{1\} \times Y$, we need a slight modification from that cited above. Now we describe just it in the following lemma 2.2 (see [APS1, Proposition 2.5 and Proposition 2.12] for the full details of the proof of our proposition 2.1).

Let $C^\infty_{comp}([0, \infty) \times Y, \hat{E})$ be the space of smooth sections u of the bundle $\hat{E}$ over $[0, \infty) \times Y$ such that each u vanishes for sufficiently large $t \gg 1$, and for each $L \in \mathscr{B}_N$, $C^\infty([0, \infty) \times Y, \hat{E}; 1 - \pi_L)(C^\infty_{comp}([0, \infty) \times Y, \hat{E}; 1 - \pi_L))$ be the space of smooth sections $u \in C^\infty([0, \infty) \times Y, \hat{E})(u \in C^\infty_{comp}([0, \infty) \times Y, \hat{E}))$ satisfying $(1 - \pi_L) \circ \gamma_0(u) = 0$.

Lemma 2.2. *Let A be the operator as above and let $L \in \mathscr{B}_N$ be fixed. Then there is an operator Q_L*

$$Q_L : C^\infty_{comp}([0, \infty) \times Y, \hat{E}) \to C^\infty([0, \infty) \times Y, \hat{E}; 1 - \pi_L) \tag{2-14}$$

such that

(i) $\left(\dfrac{\partial}{\partial t} + A\right) Q_L g = g$ *for* $g \in C^\infty_{comp}([0, \infty) \times Y, \hat{E})$

(ii) $Q_L\left(\dfrac{\partial}{\partial t} + A\right) f = f$ *for* $f \in C^\infty_{comp}([0, \infty) \times Y, \hat{E}; 1 - \pi_L)$

(iii) *Q_L extends to a continuous map from Sobolev spaces W^{l-1} to W^l_{loc} for all integers $l = 1, 2, 3, \ldots$.*

(The parametrix near the boundary $\{1\} \times Y$ for the operator B_L is obtained easily from this Q_L).

Proof. Let $\theta_1, \theta_2, \ldots, \theta_N$ be an orthonormal basis of L and put $\theta_{-k} = G(\theta_k)$ ($k = 1, 2, \ldots, N$). Let $A(\theta_k) = \sum_{i=1}^N a_{ik}\theta_i$, then $A(\theta_{-k}) = -\sum_{i=1}^N a_{ik}\theta_{-i}$, where the constant matrix (a_{ik}) is symmetric. We expand $g \in C^\infty_{comp}([0, \infty) \times Y, \hat{E}) \subset L_2([0, \infty)) \hat{\otimes} L_2(Y, E)$ (tensor product of Hilbert spaces) by the orthonormal basis $\{\phi_k\}_{|k| \geq N+1} \cup \{\theta_k\}_{|k| \leq N}$:

$$g(t, y) = \sum_{|k| > N} g_k(t)\phi_k(y) + \sum_{|k| \leq N} g_k(t)\theta_k(y) . \tag{2-15}$$

Now we want to solve the equation $B_L f = g$. This is done by solving the following equations:

$$\frac{df_k}{dt} + \lambda_k f_k = g_k \qquad |k| > N\,, \tag{2-16}$$

$$\begin{cases} \dfrac{df_k}{dt} + \sum\limits_{i=1}^{N} a_{ik} f_i = g_k\,, \\ \dfrac{df_{-k}}{dt} - \sum\limits_{i=1}^{N} a_{ik} f_{-i} = g_{-k}\,, \qquad k = 1, \dots, N\,. \end{cases} \tag{2-17}$$

Solutions of these equations are given by

$$f_k(t) = e^{-\lambda_k t}\left(\int_0^t e^{s\lambda_k} g_k(s)ds - C_k\right), \qquad |k| > N\,, \tag{2-18}$$

and

$$\begin{bmatrix} f_1(t) \\ \vdots \\ f_N(t) \end{bmatrix} = \exp(-t(a_{ik}))\left[\int_0^t \exp(s(a_{ik}))\begin{bmatrix} g_1(s) \\ \vdots \\ g_N(s) \end{bmatrix} ds - \begin{bmatrix} C_1 \\ \vdots \\ C_N \end{bmatrix}\right], \tag{2-19}$$

$$\begin{bmatrix} f_{-1}(t) \\ \vdots \\ f_{-N}(t) \end{bmatrix} = \exp(t(a_{ik}))\left[\int_0^t \exp(-s(a_{ik}))\begin{bmatrix} g_{-1}(s) \\ \vdots \\ g_{-N}(s) \end{bmatrix} ds - \begin{bmatrix} C_{-1} \\ \vdots \\ C_{-N} \end{bmatrix}\right], \tag{2-20}$$

with suitable constants $\{C_k\}_{|k|=1}^{\infty}$. Now if we take

$$C_k = 0 \qquad \text{for } k \geq 1\,, \tag{2-21}$$

$$C_k = \int_0^{\infty} e^{s\lambda_k} g_k(s)ds \qquad \text{for } k < -N\,, \tag{2-22}$$

and

$$\begin{bmatrix} C_{-1} \\ \vdots \\ C_{-N} \end{bmatrix} = \int_0^{\infty} \exp(-s(a_{ik}))\begin{bmatrix} g_{-1}(s) \\ \vdots \\ g_{-N}(s) \end{bmatrix} ds\,, \tag{2-23}$$

then for $g \in C^{\infty}_{comp}([0, \infty) \times Y, \hat{E})$

$$(Q_L g)(t, y) = \sum_{|k|>N} f_k(t)\phi_k(y) + \sum_{|k|\leq N} f_k(t)\theta_k(y) \tag{2-24}$$

converges in C^{∞}-topology and satisfies (i), (ii) and (iii) (for details see [APS1, Proposition 2.5]).

Thus we have constructed the parametrix of B_L, which implies that B_L has compact resolvent operators. The selfadjointness of B_L is proved by noting the following fact: if we denote by D_L^*,

$$D_L^* = \{\phi \in W^1(X, \hat{E}): (1 - \pi_+) \circ \gamma_0(\phi) = 0\,,\ \ (1 - \pi_L) \circ \gamma_1(\phi) = 0\}$$

then $G(D_L) = D_L^*$, which is the domain of $\left(\frac{\partial}{\partial t} + A\right)^*$. They complete the proof of the proposition 2.1.

Now we identify the space $\mathscr{B}_N$ with the unitary group $U(N)$ by the following way. Let $U^G(N)$ be a subgroup of the unitary group $U(2N)$ on $V(N)$ such that

$$U^G(N) = \{U \in U(2N): U \circ G = G \circ U\} \tag{2-25}$$

and $U_0^G(N)$ be a subgroup of $U^G(N)$, each of which preserves the subspace L_+ (= the subspace of $V(N)$ spanned by $\{\phi_k\}_{k=1}^N$). Then $U^G(N)$ acts on $\mathscr{B}_N$ transitively and the stationary subgroup at L_+ is $U_0^G(N)$. So we have the isomorphism:

$$U^G(N)/U_0^G(N) \cong \mathscr{B}_N\,, \tag{2-26}$$

through the map $U^G(N) \ni U \mapsto U(L_+) \in \mathscr{B}_N$.

Every element $U \in U^G(N)$ is expressed in the following form (2-28) with respect to the orthonormal basis $\{\psi_k\}_{|k|=1}^N$ of $V(N)$:

$$\begin{cases} \psi_k = \dfrac{1}{\sqrt{2}}(\phi_k - \sqrt{-1}\phi_{-k})\,, & 1 \le k \le N\,, \\[2ex] \psi_{-k} = \dfrac{1}{\sqrt{2}}(\phi_k + \sqrt{-1}\phi_{-k})\,, & 1 \le k \le N\,, \end{cases} \tag{2-27}$$

$$\begin{cases} U(\psi_{-k}) = \displaystyle\sum_{j=1}^N u_{jk}\psi_{-j}\,, \\[2ex] U(\psi_k) = \displaystyle\sum_{j=1}^N v_{jk}\psi_j\,. \end{cases} \tag{2-28}$$

Especially with this basis, every element $U \in U_0^G(N)$ is of the form $\begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}$ with $U \in U(N)$. So let $\alpha = \alpha_N: U(N) \to \mathscr{B}_N$ be the isomorphism defined by

$$\alpha_N: U(N) \xrightarrow{\sim} \mathscr{B}_N \tag{2-29}$$

$$U = (u_{jk}) \mapsto \begin{pmatrix} Id & 0 \\ 0 & U \end{pmatrix}(L_+)$$

$$= \text{the subspace of } V(N) \text{ spanned by } \left\{\sum_{j=1}^N u_{jk}\psi_j + \psi_{-k}\right\}_{k=1}^N.$$

In the obvious way we can see

$$\mathscr{B}_N \subset \mathscr{B}_{N+1}\,, \qquad U(N) \subset U(N+1) \tag{2-30}$$

and have the commutative diagram:

$$\begin{array}{ccc} U(N) & \xrightarrow{\alpha_N} & \mathscr{B}_N \\ \downarrow & & \downarrow \\ U(N+1) & \xrightarrow{\alpha_{N+1}} & \mathscr{B}_{N+1}\,. \end{array} \tag{2-31}$$

Let $\beta = \beta_N : \mathscr{B}_N \to \hat{\mathscr{F}}_*(L_2(X, \hat{E}))$ be defined by $\beta(B_L) = B_L \circ \sqrt{1 + B_L^2}^{-1} \equiv \tilde{B}_L$, then, finally the map $\gamma: \varinjlim U(N) = U(\infty) \to \hat{\mathscr{F}}_*(L_2(X, \hat{E}))$ is defined by

$$\gamma(U) = \beta(B_{\alpha(U)}) \tag{2-32}$$

for $U \in U(\infty)$. The proposition 2.1 implies that each operator $\gamma(U)$ is a bounded selfadjoint Fredholm operator in $L_2(X, \hat{E})$. We will prove the following theorem in the next section.

Theorem 2.3. (i) *The map $\gamma: U(\infty) \to \hat{\mathscr{F}}_*$ is continuous and one to one.*
(ii) *the map γ induces the following isomorphism for each N:*

$$\gamma_*: \pi_1(U(N)) \xrightarrow{\sim} \pi_1(\hat{\mathscr{F}}_*) . \tag{2-33}$$

We have the first spectral flow formula as a corollary of the above theorem.

Theorem 2.4. *For each loop $l: [0, 1] \to U(N)$ the spectral flow $sf\{\gamma \circ l\}$ is given by*

$$sf\{\gamma \circ l\} = \int_l \det{}^*([d\theta]) ,$$

where $[d\theta]$ is the generator of $H^1(U(1), \mathbf{Z})$.

Proof. The proof is obtained by noting that the map $\det: U(N) \to U(1)$ induces the isomorphism $\det^*: H^1(U(1), \mathbf{Z}) \cong H^1(U(N), \mathbf{Z})$.

Remark 2.2. From the proposition 2.1 our operators B_L are in the space $\hat{\mathscr{C}}_*$, the space of selfadjoint operators with compact resolvent operators and infinite numbers of both positive and negative eigenvalues. Operators in $\hat{\mathscr{C}}_*$ have different domains of definitions and necessarily are not bounded. We transform them to operators in $\hat{\mathscr{F}}_*$:

$$\hat{\mathscr{C}}_* \ni T \to T \circ (\sqrt{1 + T^2})^{-1} \in \hat{\mathscr{F}}_* . \tag{2-34}$$

Here arises a problem: Although finite numbers of eigenvalues themselves change continuously under a fairly week perturbation of operators in $\hat{\mathscr{C}}_*$ (=with respect to the topology of $\hat{\mathscr{C}}_*$ defined by the generalized convergence (see [K])), we need the continuity of the above transformation (with a suitable topology of $\hat{\mathscr{C}}_*$), because the spectral flow has the meaning just for continuous loops in $\hat{\mathscr{F}}_*$ as a topological invariant. We can show this only on the subspace $U(\infty)$ with the usual topology regarding as a subspace of $\hat{\mathscr{C}}_*$ in the above way. This is a reason why we do not consider the whole space of boundary conditions defined in [BW3].

Remark 2.3. Our boundary conditions are rewritten as follows: Let $E_\pm$ be the subbundle of E consisting of $\pm\sqrt{-1}$ eigenspaces of G, and let P be the operator $P: \Gamma(E_+) \to \Gamma(E_-)$ defined by $\psi_k \mapsto \psi_{-k}$ $(k = 1, 2, 3, \dots)$. Here $\{\psi_k\}$ $(\{\psi_{-k}\})$ is the orthonormal basis of $L_2(Y, E_+)(L_2(Y, E_-))$ defined in (2-27). Then P is a unitary pseudo-differential operator, since P is the composition of operators:

(2-35) $$\Gamma(E_+) \xrightarrow{\text{inclusion}} \Gamma(E) \xrightarrow{2\pi_+ - Id} \Gamma(E) \xrightarrow{\text{projection}} \Gamma(E_-)\,.$$

Let $U = (u_{ij}) \in U(N)$ and define the operator T_U on $L_2(Y, E_+)$ by

(2-36) $$\begin{cases} T_U(\psi_k) = \psi_k & k > N\,, \\ T_U(\psi_k) = \sum_{j=1}^{N} u_{jk}\psi_j & 1 \le k \le N\,. \end{cases}$$

Then $P \circ T_U$ is also pseudo-differential and unitary. Let $\varphi = \varphi_+ + \varphi_-$, $\varphi \in L_2(Y, E)$, $\varphi_\pm \in L_2(Y, E_\pm)$, then we have, for $f \in W^1(X, \hat{E})$, $f \in D_{\alpha(-U^{-1})}$ if and only if

(2-37) $$P(\gamma_0(f)_+) = -\gamma_0(f)_- \quad \text{and} \quad P \circ T_U((\gamma_1 f)_+) = \gamma_1(f)_-\,.$$

Boundary conditions of this form were treated in [R].

§ 3. Proof of Theorem 2.3

In this section we prove the theorem 2.3. First we show (i) of the theorem 2.3. To prove the continuity of the map γ it is enough to show the map

$$U^G(N) \to \mathscr{B}_N \to \hat{\mathscr{F}}_*$$

$$U \mapsto U(L_+) \mapsto B_{U(L_+)} \circ (1 + B^2_{U(L_+)})^{-1/2} \equiv \tilde{B}_{U(L_+)}$$

is continuous for each N, since the projection $U^G(N) \to \mathscr{B}_N$ is an open mapping. Here L_+ is the subspace in $V(N)$ spanned by $\{\phi_k\}_{k=1}^N$.

Let B_+ denote the operator B_{L_+}, and denote by D_+ the domain D_{L_+} of B_{L_+}, then

Proposition 3.1. *For any $L \in \mathscr{B}_N$, the operator B_L is unitarily equivalent to B_+ modulo bounded selfadjoint operators.*

Proof. Let $U \in U^G(N)$ be a unitary transformation of $V(N)$ such that $U(L_+) = L$, and take a smooth curve $\{u_t\}_{0 \le t \le 1}$ in $U^G(N)$ satisfying

(3-1) $$u_0 = Id\,, \qquad u_1 = U\,,$$

and we regard that for each $t \in [0, 1]$ u_t acts on $V(N)^\perp$ as the identity transformation.

Now by making use of this curve we define a unitary operator $\tilde{U}$ on $L_2(X, \hat{E})$ by

(3-2) $$(\tilde{U}\phi)(t, y) = u_t(\phi(t, y))\,, \qquad \text{for } \phi \in L_2(X, \hat{E})\,,$$

and a bounded operator $d\tilde{U}$ on $L_2(X, \hat{E})$ by

(3-3) $$(d\tilde{U}\phi)(t, y) = du_t/dt(\phi(t, y))\,, \qquad \phi \in L_2(X, \hat{E})\,.$$

Then for each $k \geq 0$ the operators $\tilde{U}$ and $d\tilde{U}$ are also continuous as operators from the Sobolev space $W^k(X, \hat{E})$ to $W^k(X, \hat{E})$, and we have

$$\tilde{U}(D_+) = D_L\,, \qquad \text{and} \tag{3-4}$$

$$\begin{aligned}(\tilde{U}^{-1} \circ B_L \circ \tilde{U}\phi)(t, y) &= (B_+\phi)(t, y) + (G \circ \tilde{U}^{-1} \circ d\tilde{U})(\phi(t, y)) \\ &\quad + (G \circ (u_t^{-1} \circ A \circ u_t - A))\phi(t, y)\,, \qquad \phi \in D_+\,.\end{aligned} \tag{3-5}$$

The last term of the right hand side of (3-5) is of the form $Id \otimes$ {finite rank operator} on $L_2(X, \hat{E}) = L_2(0, 1) \hat{\otimes} L_2(Y, E)$. Hence we have shown the proposition.

Proof of (i) *of the theorem* 2.3. The injectivity of the map γ is shown by noting the facts: (1) the function $\lambda \mapsto \lambda(1 + \lambda^2)^{-1/2}$ is strictly increasing, and (2) the uniqueness of the spectral measure for a selfadjoint operator.

Now let $V \in U^G(N)$ be fixed, and $\mathscr{U}$ be a neighborhood of V in $U^G(N)$, which will be taken so small that each $U \in \mathscr{U}$ will satisfy the following inequalities: Let take a family of smooth curves $\{u(t, U)\}_{0 \leq t \leq 1}$ in $U^G(N)$ for each $U \in \mathscr{U}$ satisfying

$$u(0, U) = Id\,, \qquad u(1, U) = U\,, \tag{3-6}$$

$$\|u(t, V) - u(t, U)\| \leq \|V - U\|\,, \tag{3-7}$$

$$\left\|\frac{d}{dt}u(t, V) - \frac{d}{dt}u(t, U)\right\| \leq C_1\|V - U\|\,. \tag{3-8}$$

Here the norms of $N \times N$ matrices should be suitably taken, and the constant $C_1 > 0$ is independent of $U \in \mathscr{U}$. The family of such curves $\{u(t, U)\}_{U \in \mathscr{U}}$ can be taken by, for example, considering a tubular neighborhood of a fixed smooth curve joining Id and V. The neighborhood $\mathscr{U}$ may depend on this tubular neighborhood.

By making use of these curves we define unitary operators $\tilde{U}$ and bounded operators $d\tilde{U}$ on $L_2(X, \hat{E})$ as the same way as in the proof of the proposition 3.1.

Then we have

$$\begin{cases} \|\tilde{V} - \tilde{U}\| \leq \|V - U\|\,, \\ \|\tilde{V}^{-1} - \tilde{U}^{-1}\| \leq \|V - U\|\,, \\ \|d\tilde{V} - d\tilde{U}\| \leq C_1\|V - U\|\,. \end{cases} \tag{3-9}$$

Let denote

$$\tilde{U}^{-1} \circ B_{U(L_+)} \circ \tilde{U} = B_+ + C_U\,, \tag{3-10}$$

where

$$\begin{aligned}(C_U\phi)(t, y) &= \left(G \circ u(t, U)^{-1} \circ \frac{d}{dt}u(t, U)\right)\phi(t, y) \\ &\quad + G \circ (u(t, U)^{-1} \circ A \circ u(t, U) - A)\phi(t, y)\,.\end{aligned} \tag{3-11}$$

Now we estimate the norm of the difference of operators (as an operator on $L_2(X, \hat{E})$)

$$B_{V(L_+)}(1 + B^2_{V(L_+)})^{-1/2} - B_{U(L_+)}(1 + B^2_{U(L_+)})^{-1/2} \tag{3-12}$$

as follows: First we rewrite this difference (3-12)

$$= \tilde{V}(B_+ + C_V)[1 + (B_+ + C_V)^2]^{-1/2}\tilde{V}^{-1} - \tilde{U}(B_+ + C_U)[1 + (B_+ + C_V)^2]^{-1/2}\tilde{U}^{-1}$$

$$= (\tilde{V} - \tilde{U})(B_+ + C_V)[1 + (B_+ + C_V)^2]^{-1/2}\tilde{V}^{-1} \tag{3-13}$$

$$+ \tilde{U}(C_V - C_U)[1 + (B_+ + C_V)^2]^{-1/2}\tilde{V}^{-1} \tag{3-14}$$

$$+ \tilde{U}(B_+ + C_U)\{[1 + (B_+ + C_V)^2]^{-1/2} - [1 + (B_+ + C_U)^2]^{-1/2}\}\tilde{V}^{-1} \tag{3-15}$$

$$+ \tilde{U}(B_+ + C_U)[1 + (B_+ + C_U)^2]^{-1/2}(\tilde{V}^{-1} - \tilde{U}^{-1}). \tag{3-16}$$

Owing to (3-6) $\sim$ (3-9) we can get easily the following inequalities:

$$\sup_{U \in \mathscr{U}} \|B_+ + C_U\|_{D_+, L_2} \le C_2 \quad \text{with some } C_2 > 0, \tag{3-17}$$

$$\|C_V - C_U\|_{D_+, L_2} \le C_3 \|V - U\| \quad \text{with some } C_3 > 0. \tag{3-18}$$

Hence also

$$\|C_V - C_U\|_{L_2, L_2} \le C_3 \|V - U\|, \tag{3-19}$$

for any $U \in \mathscr{U}$. Here D_+ has the norm as a subspace of $W^1(X, \hat{E})$, and $\|\cdot\|_{D_+, L_2}$ denotes the norm of operators from D_+ to $L_2(X, \hat{E})$, and so on.

If we have the following inequality

$$\begin{aligned} &\|[1 + (B_+ + C_V)^2]^{-1/2} - [1 + (B_+ + C_U)^2]^{-1/2}\|_{L_2, D_+} \\ &\le C_4 \|V - U\|, \qquad U \in \mathscr{U}, \quad \text{with some } C_4 > 0, \end{aligned} \tag{3-20}$$

then also the following is clear

$$\sup_{U \in \mathscr{U}} \|[1 + (B_+ + C_U)^2]^{-1/2}\|_{L_2, D_+} < +\infty. \tag{3-21}$$

Hence by (3-13) $\sim$ (3-21) we see the map $\gamma: U(\infty) \to \hat{\mathscr{F}}_*$ is continuous, which proves (i) of the theorem 2.3, and we have the map $\gamma_*: \pi_1(U(N)) \to \pi_1(\hat{\mathscr{F}}_*)$ for each N.

Proof of (3-20). First note that D_+ is closed as a subspace of $W^1(X, \hat{E})$. By the selfadjointness of $B_+ + C_U$ (or by the existence of a parametrix for the operator B_+) and the inequality (3-18) there exists a constant $C_5 > 0$ such that for any $u \in D_+$ and $U \in \mathscr{U}$

$$\|u\|_1 \le C_5(\|(B_+ + C_U)(u)\| + \|u\|), \tag{3-22}$$

where $\|u\|_1$ is the first order Sobolev norm of $u \in W^1(X, \hat{E})$ and $\|u\|$ is the L_2-norm. Now we have

(3-23)
$$[1+(B_+ + C_V)^2]^{-1/2} - [1+(B_+ + C_U)^2]^{-1/2}$$
$$= \frac{1}{2\pi i}\int_\Gamma (1+\lambda)^{-1/2}\{[\lambda-(B_+ + C_V)^2]^{-1} - [\lambda-(B_+ + C_U)^2]^{-1}\}d\lambda$$
$$= \frac{1}{2\pi i}\int_\Gamma (1+\lambda)^{-1/2}[\lambda-(B_+ + C_V)^2]^{-1}[B_+(C_V + C_U)$$
$$+ (C_V - C_U)B_+ + C_V^2 - C_U^2][\lambda-(B_+ + C_U)^2]^{-1}d\lambda\,,$$

where the path Γ is taken suitably with $|\arg\lambda| = \theta_0$, $0<\theta_0<\frac{\pi}{2}$, for $|\lambda|\gg 1$.

Note that

(3-24)
$$\|[\lambda-(B_+ + C_U)^2]^{-1}\|_{L_2,D_+} = O(|\lambda|^{-1/2})$$

uniformly with respect to $U\in\mathscr{U}$ on the line $|\arg\lambda| = \theta_0$, $|\lambda|\gg 1$, because from (3-21)

$$\|[\lambda-(B_+ + C_U)^2]^{-1}u\|_1$$
$$\le C_5(\|(B_+ + C_U)([\lambda-(B_+ + C_U)^2]^{-1}u)\| + \|[\lambda-(B_+ + C_U)^2]u\|)$$
$$\le C_5(\|(B_+ + C_U - \sqrt{\lambda})([\lambda-(B_+ + C_U)^2]^{-1}u)\|$$
$$+ \sqrt{\lambda}\|[\lambda-(B_+ + C_U)^2]^{-1}u\|) + O(|\lambda|^{-1})\|u\|$$
$$\le C_6|\lambda|^{-1/2}\|u\|\,.$$

Hence by substituting (3-24) into (3-23), the left term of (3-20)

$$\le C_7\int_\Gamma |(1+\lambda)^{-1/2}|\frac{1}{\sqrt{|\lambda|}}\|(B_+ + C_V)^2 - (B_+ + C_U)^2\|\frac{1}{\sqrt{|\lambda|}}|d\lambda|$$
$$\le C_8\|V-U\|\,.$$

Proof of (ii) *of the theorem* 2.3. Next we determine the spectral flow of the loop $\{\tilde{B}_{l(\theta)(L_+)}\}_{-\pi/2\le\theta\le\pi/2}$ corresponding to the generator of $\pi_1(\mathscr{B}_1)$, where

(3-25)
$$l(\theta) = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\in U^G(1)\,,\qquad -\pi/2\le\theta\le\pi/2\,,$$

and here L_+ is the subspace of $V(1)$ spanned by ϕ_1. $l(\theta)$ acts on $L_2(Y, E)$ as follows:

(3-26)
$$\begin{cases} l(\theta)(\phi_n) = \phi_n\,, & |n|\ge 2\,,\\ l(\theta)(\phi_{-1}) = \cos\theta\,\phi_{-1} + \sin\theta\,\phi_1\,, & \\ l(\theta)(\phi_1) = -\sin\theta\,\phi_{-1} + \cos\theta\,\phi_1\,. & \end{cases}$$

To determine the spectral flow of $\{\tilde{B}_{l(\theta)(L_+)}\}$, we solve the eigenvalue problem

(3-27)
$$B_{l(\theta)(L_+)}f = \lambda f\,,\qquad \lambda\in\mathbf{R}\,,\quad f\in W^1\,,$$

with boundary conditions

$$\pi_+ \circ \gamma_0(f) = 0\,, \tag{3-28}$$

$$\pi_{l(\theta)(L_+)} \circ \gamma_1(f) = 0\,. \tag{3-29}$$

We want to determine eigenvalues which are across zero when the parameter θ moves from $-\pi/2$ to $\pi/2$. Let $f \in W^1(X, \hat{E})$ satisfy (3-27), then f is smooth on $X \backslash \partial X = (0, 1) \times Y$. Hence we see that for any smooth $\phi \in \Gamma(E)$ the function $\rho(t) = (f(t, \cdot), \phi(\cdot))_Y$ has the L_2-derivative on $[0, 1]$, because of the inequality:

$$\begin{aligned} \int \left|\frac{d}{dt}\rho(t)\right|^2 \mathrm{d}t &= \int \left|\int \frac{\partial}{\partial t} f(t, y)\phi(y)\mathrm{d}y\right|^2 \mathrm{d}t \\ &\leq \int_0^1 \int_Y \left|\frac{\partial}{\partial t} f(t, y)\right|^2 \mathrm{d}y\mathrm{d}t \cdot \int_0^1 \int_Y |\phi(y)|^2 \mathrm{d}y\mathrm{d}t < +\infty\,. \end{aligned} \tag{3-30}$$

Consequently, if we decompose the eigensection f of (3-27) with conditions (3-28) and (3-29) into

$$f = f_0 + f_\infty\,, \qquad f_0 \in L_2(0, 1)\,\hat{\otimes}\, V(1)\,, \qquad f_\infty \in L_2(0, 1)\,\hat{\otimes}\, V(1)^\perp\,, \tag{3-31}$$

then both f_0 and f_∞ are in $D_{l(\theta)(L_+)}$. Moreover we have

$$B_{l(\theta)(L_+)}(f_0) = \lambda f_0\,, \qquad B_{l(\theta)(L_+)}(f_\infty) = \lambda f_\infty\,, \tag{3-32}$$

since

$$B_{l(\theta)(L_+)}(f_0) \in L_2(0, 1)\,\hat{\otimes}\, V(1) \quad \text{and} \quad B_{l(\theta)(L_+)}(f_\infty) \in L_2(0, 1)\,\hat{\otimes}\, V(1)^\perp\,. \tag{3-33}$$

If $f_\infty \neq 0$, then λ should be constant when the parameter moves. So we can assume $f_\infty = 0$. Now the problem we want to solve is reduced to the following:

$$G\left(\frac{\partial}{\partial t} + A\right)(a(t)\phi_{-1} + b(t)\phi_1) = \lambda(a(t)\phi_{-1} + b(t)\phi_1)\,, \qquad \text{or}$$

$$\begin{cases} b'(t) + \lambda_1 b(t) = \lambda a(t)\,, \\ -a'(t) + \lambda_1 a(t) = \lambda b(t)\,, \end{cases} \tag{3-34}$$

$$b(0) = 0\,, \qquad -a(1)\sin\theta + b(1)\cos\theta = 0\,. \tag{3-35}$$

If $\lambda_1 = 0$, then we can easily solve them with solutions for $-\pi/2 \leq \theta \leq \pi/2$,

$$\begin{cases} a(t) = c\cos\lambda t\,, \qquad b(t) = c\sin\lambda t \quad (c \neq 0)\,, \\ \lambda = \theta + n\pi\,, \qquad n = 0,\ \pm 1,\ \pm 2,\ \ldots. \end{cases} \tag{3-36}$$

Hence we have

$$sf\{\widetilde{B}_{l(\theta)(L_+)}\}_{-\pi/2 \leq \theta \leq \pi/2} = 1\,. \tag{3-37}$$

Next assume $\lambda_1 \neq 0$, namely the operator A is invertible. Then we see that

$$B_{l(\theta)(L_+)} f = 0\,, \qquad f \in D_{l(\theta)(L_+)}$$

has the only non-zero solution $f(t, y) = ce^{\lambda_1 t}\phi_{-1}(y)$, when $\theta = 0$, and for other $0 < \theta \leq \frac{\pi}{2}$ and $-\frac{\pi}{2} \leq \theta < 0$, the operator $B_{l(\theta)(L_+)}$ is invertible.

General solutions of (3-34) with (3-35) for $|\lambda| < \lambda_1$ are given by

$$\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \exp t \begin{pmatrix} \lambda_1 & -\lambda \\ \lambda & -\lambda_1 \end{pmatrix} \begin{pmatrix} a(0) \\ 0 \end{pmatrix}$$

$$= \left[\cosh \mu t \, Id + \frac{\sinh \mu t}{\mu} \begin{pmatrix} \lambda_1 & -\lambda \\ \lambda & -\lambda_1 \end{pmatrix} \right] \begin{pmatrix} a(0) \\ 0 \end{pmatrix}.$$

Here λ and θ satisfy the relation

$$\text{(3-38)} \qquad \frac{\sin \theta}{\cos \theta} = \frac{\frac{\lambda}{\mu} \sinh \mu}{\cosh \mu + \frac{\lambda_1}{\mu} \sinh \mu},$$

where $\mu = \sqrt{\lambda_1^2 - \lambda^2}$. Hence by the continuity of the simple eigenvalue under the variation of the parameter θ, we can see that the solution curve $\lambda = \lambda(\theta)$ of (3-38) through $\lambda(0) = 0$ changes the sign at $\theta = 0$ from $-$ to $+$, when θ changes sign from $-$ to $+$. Consequently in this case we also have

$$\text{(3-39)} \qquad sf\{\tilde{B}_{l(\theta)(L_+)}\} = 1.$$

Hence they complete the proof of the theorem 2.3, (ii).

§ 4. Poincaré dual of $\gamma^*([sf])$

In this section we show the following theorem and its corollary (Theorem 4.2) which gives the second spectral flow formula in terms of the intersection number cited in the introduction.

Theorem 4.1. *Let $\gamma = \gamma_N$ be considered on $U(N)$ for a fixed N.*

(i) $\gamma^{-1}(\mathscr{Z}) = \{U \in U(N): \alpha(U) \cap L_+ \neq \{0\}\} = \{U \in U(N): \det(U - 1) = 0\}$.

(ii) $\gamma^{-1}(\mathscr{Z})$ *can be seen as a codim-one singular cycle in the sense* (a), (b) *and* (c) *of the introduction, and its homology class is the Poincaré dual of the cohomology class* $\det^*([d\theta]) \in H^1(U(N), \mathbf{Z})$.

Proof. Let $L \in \mathscr{B}_N$ and assume that for some $f \in D_L$, $f \neq 0$,

$$\text{(4-1)} \qquad B_L(f) = 0.$$

If we expand f as follows;

$$\text{(4-2)} \qquad f(t, y) = \sum_{|k| \leq N} (f(t, \cdot), \varphi_k)_Y \varphi_k(y) + \sum_{|k| > N} (f(t, \cdot), \varphi_k)_Y \varphi_k(y) \equiv f_N + f_\infty,$$

then both f_N and f_∞ are in D_L. Since $(f(0, \cdot), \varphi_k)_Y = 0$, $k > 0$, and $(f(1, \cdot), \varphi_k)_Y = 0$, $k < -N$, we have $f_\infty = 0$, and f is of the following form:

$$\text{(4-3)} \qquad f(t, y) = \sum_{-N \leq k \leq -1} a_k e^{-\lambda_k t} \varphi_k(y),$$

with $(a_{-1},\dots,a_{-N})\neq(0,0,\dots,0)$. By considering the boundary condition

$$L\perp f(1,\cdot)\qquad\text{(orthogonal)}$$

$$0\neq G(f(1,\cdot))=-\sum_{k=1}^{N}a_{-k}e^{\lambda_k}\varphi_k\in L\cap L_+\,.$$

So, if $L=\alpha(U)$, $U\in U(N)$, then $0\neq\sum_{k=1}^{N}c_k\varphi_k\in L\cap L_+$, if and only if

$$\sum_{k=1}^{N}c_k\psi_k\quad\text{is an eigenvector of } U \text{ with the eigenvalue } 1\,,\tag{4-4}$$

which showes (i).

From (i), the proof of (ii) is given by the diagonalization of unitary matrices. We now sketch the construction of the manifold Y and a map $\tau\colon Y\to U(N)$ satisfying the conditions (a), (b) and (c) in the introduction. Let $U_1=\{U\in U(N)\colon \det(U-I)=0\}$ and

$$\tilde\tau\colon U(N)\times T^{N-1}\to U(N)\,,$$

$$\left\langle U,\begin{bmatrix}\lambda_2 & & 0\\ & \ddots & \\ 0 & & \lambda_N\end{bmatrix}\right\rangle\mapsto U\begin{bmatrix}1 & & & 0\\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_N\end{bmatrix}U^{-1}\tag{4-5}$$

where $\lambda_i\in\mathbf{C}$, $|\lambda_i|=1$. Then $\operatorname{Im}\tilde\tau=U_1$, and $\tilde\tau$ descends to $\bar\tau\colon(U(N)/T^N)\times T^{N-1}\to U_1$. Next, let S_{N-1} be the $(N-1)$ symmetric group and identify it with a subgroup $\tilde S_{N-1}$ of $U(N)$:

$$\tilde S_{N-1}=\left\{\tilde\sigma\in U(N)\colon\tilde\sigma=\begin{pmatrix}1&0\\0&\delta_\sigma\end{pmatrix},\ \delta_\sigma\equiv(\delta_{\sigma(i),j})\in U(N-1),\ \sigma\in S_{N-1}\right\}.\tag{4-6}$$

Then S_{N-1} acts on $(U(N)/T^N)\times T^{N-1}$ in the following way:

$$S_{N-1}\times(U(N)/T^N)\times T^{N-1}\to(U(N)/T^N)\times T^{N-1}$$

$$\left\langle\sigma;[U],\begin{bmatrix}\lambda_2 & & 0\\ & \ddots & \\ 0 & & \lambda_N\end{bmatrix}\right\rangle\mapsto\left\langle[U\tilde\sigma^{-1}],\delta_\sigma\begin{bmatrix}\lambda_2 & & 0\\ & \ddots & \\ 0 & & \lambda_N\end{bmatrix}\delta_\sigma^{-1}\right\rangle.\tag{4-7}$$

This action is free and commutes with the map $\bar\tau$. Also, since the action of $\sigma\in S_{N-1}$ on $U(N)/T^N$, $[U]\mapsto[U\tilde\sigma^{-1}]$, preserves and reverses the orientation according to the signature of the permutation σ, the action (4-7) preserves the orientation. The resulting quotient manifold $((U(N)/T^N)\times T^{N-1})/S_{N-1}$ and the map

$$\tau\colon((U(N)/T^N)\times T^{N-1})/S_{N-1}\to U_1\tag{4-8}$$

are those required in the introduction. Let $Y_r = ((U(N)/T^N) \times \mathscr{D})/S_{N-1}$, where

$$\mathscr{D} = \left\{ \begin{bmatrix} \lambda_2 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{bmatrix} \in T^{N-1} : \lambda_i \neq 1 \text{ and } \lambda_i \neq \lambda_j \text{ for } i \neq j \right\}, \tag{4-9}$$

then Y_r is open dense in Y, τ is a one to one immersion on Y_r, and $\tau(Y_r)$ is open dense in U_1. Namely the conditions (a), (b) and (c) in the introduction are satisfied by this closed orientable manifold $Y = ((U(N)/T^N) \times T^{N-1})/S_{N-1}$, the map τ and the open dense submanifold Y_r.

Now let l be a loop

$$l(\theta) = \begin{bmatrix} e^{i\theta} & & & 0 \\ & e^{i\eta_2} & & \\ & & \ddots & \\ 0 & & & e^{i\eta_N} \end{bmatrix} \qquad (0 \leq \theta \leq 2\pi) \tag{4-10}$$

in $U(N)$, where $0 < |\eta_i| < \pi$, $\eta_i \neq \eta_j$ for $i \neq j$, and $\sum_{i=2}^N \eta_i = 0$. Then this loop is a generator of $\pi_1(U(N)) \cong H_1(U(N), \mathbf{Z}) \cong \mathbf{Z}$ and intersects transversally with U_1 just once at the point

$$\begin{bmatrix} 1 & & & 0 \\ & e^{i\eta_2} & & \\ & & \ddots & \\ 0 & & & e^{i\eta_N} \end{bmatrix} \in \tau(Y_r)\,. \tag{4-11}$$

Hence we can say that $\gamma^{-1}(\mathscr{X})$ represents a generator of $(N^2 - 1)$-dim homology group of $U(N)$. The orientation of Y is taken so as to be compatible with the equality $\int_l \det^*[d\theta] = 1$. These complete the proof of the theorem 4.1.

Theorem 4.2. *For a loop l in $U(N)$ the spectral flow of the loop $\{\gamma \circ l\}$ is equal to the intersection number of $\gamma^{-1}(\mathscr{X})$ with l in $U(N)$.*

Finally we explain a relation with the Maslov index. Let $V_{\mathbf{R}}(N)$ be the $\mathbf{R}$-vector space spanned by $\{\phi_k\}_{|k| \leq N}$, then $V_{\mathbf{R}}(N)$ has a symplectic structure ω: $\omega(x, y) = (x, Gy)_Y$ for $x, y \in V_{\mathbf{R}}(N)$. The inner product $(\cdot, \cdot)_Y$, ω and the almost complex structure G on $V_{\mathbf{R}}(N)$ are compatible. We regard $V_{\mathbf{R}}(N)$ as a complex vector space by means of this almost complex structure G and denote it by $\tilde{V}_{\mathbf{R}}(N)$: for $z = a + \sqrt{-1}b \in \mathbf{C}$ and $x \in \tilde{V}_{\mathbf{R}}(N)$ $z \cdot x = ax + bG(x)$. Then $\{\phi_k\}_{k=1}^N$ is an orthonormal basis of $\tilde{V}_{\mathbf{R}}(N)$ over $\mathbf{C}$.

Let $\Lambda(N)$ denote the space of all Lagrangian subspaces of the symplectic vector space $V_{\mathbf{R}}(N)$. Let $\lambda \in \Lambda(N)$, then the complexification $\lambda \otimes \mathbf{C} = j_N(\lambda)$ are in $\mathscr{B}_N$ and also we have a commutative diagram of embeddings:

(4-12)
$$\begin{tikzcd} \Lambda(N) \arrow[r, "j_N"] \arrow[d] & \mathscr{B}_N \arrow[d] \\ \Lambda(N+1) \arrow[r, "j_{N+1}"] & \mathscr{B}_{N+1} . \end{tikzcd}$$

By considering the fact that $j_1 \colon \Lambda(1) \cong \mathscr{B}_1$, we have

Proposition 4.3. *All embeddings of* (4-12) *induce isomorphisms of their fundamental groups.*

If we identify $\Lambda(N) \cong U(N)/O(N)$ through the action on $\Lambda(N)$ of unitary transformations of $\tilde{V}_{\mathbf{R}}(N)$ by $\lambda \mapsto U(\lambda)$ for $\lambda \in \Lambda(N)$ ($O(N)$ is the stationary subgroup of $\lambda_0 \in \Lambda(N)$, the Lagrangian subspace spanned by $\{\phi_k\}_{k=1}^N$), the map $j = j_N$ is written in the form:

Proposition 4.4.

(4-13) $$j_N(U(\lambda_0)) = \alpha(U^t U),$$

where $U = (u_{jk}) = (a_{jk} + \sqrt{-1}\, b_{jk}) \in U(N)$ *and* $U(\lambda_0) =$ *Lagrangian subspace spanned by* $\{\sum_{j=1}^N a_{jk}\phi_j + \sum_{j=1}^N b_{jk}\phi_{-j}\}_{k=1}^N$.

From this proposition we have $\mathscr{M} = j_N^* \circ (\alpha_N^{-1})^* \circ \det^* ([d\theta])$ coincides with the Maslov class of $\Lambda(N)$ (=Keller-Maslov-Arnold characteristic class). Consequently we have

Theorem 4.5. *Let* $l \colon [0, 1] \to \Lambda(N)$ *be a continuous loop, then the spectral flow of the family* $\{\tilde{B}_{l(t)\otimes \mathbf{C}}\}$ *is equal to the Maslov index of the loop* l:

(4-14) $$sf\{\tilde{B}_{l(t)\otimes \mathbf{C}}\} = \int_l \mathscr{M}.$$

Now we reinterpret this in terms of intersection numbers.

Proposition 4.6.

(4-15) $$\begin{aligned} \alpha_N^{-1} \circ j_N(\Lambda(N)) &= \{U \in U(N) \colon U = {}^tU\} \\ &= \{U \in U(N) \colon U = V^tV \text{ with some } V \in U(N)\} \\ &= \{U \in U(N) \colon U \text{ can be diagonalizable by an orthogonal matrix}\} \end{aligned}$$

From the theorem 4.1 we have

Proposition 4.7.

(4-16) $$j_N^{-1} \circ \alpha_N \circ \gamma_N^{-1}(\mathscr{Z}) = \{\lambda \in \Lambda(N) \colon \lambda \cap \lambda_0 \neq \{0\}\}.$$

(λ_0: *subspace spanned by* $\{\varphi_k\}_{k=1}^N$)

This set $(\gamma \circ \alpha^{-1} \circ j)^{-1}(\mathscr{Z})$ is called the Maslov cycle and is known as the Poincaré dual of the Maslov class ([Ar], [Du]). This also can be shown in the following way by making use of the theorem 4.1 and above propositions 4.3 ~ 4.6.

Since the conjugation action of $O(N)$ on $U(N)$ leaves the image of $\alpha_N^{-1} \circ j_N$ $(=\{U \in U(N): U = {}^tU\})$ invariant, we have,

Proposition 4.8. *Let* $W=((O(N)/\mathbf{Z}_2^N) \times T^{N-1})/S_{N-1}$ *be the submanifold of* $Y=((U(N)/T^N) \times T^{N-1})/S_{N-1}$, *the map* $\rho = \tau|_W$ *and the open dense submanifold* $W_r = Y_r \cap W$ *of* W, *then these are required ones in the introduction to say that the subset* (4–16) *can be seen as a singular cycle in* $\Lambda(N)$.

By considering the intersection of $\rho(W)$ with the loop (4-10) we see that the subset (4–16) represents the generator of $H_{((N^2+N)/2)-1}(\Lambda(N), \mathbf{Z}) \cong \mathbf{Z}$. Hence we have

Corollary 4.9. *Let* l: $[0, 1] \to \Lambda(N)$ *be a continuous loop, then*

$sf\{\tilde{B}_{l(t)\otimes\mathbf{C}}\}$ *equals to the intersection number of the subset* (4-16) *with the loop* l.

§5. Another spectral flow formula defined by ordinary differential equations

In this section we show a spectral flow formula for ordinary differential operators with periodic zeroth order terms. Before going to state the formula we remark a general problem concerning how non-trivial spectral flow may arise. Let M be a closed manifold, E a Hermitian vector bundle over M and D a first order elliptic symmetric differential operator acting on the sections of E. We assume that D has infinite numbers of both positive and negative eigenvalues. Let $S(E)$ denote the space of Hermitian bundle maps of E. We denote by A the operator on $L_2(M, E)$ defined by $A \in S(E)$ for simplicity. Now let $D_A = D + A$ and γ: $S(E) \to \hat{\mathscr{F}}_*(L_2(M, E))$ be

$$(5\text{-}1) \qquad A \mapsto D_A \circ (\sqrt{1 + (D_A)^2})^{-1} = \tilde{D}_A\,.$$

Let l: $I = [0, 1] \to S(E)$ be a smooth loop, then the spectral flow of the loop of operators $\{\tilde{D}_{l(t)}\}_{0\le t\le 1}$ are zero, since $S(E)$ is a vector space. However assume that for A and $B \in S(E)$ there exists a unitary bundle map g: $E \to E$ such that

$$(5\text{-}2) \qquad g^{-1} \circ D_A \circ g = D_B\,,$$

namely D_A and D_B are unitarily equivalent by the operator g. Then taking a path joinning g and Id in the group of all unitary transformations of $L_2(M, E)$ we can make the path $\{\tilde{D}_{(1-s)A+sB}\}_{0\le s\le 1}$ into a loop. In [BW1] the spectral flow formula for such loops are proved. In this case the unitary map g plays an important role in the formula.

There is another way of making a loop from a path $\{\tilde{D}_{(1-s)A+sB}\}_{0\le s\le 1}$ under the assumptions that both D_A and D_B are invertible. In this case, since the space $\hat{\mathscr{F}}_*\backslash\mathscr{Z}$ has trivial homotopy groups (see Lemma 5.1), we can join $\tilde{D}_A$ and $\tilde{D}_B$ by a path in $\hat{\mathscr{F}}_*\backslash\mathscr{Z}$ and denote by $\tilde{l}(A, B)$ a resulting loop. Of course such loops are all homotopic. It will not be so clear to determine the spectral flow for this loop in terms of A, B and D. In [Y] for 3-dim M this case is treated. He gives a spectral flow formula in terms of the Maslov index and apply it to calculations of Floer homology groups.

Lemma 5.1.

$$\pi_k(\hat{\mathscr{F}}_*\backslash\mathscr{Z}) = 0 \qquad k = 0,\ 1,\ 2,\ \ldots. \tag{5-3}$$

Proof. Let $\hat{\mathscr{F}}_\varepsilon = \{A \in \hat{\mathscr{F}}_*\colon \lambda \in \sigma(A) \text{ then } |\lambda| \geq \varepsilon\}$ then

$$\bigcup_{\varepsilon > 0} \hat{\mathscr{F}}_\varepsilon = \hat{\mathscr{F}}_*\backslash\mathscr{Z}\,, \qquad \hat{\mathscr{F}}_{\varepsilon_1} \subset \hat{\mathscr{F}}_{\varepsilon_2}\,, \qquad \varepsilon_2 \leq \varepsilon_1\,.$$

Hence it is enough to show $\pi_k(\hat{\mathscr{F}}_\varepsilon) = 0$ for any $\varepsilon > 0$.

Let $\varepsilon > 0$ be fixed and take a continuous function $f(s, t)\colon I \times \mathbf{R} \to \mathbf{R}$ such that

$$\begin{cases} f(0, t) = t, & \\ f(s, t) = t & \text{for } |t| \geq 1, \quad 0 \leq s \leq 1, \\ |f(s, t)| \leq 1 & \text{for } |t| \leq 1, \quad 0 \leq s \leq 1, \\ f(1, t) = \begin{cases} 1 & \varepsilon \leq t, \\ 0 & |t| \leq \varepsilon/2, \\ -1 & t \leq -\varepsilon. \end{cases} & \end{cases} \tag{5-4}$$

Let $\mathbf{J} = \{J \in \hat{\mathscr{F}}_*\colon J^2 = Id\}$, then we have for $\varepsilon > 0$, $\hat{\mathscr{F}}_\varepsilon$ is a deformation retract of $\mathbf{J}$, since let $G\colon I \times \hat{\mathscr{F}}_\varepsilon \to \hat{\mathscr{F}}_\varepsilon$ be such that $G(s, A) = \int f(s, t)\mathrm{d}E_t^A$, where $\{E_t^A\}$ is the spectral measure of A, then G is continuous (see [AS]) and

$$\begin{cases} G(s, J) = J & \text{for } J \in \mathbf{J},\ 0 \leq s \leq 1\,, \\ G(1, A) \in \mathbf{J} & \text{and } G(0, A) = A \qquad \text{for } A \in \hat{\mathscr{F}}_\varepsilon\,. \end{cases} \tag{5-5}$$

Now we prove that $\mathbf{J}$ is contractible. Since $\mathbf{U} = \mathbf{U}(H)$ the group of all unitary transformations in the Hilbert space H acts on $\mathbf{J}$ transitively, it is enough to show the existance of a local section $s\colon \mathbf{J} \to \mathbf{U}$. Now we fix a $J_0 \in \mathbf{J}$ and identify $\mathbf{J} \cong \mathbf{U}/\mathbf{U}_0$, where $\mathbf{U}_0 = \{\text{stationary subgroup of } J_0\}$. Let $\pi_\pm^J$ denote the projection operator in H onto the subspaces $H_\pm^J = \{x \in H\colon Jx = \pm x\}$ for $J \in \mathbf{J}$ and denote by $\mathscr{U}_0 = \{J \in \mathbf{J}\colon \|\pi_+^J - \pi_+^{J_0}\| < 1\}$. Then a section s on $\mathscr{U}_0$ is given by

$$s(J) = ((\pi_+^{J_0}\pi_+^J + \pi_-^{J_0}\pi_-^J) \circ (\pi_+^{J_0}\pi_+^J + \pi_-^{J_0}\pi_-^J)^*)^{-1/2} \circ (\pi_+^{J_0}\pi_+^J + \pi_-^{J_0}\pi_-^J)\,.$$

By calculating $s(J)(x_\pm)$ for each element $x_\pm \in H_\pm^J$ separately, we have $s(J)J_0 s(J)^* = J$. Hence $U_0 \to U \to \mathbf{J}$ is a principal fiber bundle with the contractible fiber and total space (Kuiper's theorem), so does the space $\mathbf{J}$. Consequently we showed the lemma.

Finally we give a spectral flow formula in the second situation for ordinary differential operators with periodic coefficients in terms of intersection numbers.

Let $\mathscr{H}_N$ be the space of $N \times N$ Hermitian matrices and $A \in C^\infty(S^1, \mathscr{H}_N)$. $A = A(t)$ is also regarded as a periodic $\mathscr{H}_N$-valued C^∞ function on $\mathbf{R}$ with the period

2π. We denote by $D_A = \frac{1}{\sqrt{-1}} \frac{d}{dt} + A(t)$ an ordinary differential operator acting on $C^\infty(S^1) \otimes \mathbf{C}^N$, and $\tilde{D}_A = D_A(\sqrt{1 + D_A^2})^{-1} \in \hat{\mathscr{F}}_*(L_2(S^1) \otimes \mathbf{C}^N)$. Let $\Phi_A(t)$ be the fundamental matrix solution of D_A with the initial condition $\Phi_A(0) = Id$:

$$
\text{(5-6)} \qquad \begin{cases} \frac{1}{i} \frac{d}{dt} \Phi_A(t) + A(t)\Phi(t) = 0\,, & t \in \mathbf{R}, \\ \Phi_A(0) = Id\,. \end{cases}
$$

The matrix $\Phi_A(2\pi)$ is called the monodromy matrix for D_A. We always consider that the fundamental solution matrix has the initial value Id, and denote by $m(A)$ the corresponding monodromy matrix. Here we list fundamental properties of $\Phi_A(t)$, $m(A)$ and related quantities (see [H] for these properties):

(5-7) $\Phi_A(t)$ is unitary, for any $t \in \mathbf{R}$.

(5-8) Let $S_A \in \mathscr{H}_N$ be such that $m(A) = \exp(2\pi i S_A)$, and put $U(t, S_A) = \Phi_A(t) \exp(-itS_A)$, then $U(t, S_A)$ is unitary and periodic with the period 2π.

(5-9) $U(t, S_A)^{-1} \circ D_A \circ U(t, S_A) = D_{-S_A}$.

(5-10) D_A is invertible in $L_2(S^1) \otimes \mathbf{C}^N$ if and only if $\det(m(A) - Id) \neq 0$, that is, $m(A) \notin U_1 = \{U \in U(N) \colon \det(U - 1) = 0\}$.

Now we assume that both D_A and D_B for $A,\ B \in C^\infty(S^1, \mathscr{H}_N)$ are invertible on $L_2(S^1) \otimes \mathbf{C}^N$, and for such a pair we denote by $\tilde{l}(A, B)$ a loop made from the path $\{\tilde{D}_{(1-s)A+sB}\}_{0 \le s \le 1}$ by joinning $\tilde{D}_B$ and $\tilde{D}_A$ with a continuous path in $\hat{\mathscr{F}}_* \backslash \mathscr{Z}$. Also we denote by $\tilde{m}(A, B)$ a loop in $U(N)$ made from the path $\{m((1-s)A + sB)\}_{0 \le s \le 1}$ by joinning $m(B)$ and $m(A)$ with a curve in $U(N) \backslash U_1$. Note here that $U(N) \backslash U_1$ is diffeomorphic to an affine space through the Cayley transformation.

Then

Theorem 5.2. *Assume that both D_A and D_B are invertible, then*

$$sf\{\tilde{l}(A, B)\} = \textit{the intersection number of } U_1 \textit{ with the loop } \tilde{m}(A, B)\,.$$

Proof. First we note that it is possible to make the loop $\{U(t, S_A)\}_{0 \le t \le 2\pi}$ in $U(N)$ to be homotopic to the constant map, by exchanging S_A, and from now on we assume that $\{U(t, S_A)\}$ is homotopic to the identity, and denote by $\{U^s(t, S_A)\}_{0 \le t \le 2\pi, 0 \le s \le 1}$ a C^∞-homotopy such that

$$
\text{(5-11)} \qquad \begin{cases} U^0(t, S_A) = Id & 0 \le t \le 2\pi\,, \\ U^1(t, S_A) = U(t, S_A) & 0 \le t \le 2\pi\,, \\ U^s(0, S_A) = Id & 0 \le s \le 1\,. \end{cases}
$$

Then we have that the two paths

$$\{m((1-s)A+sB)\}_{0\le s\le 1} \qquad \text{and} \qquad \{m((1-s)S_A+sS_B)\}_{0\le s\le 1}$$

are homotopic (note that $m(A)=m(S_A)$, $m(B)=m(S_B)$), since in $C^\infty(S^1, \mathscr{H}_N)$ two paths $\{(1-s)A+sS_A\}$ and

$$\left\{\frac{1}{\sqrt{-1}}U^s(t,S_A)^{-1}\frac{d}{dt}U^s(t,S_A)+U^s(t,S_A)^{-1}A(t)U^s(t,S_A)\right\}_{0\le s\le 1}$$

are homotopic and

$$m\left(\frac{1}{\sqrt{-1}}U^s(t,S_A)^{-1}\frac{d}{dt}U^s(t,S_A)+U^s(t,S_A)^{-1}A(t)U^s(t,S_A)\right)$$
$$\equiv m(A) \qquad \text{for } 0\le s\le 1\,.$$

So the loop $\{m((1-s)A+sS_A)\}_{0\le s\le 1}$ is contractible. Also the sames hold for the case of B. Hence the intersection numbers of U_1 with paths $\{m((1-s)A+sB)\}_{0\le s\le 1}$ and $\{m((1-s)S_A+sS_B)\}_{0\le s\le 1}$ are equal. Again, since $\tilde{D}_A$ and $\tilde{D}_{S_A}$ can be joined by a path consisting of a unitarily equivalent operators $\tilde{D}_{\tilde{A}_s}$, where

$$\tilde{A}_s=\frac{1}{\sqrt{-1}}U^s(t,S_A)^{-1}\frac{d}{dt}U^s(t,S_A)+U^s(t,S_A)^{-1}A(t)U^s(t,S_A)\,.$$

Hence the spectral flows of the loop $\tilde{l}(A,B)$ and $\tilde{l}(S_A,S_B)$ are equal. Consequently the proof is reduced to that of the constant coefficient cases.

Let V (and W) $\in U(N)$ be such that

$$\text{(5-12)} \qquad V^*S_AV=\begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_N\end{bmatrix} \qquad W^*S_BW=\begin{bmatrix}\mu_1 & & 0\\ & \ddots & \\ 0 & & \mu_N\end{bmatrix},$$

then $\tilde{D}_{S_A}$ and $\tilde{D}_{V^*S_AV}$ (also $\tilde{D}_{S_B}$ and $\tilde{D}_{W^*S_BW}$) are joined by a path of unitary equivalent operators $\in \hat{\mathscr{F}}_*\backslash\mathscr{Z}$. So

$$sf\{\tilde{l}(S_A,S_B)\}=sf\{\tilde{l}(V^*S_AV,W^*S_BW)\}\,.$$

Here note that the assumption that $D_A(D_B)$ is invertible is equivalent to say that $\lambda_i\notin\mathbf{Z}(\mu_i\notin\mathbf{Z})$. The spectral flow for the loop $\tilde{l}(V^*S_AV,W^*S_BW)$ are calculated explicitly by solving the equation:

$$\left(\frac{1}{\sqrt{-1}}\frac{d}{dt}+[(1-s)\lambda_k+s\mu_k]\right)\varphi=\lambda\varphi\,.$$

Namely we have for $\lambda=\lambda_s\equiv(1-s)\lambda_k+s\mu_k \pmod{\mathbf{Z}}$, a simple eigenfunction

$$\text{(5-13)} \qquad \varphi(t)=e^{i[\lambda-(1-s)\lambda_k-s\mu_k]t}\,.$$

Summing up, for $\lambda,\ \mu \in \mathbf{R}\backslash\mathbf{Z}$ put

$$N(\lambda, \mu) = \begin{cases} \#\{n \in \mathbf{Z}: n \in (\lambda, \mu)\} & \text{if } \lambda \le \mu\,, \\ -\#\{n \in \mathbf{Z}: n \in (\mu, \lambda)\} & \text{if } \mu < \lambda\,, \end{cases} \tag{5-14}$$

then

$$sf\{\tilde{l}(V^*S_A V, W^*S_B W)\} = \sum_{k=1}^{N} N(\lambda_k, \mu_k)\,,$$

which is equal to the intersection number of U_1 with the loop

$$\tilde{m}(V^*S_A V, W^*S_B W) = \tilde{m}(S_A, S_B) = \tilde{m}(A, B)\,.$$

They complete the proof of the theorem 5.2.

Remark 5.1. Let A and B be zeroth order selfadjoint pseudo-differential operators on $L_2(S^1) \otimes \mathbf{C}^N$, and assume that both $-id/dt + A$ and $-id/dt + B$ are invertible. Then still we have a loop $\tilde{l}(A, B)$ in $\hat{\mathscr{F}}_*(L_2(S^1) \otimes \mathbf{C}^N)$ from the path $\{-id/dt + (1 - s)A + sB\}_{0 \le s \le 1}$, however we do not know how the spectral flow of this loop is characterized by A and B.

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References

[Ar] V. I. Arnold, Characteristic class entering in quantization conditions, Funct. Anal. Appl., **1** (1967), 1-13.

[AS] M. F. Atiyah and I. M. Singer, Index theory for skew-adjoint Fredholm operators, Publ. Math. Inst. Hautes Sci., **37** (1969), 5-26.

[APS1] M. F. Atiyah, V. K. Patodi and I. M. Singer, Spectral asymmetry and Riemannian geometry I, Math. Proc. Camb. Phil. Soc., **77** (1975), 43-69.

[APS2] M. F. Atiyah, V. K. Patodi and I. M. Singer, Spectral asymmetry and Riemannian geometry III, Math. Proc. Camb. Phil. Soc., **79** (1976), 71-99.

[At] M. F. Atiyah, Eigenvalues of the Dirac operators, Springer Lecture Note **1111** (1987), 251-260.

[B] B. Booss, The geometry of global elliptic boundary value problems (Decomposition of manifolds and Hilbert spaces), to appear.

[BW1] B. Booss and K. Wojciechowski, Desuspension of splitting elliptic symbols I, Ann. Glob. Anal. and Geom., **3** (1985), 337-383.

[BW2] B. Booss and K. Wojciechowdki, Desuspension of splitting elliptic symbols II, Ann. Glob. Anal. and Geom., **4** (1986), 349-400.

[BW3] B. Booss and K. Wojciechowski, Pseudo-differential projections and the topology of certain spaces of elliptic boundary value problems, Comm. Math. Phys., **121** (1989), 1-9.

[Du] J. J. Duistermaat, On the Morse index in variational calculus, Advances in Math., **21** (1976), 173-195.

[DW1] R. Douglas and K. Wojciechowski, On the spectral flow of the families of the generalized Atiyah-Patodi-Singer problem (preprint).

[DW2] R. Douglas and K. Wojciechowdki, Adiabatic limits of the η-invariant, the odd-dimensional Atiyah-Patodi-Singer problem, to appear in Comm. Math. Physics.

[F] A. Floer, A relative Morse index for the symplectic action, Comm. Pure Appl. Math., **XLI** (1988), 393–407.

[H] J. Hale, Ordinary differential equations, Wiley-Interscience, 1969.

[K] T. Kato, Perturbation theory for linear operators, 2nd edition, Springer-Verlag, 1976.

[OF] N. Otsuki and K. Furutani, Spectral flow and Maslov index arising from Lagrangian intersections, Tokyo J. of Math., **14-1** (1991), 135–150.

[R] J. Ralston, Deficiency indices of symmetric operators with elliptic boundary conditions, Comm. Pure and Appl. Math., **23** (1970), 221–232.

[T] C. Taubes, Casson's invariant and gauge theory, J. Diff. Geom., **31** (1990), 547–599.

[VW] C. Vafa and E. Witten, Eigenvalue inequalities for Fermions in gauge theories, Comm. Math. Phys., **95** (1984), 257–276.

[W1] K. Wojciechowski, A note on the space of pseudodifferential projections with the same principal symbol, J. Operator Theory, **15** (1986), 207–216.

[W2] K. Wojciechowski, The homotopy groups of pseudodifferential Grassmannians (preprint).

[Y] T. Yoshida, Floer homology and splittings of manifolds, to appear in Ann. of Math.