J. Math. Kyoto Univ. (JMKYAZ)
33-1 (1993) 171–193

BMO extension theorem for relative uniform domains

By

Yasuhiro GOTOH

§ 1. Introduction

Let D be a domain in n-dimensional Euclidean space, and $BMO(D)$ the space of all functions of (n-dimensional) bounded mean oscillation on D. We say D has BMO extension property if each $BMO(D)$ function is the restriction to D of some $BMO(\mathbf{R}^n)$ function.

In 1980, P. Jones [J] showed that a domain D has BMO extension property if and only if D is a uniform domain (cf. [GO]). For various characterizations of uniform domain, see [G]. A uniform domain is 'uniform' as a subdomain of $\mathbf{R}^n$ or $\mathbf{R}^n \cup \{\infty\}$. Here we consider relative uniformness of domains, that is, a uniformness as a subdomain of other domain, and show that this relative uniformness and the corresponding relative BMO extension property coincides with to each other, which is a generalization of Jones' result (Th. 1.)

Our method is essentially almost the same as the original one of Jones, but since we must localize his method, and for the completeness, we shall give the proofs for all our lemmas below.

§ 2. Notation, preliminary lemmas and main result

Throughout this paper we treat only 2-dimensional case for the simplicity, since the same argument holds in the case of general dimension. Let D be a domain lying in $\mathbf{R}^2$. We say that a function $u \in L^1_{loc}(D)$ is in $BMO(D)$ if

$$\|u\|_{*,D} = \sup_Q \frac{1}{m(Q)} \int_Q |u(z) - u_Q|\, dm(z) < \infty ,$$

where dm is the two dimensonal Lebesgue measure, $u_Q = m(Q)^{-1} \int_Q u\,dm$ and the supremum is taken for every closed square Q in D whose sides are parallel to the coodinate axes. Throughout this paper 'square' means a closed square whose sides are parallel to the coordinate axes, 'dyadic square' means a square $[k2^n, (k+1)2^n] \times [l2^n, (l+1)2^n]$, k, l, $n \in \mathbf{Z}$, $l(Q)$ denotes the side length of a square Q, tQ, $t > 0$ denotes the square having the same center as Q and $tl(Q)$ as its side length, $d(\cdot, \cdot)$ denotes the Euclidean distance, A_1, A_2, ... denotes

Received July 8, 1991

positive universal constants, A denotes positive universal constant which may vary from place to place.

We say that a square Q lying in D is admissible if it satisfies $d(Q, \partial D) \geq 32l(Q)$ and $\mathscr{A}(D)$ denotes the set of all admissible squares in D. We say that a sequence of admissible square $Q_0, Q_1, \ldots, Q_n$ in D is an admissible chain if it satisfies the following conditions;

$$Q_i \cap Q_{i+1} \neq \emptyset, \qquad 0 \leq i \leq n-1,$$

$$\frac{1}{2} \leq \frac{l(Q_i)}{l(Q_{i+1})} \leq 2, \qquad 0 \leq i \leq n-1,$$

and call n its length. For two admissible squares Q, Q' in D we define

$$\delta_D(Q, Q') = \min\{n \geq 1 \mid Q = Q_0, Q_1, \ldots, Q_n = Q' \text{ is an admissible chain}\}$$

and the admissible chain which attains above minimum is called geodesic admissible chain joining Q and Q'. Remark we define δ_D so that $\delta_D(Q, Q') \geq 1$ for technical reason, δ_D is not a distance function but triangle inequality holds. Further for two squares Q, Q' lying in $\mathbf{R}^2$, we define

$$\psi(Q, Q') = \log\left(1 + \frac{l(Q) + l(Q') + d(Q, Q')}{l(Q)}\right)\left(1 + \frac{l(Q) + l(Q') + d(Q, Q')}{l(Q')}\right),$$

then

Lemma 1. *Let Q, Q' be admissible squares in D then*

$$\psi(Q, Q') \leq A_1 \delta_D(Q, Q').$$

Conversely if there exists a square $\tilde{Q}$ such that $Q \cup Q' \subset \tilde{Q} \subset 2\tilde{Q} \subset D$ then

$$\delta_D(Q, Q') \leq A_2 \psi(Q, Q').$$

Especially, for all squares Q, $Q' \subset \mathbf{R}^2$, we have

$$A_1^{-1}\psi(Q, Q') \leq \delta_{\mathbf{R}^2}(Q, Q') \leq A_2\psi(Q, Q')$$

Proof. First of all we prove the first inequality. Let Q, $Q' \in \mathscr{A}(D)$. We may assume $l(Q) \leq l(Q')$. Let $Q = Q_0, Q_1, \ldots, Q_n = Q'$ be arbitrary admissible chain joining Q, Q'.

(Case 1) $d(Q_0, Q_n) \geq l(Q_n)$. In this case it suffices to show that $\log d(Q_0, Q_n)/l(Q_0) \leq An$. And since $l(Q_i) \leq 2^i l(Q_0)$, $0 \leq i \leq n$, it holds that

$$d(Q_0, Q_n) \leq \sum_{i=1}^{n-1} \sqrt{2} l(Q_i) \leq \sqrt{2} \sum_{i=1}^{n-1} 2^i l(Q_0) \leq \sqrt{2} 2^n l(Q_0).$$

(Case 2) $d(Q_0, Q_n) < l(Q_n)$. Then it suffices to show that $\log l(Q_n)/l(Q_0) \leq An$, and since $l(Q_n) \leq 2^n l(Q_0)$ this inequality holds.

Next we will prove the second inequality. We may assume $l(\tilde{Q}) \leq l(Q) + l(Q') + d(Q, Q')$ by replacing $\tilde{Q}$ with some smaller square if necessary. Let $2^m \leq$

$l(\tilde{Q})/l(Q) \leq 2^{m+1}$, $2^{m'} \leq l(\tilde{Q})/l(Q') \leq 2^{m'+1}$, then there exist two chains $Q = Q_0 \subset Q_1 \subset \cdots \subset Q_{m+1} = \tilde{Q}$, $Q' = Q'_0 \subset Q'_1 \subset \cdots \subset Q'_{m'+1} = \tilde{Q}$ which are admissible as chains lying in $\mathbf{R}^2$. Hence $Q = Q_0, Q_1, \dots, Q_m, Q_{m+1}, Q'_{m'}, Q'_{m'-1}, \dots, Q'_1, Q'_0 = Q'$ is admissible as a chain lying in $\mathbf{R}^2$ of length $m + m' + 2$. If we decompose each square into 4096 congruent subsquares by dividing its sides into 64 pieces, then each such subsquare is admissible hence we can easily construct an admissible chain joining Q, Q' whose length is at most $6 + 64(m + m' + 1) + 6$, therefore

$$\delta_D(Q, Q') \leq 64m + 64m' + 76 \leq A\left\{1 + \log\frac{l(\tilde{Q})}{l(Q)} + \log\frac{l(\tilde{Q})}{l(Q')}\right\} \leq A\psi(Q, Q').$$

Q.E.D.

Lemma 2 (cf. [S]). *There exists a decomposition of D into a family of dyadic squares $\mathscr{D}(D) = \{Q_i\}$, $Q_i^\circ \cap Q_j^\circ = \emptyset$, $(i \neq j)$, $\bigcup_i Q_i = D$ for each $\alpha \geq 2$ such that*

$$\alpha \leq \frac{d(Q_i, \partial D)}{l(Q_i)} \leq 2\alpha + 2,$$

$$\frac{1}{2} \leq \frac{l(Q_i)}{l(Q_j)} \leq 2, \qquad \textit{if } Q_i \cap Q_j \neq \emptyset.$$

Proof. First of all we decompose $\mathbf{R}^2$ into a family of dyadic square $[k, k+1] \times [l, l+1]$, $k, l \in \mathbf{Z}$. If there exists a square Q in this family such that $d(Q, \partial D) < \alpha l(Q)$, then we decompose Q into 4 congruent subsquares. Let Q' be one of such subsquares. Then

$$\frac{d(Q', \partial D)}{l(Q')} \leq \frac{2\left(d(Q, \partial D) + \frac{\sqrt{2}}{2}l(Q)\right)}{l(Q)} < 2\alpha + \sqrt{2} < 2\alpha + 2.$$

Hence by repeating above process, we can decompose Q into a family of dyadic squares Q'' which satisfies $\alpha \leq d(Q'', \partial D)/l(Q'') \leq 2\alpha + 2$.

Next, suppose there exist a dyadic square Q such that $2\alpha + 2 < d(Q, \partial D)/l(Q)$ then let Q' be the dyadic square containing Q such that $l(Q') = 2l(Q)$. We join all squares in Q' into one square Q'. Then

$$d(Q', \partial D) \geq d(Q, \partial D) - \sqrt{2}l(Q) \geq (2\alpha + 2 - \sqrt{2})l(Q) > \alpha l(Q').$$

Hence by repeating above process we obtain a family of square Q such that

$$\alpha \leq \frac{d(Q, \partial D)}{l(Q)} \leq 2\alpha + 2,$$

Finally, for such two squares Q, Q' such that $Q \cap Q' \neq \emptyset$ we have

$$l(Q') \leq \alpha^{-1}d(Q', \partial D) \leq \alpha^{-1}(d(Q, \partial D) + \sqrt{2}l(Q)) \leq \left(2 + \frac{2+\sqrt{2}}{\alpha}\right)l(Q) < 4l(Q).$$

and so $l(Q') \leq 2l(Q)$. Q.E.D.

In the following, $\mathscr{D}(D)$ denotes the family obtained by above method with $\alpha = 32$, which we call Whitney decomposition of D. Note that if $D \subset D'$ then for each $Q \in \mathscr{D}(D)$ there exists a square $Q' \in \mathscr{D}(D')$ such that $Q \subset Q'$. And note that if $Q, Q' \in \mathscr{D}(D)$ satisfy $Q' \cap 21Q \neq \emptyset$ then $l(Q') \geq l(Q)/2$. In fact, if $Q'' \in \mathscr{D}(D)$ satisfys $l(Q'') \leq l(Q)/4$ then

$$d(Q, Q'') \geq d(Q, \partial D) - d(Q'', \partial D) - \sqrt{2}l(Q'')$$
$$\geq 32l(Q) - 66 \cdot \frac{1}{4}l(Q) - \frac{\sqrt{2}}{4}l(Q) > 15l(Q).$$

hence $Q'' \cap 21Q = \emptyset$.

We say that a sequence $Q_0, Q_1, \ldots, Q_n \in \mathscr{D}(D)$ is a Whitney chain if $Q_i \cap Q_{i+1} \neq \emptyset$. Since $\mathscr{D}(D) \subset \mathscr{A}(D)$, every Whitney chain is admissible. We set

$$W_D(Q, Q') = \min\{n \geq 1 \mid Q = Q_0, Q_1, \ldots, Q_n = Q' \text{ is a Whitney chain}\}$$

and the Whitney chain which attains above minimum is called geodesic Whitney chain joining Q and Q'. It holds that $\delta_D(Q, Q') \leq W_D(Q, Q')$, $Q, Q' \in \mathscr{D}(D)$ by definition. Conversely

Lemma 3. $W_D(Q, Q') \leq A_3 \delta_D(Q, Q')$, $Q, Q' \in \mathscr{D}(D)$

Proof. Let $Q = Q_0, Q_1, \ldots, Q_n = Q'$ be a geodegic admissible chain in D. Let $\hat{Q} \in \mathscr{D}(D)$ be a square such that $\hat{Q} \cap Q_j \neq \emptyset$. Then we have $l(\hat{Q}) \geq l(Q_i)/4$, hence the number of square $\hat{Q} \in \mathscr{D}(D)$ satisfying $\hat{Q} \cap Q_j \neq \emptyset$ is at most $(4+2)^2 = 36$. It follows that $W_D(Q, Q') \leq 36(n-1) + 1 \leq 36n = 36\delta_D(Q, Q')$. Q.E.D.

Let D_2 be a domain lying in $\mathbf{R}^2$. We say that a domain $D_1 \subset D_2$ is relative uniform with respect to D_2 if it satisfies

$$\delta_{D_1}(Q, Q') \leq M\delta_{D_2}(Q, Q'), \qquad Q, Q' \in \mathscr{A}(D_1)$$

for some constant $M \geq 1$. And $\mathscr{U}(D_2, M)$ denotes the set of all subdomains of D_2 satisfying this condition. Note that if $D_1 \in \mathscr{U}(D_2, M_1)$ and $D_2 \in \mathscr{U}(D_3, M_2)$ then $D_1 \in \mathscr{U}(D_3, M_2 M_1)$. And note that if $D_1 \in \mathscr{U}(D_2, M)$ and $f: D_2 \to D_2'$ is a quasiconformal mapping then $f(D_1) \in \mathscr{U}(D_2', KM)$ where K is a constant depending only on the maximal dilatation of f. (cf. lemma 23.)

We say also a domain D lying in $\mathbf{R}^2$ is uniform (cf. [G]) if it satisfies

$$W_D(Q, Q') \leq M\psi(Q, Q'), \qquad Q, Q' \in \mathscr{D}(D)$$

for some constant $M > 0$. Lemma 1 and lemma 4 below shows that D is uniform if and only if it is uniform with respect to $\mathbf{R}^2$.

In the following, $B_1(M)$, $B_2(M)$, ... denote constants depending only on M, and $B(M)$ denotes a constant depending only on M which may vary from place to place. The relative uniformness follows from the following property which is weaker in appearance than its definition.

Lemma 4. *Let D_1 be a subdomain of D_2 such that*

$$W_{D_1}(Q, Q') \le M\delta_{D_2}(Q, Q'), \qquad Q,\ Q' \in \mathscr{D}(D_1)$$

then

$$\delta_{D_1}(Q, Q') \le A_4 M\delta_{D_2}(Q, Q') \qquad Q,\ Q' \in \mathscr{A}(D_1).$$

Proof. Let $\tilde{Q}$, $\tilde{Q}'$ be squares in $\mathscr{D}(D_1)$ which minimize $W_{D_1}(\tilde{Q}, \tilde{Q}')$ under the condition $\tilde{Q} \cap Q \ne \emptyset$, $\tilde{Q}' \cap Q' \ne \emptyset$.
(Case 1) $W_{D_1}(\tilde{Q}, \tilde{Q}') = 1$. Then there exists a square $\hat{Q}$ such that $Q \cup Q' \subset \hat{Q} \subset 2\hat{Q} \subset D_1$ therefore by lemma 1,

$$\delta_{D_1}(Q, Q') \le A_2\psi(Q, Q') \le A_1A_2\delta_{D_2}(Q, Q')$$

(Case 2) $W_{D_1}(\tilde{Q}, \tilde{Q}') \ge 2$. Then first of all we will show $\delta_{D_1}(Q, Q') \ge A \log l(\tilde{Q})/l(Q)$. If $l(Q) \ge l(\tilde{Q})/4$ then this inequality is trivial, hence we may assume $l(Q) < l(\tilde{Q})/4$. Since $Q' \cap 2\tilde{Q} = \emptyset$, if follows that

$$d(Q, Q') \ge \frac{1}{2}l(\tilde{Q}) - l(Q) \ge \frac{1}{4}l(\tilde{Q}).$$

On the other hand, let $Q = Q_0, Q_1, \dots, Q_n = Q'$ be a geodesic admissible chain in D_1. Then

$$d(Q, Q') \le \sum_{k=1}^{n-1} \sqrt{2}l(Q_k) \le \sum_{k=1}^{n-1} \sqrt{2}2^k l(Q) \le \sqrt{2}2^n l(Q)$$

Hence $l(\tilde{Q}) \le A2^n l(Q)$ and so $\delta_{D_1}(Q, Q') = n \ge A \log l(\tilde{Q})/l(Q)$. Similarly we have

$$\delta_{D_1}(Q, Q') \ge A \log\frac{l(\tilde{Q}')}{l(Q')}, \qquad \delta_{D_2}(Q, Q') \ge A \log\frac{l(\tilde{Q})}{l(Q)}, \qquad \delta_{D_2}(Q, Q') \ge A \log\frac{l(\tilde{Q}')}{l(Q')}.$$

And so

$$\begin{aligned} \delta_{D_1}(Q, Q') &\le \delta_{D_1}(Q, \tilde{Q}) + \delta_{D_1}(\tilde{Q}, \tilde{Q}') + \delta_{D_1}(\tilde{Q}', Q') \\ &\le A + A\log\frac{l(\tilde{Q})}{l(Q)} + A\log\frac{l(\tilde{Q}')}{l(Q')} + M\delta_{D_2}(\tilde{Q}, \tilde{Q}') \\ &\le A + A\delta_{D_2}(Q, Q') + M(\delta_{D_2}(\tilde{Q}, Q) + \delta_{D_2}(Q, Q') + \delta_{D_2}(Q', \tilde{Q}')) \\ &\le AM\delta_{D_2}(Q, Q'). \end{aligned}$$

Q.E.D.

Let Q, Q' be admissible squares in D. We set

$$\hat{\delta}_D(Q, Q') = \begin{cases} W_D(\tilde{Q}, \tilde{Q}') + \log\left(2 + \dfrac{l(\tilde{Q})}{l(Q)}\right)\left(2 + \dfrac{l(\tilde{Q}')}{l(Q')}\right), & \delta_D(\tilde{Q}, \tilde{Q}') \ge 2, \\ \psi(Q, Q'), & \delta_D(\tilde{Q}, \tilde{Q}') = 1, \end{cases}$$

where $\tilde{Q}$, $\tilde{Q}'$ are squares in $\mathscr{D}(D)$ which minimizes $W_D(\tilde{Q}, \tilde{Q}')$ under the condition $\tilde{Q} \cap Q \neq \emptyset$, $\tilde{Q}' \cap Q' \neq \emptyset$. (Minimizing condition is not essential.) Then above argument shows that

$$\frac{1}{A}\hat{\delta}_D(Q, Q') \le \delta_D(Q, Q') \le A\hat{\delta}_D(Q, Q').$$

Now we can state our main theorem.

Theorem 1. *Let D_2 be a domain lying in $\mathbf{R}^2$. Then the following three conditions are equivalent to each other for subdomain D_1 of D_2;*

(1) *Every $BMO(D_1)$ function is the restrinction of some $BMO(D_2)$ function.*

(2) *There exists a constant $M > 0$ such that*

$$W_{D_1}(Q, Q') \le M\delta_{D_2}(Q, Q'), \qquad Q, Q' \in \mathscr{D}(D_1).$$

(3) *D_1 is relative uniform with respect to D_2, that is, there exist a constant $M \ge 1$ such that*

$$\delta_{D_1}(Q, Q') \le M\delta_{D_2}(Q, Q'), \qquad Q, Q' \in \mathscr{A}(D_1).$$

The relative *BMO* extension property is not local property. There exists two domains $D_1 \subset D_2$ such that

(1) for every square $Q \subset D_2$ and every $u \in BMO(D_1)$, there exists a extension $\hat{u}$ of u to Q such that $\|\hat{u}\|_{*,Q} \le A\|u\|_{*,D_1}$

(2) but there exists a $BMO(D_1)$ function u which can not be extended to a $BMO(D_2)$ function.

Example 1. Let

$$S_n = \left\{0 < x < \frac{1}{n}, 0 < y < 1\right\} \cup \left\{1 - \frac{1}{n} < x < 1, 0 < y < 1\right\}$$
$$\cup \left\{0 < x < 1, 0 < y < \frac{1}{n}\right\},$$

$$T_n = \left\{\frac{1}{n} \le x < \frac{1}{4}, \frac{7}{8} < y < 1\right\}, \qquad U_n = \left\{\frac{3}{4} < x \le 1 - \frac{1}{n}, \frac{7}{8} < y < 1\right\},$$

$$V = \left\{\frac{1}{4} \le x \le \frac{3}{4}, \frac{7}{8} < y < 1\right\}, \qquad D_1^n = S_n \cup T_n \cup U_n, \qquad D_2^n = D_1^n \cup V,$$

then for every square $Q \subset D_2^n$ and every $u \in BMO(D_1^n)$, there exists a extension $\hat{u}$ of u to Q such that $\|\hat{u}\|_{*,Q} \le A\|u\|_{*,D_1^n}$. We set

$$u_n(x, y) = \begin{cases} nx, & (x, y) \in S_n, \\ 0, & (x, y) \in T_n, \\ n, & (x, y) \in U_n, \end{cases}$$

then $u_n \in BMO(D_1^n)$ and $\|u_n\|_{*,D_1^n} \leq 1$. On the other hand for every extension $\hat{u}_n$ of u_n to D_2^n it holds that $\|\hat{u}_n\|_{*,D_2^n} \to \infty$. Hence by linking these domains D_2^n in a suitable way, we obtain our requirement.

Note that $W_{D_1}(Q, Q')$, Q, $Q' \in \mathscr{D}(D_1)$ corresponds to the quasi-hyperbolic metric

$$k_{D_1}(z, z') = \inf \int_\gamma \frac{|d\zeta|}{d(\zeta, \partial D_1)}$$

where the infimum is taken for all rectifiable curves γ joining z to z', and $\delta_{D_2}(Q, Q')$, Q, $Q' \in \mathscr{D}(D_1)$ corresponds to the following metric

$$j_{D_1,D_2}(z, z') = \begin{cases} k_{D_2}(z, z') + \log \dfrac{d(z, \partial D_2)\, d(z', \partial D_2)}{d(z, \partial D_1)\, d(z', \partial D_1)}, & |z - z'| \geq d(z, \partial D_2)/2, \\ \log\left(1 + \dfrac{|z - z'|}{d(z, \partial D_1)}\right)\left(1 + \dfrac{|z - z'|}{d(z', \partial D_1)}\right), & |z - z'| < d(z, \partial D_2)/2, \end{cases}$$

Hence the condition (2) of theorem 1 implies;

$$k_{D_1}(z, z') \leq K j_{D_1,D_2}(z, z') + L, \qquad z, z' \in D_1$$

§ 3. Proof of Theorem 1

Lemma 5. *Let Q, $Q' \in \mathscr{A}(D)$ and u a $BMO(D)$ function then*

$$|u_Q - u_{Q'}| \leq A_5 \|u\|_{*,D} \delta_D(Q, Q').$$

Proof. Let $Q = Q_0, Q_1, \ldots, Q_n = Q'$ be a geodesic admissible chain in D. First of all we estimate $u_{Q_{i+1}} - u_{Q_i}$. We may assume $l(Q_{i+1}) \leq l(Q_i)$, then $Q_{i+1} \cup Q_i \subset 3Q_i \subset D$, hence

$$|u_{Q_i} - u_{3Q_i}| \leq \frac{1}{m(Q_i)} \int_{Q_i} |u - u_{3Q_i}| dm$$
$$\leq \frac{9}{m(3Q_i)} \int_{3Q_i} |u - u_{3Q_i}| dm \leq 9 \|u\|_{*,D}.$$

Similarly we have $|u_{3Q_i} - u_{Q_{i+1}}| \leq 36 \|u\|_{*,D}$, hence

$$|u_{Q'} - u_Q| \leq \sum_{i=0}^{n-1} |u_{Q_i} - u_{Q_{i+1}}| \leq \sum_{i=0}^{n-1} 45 \|u\|_{*,D} = 45 \|u\|_{*,D} \delta_D(Q, Q'). \quad \text{Q.E.D.}$$

Lemma 6 (cf. [RR], [J]). *Let $u \in L^1_{loc}(D)$ be a function which belongs to $BMO(Q)$ for every square Q in D such that $d(Q, \partial D) \geq \lambda l(Q)$ $(\lambda \geq 1)$ and $\|u\|_{*,Q} \leq K$ then u is in $BMO(D)$ and $\|u\|_{*,D} \leq A_6 K \lambda$.*

Proof. We set $[3\lambda + \sqrt{2}] + 1 = s$. Let Q be arbitrary square in D. We may assume its center is the origin. We set $l(Q) = l$. Let Q_m, $m = 1, 2, \ldots$ be squares having the origin as its center and $l(Q_m) = (1 - 2^{-m})l$. We deompose

Q_m, $m \geq 2$ into congruent subsquares with side length $2^{-m-1}l$ and $\mathscr{D}_m$ denotes the set of every such subsquare which is not contained in Q_{m-1}. Concerning Q_1, we decompose it into 4 congruent subsquares which we denotes $\mathscr{D}_1$, then $\#\mathscr{D}_m = 2^{m+3} - 12$. Further we decompose each square of $\mathscr{D}_m$ into s^2 congruent subsquares by decompose its sides into s pieces, which is denoted by $\mathscr{D}'_m = \{Q_{m,i}\}$, $1 \leq i \leq s^2(2^{m+3} - 12)$. Let $Q_{m,i} \cap Q_{m',i'} \neq \emptyset$ then $1/2 \leq l(Q_{m,i})/l(Q_{m',i'}) \leq 2$. We may assume $l(Q_{m,i}) \geq l(Q_{m',i'})$. Then $Q_{m,i} \cup Q_{m',i'} \subset 3Q_{m,i}$ and

$$\frac{d(3Q_{m,i}, \partial D)}{l(3Q_{m,i})} \geq \frac{d(Q_{m,i}, \partial D) - \sqrt{2}l(Q_{m,i})}{3l(Q_{m,i})} \geq \frac{s - \sqrt{2}}{3} \geq \lambda .$$

hence $3Q_{m,i}$ satisfy the condition of lemma. It follows $|u_{Q_{m,i}} - u_{Q_{m',i'}}| \leq 45K$ by the same argument as lemma 5. Let Q_0 be one of the square in $\{Q_{1,i}\}$ containing the origin. Then we can join every square in $Q_{m,i}$ to Q_0 by a chain which consists of at most ms squares, hence

$$\begin{aligned}\int_Q |u - u_{Q_0}|dm &\leq \sum_{m,i} \int_{Q_{m,i}} (|u - u_{Q_{m,i}}| + |u_{Q_{m,i}} - u_{Q_0}|)dm \\ &\leq \sum_{m,i} (m(Q_{m,i})K + m(Q_{m,i})45Kms) \\ &\leq \sum_{m,i} m(Q_{m,i})46Kms \\ &= \sum_{m=1}^{\infty} l^2 s^{-2} 2^{-2m-2} \cdot 46Kms \cdot s^2(2^{m+3} - 12) \\ &\leq 92 l^2 sK \sum_{m=1}^{\infty} m2^{-m} \leq AKl^2\lambda\end{aligned}$$

and so $m(Q)^{-1} \int_Q |u - u_Q|dm \leq 2m(Q)^{-1} \int_Q |u - u_{Q_0}|dm \leq AK\lambda.$ Q.E.D.

Lemma 7. *Let Q_0 be a square in $\mathscr{D}(D)$. We set a function $F_{Q_0} \in L^1_{loc}(D)$ as follows;*

$$F_{Q_0}(x) = W_D(Q, Q_0) , \qquad x \in Q \in \mathscr{D}(D) .$$

Then F_{Q_0} is a $BMO(D)$ function and $\|F_{Q_0}\|_{*,D} \leq A_7$.

Proof. Let $Q \in \mathscr{A}(D)$, then the proof of lemma 3 shows that Q intersect at most 36 squares in $\mathscr{D}(D)$, hence $\|F_{Q_0}\|_{*,Q} \leq 36$. Therefore we have $\|F_{Q_0}\|_{*,Q} \leq A_6 \cdot 36 \cdot 32$ by lemma 6. Q.E.D.

Let D_1 be a subdomain of D_2. Assume that every $BMO(D_1)$ function is the restriction to D_1 of some $BMO(D_2)$ function. Then by open mapping theorem there exists a constant $N \geq 1$ such that for every $u \in BMO(D_1)$, we can find an extension $\hat{u} \in BMO(D_2)$ of u satisfying

$$\|\hat{u}\|_{*,D_2} \leq N \|u\|_{*,D_1} .$$

$\mathscr{E}(D_2, N)$ denotes the set of all subdomains of D_2 which satisfy above condition.

Lemma 8. $\mathscr{E}(D_2, N) \subset \mathscr{U}(D_2, A_8 N)$.

Proof. Let $D_1 \in \mathscr{E}(D_2, N)$ and fix a square $Q_0 \in \mathscr{D}(D_1)$. Let F_{Q_0} be the function in lemma 7, then $\|F_{Q_0}\|_{*,D_1} \le A_7$ by lemma 7. Hence there exists an extension $\hat{F}_{Q_0}$ of F_{Q_0} such that $\|\hat{F}_{Q_0}\|_{*,D_2} \le A_7 N$ by hypothesis. Let $Q_1 \in \mathscr{D}(D_1)$ then lemma 5 shows

$$W_{D_1}(Q_1, Q_0) - 1 \le |(\hat{F}_{Q_0})_{Q_1} - (\hat{F}_{Q_0})_{Q_0}|$$
$$\le A_5 \|\hat{F}_{Q_0}\|_{*,D_2} \delta_{D_2}(Q_1, Q_0) \le A_5 A_7 N \delta_{D_2}(Q_1, Q_0),$$

hence by lemma 4

$$\delta_{D_1}(Q, Q') \le 2A_4 A_5 A_7 N \delta_{D_2}(Q, Q'), \qquad Q,\ Q' \in \mathscr{A}(D_1). \qquad \text{Q.E.D.}$$

Lemma 9. *Let $Q_0, Q_1, \ldots, Q_n$ be a geodesic Whitney chain in $D_1 \in \mathscr{U}(D_2, M)$ such that $l(Q_0) = l(Q_n)$ and $d(Q_0, Q_n) \ge B_1(M) l(Q_0)$. Further assume there exists a square $\tilde{Q}$ such that $Q_0 \cup Q_n \subset \tilde{Q} \subset 2\tilde{Q} \subset D_2$. Then there exists an integer i such that $l(Q_i) = 2l(Q_0)$.*

Proof. Let $B_1(M) > 0$ be a constant such that $t > 2\sqrt{2}MA_3A_2 \log(3 + t)$ holds for every $t \ge B_1(M)$. By lemma 1 and 3,

$$n = W_{D_1}(Q_0, Q_n) \le A_3 \delta_{D_1}(Q_0, Q_n) \le MA_3 \delta_{D_2}(Q_0, Q_n) \le MA_3 A_2 \psi(Q_0, Q_n)$$
$$= 2MA_3A_2 \log\left(3 + \frac{d(Q_0, Q_n)}{l(Q_0)}\right)$$

On the other hand if $l(Q_i) \le l(Q_0)$ for every Q_i then

$$d(Q_0, Q_n) \le \sum_{i=1}^{n-1} \sqrt{2} l(Q_i) \le n\sqrt{2} l(Q_0)$$

hence

$$\frac{d(Q_0, Q_n)}{l(Q_0)} \le \sqrt{2}n \le 2\sqrt{2}MA_3A_2 \log\left(3 + \frac{d(Q_0, Q_n)}{l(Q_0)}\right)$$

which is a contradiction. Hence there exists an integer i such that $l(Q_i) = 2l(Q_0)$. Q.E.D.

Lemma 10. *Let $Q_0, Q_1, \ldots, Q_n$ be a geodesic Whitney chain in $D_1 \in \mathscr{U}(D_2, M)$ such that $l(Q_n) = 2l(Q_0)$ and $l(Q_i) < l(Q_n)$, $0 \le i \le n-1$. Further assume there exists a square $\tilde{Q}$ such that $Q_0 \cup Q_1 \cup \cdots \cup Q_n \subset \tilde{Q} \subset 2\tilde{Q} \subset D_2$. Then we have*

$$n \le B_2(M), \qquad \frac{d(Q_0, Q_n)}{l(Q_0)} \le B_3(M).$$

Proof. Since $l(Q_{n-1}) = l(Q_0)$, we have $d(Q_0, Q_{n-1}) < B_1(M) l(Q_0)$ by applying lemma 9 to the geodesic Whitney chain $Q_0, Q_1, \ldots, Q_{n-1}$. Hence

$$d(Q_0, Q_n) \le d(Q_0, Q_{n-1}) + \sqrt{2} l(Q_{n-1}) \le B(M) l(Q_0).$$

Further

$$n = W_{D_1}(Q_0, Q_n) \le MA_3A_2\psi(Q_0, Q_n)$$
$$\le 2MA_3A_2 \log\left(4 + \frac{d(Q_0, Q_n)}{l(Q_0)}\right) = 2MA_3A_2 \log(4 + B(M)). \quad \text{Q.E.D.}$$

Lemma 11. *Let $Q_0, Q_1, \dots, Q_n$ be a geodesic Whitney chain in $D_1 \in \mathcal{U}(D_2, M)$ such that $l(Q_i) < l(Q_n)$, $0 \le i \le n-1$. Further assume there exists a square $\tilde{Q}$ such that $Q_0 \cup Q_1 \cup \cdots \cup Q_n \subset \tilde{Q} \subset 2\tilde{Q} \subset D_2$. Then we have*

$$n \le B_4(M) \log \frac{l(Q_n)}{l(Q_0)}, \qquad \frac{d(Q_0, Q_n)}{l(Q_n)} \le B_5(M).$$

Proof. Let $l(Q_n) = 2^m l(Q_0)$ and set

$$s_k = \min\{i \mid l(Q_i) = 2^k l(Q_0)\}, \qquad 0 \le k \le m.$$

By applying lemma 9 to the geodesic Whitney chain $Q_{s_k}, Q_{s_k+1}, \dots, Q_{s_{k+1}}$ we have

$$s_{k+1} - s_k \le B_2(M), \qquad \frac{d(Q_{s_{k+1}}, Q_{s_k})}{l(Q_{s_k})} \le B_3(M),$$

hence

$$n = \sum_{k=0}^{m-1} (s_{k+1} - s_k) \le mB_2(M) \le B(M) \log \frac{l(Q_n)}{l(Q_0)}.$$

And so

$$d(Q_0, Q_n) \le \sum_{k=0}^{m-1} d(Q_{s_k}, Q_{s_{k+1}}) + \sum_{k=1}^{m-1} \sqrt{2}\, l(Q_{s_k})$$
$$\le B_3(M) \sum_{k=0}^{m-1} l(Q_{s_k}) + \sum_{k=1}^{m-1} \sqrt{2}\, l(Q_{s_k})$$
$$\le (B_3(M) + \sqrt{2}) \sum_{k=0}^{m-1} 2^k l(Q_0) \le B(M) l(Q_n). \qquad \text{Q.E.D.}$$

Lemma 12. *Let $Q_0, Q_1, \dots, Q_n$ be a geodesic Whitney chain in $D_1 \in \mathcal{U}(D_2, M)$ and $\hat{Q}$ one of the largest square in this chain. Then*

$$\log\left(2 + \frac{l(\hat{Q})}{l(Q_0)}\right)\left(2 + \frac{l(\hat{Q})}{l(Q_n)}\right) \le A_9 n.$$

Further if there exists a square $\tilde{Q}$ such that $Q_0 \cup Q_1 \cup \cdots \cup Q_n \subset \tilde{Q} \subset 2\tilde{Q} \subset D_2$ then

$$n \le B_6(M) \log\left(2 + \frac{l(\hat{Q})}{l(Q_0)}\right)\left(2 + \frac{l(\hat{Q})}{l(Q_n)}\right), \qquad d(Q_0, Q_n) \le B_7(M) l(\hat{Q}).$$

Proof. The first inequality is trivial since $1/2 \le l(Q_{i+1})/l(Q_i) \le 2$. Next assume there exists a square $\tilde{Q}$ such that $Q_0 \cup Q_1 \cup \cdots \cup Q_n \subset \tilde{Q} \subset 2\tilde{Q} \subset D_2$. We set $S = \{i | 0 \le i \le n,\ l(Q_j) = l(\hat{Q})\}$, $i_1 = \min S$, $i_2 = \max S$. Lemma 11 shows that

$$i_1 \le B_4(M) \log \frac{l(Q_{i_1})}{l(Q_0)}, \qquad \frac{d(Q_0, Q_{i_1})}{l(Q_{i_1})} \le B_5(M).$$

$$n - i_2 \le B_4(M) \log \frac{l(Q_{i_2})}{l(Q_n)}, \qquad \frac{d(Q_{i_2}, Q_n)}{l(Q_{i_2})} \le B_5(M),$$

and lemma 9 shows that

$$\frac{d(Q_{i_1}, Q_{i_2})}{l(Q_{i_1})} < B_1(M).$$

Hence

$$\begin{aligned} i_2 - i_1 &= W_{D_1}(Q_{i_1}, Q_{i_2}) \le MA_3\delta_{D_2}(Q_{i_1}, Q_{i_2}) \le MA_3A_2\psi(Q_{i_1}, Q_{i_2}) \\ &= 2MA_3A_2 \log\left(3 + \frac{d(Q_{i_1}, Q_{i_2})}{l(Q_{i_1})}\right) \le 2MA_3A_2 \log(3 + B_1(M)) \le B(M). \end{aligned}$$

And so

$$\begin{aligned} n &= (n - i_2) + (i_2 - i_1) + i_1 \\ &\le B_4(M) \log \frac{l(Q_{i_2})}{l(Q_n)} + B(M) + B_4(M) \log \frac{l(Q_{i_1})}{l(Q_0)} \\ &\le B(M) \log\left(2 + \frac{l(\hat{Q})}{l(Q_0)}\right)\left(2 + \frac{l(\hat{Q})}{l(Q_n)}\right), \end{aligned}$$

further

$$\begin{aligned} d(Q_0, Q_n) &\le d(Q_0, Q_{i_1}) + \sqrt{2}l(Q_{i_1}) + d(Q_{i_1}, Q_{i_2}) + \sqrt{2}l(Q_{i_2}) + d(Q_{i_2}, Q_n) \\ &\le B_5(M)l(Q_{i_1}) + \sqrt{2}l(Q_{i_1}) + B_1(M)l(Q_{i_1}) + \sqrt{2}l(Q_{i_2}) + B_5(M)l(Q_{i_2}) \\ &\le B(M)l(\hat{Q}). \end{aligned}$$

Q.E.D.

Corollary 1. *Let $D \in \mathscr{U}(\mathbf{R}^2, M)$ and $Q, Q' \in \mathscr{D}(D)$. And Q is the largest square in a ginven geodesic Whitney chain joining Q and Q'. Then*

$$B_6(M)^{-1} W_D(Q, Q') \le \log\left(2 + \frac{l(\hat{Q})}{l(Q)}\right)\left(2 + \frac{l(\hat{Q})}{l(Q')}\right) \le A_9 W_D(Q, Q').$$

Lemma 13. *Let $D_1 \in \mathscr{U}(D_2, M)$ and $Q, Q' \in \mathscr{D}(D_1)$. Assume there exists a square $\tilde{Q}$ such that*

$$Q \cup Q' \subset \tilde{Q} \subset 6\tilde{Q} \subset D_2, \qquad d(Q, Q') \ge \frac{1}{4} l(\tilde{Q}),$$

Let $Q = Q_0, Q_1, \ldots, Q_n = Q'$ be a geodesic Whitney chain in D_1. Then there exists an integer i satisfying

$$l(Q_i) \ge B_8(M) l(\tilde{Q}), \qquad Q_i \subset 3\tilde{Q}.$$

Proof. In the case that $Q_0 \cup Q_1 \cup \cdots \cup Q_n \subset 3\tilde{Q}$, since $2(3\tilde{Q}) = 6\tilde{Q} \subset D_2$, lemma 12 shows that there exists an integer i such that

$$l(Q_i) \geq \frac{d(Q_0, Q_n)}{B_7(M)} \geq \frac{l(\tilde{Q})}{4B_7(M)} .$$

Next in the case that $Q_0 \cup Q_1 \cup \cdots \cup Q_n \not\subset 3\tilde{Q}$ then there exists an integer m such that

$$Q_0 \cup Q_1 \cup \cdots \cup Q_m \subset 3\tilde{Q}, \qquad Q_{m+1} \not\subset 3\tilde{Q}$$

If $l(Q_m) < l(\tilde{Q})/12$ then

$$\begin{aligned} l(\tilde{Q}) &\leq d(Q_0, \partial(3\tilde{Q})) \leq d(Q_0, Q_m) + \sqrt{2}l(Q_m) + \sqrt{2}l(Q_{m+1}) \\ &\leq d(Q_0, Q_m) + \frac{\sqrt{2}l(\tilde{Q})}{12} + \frac{2\sqrt{2}l(\tilde{Q})}{12} = d(Q_0, Q_m) + \frac{\sqrt{2}l(\tilde{Q})}{4} \end{aligned}$$

hence

$$\left(1 - \frac{\sqrt{2}}{4}\right) l(\tilde{Q}) \leq d(Q_0, Q_m) .$$

And so by applying lemma 12 to $Q_0 \cup Q_1 \cup \cdots \cup Q_m$ it follows that there exists an integer i, $0 \leq i \leq m$ such that $d(Q_0, Q_m) \leq B_7(M) l(Q_i)$ therefore

$$l(Q_i) \geq \frac{d(Q_0, Q_m)}{B_7(M)} \geq \frac{\left(1 - \frac{\sqrt{2}}{4}\right) l(\tilde{Q})}{B_7(M)} \geq \frac{l(\tilde{Q})}{4B_7(M)} .$$

It follows that the constant

$$B_8(M) = \min\left\{\frac{1}{4B_7(M)}, \frac{1}{12}\right\}$$

satisfys our assertion. Q.E.D.

Let $D_1 \in \mathscr{U}(D_2, M)$. In the following we set $D' = D_2 \backslash \overline{D_1}$ and $\mathscr{D}(D')$ denotes its Whitney decomposition.

Lemma 14. *Let $D_1 \in \mathscr{U}(D_2, M)$ and Q a square in D_2. Then there exists a square $\tilde{Q}' \in \mathscr{D}(D_1) \cup \mathscr{D}(D')$ and a dyadic square Q' such that*

$$Q' \subset \tilde{Q}' \cap Q, \qquad l(Q') \geq B_9(M) l(Q)$$

Proof. We may assume that $2Q \subset D_2$ by considering $(1/2)Q$ instead of Q if necessary. Let $Q = [a, a + l] \times [b, b + l]$ and set

$$Q_\alpha = \left[a + \frac{1}{3}l, a + \frac{5}{12}l\right] \times \left[b + \frac{1}{3}l, b + \frac{5}{12}l\right],$$

$$Q_\beta = \left[a + \frac{7}{12}l, a + \frac{2}{3}l\right] \times \left[b + \frac{7}{12}l, b + \frac{2}{3}l\right].$$

In the case that $Q_\alpha{}^\circ \subset D_1$, let Q' be the dyadic square in $\mathscr{D}(Q_\alpha)$ containing the center of Q_α. Then

$$l(Q') \geq \frac{1}{66} d(Q', \partial Q_\alpha) \geq \frac{1}{66}\left(\frac{l(Q_\alpha)}{2} - l(Q')\right)$$

and so $l(Q') \geq \frac{1}{134} l(Q_\alpha) = \frac{1}{1608} l(Q)$. And since $Q_\alpha{}^\circ \subset D_1$ there exists a square $\tilde{Q}' \in \mathscr{D}(D_1)$ containing Q'.

We can prove similarly in the case that $Q_\alpha{}^\circ \subset D'$, $Q_\beta{}^\circ \subset D_1$, $Q_\beta{}^\circ \subset D'$, hence we may assume that $Q_\alpha{}^\circ \cap \partial D_1 \neq \emptyset$ and $Q_\beta{}^\circ \cap \partial D_1 \neq \emptyset$. In this case we can find two squares $Q'_\alpha \in \mathscr{D}(D_1)$, $Q'_\beta \in \mathscr{D}(D_1)$ such that $Q'_\alpha \subset Q_\alpha$, $Q'_\beta \subset Q_\beta$. Hence if we set $\hat{Q} = \frac{1}{3} Q$ then

$$Q'_\alpha \cup Q'_\beta \subset Q_\alpha \cup Q_\beta \subset \hat{Q} \subset 6\hat{Q} \subset D_2\,,$$

$$d(Q'_\alpha, Q'_\beta) \geq d(Q_\alpha, Q_\beta) = \frac{\sqrt{2}}{2} l(\hat{Q}) \geq \frac{1}{4} l(\hat{Q})\,,$$

and so lemma 13 implies that there exists a square $Q'(=\tilde{Q}') \in \mathscr{D}(D_1)$ such that

$$Q' \subset 3\hat{Q} = Q\,, \qquad l(Q') \geq B_8(M) l(\hat{Q}) = \frac{B_8(M)}{3} l(Q)\,, \qquad \text{Q.E.D.}$$

Especially no point of $(\partial D_1) \cap D_2$ is the density point for $(\partial D_1) \cap D_2$, hence

Corollary 2. *Let $D_1 \in \mathscr{U}(D_2, M)$ then $m((\partial D_1) \cap D_2) = 0$.*

Lemma 15. *Let $D_1 \in \mathscr{U}(D_2, M)$ and $\mathscr{D}(D_1)$ contain arbitrary large square, then for every square $Q' \in \mathscr{D}(D')$ such that $d(Q', \partial D_2) \geq B_{10}(M) d(Q', \partial D' \cap D_2)$ there exists a square $Q \in \mathscr{D}(D_1)$ and a square $\tilde{Q} \subset D_2$ such that*

$$l(Q) = l(Q')\,, \qquad d(Q, Q') \leq B_{11}(M) l(Q') \qquad Q \cup Q' \subset \tilde{Q} \subset 2\tilde{Q} \subset D_2\,,$$

Proof. We set $L(M) = \max\{4B_7(M), 300\}$ and choose two constants $B_{10}(M) \geq 1$, $B_{11}(M) > 0$ so that

$$32B_{10}(M) - 132 - \frac{\sqrt{2}}{2} > \sqrt{2} L(M)\,, \qquad B_{11}(M) > \sqrt{2} L(M)\,.$$

Then

$$d(Q', \partial D_2) \geq B_{10}(M) d(Q', \partial D' \cap D_2) \geq d(Q', \partial D' \cap D_2)$$

and so $d(Q', \partial D' \cap D_2) = d(Q', \partial D')$. Hence there exists a square $Q_0 \in \mathscr{D}(D_1)$ such that

$$d(Q', Q_0) \leq 2d(Q', \partial D')\,, \qquad l(Q_0) \leq l(Q')\,,$$

therefore by lemma 2

$$d(Q', Q_0) \leq 2d(Q', \partial D') \leq 132 l(Q').$$

Let $\tilde{Q}$ be a square having the same center z_0 as Q_0 and $l(\tilde{Q}) = L(M)l(Q')$. Then

$$d(z_0, (D_2)^c) \geq d((D_2)^c, Q') - d(Q', Q_0) - \frac{\sqrt{2}}{2} l(Q_0)$$

$$\geq \left(32B_{10}(M) - 132 - \frac{\sqrt{2}}{2}\right) l(Q') > \frac{\sqrt{2}}{2} 2L(M)l(Q') = \frac{\sqrt{2}}{2} l(2\tilde{Q}).$$

hence $2\tilde{Q} \subset D_2$, further

$$\frac{L(M)}{2} l(Q') = \frac{1}{2} l(\tilde{Q}) = d(z_0, (\tilde{Q})^c) \leq \frac{\sqrt{2}}{2} l(Q_0) + d(Q_0, Q') + \sqrt{2} l(Q') + d(Q', (\tilde{Q})^c)$$

$$\leq \frac{\sqrt{2}}{2} l(Q') + 132 l(Q') + \sqrt{2} l(Q') + d(Q', (\tilde{Q})^c)$$

hence

$$d(Q', (\tilde{Q})^c) \geq \left(\frac{L(M)}{2} - 132 - \frac{3\sqrt{2}}{2}\right) l(Q') > 0$$

and so $Q' \subset \tilde{Q}$. Since $\mathscr{D}(D_1)$ contains arbitrary large square there exists a geodesic Whitney chain $Q_0, Q_1, \dots, Q_n$ in D_1 such that

$$Q_0, Q_1, \dots, Q_{n-1} \subset \tilde{Q}, \qquad Q_n \not\subset \tilde{Q}$$

We will show that $l(Q_i) = l(Q')$ for some integer i, $0 \leq i \leq n-1$. We may assume $l(Q_{n-1}) < l(Q')$. Then $l(Q_n) \leq l(Q')$ and so

$$\frac{L(M)}{2} l(Q') = \frac{1}{2} l(\tilde{Q}) = d(z_0, \partial\tilde{Q}) \leq \frac{\sqrt{2}}{2} l(Q_0) + d(Q_0, Q_{n-1}) + \sqrt{2} l(Q_{n-1}) + \sqrt{2} l(Q_n)$$

$$\leq \frac{5\sqrt{2}}{2} l(Q') + d(Q_0, Q_{n-1})$$

hence

$$d(Q_0, Q_{n-1}) \geq \left(\frac{L(M)}{2} - \frac{5\sqrt{2}}{2}\right) l(Q') \geq \frac{L(M)}{4} l(Q').$$

Therefore by applying lemma 12 to $Q_0, Q_1, \dots, Q_{n-1} \subset \tilde{Q} \subset 2\tilde{Q} \subset D_2$, there exists some j, $0 \leq j \leq n-1$ such that

$$l(Q_j) \geq \frac{d(Q_0, Q_{n-1})}{B_7(M)} \geq \frac{L(M)}{4B_7(M)} l(Q') \geq l(Q').$$

and so there exist some i, $0 \leq i \leq n-1$ such that $l(Q_i) = l(Q')$.

Moreover since $Q' \cup Q_i \subset \tilde{Q}$ we have

$$d(Q', Q_i) \leq \sqrt{2} l(\tilde{Q}) \leq \sqrt{2} L(M) l(Q') \leq B_{11}(M) l(Q') \qquad \text{Q.E.D.}$$

For domains $D_1 \in \mathscr{U}(D_2, M)$ such that $\mathscr{D}(D_1)$ contain arbitrary large square, we set

$$\mathscr{D}(D')_\alpha = \{Q' \in \mathscr{D}(D') | d(Q', \partial D_2) \geq B_{10}(M) d(Q', \partial D' \cap D_2)\},$$

$$\mathscr{D}(D')_\beta = \mathscr{D}(D') \backslash \mathscr{D}(D')_\alpha, \qquad D'_\alpha = \bigcup_{Q \in \mathscr{D}(D')_\alpha} Q, \qquad D'_\beta = \bigcup_{Q \in \mathscr{D}(D')_\beta} Q.$$

And for each $Q' \in \mathscr{D}(D)_\alpha$, $\tau(Q')$ denotes one of the square Q in $\mathscr{D}(D_1)$ obtained by above lemma.

Let $u \in L^1_{loc}(D_1)$. We extend this function to $D_2 \backslash D'_\beta$ by setting

$$\tilde{u}(z) = u_{\tau(Q')}, \qquad z \in Q' \in \mathscr{D}(D')_\alpha$$

on D'_α. Note that $\tilde{u}$ is defined almost everywhere on $D_2 \backslash D'_\beta$ by corollary 2.

Lemma 16. *Let $D_1 \in \mathscr{U}(D_2, M)$ and $\mathscr{D}(D_1)$ contain arbitrary large square. Let $u \in BMO(D_1)$. Then*

$$|\tilde{u}_{Q_2} - \tilde{u}_{Q_1}| \leq B_{12}(M) \|u\|_{*, D_1} \delta_{D_2}(Q_2, Q_1), \qquad Q_1, Q_2 \in \mathscr{D}(D_1) \cup \mathscr{D}(D')_\alpha.$$

Proof. We will prove this lemma only in the case $Q_1, Q_2 \in \mathscr{D}(D')_\alpha$, since we can treat the other case as the same way.

Because of lemma 1 and 15

$$\delta_{D_2}(Q_1, \tau(Q_1)) \leq A_2 \psi(Q_1, \tau(Q_1)) = 2A_2 \log\left(3 + \frac{d(Q_1, \tau(Q_1))}{l(Q_1)}\right)$$
$$\leq 2A_2 \log(3 + B_{11}(M)) \leq B(M).$$

Similarly we have $\delta_{D_2}(Q_2, \tau(Q_2)) \leq B(M)$, therefore

$$\delta_{D_2}(\tau(Q_2), \tau(Q_1)) \leq \delta_{D_2}(\tau(Q_2), Q_2) + \delta_{D_2}(Q_2, Q_1) + \delta_{D_2}(Q_1, \tau(Q_1))$$
$$\leq B(M) + \delta_{D_2}(Q_2, Q_1) \leq B(M) \delta_{D_2}(Q_2, Q_1).$$

And so by lemma 5 we have

$$|\tilde{u}_{Q_2} - \tilde{u}_{Q_1}| = |u_{\tau(Q_2)} - u_{\tau(Q_1)}| \leq A_5 \|u\|_{*, D_1} \delta_{D_1}(\tau(Q_2), \tau(Q_1))$$
$$\leq A_5 \|u\|_{*, D_1} M \delta_{D_2}(\tau(Q_2), \tau(Q_1)) \leq A_5 \|u\|_{*, D_1} M B(M) \delta_{D_2}(Q_2, Q_1)$$

Q.E.D.

Lemma 17. *Let $D_1 \in \mathscr{U}(D_2, M)$ and $\mathscr{D}(D_1)$ contain arbitrary large square. Let Q be a dyadic square such that $Q \subset D_2 \backslash D'_\beta$ and $2Q \subset D_2$. Let $u \in BMO(D_1)$. Then*

$$\frac{1}{m(Q)} \int_Q |\tilde{u} - \tilde{u}_Q| dm \leq B_{13}(M) \|u\|_{*, D_1}.$$

Proof. Let $s > 0$ be the smallest integer such that $2^s B_9(M) > 1$. We decompose Q into 2^{2s} congruent dyadic subsquares by dividing its each side into $N = 2^s$ pieces. Then by lemma 14, at least one of such N^2 subsquares $\hat{Q}$ satisfys the condition

$$\hat{Q} \subset Q' \cap Q\,, \qquad l(\hat{Q}) = \frac{1}{N} l(Q)$$

for some $Q' \in \mathscr{D}(D_1) \cup \mathscr{D}(D')_\alpha$. Let $\{Q_{j_1}\}_{j_1=1,2,\dots,N^2-1}$ be the set of all such subsquares except for $\hat{Q}$. Further, we decompose each Q_{j_1} into N^2 congruent dyadic subsquares by dividing its each side into N pieces similarly, and $\hat{Q}_{j_1}$ and $Q'_{j_1} \in \mathscr{D}(D_1) \cup \mathscr{D}(D')_\alpha$ are two squares which satisfy

$$\hat{Q}_{j_1} \subset Q'_{j_1} \cap Q_{j_1}\,, \qquad l(\hat{Q}_{j_1}) = \frac{1}{N} l(Q_{j_1})\,.$$

Let $\{Q_{j_1 j_2}\}_{j_2=1,2,\dots,N^2-1}$ be the set of all such subsquares of Q_{j_1} except for $\hat{Q}_{j_1}$. And by repeating this process, we obtain three families of dyadic squares $Q'_{j_1 j_2 \dots j_n} \in \mathscr{D}(D_1) \cup \mathscr{D}(D')_\alpha$ and $\hat{Q}_{j_1 j_2 \dots j_n}$, $Q_{j_1 j_2 \dots j_n}$ such that

$$Q_{j_1 j_2 \dots j_{n-1}} = \bigcup_{j_n} Q_{j_1 j_2 \dots j_n} \cup \hat{Q}_{j_1 j_2 \dots j_{n-1}}\,,$$

$$\hat{Q}_{j_1 j_2 \dots j_n} \subset Q'_{j_1 j_2 \dots j_n} \cap Q_{j_1 j_2 \dots j_n}\,, \qquad l(\hat{Q}_{j_1 j_2 \dots j_n}) = \frac{1}{N} l(Q_{j_1 j_2 \dots j_n})\,.$$

Then

$$\sum_{j_1 j_2 \cdots j_n} m(Q_{j_1 j_2 \dots j_n}) = \left(1 - \frac{1}{N^2}\right) \sum_{j_1 j_2 \cdots j_{n-1}} m(Q_{j_1 j_2 \dots j_{n-1}}) = \cdots$$

$$= \left(1 - \frac{1}{N^2}\right)^{n-1} \sum_{j_1} m(Q_{j_1}) = \left(1 - \frac{1}{N^2}\right)^n m(Q)\,.$$

hence, by regarding $Q_{j_1 j_2 \dots j_n} = Q$, $\hat{Q}_{j_1 j_2 \dots j_n} = \hat{Q}$ when $n = 0$, we have

$$\sum_{j_1 j_2 \cdots j_n} m(\hat{Q}_{j_1 j_2 \dots j_n}) = \frac{1}{N^2} \sum_{j_1 j_2 \cdots j_n} m(Q_{j_1 j_2 \dots j_n}) = \frac{1}{N^2}\left(1 - \frac{1}{N^2}\right)^n m(Q)\,.$$

Therefore

$$\sum_{n=0}^{\infty} \sum_{j_1 j_2 \cdots j_n} m(\hat{Q}_{j_1 j_2 \dots j_n}) = m(Q)\,.$$

Hence the family $\{\hat{Q}_{j_1 j_2 \dots j_n}\}$ make a decomposition of Q.

Here we will show that $|\tilde{u}_{\hat{Q}_{j_1 j_2 \dots j_n}} - \tilde{u}_{\hat{Q}_{j_1 j_2 \dots j_{n-1}}}| \le B(M) \|u\|_{*.D_1}$.

(Case 1) $Q_{j_1 j_2 \dots j_{n-1}} \subset Q'_{j_1 j_2 \dots j_{n-1}}$. When $Q'_{j_1 j_2 \dots j_{n-1}} \in \mathscr{D}(D_1)$ then

$$\hat{Q}_{j_1 j_2 \dots j_n} \cup \hat{Q}_{j_1 j_2 \dots j_{n-1}} \subset Q'_{j_1 j_2 \dots j_{n-1}} \subset 2Q'_{j_1 j_2 \dots j_{n-1}} \subset D_1$$

hence by lemma 5 and 1

$$|\hat{u}_{\hat{Q}_{j_1 j_2 \dots j_n}} - \hat{u}_{\hat{Q}_{j_1 j_2 \dots j_{n-1}}}| \le A_5 \|u\|_{*,D_1} \delta_{D_1}(\hat{Q}_{j_1 j_2 \dots j_n}, \hat{Q}_{j_1 j_2 \dots j_{n-1}})$$

$$\le A_5 A_2 \|u\|_{*,D_1} \psi(\hat{Q}_{j_1 j_2 \dots j_n}, \hat{Q}_{j_1 j_2 \dots j_{n-1}}) \le A \|u\|_{*.D_1}\,.$$

And when $Q'_{j_1j_2\dots j_{n-1}} \in \mathscr{D}(D')_\alpha$ then $\hat{u}$ is a constant function on $Q_{j_1j_2\dots j_{n-1}}$ hence it follows that $|\hat{u}_{\hat{Q}_{j_1j_2\dots j_n}} - \hat{u}_{\hat{Q}_{j_1j_2\dots j_{n-1}}}| = 0$.
(Case 2) $Q'_{j_1j_2\dots j_{n-1}} \subset Q_{j_1j_2\dots j_{n-1}}$. Then $Q'_{j_1j_2\dots j_n} \subset Q_{j_1j_2\dots j_{n-1}}$, hence

$$Q'_{j_1j_2\dots j_n} \cup Q'_{j_1j_2\dots j_{n-1}} \subset Q_{j_1j_2\dots j_{n-1}} \subset Q \subset 2Q \subset D_2 .$$

When $Q'_{j_1j_2\dots j_n} \in \mathscr{D}(D_1)$ then

$$\begin{aligned} |\tilde{u}_{\hat{Q}_{j_1j_2\dots j_n}} - \tilde{u}_{Q'_{j_1j_2\dots j_n}}| &= |u_{\hat{Q}_{j_1j_2\dots j_n}} - u_{Q'_{j_1j_2\dots j_n}}| \\ &\le \frac{1}{m(\hat{Q}_{j_1j_2\dots j_n})} \int_{\hat{Q}_{j_1j_2\dots j_n}} |u - u_{Q'_{j_1j_2\dots j_n}}| dm \\ &\le N^4 \frac{1}{m(Q'_{j_1j_2\dots j_n})} \int_{Q'_{j_1j_2\dots j_n}} |u - u_{Q'_{j_1j_2\dots j_n}}| dm \le N^4 \|u\|_{*,D_1} . \end{aligned}$$

And when $Q'_{j_1j_2\dots j_n} \in \mathscr{D}(D')_\alpha$ then $|\tilde{u}_{\hat{Q}_{j_1j_2\dots j_n}} - \tilde{u}_{Q'_{j_1j_2\dots j_n}}| = 0$. Therefore we have

$$|\tilde{u}_{\hat{Q}_{j_1j_2\dots j_n}} - \tilde{u}_{Q'_{j_1j_2\dots j_n}}| \le N^4 \|u\|_{*,D_1}$$

in either case. Similarly we obtain

$$|\tilde{u}_{\hat{Q}_{j_1j_2\dots j_{n-1}}} - \tilde{u}_{Q'_{j_1j_2\dots j_{n-1}}}| \le N^2 \|u\|_{*,D_1}$$

Further by lemma 16 and 1

$$|\tilde{u}_{Q'_{j_1j_2\dots j_n}} - \tilde{u}_{Q'_{j_1j_2\dots j_{n-1}}}| \le B_{12}(M) A_2 \|u\|_{*,D_1} \psi(Q'_{j_1j_2\dots j_n}, Q'_{j_1j_2\dots j_{n-1}}) \le B(M) \|u\|_{*,D_1} .$$

It follows that

$$\begin{aligned} |\tilde{u}_{\hat{Q}_{j_1j_2\dots j_n}} - \tilde{u}_{\hat{Q}_{j_1j_2\dots j_{n-1}}}| &\le |\tilde{u}_{\hat{Q}_{j_1j_2\dots j_n}} - \tilde{u}_{Q'_{j_1j_2\dots j_n}}| + |\tilde{u}_{Q'_{j_1j_2\dots j_n}} - \tilde{u}_{Q'_{j_1j_2\dots j_{n-1}}}| \\ &\quad + |\tilde{u}_{Q'_{j_1j_2\dots j_{n-1}}} - \tilde{u}_{\hat{Q}_{j_1j_2\dots j_{n-1}}}| \\ &\le B(M) \|u\|_{*,D_1} . \end{aligned}$$

And so

$$|\tilde{u}_{\hat{Q}} - \tilde{u}_{\hat{Q}_{j_1j_2\dots j_n}}| \le \sum_{i=1}^{n} |\tilde{u}_{\hat{Q}_{j_1j_2\dots j_i}} - \tilde{u}_{\hat{Q}_{j_1j_2\dots j_{i-1}}}| \le nB(M) \|u\|_{*,D_1} .$$

hence

$$\begin{aligned} \int_Q |\tilde{u} - \tilde{u}_{\hat{Q}}| dm &\le \sum_{n=0}^{\infty} \sum_{j_1j_2\dots j_n} \int_{\hat{Q}_{j_1j_2\dots j_n}} (|\tilde{u} - \tilde{u}_{\hat{Q}_{j_1j_2\dots j_n}}| + |\tilde{u}_{\hat{Q}_{j_1j_2\dots j_n}} - \tilde{u}_{\hat{Q}}|) dm \\ &\le \sum_{n=0}^{\infty} \sum_{j_1j_2\dots j_n} (\|u\|_{*,D_1} + nB(M)\|u\|_{*,D_1}) m(\hat{Q}_{j_1j_2\dots j_n}) \\ &\le \sum_{n=0}^{\infty} nB(M) \|u\|_{*,D_1} \frac{1}{N^2} \left(1 - \frac{1}{N^2}\right)^n m(Q) \le B(M) \|u\|_{*,D_1} m(Q) . \end{aligned}$$

Q.E.D.

Lemma 18. *Let $u \in L^1_{loc}(Q)$ satisfy the following condition;*

$$\frac{1}{m(Q')}\int_{Q'}|u-u_{Q'}|dm \le K, \qquad Q' \in \mathscr{D}(Q),$$

$$|u_{Q'}-u_{Q''}| \le K, \qquad \text{if } Q',\ Q'' \in \mathscr{D}(Q),\quad Q' \cap Q'' \neq \emptyset.$$

then $u \in L^1(Q)$ and

$$\frac{1}{m(Q)}\int_{Q}|u-u_{Q}|dm \le A_{10}K.$$

Proof. Let Q_0 be the largest square in $\mathscr{D}(Q)$. Set

$$\mathscr{F}_m = \left\{Q' \in \mathscr{D}(Q) | l(Q') = \frac{1}{2^{m-1}} l(Q_0)\right\}, \qquad 1 \le m < \infty,$$

then we can show the following estimate easily;

$$\sum_{Q' \in \mathscr{F}_m} m(Q') \le A2^{-m}m(Q),$$

$$W_Q(Q', Q_0) \le Am, \qquad Q' \in \mathscr{F}_m.$$

hence

$$\begin{aligned}\int_Q |u-u_{Q_0}|dm &\le \sum_{m=1}^{\infty}\sum_{Q' \in \mathscr{F}_m}\int_{Q'}(|u-u_{Q'}|+|u_{Q'}-u_{Q_0}|)dm \\ &\le \sum_{m=1}^{\infty}\sum_{Q' \in \mathscr{F}_m}(K+KAm)m(Q') \\ &\le AK\sum_{m=1}^{\infty}\frac{m}{2^m}m(Q) \le AKm(Q),\end{aligned}$$

and so $m(Q)^{-1}\int_Q |u-u_Q|dm \le 2m(Q)^{-1}\int_Q |u-u_{Q_0}|dm \le AK$. Q.E.D.

Lemma 19. *Let $D_1 \in \mathscr{U}(D_2, M)$ and $\mathscr{D}(D_1)$ contain arbitrary large square. Let Q be a square such that $Q \subset D_2 \backslash D'_\beta$ and $u \in BMO(D_1)$. Then*

$$\frac{1}{m(Q)}\int_Q |\tilde{u}-\tilde{u}_Q|dm \le B_{14}(M)\|u\|_{*,D_1}.$$

Proof. Let Q_1, $Q_2 \in \mathscr{D}(Q)$, $Q_1 \cap Q_2 \neq \emptyset$. By lemma 17 and 18, it suffices to show that $|\tilde{u}_{Q_1}-\tilde{u}_{Q_2}| \le B(M)\|u\|_{*,D_1}$. By the proof of lemma 17, there exist two dyadic squares $\hat{Q}_i$, $(i=1, 2)$ in Q_i and two squares $Q'_i \in \mathscr{D}(D_1) \cup \mathscr{D}(D')_\alpha$ such that

$$\hat{Q}_i \subset Q'_i \cap Q_i, \qquad l(\hat{Q}_i) = \frac{1}{N}l(Q_i).$$

(Case 1) If $l(Q'_1) \geq 4l(Q_1)$ and $Q'_1 \in \mathscr{D}(D_1)$ then since $Q_2 \subset 2Q'_1$ it holds that $Q'_1 \cap Q'_2 \neq \emptyset$ and $Q_2 \subset Q'_2$. Hence Q_1, Q_2 is an admissible chain in D_1 and by lemma 5

$$|\tilde{u}_{Q_1} - \tilde{u}_{Q_2}| \leq A_5 \|u\|_{*,D_1} .$$

(Case 2) If $l(Q'_1) \geq 4l(Q_1)$ and $Q'_1 \in \mathscr{D}(D')_\alpha$ the same argument as in case 1 shows

$$Q'_1 , Q'_2 \in \mathscr{D}(D')_\alpha , \qquad Q_1 \subset Q'_1 , \qquad Q_2 \subset Q'_2 , \qquad Q'_1 \cap Q'_2 \neq \emptyset .$$

hence by lemma 16

$$|\tilde{u}_{Q_1} - \tilde{u}_{Q_2}| = |\tilde{u}_{Q'_1} - \tilde{u}_{Q'_2}| \leq B_{12}(M) \|u\|_{*,D_1} .$$

which also proves this lemma in the case $l(Q'_2) \geq 4l(Q_2)$, and finally
(Case 3) Assume $l(Q'_1) \leq 2l(Q_1)$ and $l(Q'_2) \leq 2l(Q_2)$. In the case $Q'_1 \subset Q_1$ then by lemma 17

$$|\tilde{u}_{Q'_1} - \tilde{u}_{Q_1}| \leq \frac{1}{m(Q'_1)} \int_{Q'_1} |\tilde{u} - \tilde{u}_{Q_1}| dm \leq \frac{N^2}{m(Q_1)} \int_{Q_1} |\tilde{u} - \tilde{u}_{Q_1}| dm \leq N^2 B_{13}(M) \|u\|_{*,D_1} .$$

Next in the case $Q_1 \subset Q'_1$ then also by lemma 17, $|\tilde{u}_{Q'_1} - \tilde{u}_{Q_1}| \leq 4B_{13}(M) \|u\|_{*,D_1}$. Therefore $|\tilde{u}_{Q'_1} - \tilde{u}_{Q_1}| \leq B(M) \|u\|_{*,D_1}$ holds in either case. Similarly we have $|\tilde{u}_{Q'_2} - \tilde{u}_{Q_2}| \leq B(M) \|u\|_{*,D_1}$. Moreover since

$$\frac{1}{N} l(Q_i) \leq l(Q'_i) \leq 2l(Q_i) , \quad l(Q'_i) \leq 4l(Q_{1-i}) , \quad d(Q'_i, Q'_{1-i}) \leq 3\sqrt{2} l(Q_i) , \quad i = 0, 1 .$$

it holds that

$$\psi(Q'_1, Q'_2) \leq 2 \log \{1 + N(4 + 2 + 3\sqrt{2})\} \leq B(M) .$$

and since $Q'_1 \cup Q'_2 \subset 9Q_1 \subset 18Q_1 \subset D_2$ lemma 16 and 1 show that

$$\begin{aligned} |\tilde{u}_{Q'_1} - \tilde{u}_{Q'_2}| &\leq B_{12}(M) \|u\|_{*,D_1} \delta_{D_2}(Q'_1, Q'_2) \\ &\leq B_{12}(M) \|u\|_{*,D_1} A_2 \psi(Q'_1, Q'_2) \leq B(M) \|u\|_{*,D_1} . \end{aligned}$$

Hence

$$|\tilde{u}_{Q_1} - \tilde{u}_{Q_2}| \leq |\tilde{u}_{Q_1} - \tilde{u}_{Q'_1}| + |\tilde{u}_{Q'_1} - \tilde{u}_{Q'_2}| + |\tilde{u}_{Q'_2} - \tilde{u}_{Q_2}| \leq B(M) \|u\|_{*,D_1} .$$

Q.E.D.

Lemma 20. *Let $D_1 \in \mathscr{U}(D_2, M)$ and $\mathscr{D}(D_1)$ contain arbitrary large square. Let $u \in BMO(D_1)$ and $Q \in \mathscr{D}(D')_\beta$. We set*

$$S(u, Q) = \sup \{\tilde{u}(Q') - B_{12}(M) \|u\|_{*,D_1} \delta_{D'}(Q, Q') | Q' \in \mathscr{D}(D')_\alpha\}$$

where $B_{12}(M)$ is the constant in lemma 16, *then $S(u, Q) < \infty$ and if we define the extension $\hat{u}$ of $\tilde{u}$ to D_2 by setting*

$$\hat{u}(z) = S(u, Q) , \qquad z \in Q \in \mathscr{D}(D')_\beta ,$$

on D'_β, *then*

$$|\hat{u}_{Q_2} - \hat{u}_{Q_1}| \le B_{12}(M)\|u\|_{*,D_1} W_{D'}(Q_2, Q_1), \qquad Q_1, Q_2 \in \mathscr{D}(D').$$

Proof. Let $Q \in \mathscr{D}(D')_\beta$. First of all we show $S(u, Q) < \infty$. Let Q_0, Q_1 be arbitrary squares in $\mathscr{D}(D')_\alpha$, then by lemma 15

$$\begin{aligned}\tilde{u}_{Q_1} - \tilde{u}_{Q_0} &\le B_{12}(M)\|u\|_{*,D_1}\delta_{D_2}(Q_1, Q_0) \le B_{12}(M)\|u\|_{*,D_1}\delta_{D'}(Q_1, Q_0)\\ &\le B_{12}(M)\|u\|_{*,D_1}(\delta_{D'}(Q_1, Q) + \delta_{D'}(Q, Q_0))\end{aligned}$$

hence

$$\tilde{u}_{Q_1} - B_{12}(M)\|u\|_{*,D_1}\delta_{D'}(Q_1, Q) \le \tilde{u}_{Q_0} + B_{12}(M)\|u\|_{*,D_1}\delta_{D'}(Q, Q_0).$$

and so $S(u, Q) \le \tilde{u}_{Q_0} + B_{12}(M)\|u\|_{*,D_1}\delta_{D'}(Q, Q_0) < \infty$.

Next let Q_1, $Q_2 \in \mathscr{D}(D')_\beta$ be squares which adjacent to each other. Let $Q' \in \mathscr{D}(D')_\alpha$, then $\hat{u}_{Q_1} \ge \tilde{u}(Q') - B_{12}(M)\|u\|_{*,D_1}\delta_{D'}(Q_1, Q')$ hence

$$\begin{aligned}\hat{u}_{Q_1} + B_{12}(M)\|u\|_{*,D_1} &\ge \tilde{u}(Q') - B_{12}(M)\|u\|_{*,D_1}(\delta_{D'}(Q_1, Q') - \delta_{D'}(Q_1, Q_2))\\ &\ge \tilde{u}(Q') - B_{12}(M)\|u\|_{*,D_1}\delta_{D'}(Q', Q_2).\end{aligned}$$

and so $\hat{u}_{Q_1} + B_{12}(M)\|u\|_{*,D_1} \ge \hat{u}_{Q_2}$. Therefore by the symmetry for Q_1, Q_2

$$|\hat{u}_{Q_1} - \hat{u}_{Q_2}| \le B_{12}(M)\|u\|_{*,D_1}.$$

Next let $Q_1 \in \mathscr{D}(D')_\beta$, $Q_2 \in \mathscr{D}(D')_\alpha$ be squares which adjacent to each other. Then

$$\hat{u}_{Q_1} \ge \tilde{u}_{Q_2} - B_{12}(M)\|u\|_{*,D_1}\delta_{D'}(Q_1, Q_2) = \tilde{u}_{Q_2} - B_{12}(M)\|u\|_{*,D_1}.$$

Let Q' be an arbitrary square in $\mathscr{D}(D')_\alpha$. Then by applying lemma 16

$$\begin{aligned}&\tilde{u}_{Q'} - B_{12}(M)\|u\|_{*,D_1}\delta_{D'}(Q_1, Q')\\ &\le (\tilde{u}_{Q_2} + B_{12}(M)\|u\|_{*,D_1}\delta_{D'}(Q_2, Q')) - B_{12}(M)\|u\|_{*,D_1}\delta_{D'}(Q_1, Q')\\ &\le (\tilde{u}_{Q_2} + B_{12}(M)\|u\|_{*,D_1}.\end{aligned}$$

hence $\hat{u}_{Q_1} \le \tilde{u}_{Q_2} + B_{12}(M)\|u\|_{*,D_1}$ and so $|\hat{u}_{Q_1} - \hat{u}_{Q_2}| \le B_{12}(M)\|u\|_{*,D_1}$.

Hence by combining lemma 16 we obtain $|\hat{u}_{Q_1} - \hat{u}_{Q_2}| \le B_{12}(M)\|u\|_{*,D_1}$ for all Q_1, $Q_2 \in \mathscr{D}(D')$ which are adjacent to each other. Therefore

$$|\hat{u}_{Q_2} - \hat{u}_{Q_1}| \le B_{12}(M)\|u\|_{*,D'} W_{D'}(Q_2, Q_1), \qquad Q_1, Q_2 \in \mathscr{D}(D'). \qquad \text{Q.E.D.}$$

Lemma 21. *Let* $D_1 \in \mathscr{U}(D_2, M)$ *and* $\mathscr{D}(D_1)$ *contain arbitrary large square. Let* $u \in BMO(D_1)$. *Then* $\hat{u}$ *belongs to* $BMO(D_2)$ *and it holds that*

$$\|\hat{u}\|_{*,D_2} \le B_{15}(M)\|u\|_{*,D_1}$$

Proof. We set

$$L(M) = 4(66B_{10}(M) + \sqrt{2}).$$

Let $Q \subset D_2$ be a square such that $L(M)l(Q) \le d(Q_1, \partial D_2)$.
(Case 1) If $Q \subset D_2 \backslash D'_\beta$ then by lemma 19

$$\frac{1}{m(Q)} \int_Q |\hat{u} - \hat{u}_Q| dm \le B_{14}(M) \|u\|_{*, D_1} .$$

(Case 2) If $Q \not\subset D_2 \backslash D'_\beta$ then there exists a square $Q_0 \in \mathscr{D}(D')_\beta$ such that $Q_0 \cap Q \neq \emptyset$. When $d(Q_0, \partial D_2) \le d(Q_0, \partial D' \cap D_2)$ then

$$d(Q_0, \partial D_2) = d(Q_0, \partial D') \le 66 l(Q_0) .$$

and when $d(Q_0, \partial D_2) > d(Q_0, \partial D' \cap D_2)$ then

$$d(Q_0, \partial D_2) \le B_{10}(M) d(Q_0, \partial D' \cap D_2) = B_{10}(M) d(Q_0, \partial D') \le 66 B_{10}(M) l(Q_0) .$$

Hence it holds that $d(Q_0, \partial D_2) \le 66 B_{10}(M) l(Q_0)$ in either case. And so

$$66 B_{10}(M) l(Q_0) \ge d(Q_0, \partial D_2) \ge d(Q, \partial D_2) - \sqrt{2} l(Q_0) \ge L(M) l(Q) - \sqrt{2} l(Q_0) .$$

hence

$$l(Q_0) \ge \frac{L(M)}{66 B_{10}(M) + \sqrt{2}} l(Q) \ge 4 l(Q) .$$

therefore Q is covered by at most 4 squares in $\mathscr{D}(D')$, hence by lemma 20

$$\frac{1}{m(Q)} \int_Q |\hat{u} - \hat{u}_Q| dm \le \sup_{z_1, z_2 \in Q} |\hat{u}(z_2) - \hat{u}(z_1)| \le 4 B_{12}(M) \|u\|_{*, D_1} .$$

And so by lemma 6 $\hat{u}$ belongs to $BMO(D_2)$ and

$$\|u\|_{*, D_2} \le A_6 \max \{B_{14}(M), 4 B_{12}(M)\} L(M) \|u\|_{*, D_1} . \qquad \text{Q.E.D.}$$

To remove the restriction for domain D_1, we need several lemmas below.

Lemma 22. *Let $D_1 \in \mathscr{U}(D_2, M)$ and $z_0 \in D_1$. We set $D'_1 = D_1 \backslash \{z_0\}$, $D'_2 = D_2 \backslash \{z_0\}$ then $D'_1 \in \mathscr{U}(D'_2, A_{11} M)$.*

Proof. Let u be a function in $BMO(D'_1)$ then we can easily show that u is in $BMO(D_1)$ and $\|u\|_{*, D_1} \le A \|u\|_{*, D'_1}$, which implies $D'_1 \in \mathscr{E}(D_1, A)$ (cf. [RR].) Hence for $Q_1, Q_2 \in \mathscr{A}(D'_1)$ we have

$$\delta_{D'_1}(Q_1, Q_2) \le A A_8 \delta_{D_1}(Q_1, Q_2) \le A A_8 M \delta_{D_2}(Q_1, Q_2) \le A A_8 M \delta_{D'_2}(Q_1, Q_2)$$

by lemma 8. Hence $D'_1 \in \mathscr{U}(D'_2, A A_8 M)$. Q.E.D.

Lemma 23. *Let $f: D \to D'$ be a conformal map, Q_i, $(i = 1, 2)$ admissible squares in D having z_i as its center. Let Q'_i, $(i = 1, 2)$ be admissible squares in D' having $f(z_i)$ as its center satisfying $d(Q'_i, \partial D')/l(Q'_i) = d(Q_i, \partial D)/l(Q_i)$ then*

$$\frac{1}{A_{12}} \delta_D(Q_1, Q_2) \le \delta_{D'}(Q'_1, Q'_2) \le A_{12} \delta_D(Q_1, Q_2)$$

Proof. Let Q be arbitrary admissible squares in D having z_0 as its center and let Q' be the admissible squares in D' having $f(z_0)$ as its center and satisfying $d(Q', \partial D')/l(Q') = d(Q, \partial D)/l(Q)$ then Koebe's distortion theorem shows that

$$\frac{1}{A}Q' \subset f(Q) \subset AQ'$$

hence we can easily prove our assertion. Q.E.D.

Lemma 24. *Let $D_1 \in \mathscr{U}(D_2, M)$ and $f: D_2 \to D_2'$ a conformal map, we set $D_1' = f(D_1)$ then $D_1' \in \mathscr{U}(D_2', A_{13}M)$*

Proof. Let $Q_1', Q_2' \in \mathscr{A}(D')$ and Q_1, Q_2 admissible squares in D corresponding to Q_1', Q_2' in lemma 23. Then

$$\delta_{D_1'}(Q_1', Q_2') \leq A_{12}\delta_{D_1}(Q_1, Q_2) \leq A_{12}M\delta_{D_2}(Q_1, Q_2) \leq A_{12}{}^2M\delta_{D_2'}(Q_1', Q_2')$$

hence $D_1' \in \mathscr{U}(D_2', A_{12}{}^2M)$. Q.E.D.

Proposition 1 ([R], [J]). *Let $f: D \to D'$ be a conformal map, then for every $u \in BMO(D')$, $u \circ f$ belong to $BMO(D)$ and $\|u \circ f\|_{*,D} \leq A_{14}\|u\|_{*,D'}$.*

Lemma 25. $\mathscr{U}(D_2, M) \subset \mathscr{E}(D_2, B_{16}(M))$.

Proof. Let $D_1 \in \mathscr{U}(D_2, M)$. Let $z_0 \in D_1$ and set $D_1' = D_1 \backslash \{z_0\}$, $D_2' = D_2 \backslash \{z_0\}$ then by lemma 22, $D_1' \in \mathscr{U}(D_2', A_{11}M)$. We set

$$f(z) = \frac{1}{z - z_0}, \qquad D_1'' = f(D_1'), \qquad D_2'' = f(D_2')$$

then by lemma 24, $D_1'' \in \mathscr{U}(D_2'', A_{13}A_{11}M)$. Let $u \in BMO(D_1)$, then by proposition 1

$$\|u \circ f^{-1}\|_{*,D_1''} \leq A_{14}\|u\|_{*,D_1'} \leq A_{14}\|u\|_{*,D_1}$$

and further by lemma 21 there exist some extension v of $u \circ f^{-1}$ to D_2'' such that

$$\|v\|_{*,D_2''} \leq B_{15}(A_{13}A_{11}M)\, \|u \circ f^{-1}\|_{*,D_1''}\,.$$

hence $\hat{u} = v \circ f$ is a extension of u to D_2 such that

$$\|\hat{u}\|_{*,D_2} \leq A\, \|\hat{u}\|_{*,D_2'} \leq AA_{14}\|v\|_{*,D_2''} \leq AA_{14}B_{15}(A_{13}A_{11}M)A_{14}\|u\|_{*,D_1}$$

which implies the assertion. Q.E.D.

Remark 1. Let $D_1 \in \mathscr{U}(D_2, M)$ then we constructed a non linear extension operator on $BMO(D_1)$ to $BMO(D_2)$. I don't know whether we can construct such linear operator or not.

DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY

References

[G] F. W. Gehring, Uniform domains and the Ubiquitous Quasidisk, Jahresber. Deutsch. Math.-Verein., **89** (1987), 88–103.

[GO] F. W. Gehring and B. G. Osgood, Uniform domains and the quasi-hyperbolic metric, J. Analyse. Math., **36** (1979), 50–74.

[J] P. Jones, Extension theorems for BMO, Indiana Univ. Math. J., **29** (1980), 41–66.

[R] H. M. Reimann, Functions of bounded mean oscillation and quasiconformal mappings, Comm. Math. Helv., **49** (1974), 260–276.

[RR] H. M. Reimann and T. Rychener, Funktionen beschränkter mittelerer Oszillation, Lecture Notes in Math. 489, Springer, 1975.

[S] E. M. Stein, Sngular Integrals and differentiability properties of Functions, Princeton Univ. Press, 1970.