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On calculation of $L_K(1, \chi)$ for some Hecke characters

By

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§ 1. Introduction

Let $L_K(s, \chi)$ be the Hecke L-function for a non-trivial character χ of an ideal class group of an algebraic number field K. The purpose of the present paper is to express $L_K(1, \chi)$ in a form effective for numerical calculation by computers in the case where K and χ satisfy the following two conditions:

(1.1) (i) K is a quadratic extension field of a totally real algebraic number field and K has exactly two real places.
(ii) χ ramifies at all the two real places.

The methods of ours are the same as [Ko], [Ka] and [G], i.e., a generalization of the classical Kronecker's limit formula, and Hecke's method described in § 3 and § 5 of [Si1]. However, we calculate, rather than $L_K(1, \chi)$ itself, a suitable coefficient $\kappa_K(C)$ of $\zeta_K(s, C)$ in the Taylor expansion at $s=0$, where $\zeta_K(s, C)$ is the zeta function of an ideal class C of K (see (2.1)). In fact, it suffices to obtain $\kappa_K(C)$ for our purpose, so the statements will be described for $\kappa_K(C)$.

Stark-Shintani conjecture, which is one of our motivation of the present paper, predicts a kind of arithmeticity for $L_K(1, \chi)$ ([St1], [St2], [Sh5] and [T]). However, the conjecture has been unsolved yet except some special cases. In fact, the case where K has exactly one imaginary place and χ ramifies at all real places, which is described in [St2], is unsolved one. If K is of degree 4 over $\boldsymbol{Q}$, this case is contained in the case (1.1) and one can use our results to give numerical datas in the case.

Let us explain the contents in more detail. In § 2, we recall some facts on the order of $\zeta_k(s, C)$ at $s=0$ for a general algebraic number field k and prove a kind of transformation formula for $\zeta_k(s, C)$ (Proposition 1). It plays the same role as the formula (92) in [Si1, p. 140], which has been used in [Ko], [Ka] and [G]. By virture of this formula, we can despense with Gauss sums and the assumption on the primitivity of χ, so the value $\kappa_K(C)$ seems to be the more natural in our computation than $L_K(1, \chi)$. In § 3, we define a harmonic Hilbert modular function $f_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)$ in (3.1), which is a generalization of the

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logarithm of the Siegel function $\phi(a, b; z)$ in (4.7). It appear as the constant term at $s=0$ of a non-holomorphic Eisenstein series $\boldsymbol{E}(z, s)$ defined in (3.2) (Proposition 2). The term has already been calculated in [Ko], [Ka], [G] and [A]. However, their results contain some numerical mistakes in the term. Here we recalculate it in a suitable form for us in this section. In § 4, we describe some properties of $f_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)$ and its related functions. In § 5, we define some kinds of periods of $f_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)$ on cycles Φ_ω with respect o arithmetic congruence subgroups of Hilbert modular groups, which will be denoted by $\langle\Phi_\omega, \boldsymbol{d}_\Delta f\rangle$. We generalize Hecke's method and express the Dirichlet series in Proposition 1 by $\langle\Phi_\omega, \boldsymbol{d}_\Delta \boldsymbol{E}_s\rangle$, i.e., a period of $\boldsymbol{E}(z, s)$ defined similarly (Proposition 3). In Chapter 2 of [Si1], the Kronecker's limit formulas for $L_K(s, \chi)$ has been given for the following cases: (1) K is an imaginary quadratic field, (2) K is a real quadratic field and χ is unramified at the two real places, (3) K is the same as (2) and χ ramifies at two real places. We can extend them in such cases where K is any quadratic extension field of a totally real algebraic number field F and the archimedean conductor of χ is stable under the action of $\mathrm{Gal}(K/F)$, i.e., the most general cases where all the above three cases are contained. We introduce periods $\langle\Phi_\omega, \boldsymbol{d}_\Delta f\rangle$ as a natural generalization of those in (2) and (3). In § 6, we complete our calculation of $\kappa_K(C)$ and express it by $\langle\Phi_\omega, \boldsymbol{d}_\Delta f\rangle$ in the most general cases as above (Theorem 1 and Corollary 1, which contain the results of [Ko], [Ka] and [G] as special cases). We restate the theorem as Corollary 2 and 3 in the special case where K is a CM-field, hence $\langle\Phi_\omega, \boldsymbol{d}_\Delta f\rangle$ is expressed as a special value of $f_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)$ at a CM-point. Our main purpose is attained by Corollary 5 and 6, which treat the case (1.1). In this case, $\langle\Phi_\omega, \boldsymbol{d}_\Delta f\rangle$ is expressed as a difference of the special values of $f_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)$ at suitable two points. One can calculate the value of $\kappa_K(C)$ effectively by computers by means of this expression. In § 7, we give two numerical examples in the case (1.1) by using Corollary 6 (the case (1) in this section) and, in addition, give four examples in the case of Corollary 3, i.e., the case of [Ko] (the case (2)). Each $L_K(s, \chi)$ in the examples is described as a product of other L-functions whose special values are calculated by some classical formulas. Calculating $\kappa_K(C)$ or $L_K(1, \chi)$ by means of another formula, we ensure our formula from mistakes even in the numerical sense. We remark that our formulas are suitable for calculation of $L_K(1, \chi)$. In fact, we give each numerical data exactly with the order of 10^{-45}, while the time we need is a few seconds in the shortest case and less than 13 minutes even for the most longest by means of a computer with 32 bit. As an appendix, in § 8, we consider the case where two L-functions, one of which satisfies the conditons for K and χ described in the above paragraph for § 5, and the other of which does not, coincide (Theorem 2). Here one can know that there are more cases, different from those of Theorem 1, in which our formulas can be used. This section is based on [Sh 4] and is related to [I].

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Notation. We denote by $\Re(z)$, $\Im(z)$, $\bar{z}$, $|z|$ and $\|z\|$ the real part, the imaginary part, the complex conjugation, the absolute value and the square of absolute value of $z \in \boldsymbol{C}$ respectively. Let $i=\sqrt{-1}$ and $\boldsymbol{e}(z)=e^{2\pi iz}$ for $z \in \boldsymbol{C}$. Let $\mathrm{sgn}(t)=t/|t|$ for $t \in \boldsymbol{R}$, $t \neq 0$, and $\boldsymbol{R}_+^{\times}=\{t \in \boldsymbol{R} | t>0\}$. We denote by γ_0 the Euler constant, i.e., $\lim_{n \to \infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\log n\right)$. For a ring I, $I^{\times}$ denotes the group of all invertible elements of I. For two modules A, B, and for an element $b \in B$, $A+B$ and $A+b$ denote the module generated by A and B, and the set consisting of elements $x+b$ with $x \in A$ respectively. For sets A, B, and for an element b, $A-B$ and $A-b$ denote the set $A \cap B^c$ and $A \cap \{b\}^c$ respectively. For a set A, $\#(A)$ denotes the order of A. The other notations will be defined in each section.

§ 2. Preliminaries

For an algebraic number field k, Tr_k, N_k, d_k, $\mathfrak{d}_k$, I_k, E_k, R_k and W_k denote the trace with respect to $k/\boldsymbol{Q}$, the norm of an ideal with respect to $k/\boldsymbol{Q}$ (also the norm of an element, in the present paper, should be considered as that of the ideal generated by it, which is a positive rational number), the absolute value of the discriminant of k, the different of k, the ring of integers of k, the group of all units of k, the regulator of k, and the group of roots of unity in $k^{\times}$ (or its order), respectively. For $\lambda \in k$, $\boldsymbol{e}_k(\lambda)$ denotes $\boldsymbol{e}(Tr_k(\lambda))$ and $\lambda \gg 0$ means that λ is positive at all real places of k. For an integral ideal $\mathfrak{f}$ and a set Ω consisting of several real places of k, $E_k(\mathfrak{f}\Omega)$ and $W_k(\mathfrak{f}\Omega)$ denote the subgroup of E_k and W_k consisting of ε satisfying $\varepsilon \equiv 1 \pmod{\mathfrak{f}}$ and $\varepsilon_l>0$ for $l \in \Omega$, and $H_k(\mathfrak{f}\Omega)$ denotes the ideal ray class group of k with the conductor $\mathfrak{f}\Omega$. If $f=I_k$ or $\Omega=\phi$, we drop it in each notation (for example, $E_k(\Omega)$, $H_k(\mathfrak{f})$ mean $E_k(I_k\Omega)$, $H_k(\mathfrak{f}\phi)$, respectively). Further, if Ω consists of all the real places of k, we use the notations E_k^+ and $E_k^+(\mathfrak{f})$ in place of $E_k(\Omega)$ and $E_k(\mathfrak{f}\Omega)$ respectively. We denote by h_k the class number of k in wide sense, i.e., $\#(H_k(I_k))$. For an ideal $\mathfrak{A}$ of k prime to $\mathfrak{f}$ and for $C \in H_k(\mathfrak{f}\Omega)$, $\mathfrak{A}C$ denotes the element of $H_k(\mathfrak{f}\Omega)$ containing $\mathfrak{A}\mathfrak{B}$, where $\mathfrak{B} \in C$. For Ω, τ_Ω denotes the character of $k^{\times}$ defined by $\tau_\Omega(\lambda)=\prod_{l \in \Omega}\mathrm{sgn}(\lambda_l)$.

Let r_1, r_2 be the number of real and imaginary places of k ($r_1+2r_2=[k:\boldsymbol{Q}]$), and put $e_1=\#(\Omega)$, $e_1'=r_1-e_1$. Now, we show that $\zeta_k(s, C)$ for $C \in H_k(\mathfrak{f}\Omega)$ has the following expansion:

$$\frac{\zeta_k(s, C)}{s^{e_1'+r_2}}=\left[-\frac{R_k}{W_k}\frac{1}{s}\right]^{\dagger\dagger}+\kappa_k(C)+O(s), \tag{2.1}$$

where the term with the symbol †† appears only when $\mathfrak{f}=I_k$ and $\Omega=\phi$.

Since $\zeta_k(s, C)$ is rewritten in the form of a linear combination of $L_k(s, \chi)$'s for all characters χ of $H_k(\mathfrak{f}\Omega)$, we may cosider the Taylor expansion of $L_k(s,$

χ) at $s=0$ to prove (2.1). Put

$$(2.2)\qquad G_k(s,\mathfrak{f}\Omega)=(d_kN_k\mathfrak{f})^{\frac{s}{2}}\left(\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\right)^{e_1'}\left(\pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)\right)^{e_1}((2\pi)^{-s}\Gamma(s))^{r_2}.$$

If χ is primitive, set $\Lambda_k(s,\chi)=G_k(s,\mathfrak{f}\Omega)L_k(s,\chi)$. Then, we have

$$(2.3)\qquad \Lambda_k(s,\chi)=\frac{\mathfrak{g}(\chi)}{i^{e_1}\sqrt{N_k\mathfrak{f}}}\cdot\Lambda_k(1-s,\bar{\chi}),$$

where $\mathfrak{g}(\chi)$ is the Gauss sum associated with χ. By means of this equation, we can determine the order of $L_k(s,\chi)$ at $=0$ from the one at $s=1$. Since the latter is 1 or 0 as well known, the former is $e_1'+r_2-1$ or $e_1'+r_2$ according as χ is trivial or not. If χ is not primitive, $L_k(s,\chi)$ can be written as $L_k(s,\chi_0)$ times Euler factors $1-\chi_0(\mathfrak{p})N_k\mathfrak{p}^{-s}$ for finitely many primes $\mathfrak{p}$, where χ_0 is the primitive character associated with χ. Then we know that the order we consider is equal to or greater than that of $L_k(s,\chi_0)$, and that in particular, if χ is trivial, the order is equal to $r_1+r_2+\rho-1$, where ρ is the number of different prime factors of $\mathfrak{f}$. Summing up, the order of $L_k(s,\chi)$ at $s=0$ is

$$(2.4)\qquad \left\{\begin{array}{ll} r_1+r_2+\rho-1 & (\text{if } \chi \text{ is trivial}) \\ \geqq e_1'+r_2 & (\text{if } \chi \text{ is non-trivial}) \end{array}\right\}.$$

If $\rho=0$ and $e_1'=r_1$, i.e., $\mathfrak{f}=I_k$ and $\Omega=\phi$, he term with †† in (2.1) is derived from the first term on the right hand side of

$$(2.5)\qquad \frac{\zeta_k(s)}{s^{r_1+r_2}}=-\frac{R_kh_k}{W_k}\frac{1}{s}+O(1).$$

This is a modification of Dirichlet's residue formula which is obtained from (2.2) and (2.3) for the trivial charahoer by the fact that $\mathfrak{g}(\chi)=1$ in this case. Lastly (2.1) is deduced from (2.4) and (2.5).

Next, we show the following lemma related to (2.1).

Lemma 1. *Assume that* $e_1\neq 1$. *For* $C\in H_k(\mathfrak{f}\Omega)$ *and for* $\lambda\in k^\times$ *which is multiplicatively congruent to* 1 *modulo* $\mathfrak{f}$, *we have*

$$(2.6)\qquad \tau_\Omega(\lambda)\zeta_k(s,(\lambda)C)=\zeta_k(s,C)+O(s^{e_1'+r_2+1}).$$

Proof. The case $e_1=0$ s trivial. Assume that $e_1\geqq 2$. Let λ_1, $\lambda_2\in k^\times$ be elements of $k^\times$ multiplicatively congruent to 1 modulo $\mathfrak{f}$ such that $(\lambda_1/\lambda_2)_{\tilde{l}}<0$, $(\lambda_1/\lambda_2)_l>0$ ($\forall l\in\tilde{\Omega}=\Omega-\tilde{l}$), where $\tilde{l}$ is an fixed element in Ω. We may consider that $\tilde{C}(\lambda_1)C\cup(\lambda_2)C\in H_k(\mathfrak{f}\tilde{\Omega})$. By (2.1), we have

$$\zeta_k(s,(\lambda_1)C)+\zeta_k(s,(\lambda_2)C)=\zeta_k(s,\tilde{C})=O(s^{e_1'+r_2+1}).$$

Sine $\tau_\Omega(\lambda_1)=-\tau_\Omega(\lambda_2)$, this can be rewritten as

$$\tau_\Omega(\lambda_1)\zeta_k(s, (\lambda_1)C) = \tau_\Omega(\lambda_2)\zeta_k(s, (\lambda_2)C) + O(s^{e_1'+r_2+1}) .$$

Hence we obtain (2.6) from the above through the induction with respect to the number of $l \in \Omega$ such that $\lambda_l < 0$.

Before ending this section, we show a functional equation between zeta functions of ideal classes analogous to (2.3), which will be used in the proof of Theorem 1.

Proposition 1. *Let $\mathfrak{A}$, $\mathfrak{B}$ be ideals of k which satisfy $\mathfrak{A}\mathfrak{B} = (\mathfrak{f}\mathfrak{d}_k)^{-1}$ and $1 \in \mathfrak{A}$ (i.e., $\mathfrak{B} \subset (\mathfrak{f}\mathfrak{d}_k)^{-1}$). Then, we have*

$$\text{(2.7)} \qquad G_k(s, \mathfrak{f}\Omega) \sum_{\lambda \in \mathfrak{A}\mathfrak{f}+1/E_k(\mathfrak{f}\Omega)} \frac{\tau_\Omega(\lambda)}{N_k((\lambda)\mathfrak{A}^{-1})^s}$$

$$= \frac{1}{i^{e_1}\sqrt{N_k\mathfrak{f}}} \cdot G_k(1-s, \mathfrak{f}\Omega) \sum_{\lambda \in \mathfrak{B}-0/E_k(\mathfrak{f}\Omega)} \frac{\tau_\Omega(\lambda) e_k(\lambda)}{N_k((\lambda)\mathfrak{B}^{-1})^{1-s}} ,$$

where both sides should be considered as the functions of s after analytic continuation to the whole s-plane.

Proof. For $t \in \boldsymbol{R}^{\times r_1+r_2}$ and $\lambda \in k$, put $g(\lambda, t) = \prod_{l \in \Omega}(k_l t_l^{\frac{1}{2}}) \prod_{p \in \Omega^a} e^{-\rho_p \pi \|\lambda_p\| t_p}$, where Ω^a denotes the set of all archimedean places of k and $\rho_p = 1$ or 2 according as p is real or imaginary. Applying Poisson's summation formula to $g(\lambda, t)$, we have

$$\text{(2.8)} \qquad \sum_{\lambda \in \mathfrak{A}\mathfrak{f}+1} g(\lambda, t^{-1}) = \frac{u^{\frac{1}{2}}}{i^{e_1}\sqrt{d_k}\, N_k(\mathfrak{A}\mathfrak{f})} \sum_{\lambda \in \mathfrak{B}} e_k(\lambda) g(\lambda, t) ,$$

where $u = \prod_{p \in \Omega^a} t_p^{\rho_p}$. Define the action of $E_k(\mathfrak{f}\Omega)$ on $\boldsymbol{R}_+^{\times r_1+r_2}$ by $(\varepsilon \cdot t)_p = \|\varepsilon_p\| t_p$ ($p \in \Omega_a$) for $\varepsilon \in E_k(\mathfrak{f}\Omega)$ and $t \in \boldsymbol{R}_+^{\times r_1+r_2}$. Take a fundamental domain D of $\boldsymbol{R}_+^{\times r_1+r_2}$ by $E_k(\mathfrak{f}\Omega)$ such that $\xi D \subset D$ for $\xi \in \boldsymbol{R}_+^\times$. By Mellin transformation under the assumption that $\Re(s) > 1$ on the left and $\Re(s) < 0$ on the right in (2.7) respectively, we have

$$\text{(2.9)} \qquad \text{l.h.s.} = (d_k N_k\mathfrak{f})^{\frac{s}{2}} N_k\mathfrak{A}^s \cdot \int_D \sum_{\lambda \in \mathfrak{A}\mathfrak{f}+1} g(\lambda, t) \cdot u^{\frac{s}{2}} d^\times t ,$$

$$\text{(2.10)} \qquad \text{r.h.s.} = \frac{(d_k N_k\mathfrak{f})^{\frac{s-1}{2}} N_k\mathfrak{A}^{s-1}}{i^{e_1}\sqrt{N_k\mathfrak{f}}} \cdot \int_D \sum_{\lambda \in \mathfrak{B}-0} e_k(\lambda) g(\lambda, t) \cdot u^{\frac{1-s}{2}} d^\times t ,$$

where $d^\times t = \prod_{p \in \Omega^a} \frac{dt_p}{t_p}$. Divide D into $D_{<1}$ and $D_{>1}$, where the former denotes the part satisfying the condition $u < 1$ and the latter $u \geqq 1$, and apply (2.8) to the summation in the $\int_{D_{<1}}$-part of (2.9) and (2.10), changing t into t^{-1} if necessary. Then, both right hand sides of (2.9) and (2.10) can be rewritten as

$$\frac{(d_k N_k \mathfrak{f})^{\frac{s-1}{2}} N_k \mathfrak{A}^{s-1}}{i^{e_1}\sqrt{N_k \mathfrak{f}}} \cdot \int_{D>1} \sum_{\lambda \in \mathfrak{B}-0} e_k(\lambda) g(\lambda, t) \cdot u^{\frac{1-s}{2}} d^{\times} t$$

$$+\left[\frac{(d_k N_k \mathfrak{f})^{\frac{s-1}{2}} N_k \mathfrak{A}^{s-1}}{i^{e_1}\sqrt{N_k \mathfrak{f}}} \cdot \frac{2^{r_1} R_k}{s-1}\right]$$

$$+(d_k N_k \mathfrak{f})^{\frac{s}{2}} N_k \mathfrak{A}^{s} \cdot \int_{D>1} \sum_{\lambda \in \mathfrak{A}\mathfrak{f}+1} g(\lambda, t) \cdot u^{\frac{s}{2}} d^{\times} t ,$$

where the term with the symbol ‡ appears only when $\Omega=\phi$. It is a meromorphic function on the whole complex s-plane and hence this completes the proof.

3. Kronecker's limit formula

In the rest of the paper, let F be a totally real algebraic number field of degree n over $\boldsymbol{Q}$. In this section, we recall the generalized Kronecker's limit formula, which has been treated in Konno [Ko], Katayama [Ka] and Goldstein [G].

Let $\Omega_0{}^a$ be the set of all (real) archimedean places of F, and we use the notation g to express a general element of $\Omega_0{}^a$. For a set X, we denote by X^a the direct product of n-copies of X indexed by $g \in \Omega_0{}^a$. We set $\mathfrak{H}=\{z \in \boldsymbol{C} | \mathfrak{I}(z)>0\}$. For $z \in (\boldsymbol{C}-\boldsymbol{R})^a$, x, y denote the elements of $\boldsymbol{R}^a$ such that $x_g=(\mathfrak{R}(z))_g$, $y_g=(\mathfrak{I}(z))_g$. We denote $\prod_g z_g$, $\prod_g \|z_g\|$ and $\prod_g |y_g|$ by Nz, $N\|z\|$ and Ny respectively. Further, for $z \in (\boldsymbol{C}-\boldsymbol{R})^a$, we denote by z^+ the elements of $\mathfrak{H}^a$ such that $(z^+)_g=z_g$ or $\overline{z_g}$ according as $y_g>0$ or <0 respectively.

Let $\mathfrak{a}$ and $\mathfrak{b}$ be integral ideals of F. Put $\mathfrak{a}_1=\mathfrak{a}\mathfrak{d}_F{}^{-1}$, $\mathfrak{b}_1=\mathfrak{b}\mathfrak{d}_F{}^{-1}$, $\mathfrak{c}=\mathfrak{d}_F\mathfrak{a}\mathfrak{b}^{-1}$, $\mathfrak{c}_0=\mathfrak{d}_F\mathfrak{a}_1\mathfrak{b}_1$ and $\mathfrak{c}_1=\mathfrak{d}_F\mathfrak{a}^{-1}\mathfrak{b}$. For $(a, b) \in F \times F$, put $\mathfrak{a}_a=(a)\mathfrak{a}$ and $\mathfrak{b}_b=(b)\mathfrak{b}$. Here assume that $\mathfrak{a}_a=\mathfrak{a}$ if $a=0$ and $\mathfrak{b}_b=\mathfrak{b}$ if $b=0$. Let $\mathfrak{m}$, $\mathfrak{n}$ be the denominator part of $\mathfrak{a}_a$, $\mathfrak{b}_b$, respectively. We use the nontations $\zeta_F(s, \mathfrak{a}_a)$, $\kappa_F(\mathfrak{b}_b)$ to denote $\zeta_F(s, C_1)$, $\kappa_F(C_2)$ with C_1, C_2 such that $\mathfrak{a}_a\mathfrak{m} \in C_1 \in H_F(\mathfrak{m})$, $\mathfrak{b}_b\mathfrak{n} \in C_2 \in H_F(\mathfrak{n})$, respectively.

In this section, we drop the subscript F of E_F and denote it simply by E. For a subgroup U of E with finite index, we denote $|E: U|$ by $\mathrm{Ind}(U)$.

For $\mathfrak{a}$, $\mathfrak{b}$ and (a, b), define a function $\mathfrak{h}^a \ni z \to f_{\mathfrak{a}\times\mathfrak{b}}(a, b; z) \in \boldsymbol{C}$ by

$$f_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)=\frac{1}{\mathrm{Ind}(E(\mathfrak{m}))} \cdot \frac{\zeta_F(-1, \mathfrak{a}_a)}{N_F(\mathfrak{m}\mathfrak{c})} \cdot (-2\pi)^n Ny \tag{3.1}$$

$$+\frac{1}{\mathrm{Ind}(U)} \cdot \sum_{\beta \in \mathfrak{b}_1-0/U} \sum_{0 \neq \alpha \in \mathfrak{a}^{-1}+a} \frac{e_F(\beta b+(\alpha\beta z)^+)}{N_F((\beta)\mathfrak{b}_1{}^{-1})}$$

$$+\left[\frac{1}{\mathrm{Ind}(E(\mathfrak{n}))} \cdot \frac{2^n \kappa_F(\mathfrak{b}_b)}{\sqrt{d_F}}\right]^{\ddagger},$$

where U is a subgroup of $E(\mathfrak{m}) \cap E(\mathfrak{n})$ with finite index (the above definition is well defined with respect to U), and the term with the symbol ‡ is considered only when $a\in\mathfrak{a}^{-1}$.

As will be shown in Proposition 2, the summations on the right hand side of (3.1) are uniformly absolutely convergent on any compact subset in $\mathfrak{H}^a$ with respect to z. Note that $f_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)$ depends only on the class (a, b) mod $\mathfrak{a}^{-1}\times\mathfrak{b}^{-1}$, and in fact, $f_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)$ is real valued (see § 4).

Further, we define Eisenstein series associated with an integral ideal $\mathfrak{f}_0$ of F, $\mathfrak{a}$ and $\mathfrak{b}$, and $(a, b)\in\mathfrak{f}_0^{-1}\mathfrak{a}^{-1}\times\mathfrak{f}_0^{-1}\mathfrak{b}^{-1}$ as

$$(3.2)\qquad \boldsymbol{E}(z, s)=\frac{1}{\mathrm{Ind}(E(\mathfrak{f}_0))}\cdot\frac{(N_F\mathfrak{c}_0Ny)^s}{\pi^n}\sum_{(\alpha,\beta)\in\mathfrak{a}_1\times\mathfrak{b}_1-(0,0)/E(\mathfrak{f}_0)}\frac{e_F(\alpha a-\beta b)}{N\|\alpha+\beta z\|^s}.$$

Proposition 2. *For each $z\in\mathfrak{H}^a$, $\boldsymbol{E}(z, s)$ is absolutely convergent for $\Re(s)>1$, and can be meromorphically continued to the whole complex s-plane. The right hand side of* (3.1) *is absolutely convergent for $z\in\mathfrak{H}^a$, and $\boldsymbol{E}(z, 1+s)$ has, after the analytic continuation, the following expansion at $s=0$:*

$$(3.3)\qquad \boldsymbol{E}(z, 1+s)=\left[\frac{2^{n-2}R_F}{s}\right]^{\dagger}+\sqrt{d_F}\cdot f_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)$$
$$-[2^{n-2}R_F\cdot\{\log(N_F\mathfrak{c}_1Ny)-2n(\log\pi+\gamma_0)\}]^{\dagger}+O(s),$$

where the terms with the symbol † appears only when $(a, b)\in\mathfrak{a}^{-1}\times\mathfrak{b}^{-1}$.

Before proving Proposition 2, we shall give some facts we need.

In the first place, we consider the integral $\int\frac{e(\xi t)}{(t^2+1)^s}dt$ for $s\in\boldsymbol{C}$ and $\xi\in\boldsymbol{R}$, and the path P ($=P^+$ or P^- according as $\xi>0$ or $\xi<0$ respectively) associated with the integral. P^+ is defined as follows. Let ε be a positive real number. Firstly go straight on from ∞i to $(1+\varepsilon)i$ along the imaginary axis, secondly turn around i counterclockwise with the distance ε, and lastly go back to ∞i along the imaginary axis. P^- is defined as the symmetric path of P^+ with respect to the real axis. We have the following lemma (see [Si1], [Ko]).

Lemma 2. *For $\int_{-\infty}^{\infty}\frac{e(\xi t)}{(t^2+1)^s}dt$ and $\int_P\frac{e(\xi t)}{(t^2+1)^s}dt$, which are integrable for $\Re(s)>\frac{1}{2}$ and an arbitrary $s\in\boldsymbol{C}$ respectively, the followings hold.*

$$(1)\quad \int_{-\infty}^{\infty}\frac{e(\xi t)}{(t^2+1)}dt=\left\{\begin{array}{ll}\dfrac{\sqrt{\pi}\,\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} & (\xi=0)\\ \displaystyle\int_P\frac{e(\xi t)}{(t^2+1)^s}dt & (\xi\neq0)\end{array}\right\}\quad \text{for } \Re(s)>\frac{1}{2}.$$

(2) *Assume that $\xi\neq0$. For any compact subset $D\subset\boldsymbol{C}$, there exists a positive*

real number M such that

$$\left|\int_P \frac{e(\xi t)}{(t^2+1)^s}dt\right| < \frac{M}{|\xi| e^{\pi|\xi|}} \qquad (\forall s \in D).$$

(3) $$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e(\xi t)}{t^2+1}dt = e(|\xi| i) \quad \textit{for} \quad \xi \in \boldsymbol{R}.$$

Next, we describe the following lemma which we need to show convergence of some infinite sums and integrals.

Lemma 3.(1) *For a lattice L in $\boldsymbol{R}^a$ and a positive real number A, there exists a positive real number M satisfying the inequality*

$$\sum_{\lambda \in \frac{1}{u}L+w} \prod_g \frac{1}{\lambda_g{}^2+1} < M \cdot \prod_g u_g\,,$$

for $u \in \boldsymbol{R}_+^{\times a}$ such that $u_g > A$ $(g \in \Omega_0{}^a)$ and $w \in \boldsymbol{R}^a$.

(2) *For a F-lattice L in $\boldsymbol{R}^a$, $w \in F$, real numbers $A_1 > 1$ and $A_2 > 1$, there exist positive real numbers M_0, M_1, $M_2 (M_2 > 1)$ satisfying the inequality*

$$\sum_{0 \neq \lambda \in L+w} \prod_g \frac{1}{A_2{}^{u_g|\lambda_g|}} < \frac{M_0 \prod_g u_g + M_1}{\prod_g (u_g M_2{}^{u_g})}\,,$$

for $u \in \boldsymbol{R}_+^{\times a}$ such that $u_{g_1}/u_{g_2} < A_1$ $(\forall g_1, \forall g_2 \in \boldsymbol{X}_0{}^a)$.

Proof. For $\sigma \in \{1, -1\}^a$, put $\boldsymbol{R}^\sigma = \{v \in \boldsymbol{R}^a | \mathrm{sgn}(v_g) = \sigma_g (g \in \Omega_0{}^a)$ or $u_g = 0$ $(g \in \Omega_0{}^a)\}$, $V = \frac{1}{u}L + w$ (resp. $L + w$), and $V^\sigma = V \cap \boldsymbol{R}^\sigma$. For $L = \boldsymbol{Z}v_1 + \cdots \boldsymbol{Z}v_n$, choose $v_0 = N_1 v_1 + \cdots + N_n v_n \in \boldsymbol{R}^\sigma$ $(N_1, \cdots, N_n \in \boldsymbol{Z})$ and $N_0 \in \boldsymbol{Z}$ $(N_0 > 1)$ such that $v_j' = v_j + N_0 v_0 \in \boldsymbol{R}^\sigma$ for $j = 1, \cdots, n$. Then, $L' = \boldsymbol{Z}v_1' + \cdots \boldsymbol{Z}v_n'$ is a sublattice of L, and $|L : L'|$ depends only on L and σ. Thus, to prove the lemma, it is sufficient to show the inequalities which are obtained by changing $\sum_{\lambda \in V}$ in (1) and (2) to $\sum_{\lambda \in V^\sigma}$ under the assumption that L has generators $\{v_1, \cdots, v_n\}$ such that $v_1, \cdots, v_n \in \boldsymbol{R}^\sigma$ and $w \in \boldsymbol{R}^\sigma$. For example, consider the case where $\sigma_g = 1$ for all $g \in \Omega_0{}^a$ (also the other cases can be treated similarly). For the above $\{v_1, \cdots, v_n\}$, we can take a positive real number Q_0 such that $D = \{\sum_g \lambda_g v_g \in \boldsymbol{R}^a | 0 \leqq \lambda_g < 1\} \subset [0, Q_0]^a$, and denote by $\mathrm{Cov}(L)$ the volume of D. The inequality in (1) is derived from the following:

$$\frac{\mathrm{Cov}(L)}{\prod_g u_g} \cdot \sum_{\lambda \in V^\sigma} \prod_g \frac{1}{\lambda_g{}^2+1} < \prod_g \int_{-\frac{1}{u_g}Q_0}^{\infty} h(x_g)\,dx_g\,,$$

where $h(x) = (x^2+1)^{-1}$ or 1 according as $x \geqq 0$ or < 0. As for (2), put $V_0{}^\sigma = V^\sigma \cap [0, 2nQ_0A_1)^a$, $V_g{}^\sigma = \{\lambda \in V^\sigma | \lambda_g \geqq 2nQ_0A_1\}$ for $g \in \Omega_0{}^a$, and $V_1{}^\sigma = V^\sigma - V_0{}^\sigma$ $(= \cup_{g \in \Omega_0{}^a} V_g{}^\sigma)$. Then, we have $\sum_{\lambda \in V^\sigma} < \sum_{\lambda \in V_0{}^\sigma} + \sum_{g \in \Omega_0{}^a} \sum_{\lambda \in V_g{}^\sigma}$. Choose a positive real numbr Q_1 such that $[-Q_1, Q_1]^a \subset \{\sum_g \lambda_g v_g \in \boldsymbol{R}^a \mid |\lambda_g|<1\}$, and put $Q_2=\mathrm{Min}\{\lambda_g \mid \lambda \in V_0^\sigma, g \in \Omega_0^a\}$. Since $\#(V_0^\sigma)<(2N+1)^n$, where N is a positive integer greater than $\frac{2nQ_0A_1}{Q_1}$, we have

$$\sum_{\lambda \in V_0^\sigma} \prod_g \frac{1}{A_2^{u_g|\lambda_g|}} < (2N+1)^n \cdot \frac{1}{\prod_g A_2^{Q_2 u_g}} .$$

On the other hand, by the assumption, for $\lambda \in V_g^\sigma$, we have

$$\sum_g u_g|\lambda_g| > \frac{u_{\tilde{g}}}{A_1} \sum_g \lambda_g = \frac{u_{\tilde{g}}}{A_1}((\lambda_{\tilde{g}}-2(n-1)Q_0A_1) + \sum_{g \neq \tilde{g}} (\lambda_g + 2Q_0A_1)) .$$

Since $\lambda_{\tilde{g}}-2(n-1)Q_0A_1>2Q_0$ and $\lambda_g+2Q_0A_1>2Q_0$ for $g \neq \tilde{g}$, we obtain

$$\mathrm{Cov}(L) \cdot \sum_{\lambda \in V_g^\sigma} \prod_g \frac{1}{A_2^{u_g|\lambda_g|}} < \int_{[Q_0,\infty)^a} A_2^{-\frac{u_g}{A_1}\Sigma_g x_g} dx$$

$$= \frac{1}{(\log A_2)^n} \cdot \frac{1}{\prod_g (u_g A_2^{\frac{Q_0}{A_1} u_g})} .$$

By the above estimation, the inequality in (2) is obtained.

Remark. We can take a fundamental domain D of $\boldsymbol{R}_+^{\times a}$ by the action of U (a subgroup of E^+ with finite index) such that $D \subset \{u \in \boldsymbol{R}_+^{\times a} \mid u_{g_1}/u_{g_2} < A_1 (\forall g_1, \forall g_2 \in \Omega_0^a\}$ with a suitable constant A_1 depending only on U. This can be seen, for example, by [Sh1]. Further, for such and an ideal $\mathfrak{A}$ of F, $\mathfrak{A} \cap D \subset [A_2, \infty)^a$ with a suitable constant A_2 depending only on U and $\mathfrak{A}$.

The following expansions of $\Gamma(s)$ will be used in the proof, i.e.,

(3.4) $$\Gamma(1+\xi s)^m = 1 - \xi m \gamma_0 \cdot s + O(s^2) ,$$

(3.5) $$\left(\frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}+\xi s\right)\right)^m = 1 - \xi m(\gamma_0 + 2\log 2) \cdot s + O(s^2) .$$

Proof of Proposition 2. Let the meanings of †, ‡ be as in (3.3), (3.1), respectively. Assume that $\Re(s)>1$. We begin with the following equation:

(3.6) $$\sum_{(\alpha,\beta) \in \mathfrak{a}_1 \times \mathfrak{b}_1 - (0,0)/E(\mathfrak{f}_0)} \frac{e_F(\alpha a - \beta b)}{N\|\alpha + \beta z\|^s} = \frac{1}{N_F \mathfrak{a}_1^{2s}} \sum_{\alpha \in \mathfrak{a}_1 - 0/E(\mathfrak{f}_0)} \frac{e_F(\alpha a)}{N_F((\alpha)\mathfrak{a}_1^{-1})^{2s}}$$
$$+ \frac{1}{Ny^{2s}} \sum_{\beta \in \mathfrak{b}_1 - 0/E(\mathfrak{f}_0)} \frac{e_F(-\beta b - \beta a x)}{N_F(\beta)^{2s}}$$
$$\times \sum_{\alpha_1 \in \frac{1}{\beta y}\mathfrak{a}_1} \frac{e_F\left(\left(\alpha_1 + \frac{x}{y}\right)\beta a y\right)}{N\left(\left(\alpha_1+\frac{x}{y}\right)^2+1\right)^s} ,$$

where $\alpha+\beta z=\beta y\left(\alpha_1+\frac{x}{y}+i\right)$ and $\alpha_1=\frac{\alpha}{\beta y}$. By (1) of Lemma 3 with its remark, we see that the sums on the right hand side of (3.6) are uniformly absolutely convergent on any compact subset in the right half plane $\{s\in \boldsymbol{C}|\Re(s)>1\}$. Applying Poisson's summation formla to the inner sum of the second term on the right hand side of (3.6), we have

$$\sum_{\alpha_1\in\frac{1}{\beta y}\mathfrak{a}_1}\frac{e_F\left(\left(\alpha_1+\frac{x}{y}\right)\beta a y\right)}{N\left(\left(\alpha_1+\frac{x}{y}\right)^2+1\right)^s}$$

$$=\frac{Ny\cdot N_F(\beta)}{\sqrt{d_F}N_F\mathfrak{a}_1}\sum_{\alpha_2\in\beta y\mathfrak{a}^{-1}}e_F\left(-\alpha_2\frac{x}{y}\right)\int_{\boldsymbol{R}^a}\frac{e_F(\beta a y t+\alpha_2 t)}{N(t^2+1)^s}dt\,.$$

The sum $\sum_{\alpha_2}$ is rewritten by setting $\alpha=\frac{1}{\beta y}\alpha_2+a$ as

$$=\frac{Ny\cdot N_F(\beta)}{\sqrt{d_F}N_F\mathfrak{a}_1}e_F(\beta a x)\sum_{\alpha\in\mathfrak{a}^{-1}+a}e_F(-\alpha\beta x)\prod_g\int_{-\infty}^{\infty}\frac{e(\alpha_g\beta_g y_g t_g)}{(t_g{}^2+1)^s}dt_g\,.$$

By means of (1) of Lemma 2, the integrals $\int_{-\infty}^{\infty}$ may be rewritten as $\int_P$ if $\alpha\neq 0$, and as $(\sqrt{\pi}\Gamma(s-1/2)\Gamma(s)^{-1})^n$ if $\alpha=0$, the latter of which appears only when $a\in\mathfrak{a}^{-1}$. Here the criterions of Poisson's summation formula are satisfied by (2) of Lemma 2 and (2) of Lemma 3, and so the above transformation is ensured. Then, (3.6) is rewritten as

$$\text{(3.7)}\qquad \frac{1}{\mathrm{Ind}(E(\mathfrak{f}_0))}\sum_{(\alpha,\beta)}\frac{e_F(\alpha a-\beta b)}{N\|\alpha+\beta z\|^s}=\text{(i)}+\text{(ii)}+\text{(iii)}^{\ddagger}\,,$$

where

$$\text{(i)}=\frac{1}{N_F\mathfrak{a}^{2s}}\frac{1}{\mathrm{Ind}(E(\mathfrak{m}))}\sum_{\alpha\in\mathfrak{a}_1-0/E(\mathfrak{m})}\frac{e_F(\alpha a)}{N_F((\alpha)\mathfrak{a}_1{}^{-1})^{2s}}\,,$$

$$\text{(ii)}=\frac{1}{\sqrt{d_F}N_F\mathfrak{a}_1 N_F\mathfrak{b}_1{}^{2s-1}Ny^{2s-1}}\frac{1}{\mathrm{Ind}(E(\mathfrak{f}_0))}$$

$$\times\sum_{\beta\in\mathfrak{b}_1-0/E(\mathfrak{f}_0)}\frac{e_F(-\beta b)}{N_F((\beta)\mathfrak{b}_1{}^{-1})^{2s-1}}\sum_{0\neq\alpha\in\mathfrak{a}^{-1}+a}e_F(-\alpha\beta x)\prod_g\int_P\frac{e(\alpha_g\beta_g y_g t_g)}{(t_g{}^2+1)^s}dt_g\,,$$

$$\text{(iii)}=\frac{1}{\sqrt{d_F}N_F\mathfrak{a}_1 N_F\mathfrak{b}_1{}^{2s-1}Ny^{2s-1}}\left(\frac{\sqrt{\pi}\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\right)^n$$

$$\times\frac{1}{\mathrm{Ind}(E(\mathfrak{n}))}\sum_{\beta\in\mathfrak{b}_1-0/E(\mathfrak{n})}\frac{e_F(\beta b)}{N_F((\beta)\mathfrak{b}_1{}^{-1})^{2s-1}}\,.$$

By Proposition 1, we obtain the following two equations under the assumptions that $\Re(s)>\frac{1}{2}$ and $\Re(s)>1$:

$$\sum_{\alpha\in\mathfrak{a}_1-0/E(\mathfrak{m})}\frac{e_F(\alpha a)}{N_F((\alpha)\mathfrak{a}_1^{-1})^{2s}}$$

$$=(d_F^{\frac{1}{2}-2s}N_F\mathfrak{m}^{1-2s})\cdot(2\sin\pi s\cdot(2\pi)^{-(1-2s)}\cdot\Gamma(1-2s))^n\cdot\zeta_F(1-2s,\mathfrak{a}_a)\,,$$

$$\sum_{\beta\in\mathfrak{b}_1-0/E(\mathfrak{n})}\frac{e_F(\beta b)}{N_F((\beta)\mathfrak{b}_1^{-1})^{2s-1}}$$

$$=(d_F^{\frac{3}{2}-2s}N_F\mathfrak{n}^{2-2s})\cdot(-2\cos\pi s\cdot(2\pi)^{-(2-2s)}\cdot\Gamma(2-2s))^n\cdot\zeta_F(2-2s,\mathfrak{b}_b)\,.$$

We can modify (i) and (iii) by them. Then, after changing s into $1+s$, (3.7) may be rewritten as

$$\text{(3.8)}\qquad \boldsymbol{E}(z,1+s)=\text{(i)'}+\text{(ii)'}+\text{(iii)'}^{\dagger}\,,$$

where

$$\text{(i)'}=\frac{\sqrt{d_F}}{\mathrm{Ind}(E(\mathfrak{m}))}\frac{(-2\pi)^n}{N_F(\mathfrak{m}\mathfrak{c})}\zeta_F(1-2s,\mathfrak{a}_a)$$

$$\times\left(\frac{Ny}{N_F(\mathfrak{m}^2\mathfrak{c})}\right)^s\left((2\pi)^{2s}\frac{\sin\pi s}{\pi s}\,\frac{\Gamma(1-2s)}{(2s+1)}\right)^n,$$

$$\text{(ii)'}=\frac{\sqrt{d_F}}{\mathrm{Ind}(E(\mathfrak{f}_0))}\sum_{\beta\in\mathfrak{b}_1-0/E(\mathfrak{f}_0)}\ \sum_{0\neq\alpha\in\mathfrak{a}^{-1}+a}\frac{e_F(-\beta b-\alpha\beta x)}{N_F((\beta)\mathfrak{b}_1^{-1})^{2s+1}}$$

$$\times\prod_g\frac{1}{\pi}\int_P\frac{e(\alpha_g\beta_g y_g t_g)}{(t_g{}^2+1)^{s+1}}dt_g\cdot\left(\frac{N_F\mathfrak{c}}{Ny}\right)^s,$$

$$\text{(iii)'}=\frac{1}{\mathrm{Ind}(E(\mathfrak{n}))}\frac{\zeta_F(-2s,\mathfrak{b}_b)}{(-s)^n}(N_F(\mathfrak{n}^2\mathfrak{c}_1)Ny(2\pi)^{-2n})^{-s}$$

$$\times\left(\cos\pi s\frac{\Gamma\left(\frac{1}{2}+s\right)}{\sqrt{\pi}}\,\frac{\Gamma(1-2s)}{\Gamma(1+s)}\right)^n.$$

By means of (2) of Lemma 2 ad (2) of Lemma 3 with its remark, (i)', (ii)' and (iii)' can be meromorphically continued to the whole s-plane, and (ii)' becomes an entire function. Moreover, for each $s\in\boldsymbol{C}$, it can be seen similarly that the right hand side of (3.8) is uniformly absolutely convergent on any compact subset in $\mathfrak{H}^a$ with respect to z.

The terms (i)' and (ii)' are holomorphic at $s=0$, and the values of the integrals in (ii)' at $s=0$ are obtained from (3) of Lemma 2. They give (3.3) in the case where $a\notin\mathfrak{a}^{-1}$. Assume that $a\in\mathfrak{a}^{-1}$. To obtain the terms with † on the right hand side of (3.3) and the term with ‡ in (3.1), we need to calculate the Taylor expansion of (iii)' at $s=0$.

By using (2.1), and using (3.4), (3.5), we have

$$\frac{\zeta_F(-2s, \mathfrak{b}_b)}{(-s)^n}=\left[\frac{2^{n-2}R_F}{s}\right]^{\dagger}+2^n\kappa_F(\mathfrak{b}_b)+O(s)\,,$$

$$\left(\cos\pi s\frac{\Gamma\left(\frac{1}{2}+s\right)}{\sqrt{\pi}}\frac{\Gamma(1-2s)}{\Gamma(1+s)}\right)^n=1+2n(\gamma_0-\log 2)\cdot s+O(s^2)\,,$$

respectively. From the above two expansions with

$$\begin{aligned}&(N_F(\mathfrak{n}^2\mathfrak{c}_1)Ny(2\pi)^{-2n})^{-s}\\&\quad=1-\{\log(N_F(\mathfrak{n}^2\mathfrak{c}_1)Ny)-2n\log 2\pi\}\cdot s+O(s^2)\,,\end{aligned}$$

and with the remark that $\mathfrak{n}=I_F$ and $E(\mathfrak{n})=E$ if $(a, b)\in\mathfrak{a}^{-1}\times\mathfrak{b}^{-1}$, we obtain the following expansion of (iii)':

$$\begin{aligned}\text{(iii)'}=&\left[\frac{2^{n-2}R_F}{s}\right]^{\dagger}+2^n\kappa_F(\mathfrak{b}_b)\\&-[2^{n-2}R_F\{\log(N_F\mathfrak{c}_1Ny)-2n(\log\pi+\gamma_0)\}]^{\dagger}+O(s)\,,\end{aligned}$$

which gives the remaining parts of the right hand side of (3.3). Hence we obtain our proposition.

§ 4. Function $f^{\sigma}_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)$ and its properties

In this section, we describe several properties and Fourier expansion of $f_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)$ and some functions associated with $f_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)$.

Besides the notations in the previous sections, we define more several notations associated with $(\boldsymbol{C}-\boldsymbol{R})^{\mathfrak{a}}$. Let $\mathfrak{S}=\{1, -1\}$ be the multiplicative group of order 2. We denote by $+$ the unit element of $\mathfrak{S}^{\mathfrak{a}}$, by $-\sigma$ the inverse element of $\sigma\in\mathfrak{S}^{\mathfrak{a}}$, and put $\mathrm{sgn}(\sigma)=\prod_g\sigma_g(\sigma\in\mathfrak{S}^{\mathfrak{a}})$. For $z\in(\boldsymbol{C}-\boldsymbol{R})^{\mathfrak{a}}$, let $\varpi(z)$ be the element of $\mathfrak{S}^{\mathfrak{a}}$ such that $(\varpi(z))_g=\mathrm{sgn}(\mathfrak{T}(z_g))$. For $\sigma\in\mathfrak{S}^{\mathfrak{a}}$, put $\mathfrak{H}^{\sigma}=\{z\in(\boldsymbol{C}-\boldsymbol{R})^{\mathfrak{a}}|\varpi(z)=\sigma\}$ (in particular, $\mathfrak{H}^{+}=\mathfrak{H}^{\mathfrak{a}}$), and, for $\in(\boldsymbol{C}-\boldsymbol{R})^{\mathfrak{a}}$, define $z^{\sigma}\in\mathfrak{H}^{\sigma}$ by $(z^{\sigma})_g=z_g$ or $\overline{z_g}$ according as $(\varpi(z))_g=\sigma_g$ or $-\sigma_g$. This definition is consistent with z^{+} in § 3.

For each $\sigma\in\mathfrak{S}^{\mathfrak{a}}$, define a holomorphic function $\mathfrak{H}^{\sigma}\ni z\mapsto f^{\sigma}_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)\in\boldsymbol{C}$ associated with $f_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)$ by

$$\text{(4.1)}\qquad \begin{aligned}f^{\sigma}_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)=&\frac{1}{\mathrm{Ind}(E(m))}\frac{\zeta_F(-1, \mathfrak{a}_a)}{N_F(\mathfrak{m}\mathfrak{c})}\mathrm{sgn}(\sigma)(\pi i)^n Nz\\&+\frac{1}{\mathrm{Ind}(U)}\sum_{\beta\in\mathfrak{b}_1-0/U}\ \sum_{\alpha\in\mathfrak{a}^{-1}+a,\,\varpi(\alpha\beta)=\sigma}\frac{e_F(\beta b+\alpha\beta z)}{N_F((\beta)\mathfrak{b}_1^{-1})}\end{aligned}$$

$$+\left[\frac{1}{\mathrm{Ind}(E(\mathfrak{n}))}\cdot\frac{\kappa_F(\mathfrak{b}_b)}{\sqrt{d_F}}\right]^{\ddagger},$$

where the notations are as in (3.1). Then, (3.1) is written as

$$f_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)=\sum_{\sigma\in\mathfrak{S}^a} f^{\sigma}_{\mathfrak{a}\times\mathfrak{b}}(a, b; z^{\sigma}). \tag{4.2}$$

Since $f_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)$ is absolutely convergent on $\mathfrak{H}^a$ by Proposition 2, $f^{\sigma}_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)$ is also absolutely convergent on $\mathfrak{H}^{\sigma}$. The function $f^{\sigma}_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)$ depends only on the class (a, b) mod $\mathfrak{a}^{-1}\times\mathfrak{b}^{-1}$ as $f_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)$ does so. Further, for $z\in\mathfrak{H}^a$, put

$$f^{\Omega_0}_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)=\sum_{\sigma\in\mathfrak{S}^a,\, \sigma_e=1 (e\in\Omega_0)} f^{\sigma}_{\mathfrak{a}\times\mathfrak{b}}(a, b; z^{\sigma}), \tag{4.3}$$

which is holomorphic with respect to z_e $(e\in\Omega_0)$. If $(a, b)\in\mathfrak{a}^{-1}\times\mathfrak{b}^{-1}$, we denote $f^{\sigma}_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)$, $f_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)$ and $f^{\Omega_0}_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)$ simply by $f^{\sigma}_{\mathfrak{a}\times\mathfrak{b}}(z)$, $f_{\mathfrak{a}\times\mathfrak{b}}(z)$ and $f^{\Omega_0}_{\mathfrak{a}\times\mathfrak{b}}(z)$ respectively. In this case, (4.1) and (3.1) is rewritten as

$$\begin{aligned} f^{\sigma}_{\mathfrak{a}\times\mathfrak{b}}(z)=&\,\mathrm{sgn}(\sigma)\frac{\zeta_F(-1, \mathfrak{a})}{N_F\mathfrak{c}}(\pi i)^n Nz \\ &+\sum_{\lambda\in\mathfrak{c}^{-1},\, \varpi(\lambda)=\sigma} \boldsymbol{o}^{-1}((\lambda)\mathfrak{c}, \mathfrak{b}_1^{-1})\boldsymbol{e}_F(\lambda z)+\frac{\kappa_F(\mathfrak{b})}{\sqrt{d_F}}, \end{aligned} \tag{4.4}$$

$$\begin{aligned} f_{\mathfrak{a}\times\mathfrak{b}}(z)=&\,\frac{\zeta_F(-1, \mathfrak{a})}{N_F\mathfrak{c}}(-2\pi)^n Ny \\ &+\sum_{0\neq\lambda\in\mathfrak{c}^{-1}} \boldsymbol{o}^{-1}((\lambda)\mathfrak{c}, \mathfrak{b}_1^{-1})\boldsymbol{e}_F((\lambda z)^{+})+\frac{2^n\kappa_F(\mathfrak{b})}{\sqrt{d_F}}, \end{aligned} \tag{4.5}$$

where $\boldsymbol{o}^m(\mathfrak{A}, \mathfrak{B})=\sum_{\mathfrak{C}|\mathfrak{A},\mathfrak{C}\sim\mathfrak{B}} N_F(\mathfrak{C})^m$, and $\mathfrak{C}\sim\mathfrak{B}$ means $\mathfrak{C}$ and $\mathfrak{B}$ belong to the same class in $H_F(I_F)$. In the case where $n=1$, i.e., $F=\boldsymbol{Q}$, for $(a, b)\in\boldsymbol{Q}\times\boldsymbol{Q}$, we have

$$f^{+}_{\boldsymbol{Z}\times\boldsymbol{Z}}(a, b; z)=\begin{cases} -\log(\sqrt{2\pi}\,\eta(z)) & (\text{if } (a, b)\in\boldsymbol{Z}\times\boldsymbol{Z}) \\ -\frac{1}{2}\log\phi(a, b; z) & (\text{if } 0<a<1) \end{cases}. \tag{4.6}$$

Here $\eta(z)$ and $\phi(a, b; z)$ are the Dedekind η-function and the Siegel function defined by

$$\begin{cases} \eta(z)=q_z^{\frac{1}{24}}\cdot\prod_{n\in N}(1-q_z^{\,n}) \\ \phi(a, b; z)=q_z^{\frac{1}{2}B_2(a)}(1-q_x)\cdot\prod_{n\in N}(1-q_z^{\,n}q_x)(1-q_z^{\,n}q_x^{-1}), \end{cases} \tag{4.7}$$

where $q_z=e(z)$, $q_x=e(x)$, $x=az+b$ and $B_2(a)=a^2-a+\frac{1}{6}$. The latter is well known as a modular unit (see [K-L]). Our function $f^{\sigma}_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)$ is viewed as a generalization of their logarithm.

By the definition, the following can easily be seen:

$$f^{(-\sigma)}_{\mathfrak{a}\times\mathfrak{b}}(a, b; z^{(-\sigma)})=\overline{f^{\sigma}_{\mathfrak{a}\times\mathfrak{b}}(a, b; z^{\sigma})} \quad \text{for} \quad z\in(\boldsymbol{C}-\boldsymbol{R})^{a}, \tag{4.8}$$

$$f^{\sigma}_{(\alpha)\mathfrak{a}\times(\beta)\mathfrak{b}}(a, b; z)=f^{\operatorname{sgn}(\beta\alpha^{-1})\sigma}_{\mathfrak{a}\times\mathfrak{b}}(\alpha a, \beta b; \beta\alpha^{-1}z) \quad \text{for} \quad z\in\mathfrak{H}^{\sigma}, \tag{4.9}$$

where $\alpha, \beta\in F^{\times}$. We have also the following distribution relation, i.e.,

$$\sum_{(a,b)} f^{\sigma}_{\mathfrak{a}\times\mathfrak{b}}(a_0+a, b_0+b; z)=f^{\sigma}_{\tilde{\mathfrak{a}}\times\tilde{\mathfrak{b}}}(a_0, b_0; z)-\left[\frac{R_F}{2\sqrt{d_F}}\log N_F(\tilde{\mathfrak{b}}\mathfrak{b}^{-1})\right]^{\dagger}, \tag{4.10}$$

where $\tilde{\mathfrak{a}}\subset\mathfrak{a}$, $\tilde{\mathfrak{b}}\subset\mathfrak{b}$, (a, b) runs over $\tilde{\mathfrak{a}}^{-1}\times\tilde{\mathfrak{b}}^{-1}/\mathfrak{a}^{-1}\times\mathfrak{b}^{-1}$, and the term with the symbol $\dagger$ appears only when $(a_0, b_0)\in\tilde{\mathfrak{a}}^{-1}\times\tilde{\mathfrak{b}}^{-1}$. The equation (4.10) is rewritten by (4.9) as

$$\sum_{(a,b)} f^{\sigma}_{\mathfrak{a}\times\mathfrak{b}}(a_0+a, b_0+b; z)=f^{\sigma}_{\mathfrak{a}\times\mathfrak{b}}(\alpha a_0, \beta b_0; \beta\alpha^{-1}z)-\left[\frac{R_F}{2\sqrt{d_F}}\log N_F(\beta)\right]^{\dagger}, \tag{4.11}$$

where $\alpha, \beta\in I_F$, $\beta\alpha^{-1}\gg 0$ and (a, b) runs over $(\alpha)^{-1}\mathfrak{a}^{-1}\times(\beta)^{-1}\mathfrak{b}^{-1}/\mathfrak{a}^{-1}\times\mathfrak{b}^{-1}$, and the term with the symbol $\dagger$ appears only when $(a_0, b_0)\in(\alpha)^{-1}\mathfrak{a}^{-1}\times(\beta)^{-1}\mathfrak{b}^{-1}$. (4.10) and (4.11) are viewed as a generalization of a distribution relation of Siegel functions in [K-L].

To describe automorphy of $f_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)$, we introduce some more notations. Put $\boldsymbol{G}=GL_2(\boldsymbol{R})$. For $A=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\boldsymbol{G}$ and $z\in\mathfrak{H}$ (resp. $\boldsymbol{C}\cup\{\infty\}$), we define $Az\in\mathfrak{H}$ (resp. $\boldsymbol{C}\cup\{\infty\}$) by

$$Az=\begin{cases}(\alpha z+\beta)(\gamma z+\delta)^{-1} & (\text{if } \det A>0)\\ (\alpha\bar{z}+\beta)(\gamma\bar{z}+\delta)^{-1} & (\text{if } \det A<0)\,.\end{cases} \tag{4.12}$$

We define the action of $\boldsymbol{G}$ on $\mathfrak{H}$ (resp. $\boldsymbol{C}\cup\{\infty\}$) by $Az\in\mathfrak{H}$ (resp. $\boldsymbol{C}\cup\{\infty\}$) for $A\in\boldsymbol{G}$ and $z\in\mathfrak{H}$ (resp. $\boldsymbol{C}\cup\{\infty\}$), and the action of $\boldsymbol{G}^{a}$ on $\mathfrak{H}^{a}$ (resp. $(\boldsymbol{C}\cup\{\infty\})^{a}$) componentwisely. For two ideals $\mathfrak{a}, \mathfrak{b}$ and an integral ideal $\mathfrak{f}_0$ of F, we define

$$\Gamma_{\mathfrak{a}\times\mathfrak{b}}=\left\{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in GL_2(F)\,\middle|\,\alpha, \delta\in I_F,\ \beta\in\mathfrak{a}\mathfrak{b}^{-1},\ \gamma\in\mathfrak{a}^{-1}\mathfrak{b},\ \alpha\delta-\beta\gamma\in E_F\right\}, \tag{4.13}$$

$$\Gamma_{\mathfrak{a}\times\mathfrak{b}}(\mathfrak{f}_0)=\left\{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\Gamma_{\mathfrak{a}\times\mathfrak{b}}\,\middle|\,\alpha, \delta\in\mathfrak{f}_0+1,\ \beta\in\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{f}_0,\ \gamma\in\mathfrak{a}^{-1}\mathfrak{b}\mathfrak{f}_0\right\}. \tag{4.14}$$

Now automorphy of $f_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)$, which is derived from that of $\boldsymbol{E}(z, s)$, is described as

(4.15) $$f_{\mathfrak{a}\times\mathfrak{b}}((a, b); Az)=f_{\mathfrak{a}\times\mathfrak{b}}((a, b)A; z)$$

$$\text{for}\quad (a, b)\notin\mathfrak{a}^{-1}\times\mathfrak{b}^{-1}, \qquad A\in\Gamma_{\mathfrak{a}\times\mathfrak{b}},$$

(4.16) $$f_{\mathfrak{a}\times\mathfrak{b}}((a, b); Az)=f_{\mathfrak{a}\times\mathfrak{b}}((a, b); z)$$

$$\text{for}\quad (a, b)\in\mathfrak{f}_0^{-1}\mathfrak{a}^{-1}\times\mathfrak{f}_0^{-1}\mathfrak{b}^{-1}, \quad (a, b)\notin\mathfrak{a}^{-1}\times\mathfrak{b}^{-1}, \quad A\in\Gamma_{\mathfrak{a}\times\mathfrak{b}}(\mathfrak{f}_0),$$

(4.17) $$f_{\mathfrak{a}\times\mathfrak{b}}(Az)=f_{\mathfrak{a}\times\mathfrak{b}}(z)-\frac{2^{n-2}R_F}{\sqrt{d_F}}\log N\|j(A, z)\|$$

$$\text{for}\quad A\in\Gamma_{\mathfrak{a}\times\mathfrak{b}},$$

where $j(A, z)=\gamma z+\delta$ for $A=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$.

Remark. A transformation law with respect to $f^{\sigma}_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)$ has been calculated in [G-T] (though in a much more complicated form).

Note that the function $\mathfrak{F}^{\sigma}_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)$ on $\mathfrak{H}^{\sigma}$ defined by

(4.18) $$\mathfrak{F}^{\sigma}_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)=\operatorname{sgn}(-\sigma)N_F\mathfrak{c}\cdot\partial_{\Omega_0{}^a}f^{\sigma}_{\mathfrak{a}\times\mathfrak{b}}(a, b; z),$$

where

(4.19) $$\partial_{\Omega_0{}^a}=\prod_g\partial_g, \qquad \partial_g=-\frac{1}{2\pi i}\frac{\partial}{\partial z_g}, \quad \frac{\partial}{\partial z_g}=\frac{1}{2}\left(\frac{\partial}{\partial x_g}-i\cdot\frac{\partial}{\partial y_g}\right),$$

in a holomorphic modular form of weight 2 with respect to $\Gamma_{\mathfrak{a}\times\mathfrak{b}}(\mathfrak{f}_0)$ except the case where $n=1$ and $(a, b)\in\mathfrak{a}^{-1}\times\mathfrak{b}^{-1}$. This can easily be seen by (4.16), (4.17) and (4.2). For the convenience of the reader, we derive the following Fourier expansions from (4.1) and (4.4):

(4.20) $$\mathfrak{F}^{\sigma}_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)=\frac{1}{\operatorname{Ind}(E(\mathfrak{m}))}\frac{\zeta_F(-1, \mathfrak{a}_a)}{2^nN_F\mathfrak{m}}$$

$$+\frac{1}{\operatorname{Ind}(U)}\cdot\sum_{\beta\in\mathfrak{b}^{-1}-0/U}\ \sum_{\alpha\in\mathfrak{a}^{-1}+a,\,\varpi(\alpha\beta)=\sigma}N_F((\alpha)\mathfrak{a})\boldsymbol{e}_F(\beta b+\alpha\beta z),$$

and if $(a, b)\in\mathfrak{a}^{-1}\times\mathfrak{b}^{-1}$,

(4.21) $$\mathfrak{F}^{\sigma}_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)=\frac{\zeta_F(-1, \mathfrak{a})}{2^n}+\sum_{\lambda\in\mathfrak{c}^{-1},\,\varpi(\lambda)=\sigma}\boldsymbol{o}^1((\lambda)\mathfrak{c}, \mathfrak{a})\boldsymbol{e}_F(\lambda z).$$

§ 5. Period $\langle\boldsymbol{\Phi}_{\omega}, d_{\Delta}f\rangle$ and generalization of Hecke's method

Let K be a quadratic extension of F. We denote the conjugate of $\lambda\in K$ over F by λ^{ι} and the trace of λ with respect to K/F by $Tr_{K/F}(\lambda)$. Let Ω^a, Ω^r

and Ω^c be the set of all archimedean, real and imaginary places of K respectively, and divide the set $\Omega_0{}^a$ of all archimedean places of F into the disjoint union of $\Omega_0{}^r$ and $\Omega_0{}^c$ lying below Ω^r and Ω^c. Moreover we divide $\Omega_0{}^r$ into two subsets Ω_0 and Ω_0' arbitrarily and consider the disjoint union $\Omega_0{}^a=\Omega_0\cup\Omega_0'\cup\Omega_0^c$ as an ordered partition of $\Omega_0{}^a$, denoted by $\rho=(\Omega_0, \Omega_0', \Omega_0^c)$. Here we may allow one or two of Ω_0, Ω_0' and Ω_0^c to be empty. In the rest of the paper, as g for $\Omega_0{}^a$ in § 3 and 4, we use the notations, r, e, e' and c to express general elements of $\Omega_0{}^r$, Ω_0, Ω_0' and $\Omega_0{}^c$ respectively. We denote by r_0, e_0, e_0' and c_0 the number of elements contained in $\Omega_0{}^r$, Ω_0, Ω_0' and $\Omega_0{}^c$ respectively ($e_0+e_0'=r_0$, $r_0+c_0=n$). For a set X, as X^a, we denote by X^r and X^c the direct product of r_0- and c_0-copies of X indexed by $r\in\Omega_0{}^r$ and $c\in\Omega_0{}^c$ respectively.

Take $\omega\in K-F$. For each $r\in\Omega_0{}^r$, choose and fix one of the two real places of K above r (denoted by r again) in such a way as $\omega_r>\omega^{\iota}{}_r$. Similarly, for each $c\in\Omega_0{}^c$, choose and fix one of the two embeddings of into $\boldsymbol{C}$ which coincide with c on F (denoted by c again) in such a way as $\mathfrak{I}(\omega_c)>0$. With the above choices, we consider the embedding $K\ni\lambda\hookrightarrow(\cdots, \lambda_r, \cdots, \lambda_c, \cdots)\in\boldsymbol{C}^a$, and regard elemnts of K as those of $\boldsymbol{C}^a$ through this injection. Note that this injection depends on ω.

For $\omega\in K-F$, put

$$(5.1)\qquad dz_{\Omega_0}=\bigwedge_e dz_e\,,\quad d_\omega z_{\Omega_0'}=\bigwedge_e d_\omega z_{e'}\,,\quad d_\omega z_{e'}=\frac{-(\omega_{e'}-\omega^{\iota}{}_{e'})dz_{e'}}{2(z_{e'}-\omega_{e'})(z_{e'}-\omega^{\iota}{}_{e'})}\,,$$

$$(5.2)\qquad \partial_{\Omega_0}=\prod_e\partial_e\qquad(\text{see (4.19) as to }\partial_e)\,.$$

For a function f on $\mathfrak{H}^a$, we define an r_0-differential form $d_\Delta f$ ($\Delta=(\omega, \rho)$, $\rho=(\Omega_0, \Omega_0', \Omega_0^c)$) on $\mathfrak{H}^a$ by

$$(5.3)\qquad \boldsymbol{d}_\Delta f=\partial_{\Omega_0}f(\cdots, z_r, \cdots, \omega_c, \cdots)dz_{\Omega_0}\wedge d_\omega z_{\Omega_0'}\,.$$

Note that if $\Omega_0=\Omega_0'=\phi$, then $d_\Delta f=f(\omega)$, which is an element of $\boldsymbol{C}$.

Put $\boldsymbol{G}_\omega{}^a=\{A\in\boldsymbol{G}^a|A\omega=\omega,\ A\omega^\iota=\omega^\iota\}$ (see (4.12) as to $A\omega$). In general, for a subgroup H of $\boldsymbol{G}^a$, we set $H_\omega=H\cap\boldsymbol{G}_\omega{}^a$, $H^+=\{A\in H|\det A_g>0\ (g\in\Omega_0{}^a)\}$, $H_\omega{}^+=H^+\cap\boldsymbol{G}_\omega{}^a$, and denote by $\bar{H}$ the image of H by the natural projection $\boldsymbol{G}^a\to\boldsymbol{G}^a/$(the center of $\boldsymbol{G}^a$). Let Γ_0 be a discrete subgroup of $\boldsymbol{G}^a$. Put

$$(5.4)\qquad \mathfrak{M}(\Gamma_0)=\bigcup_\Gamma\mathfrak{J}(\Gamma)\,,\quad \mathfrak{J}(\Gamma)=\{f:\mathfrak{H}^a\xrightarrow{\text{harmonic}}\boldsymbol{C}|f(Az)=f(z)\text{ for }A\in\Gamma\}\,,$$

where Γ runs over all subgroups of Γ_0 with finite index, and 'harmonic' means real C^∞-class and vanishing by the operation of $\dfrac{\partial^2}{\partial z_g\partial\bar{z}_a}$ for all $g\in\Omega_0{}^a$. We can all element of $\mathfrak{J}(\Gamma)$ a harmonic modular function on $\mathfrak{H}^a$ with respect to Γ. We define a set $\Phi_\omega(\Gamma_0)$ of η_0-chains on $\mathfrak{H}^a$ by

$$\Phi_\omega(\Gamma_0)=\left\{Z(\Gamma, t, D)=\frac{1}{|(\Gamma_0)_\omega : \Gamma_\omega|}\cdot D \,\middle|\, \Gamma, t, D\right\}, \tag{5.5}$$

where Γ is a subgroup of $(\Gamma_0)^+$ with a fundamental domain of the orbit $\boldsymbol{G}_\omega^{a}t$ of t by Γ_ω. Let f be an element of $\mathfrak{M}(\Gamma_0)$. For Γ such that $f\in\mathfrak{J}(\Gamma)$ and for $Z=Z(\Gamma, t, D)\in\Phi_\omega(\Gamma_0)$, we set

$$\langle Z, \boldsymbol{d}_\Delta f\rangle=\int_Z d_\Delta f \text{ (if } r_0>0) \quad\text{or}\quad \frac{1}{\#((\Gamma_0)_\omega)}\cdot \boldsymbol{d}_\Delta f \text{ (if } r_0=0), \tag{5.6}$$

which depends only on t in the former case and is independent of Z in the latter.

If we put $f(z)=f_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)$, which belongs to $\mathfrak{M}(\Gamma_{\mathfrak{a}\times\mathfrak{b}})$, the value $\langle Z, \boldsymbol{d}_\Delta f\rangle$ for $Z\in\Phi_\omega(\Gamma_{\mathfrak{a}\times\mathfrak{b}})$ is independent also of the choice of t (see, Proposition 3). Here we denote it also by $\langle\Phi_\omega, \boldsymbol{d}_\Delta f\rangle$.

In the rest of this section, we shall generalize Hecke's method which is developed in Chapter 2 of [Si1].

Put $I=\left\{\theta\in\boldsymbol{R}\,\middle|\, |\theta|<\frac{1}{4}\right\}$, $S=\{\zeta= \boldsymbol{e}(\theta)|\theta\in I\}$ and $\mathfrak{S}=S\times\boldsymbol{R}_+^\times$, which may be regarded as the complex right half plane through the correspondence of $(\zeta, \xi)\in\mathfrak{S}$ to $\zeta\xi$. Put $K_a{}^\times=(\boldsymbol{R}^\times\times\boldsymbol{R}^\times)^r\times\boldsymbol{C}^{\times c}$. For $\eta=(\cdots, \eta_r, \cdots, \eta_c, \cdots)\in K_a{}^\times$, let $\eta_{g1}=\eta_{r1}$ or η_c, $\eta_{g2}=\eta_{r2}$ or $\overline{\eta_c}$ according as $g=r$ or c, where $\eta_r=(\eta_{r1}, \eta_{r2})$. Through the injection $K^\times\ni\lambda\hookrightarrow\eta=(\cdots, (\lambda_r, \lambda^\iota{}_r), \cdots, \lambda_c, \cdots)\in K_a{}^\times$ (r, c are defined as before), we identify elements of $K^\times$ with those of $K_a{}^\times$. We define the action of $K_a{}^\times$ on $\mathfrak{S}^a$ by $\eta\cdot u=(\cdots, (\eta_{r1}/\eta_{r2})u_r, \cdots, u_c, \cdots)$ or $(\cdots, -(\eta_{r1}/\eta_{r2})\overline{u_r}, \cdots, u_c, \cdots)$ ($\eta\in K_a{}^\times$, $u\in\mathfrak{S}^a$) according as $\eta_{r1}/\eta_{r2}>0$ or <0.

Next, for $\omega\in K-F$, define φ_ω: $\mathfrak{S}^a\ni u\mapsto z\in\mathfrak{H}^a$ by

$$\begin{cases} z_r=\dfrac{\omega_r u_r i+\omega^\iota{}_r}{u_r i+1} & \left(u_r i=-\dfrac{z_r-\omega^\iota{}_r}{z_r-\omega_r}\right) \qquad (r\in\Omega_0{}^r) \\ z_c=\omega_c & (c\in\Omega_0{}^c), \end{cases} \tag{5.7}$$

and the embedding ψ_ω: $K_a{}^\times\ni\eta\mapsto A\in\boldsymbol{G}^a$ by

$$\begin{pmatrix}\alpha_g & \beta_g\\ \gamma_g & \delta_g\end{pmatrix}\begin{pmatrix}\omega_g & \omega^\iota{}_g\\ 1 & 1\end{pmatrix}=\begin{pmatrix}\omega_g & \omega^\iota{}_g\\ 1 & 1\end{pmatrix}\begin{pmatrix}\eta_{g1} & 0\\ 0 & \eta_{g2}\end{pmatrix}, \tag{5.8}$$

where $A_g=\begin{pmatrix}\alpha_a & \beta_g\\ \gamma_g & \delta_g\end{pmatrix}$. It can easily be seen that $\psi_\omega(K_a{}^\times)=\boldsymbol{G}_\omega{}^a$, and the actions of $K_a{}^\times$ on $\mathfrak{S}^a$ and $\boldsymbol{G}^a$ on $\mathfrak{H}^a$ are compatible with respect to φ_ω, namely, $\varphi_\omega(\eta\cdot u)=\psi_\omega(\eta)\varphi_\omega(u)$.

Let $\mathfrak{a}$, $\mathfrak{b}$ be ideals of F. If $\mathfrak{a}+\mathfrak{b}\omega$ is an ideal of K, then $\psi_\omega(E_K)=(\Gamma_{\mathfrak{a}\times\mathfrak{b}})_\omega$ and $\psi_\omega(E_F)=(\Gamma_{\mathfrak{a}\times\mathfrak{b}})_\omega\cap$ (the center of $\boldsymbol{G}^a$). For an integral ideal $\mathfrak{f}_0$ of F , let $E_K^*=\{\ \varepsilon\in E_K|\varepsilon\varepsilon^\iota\gg0\}$ and $E_K^*(\mathfrak{f}_0)=\{\varepsilon\in E_K^*|\varepsilon\equiv1(\mathrm{mod}\ \mathfrak{f}_0)\}$. Then, we have $\psi_\omega(E_K^*)=(\Gamma_{\mathfrak{a}\times\mathfrak{b}})_\omega{}^+$ and $\psi_\omega(E_K^*(\mathfrak{f}_0))=\Gamma_{\mathfrak{a}\times\mathfrak{b}}(\mathfrak{f}_0)_\omega{}^+$. In particular, we have $|E_K$:

$E_FE_K^*(\mathfrak{f}_0)|=|E_K/E_F\colon E_FE_K^*(\mathfrak{f}_0)/E_F|=|\overline{(\Gamma_{\mathfrak{a}\times\mathfrak{b}})_\omega}\colon \overline{\Gamma_{\mathfrak{a}\times\mathfrak{b}}(\mathfrak{f}_0)_\omega{}^+}|$.

For $(\alpha, \beta)\in F\times F-(0,0)$, put $\tilde{\lambda}=\alpha+\beta\omega$ and $v=\tilde{\lambda}\cdot u$. Then we have

$$(5.9)\qquad \left(\frac{y_{e'}}{\|\alpha_{e'}\beta_{e'}z_{e'}\|}\right)^s d_\omega z_{e'}=\frac{1}{4}\left(\frac{\omega_{e'}-\omega^\iota{}_{e'}}{|\tilde{\lambda}_{e'}\tilde{\lambda}^\iota_{e'}|}\right)^s\cdot 2\left(\frac{1}{v_{e'}+v_{e'}^{-1}}\right)^s\frac{dv_{e'}}{v_{e'}},$$

$$(5.10)\qquad \partial_e\left(\frac{y_e}{\|\alpha_e+\beta_e z_e\|}\right)^s dz_e=\frac{1}{4\pi}\left(\frac{\omega_e-\omega^\iota{}_e}{|\tilde{\lambda}_e\tilde{\lambda}^\iota_e|}\right)^s\cdot s\left(\frac{1}{v_e+v_e{}^{-1}}\right)^{s-1}$$

$$\times\frac{v_e i}{(v_e i+\mathrm{sgn}(\tilde{\lambda}_e\tilde{\lambda}_e^\iota))^2}\frac{dv_e}{v_e},$$

$$(5.11)\qquad \int_0^{\zeta_{e'}\infty}2\left(\frac{1}{v_{e'}+v_{e'}^{-1}}\right)^s\frac{dv_{e'}}{v_{e'}}=\frac{\Gamma\left(\frac{s}{2}\right)^2}{\Gamma(s)},$$

$$(5.12)\qquad \int_0^{\zeta_e\infty}s\left(\frac{1}{v_e+v_e{}^{-1}}\right)^{s-1}\frac{v_e i}{(v_e i+\mathrm{sgn}(\tilde{\lambda}_e\tilde{\lambda}^\iota{}_e))^2}\frac{dv_e}{v_e}=\mathrm{sgn}(\tilde{\lambda}_e\tilde{\lambda}_e^\iota)\frac{\Gamma\left(\frac{s+1}{2}\right)^2}{\Gamma(s)},$$

where $\zeta_{e'}$, $\zeta_e\in S$ (the right hand side of (5.11), (5.12) are independent of $\zeta_{e'}$, $\zeta_e\in S$), see [Si1].

Proposition 3. *Let $\mathfrak{B}$ be an ideal of K, $\mathfrak{f}$ an integral ideal of K, and put $\mathfrak{f}_0=\mathfrak{f}\cap F$. Assume that $\mathfrak{B}\subset(\mathfrak{f}\mathfrak{d}_K)^{-1}$ and $\mathfrak{B}\mathfrak{d}_F=\mathfrak{a}\nu_0+\mathfrak{b}\mu_0$ with two ideals $\mathfrak{a}$, $\mathfrak{b}$ of F and ν_0, $\mu_0\in K$. Put $a=Tr_{K/F}(\nu_0)$, $b=-Tr_{K/F}(\mu_0)$ and $\omega=\frac{\mu_0}{\nu_0}$. For a disjoint union $\Omega_0{}^r=\Omega_0\cup\Omega_0'$, put $\rho=(\Omega_0, \Omega_0', \Omega_0{}^c)$ and $\Delta=(\omega, \rho)$. Further, for $\boldsymbol{E}(z, s)$ in (3.2) associated with the above $\mathfrak{f}_0$, $\mathfrak{a}$, $\mathfrak{b}$, a and b, define the function $\boldsymbol{E}_s\colon \mathfrak{H}^a\to\boldsymbol{C}$ by $z\mapsto\boldsymbol{E}(z, s)$. Then, we have*

$$(5.13)\qquad \frac{1}{|E_K\colon E_K(\mathfrak{f}\Omega)|}\sum_{\lambda\in\mathfrak{B}-0/E_K(\mathfrak{f}\Omega)}\frac{\tau_\Omega(\lambda)\boldsymbol{e}_K(\lambda)}{N_K((\lambda)\mathfrak{B}^{-1})^s}$$

$$=\tau_\Omega(\nu_0)\frac{4^{r_0}N_K\mathfrak{f}^{\frac{s}{2}}\pi^{n(1-s)}\Gamma(s)^n}{G_K(s, \mathfrak{f}\Omega)}\langle\Phi_\omega, \boldsymbol{d}_\Delta\boldsymbol{E}_s\rangle .$$

Here $\langle\Phi_\omega, \boldsymbol{d}_\Delta\boldsymbol{E}_s\rangle$ is independent of the choice of $Z\in\Phi_\omega$.

Proof. For $(\alpha, \beta)\in\mathfrak{a}_1\times\mathfrak{b}_1$, put $\lambda=\alpha\nu_0+\beta\mu_0\in\mathfrak{B}$, $\tilde{\lambda}=\lambda/\nu_0$, and

$$h(u, \tilde{\lambda})=\prod_{e'}2(u_{e'}+u_{e'}{}^{-1})^{-s}\prod_e s(u_e+u_e{}^{-1})^{1-s}u_e i(u_e i+\mathrm{sgn}(\tilde{\lambda}_e\tilde{\lambda}_e^\iota))^{-2}\bigwedge_r\frac{du_r}{u_r}.$$

Here we consider that $h(u, \tilde{\lambda})=1$ if $r_0=0$. By (5.9) and (5.10), we have

$$(5.14)\qquad \boldsymbol{d}_\Delta\left[\prod_g\left(\frac{y_g}{\|\alpha_g+\beta_g z_g\|}\right)^s\right]=\frac{1}{4^{r_0}\pi^{e_0}2^{c_0 s}}\left(\frac{\prod_g|\omega_g-\omega^\iota{}_g|}{N_K((\tilde{\lambda}))}\right)^s h(\tilde{\lambda}\cdot u, \tilde{\lambda}).$$

Choose $Z=Z(\Gamma_{\mathfrak{a}\times\mathfrak{b}}(\mathfrak{f}_0)^+, t_0, D)\in\Phi_\omega$ and put $U=\varphi_\omega^{-1}(Z)$, $u_0=(\zeta_0, \xi_0)=\varphi_\omega^{-1}(t_0)$. Then, we obtain the following from (5.14) with $\boldsymbol{e}_F(\alpha a-\beta b)=\boldsymbol{e}_K(\lambda)$ and $\sqrt{d_K}$ $N_K((\nu_0)^{-1}\mathfrak{B})=N_F\mathfrak{c}_0\prod_g|\omega_g-\omega^{\iota}{}_g|$:

(5.15) $\quad \langle Z, \boldsymbol{d}_\Delta \boldsymbol{E}_s\rangle$

$$=\frac{1}{|E_F\colon E_F(\mathfrak{f}_0)|}\frac{d_K^{\frac{s}{2}}}{4^{r_0}\pi^{n+e_0}2^{c_0 s}}\sum_{\lambda\in\mathfrak{B}-0/E_F(\mathfrak{f}_0)}\frac{\boldsymbol{e}_K(\lambda)}{N_K((\lambda)\mathfrak{B}^{-1})^s}\int_U h(\tilde{\lambda}\cdot u, \tilde{\lambda}).$$

Here we should consider that the integral $\int_U h(\tilde{\lambda}\cdot u, \tilde{\lambda})$ is equal to $|E_K\colon E_F|^{-1}$ if $r_0=0$. The sum $\sum_{\lambda\in\mathfrak{B}-0/E_F(\mathfrak{f}_0)}$ on the right hand side of (5.15) may be considered as $\sum_{\lambda\in\mathfrak{B}-0/E_K{}^*(\mathfrak{f}_0)}\sum_{\varepsilon\in E_K{}^*(\mathfrak{f}_0)/E_F(\mathfrak{f}_0)}$ changing λ into $\lambda\varepsilon$. Then, we have

$$\sum_{\varepsilon\in E_K{}^*(\mathfrak{f}_0)/E_F(\mathfrak{f}_0)}\int_U h(\tilde{\lambda}\cdot(\varepsilon\cdot u), \tilde{\lambda}\cdot\varepsilon)=\frac{1}{|E_K\colon E_F E_K^*(\mathfrak{f}_0)|}\int_{\zeta_0 \boldsymbol{R}_+{}^{\times r}}h(u, \tilde{\lambda})$$

$$=\frac{\tau_\Omega(\lambda)\tau_\Omega(\nu_0)}{|E_K\colon E_F E_K^*(\mathfrak{f}_0)|}\frac{\Gamma\left(\frac{s}{2}\right)^{2e_0'}\Gamma\left(\frac{s+1}{2}\right)^{2e_0}}{\Gamma(s)^{r_0}},$$

by (5.11) and (5.12). Since $E_K^*(\mathfrak{f}_0)\cap E_F=E_F(\mathfrak{f}_0)$ and so $|E_K\colon E_F E_K^*(\mathfrak{f}_0)||E_F\colon E_F(\mathfrak{f}_0)|=|E_K\colon E_K^*(\mathfrak{f}_0)|$, the right hand side of (5.15) can be rewritten as

$$\frac{\tau_\Omega(\nu_0)}{|E_K\colon E_K^*(\mathfrak{f}_0)|}\frac{d_K^{\frac{s}{2}}}{4^{r_0}\pi^{n+e_0}2^{c_0 s}}\frac{\Gamma\left(\frac{s}{2}\right)^{2e_0'}\Gamma\left(\frac{s+1}{2}\right)^{2e_0}}{\Gamma(s)^{r_0}}\sum_{\lambda\in\mathfrak{B}-0/E_K{}^*(\mathfrak{f}_0)}\frac{\tau_\Omega(\lambda)\boldsymbol{e}_K(\lambda)}{N_K((\lambda)\mathfrak{B}^{-1})^s}.$$

Here we may change $E_K^*(\mathfrak{f}_0)$ into $E_K(\mathfrak{f}\Omega)$, and by the definition of $G_K(s, \mathfrak{f}\Omega)$ in (2.2), we can obtain (5.13).

§ 6. Description of $\kappa_K(C)$ by $f_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)$

Let K/F be as in § 5. We denote by $\mathfrak{d}_{K/F}$ the relative different with respect to K/F, and put $R_{K/F}=R_K/(2^{n-2}R_F W_K)$, which is equal to $|E_K\colon E_F|$ if K is a CM-field. Let Ω be a subset of Ω^r, $\mathfrak{f}$ an ideal of K, and put $\mathfrak{f}_0=\mathfrak{f}\cap F$. Throughout this section, we keep in mind that terms with the symbol †, †† are considered only when $\mathfrak{f}=I_K$, $\mathfrak{f}=I_K$ and $\Omega=\phi$, respectively. For $C\in H_K(\mathfrak{f}\Omega)$, (2.1) is rewritten as follows:

(6.1) $$\frac{\zeta_K(s, C)}{s^{2e_0'+c_0}}=\left[-\frac{R_K}{W_K}\cdot\frac{1}{s}\right]^{\dagger\dagger}+\kappa_K(C)+O(s).$$

We use the notation $\tilde{\kappa}_K(C)=|E_K\colon E_K(\mathfrak{f}\Omega)|^{-1}\kappa_K(C)$.

Theorem 1. *Assume that Ω is stable under the action of* Gal(K/F) *and consider $\rho=(\Omega_0, \Omega_0', \Omega_0{}^c)$ for the subsets Ω_0 and Ω_0' of $\Omega_0{}^r$ lying below Ω and $\Omega^r-\Omega$ respectively. For $C\in H_K(\mathfrak{f}\Omega)$, choose $\mathfrak{A}\in C^{-1}$ such that $1\in\mathfrak{A}$, and take* *ideals* $\mathfrak{a}, \mathfrak{b}$ *of* F *and elements* μ, ν *of* K *such that* $\mathfrak{A}\mathfrak{f}=\mathfrak{a}^{-1}\mu+\mathfrak{b}^{-1}\nu$. *In the case where* $\mathfrak{f}=I_K$ *and* $\Omega=\phi$, *let* $\mathfrak{A}_1=\mathfrak{A}(\mathfrak{b}^{-1}(\nu))^{-1}$. *Put* $\omega=\mu/\nu$, $\Delta=(\omega, \rho)$, *and let* (a, b) *be the element of* $\mathfrak{f}_0^{-1}\mathfrak{a}^{-1}\times\mathfrak{f}_0^{-1}\mathfrak{b}^{-1}$ *determined by* $1=a\mu+b\nu$. *Then, with* $\Phi_\omega=\Phi_\omega(\Gamma_{\mathfrak{a}\times\mathfrak{b}})$ *and* $f(z)=f_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)$, $\kappa_K(C)$ *can be expressed as follows:*

$$(6.2)\qquad \tau_\Omega(\nu)\cdot\hat{\kappa}_K(C)=\sqrt{d_F}\langle\Phi_\omega, \boldsymbol{d}_\Delta f\rangle-[2^{n-2}R_F\cdot T]^{\dagger},$$

where

$$(6.3)\qquad T=\begin{cases} R_{K/F}\log N_K\mathfrak{A}_1+\sum_{e'\in\Omega_0'}\left\langle\Phi_\omega, \boldsymbol{d}_\Delta\log\left(\frac{\mathfrak{T}(z_{e'})}{\omega_{e'}-\omega^{\iota}_{e'}}\right)\right\rangle & (\textit{if } \Omega_0=\phi)\\ \frac{1}{2}\langle\Phi_\omega, \boldsymbol{d}_\Delta\log(z_e-\omega_e)(z_e-\omega^{\iota}_e)\rangle & (\textit{if } \Omega_0=\{e\})\\ 0 & (\textit{if } e_0\geqq 2)\,. \end{cases}$$

Remark. By means of the definition of $\boldsymbol{d}_\Delta$ in (5.3) with (4.2), (4.3), we can substitute $f^{\Omega_0}_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)$ as f for $f_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)$ in (6.2). Recall that $f^{\Omega_0}_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)$ is holomorphic with respect to $z_e (e\in\Omega_0)$.

Proof. Put $\mathfrak{B}=(\mathfrak{A}\mathfrak{f}\mathfrak{d}_K)^{-1}$, $\mu_0=\mu^{\iota}/(\mu\nu^{\iota}-\mu^{\iota}\nu)$ and $\nu_0=\nu^{\iota}/(\mu\nu^{\iota}-\mu^{\iota}\nu)$. It can easily be seen that the conditions $1\in\mathfrak{A}$, $\mathfrak{A}\mathfrak{f}=\mathfrak{a}^{-1}\mu+\mathfrak{b}^{-1}\nu$, $1=a\mu+b\nu$, and $\omega=\mu/\nu$ induce $\mathfrak{B}\subset(\mathfrak{f}\mathfrak{d}_K)^{-1}$, $\mathfrak{B}=\mathfrak{a}_1\nu_0+\mathfrak{b}_1\mu_0$, $a=T_{K/F}(\nu_0)$ and $b=-T_{K/F}(\mu_0)$, and $\omega^{\iota}=\mu_0/\nu_0$, respectively, which are the conditions of Proposition 3 considering ω^{ι} as ω. Note that $\tau_\Omega(\nu_0)=(-1)^{e_0}\tau_\Omega(\nu)$ since $\nu_0=\nu^{-1}(\omega-\omega^{\iota})^{-1}$.

By means of Proposition 1 with (2.1) and Lemma 1, we have

$$(6.4)\qquad \left[-\frac{R_K}{W_K}\frac{1}{s}\right]^{\dagger\dagger}+\kappa_K(C)+O(s)$$

$$=\frac{1}{(-4)^{e_0}N_K\mathfrak{f}^{\frac{1}{2}}}\frac{G_K(1-s, \mathfrak{f}\Omega)}{s^{2e_0'+c_0}G_K(s, \mathfrak{f}\Omega)}\sum_{\lambda\in\mathfrak{B}-0/E_K(\mathfrak{f}\Omega)}\frac{\tau_\Omega(\lambda)e_K(\lambda)}{N_K((\lambda)\mathfrak{B}^{-1})^{1-s}}\,.$$

Then, by Proposition 3 and (2.2), (6.4) can be rewritten as

$$(6.5)\qquad \left[-\frac{R_K}{W_K}\cdot\frac{1}{s}\right]^{\dagger\dagger}+\tau_\Omega(\nu)\hat{\kappa}_K(C)+O(s)$$

$$=\langle\Phi_\omega, \boldsymbol{d}_\Delta\boldsymbol{E}_{1-s}\rangle\left(\frac{2^{c_0}\pi^{2n}}{\sqrt{d_K}N_K\mathfrak{f}}\right)^{s}\Gamma\left(1+\frac{s}{2}\right)^{-2e_0'}\left(\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}+\frac{s}{2}\right)\right)^{-2e_0}$$

$$\times\Gamma(1+s)^{-c_0}\Gamma(1-s)^{n}\,.$$

Here we calculate the Taylor expansion on the right hand side of (6.5). Applying Proposition 2 to $\langle\Phi_\omega, \boldsymbol{d}_\Delta\boldsymbol{E}_{1-s}\rangle$, we obtain

$$\langle\Phi_\omega, \boldsymbol{d}_\Delta\boldsymbol{E}_{1-s}\rangle=\left[-2^{n-2}R_F\langle\Phi_\omega, \boldsymbol{d}_D\mathbf{1}\rangle\cdot\frac{1}{s}\right]^{\dagger}$$

$$+\sqrt{d_F}\langle \Phi_\omega, \boldsymbol{d}_\Delta f\rangle-[2^{n-2}R_F\sum_g\langle \Phi_\omega, \boldsymbol{d}_\Delta \log y_g\rangle]^\dagger$$

$$-[2^{n-2}R_F(\log N_F\mathfrak{c}_1-\log\pi^{2n}-2n\gamma_0)\cdot\langle \Phi_\omega, \boldsymbol{d}_\Delta 1\rangle]^\dagger+O(s)\,.$$

On the other hand, applying (3.4) and (3.5) to the remaining part, we have

$$(\text{second line of r.h.s. of (6.5)})=1+\left(\log\frac{2^{2e_0+c_0}\pi^{2n}}{\sqrt{d_K}N_K\mathfrak{f}}+2n\gamma_0\right)\cdot s+O(s^2)\,.$$

Then, we obtain the following expansion of (6.5) as follows:

$$\text{(6.6)}\qquad \left[-\frac{R_K}{W_K}\cdot\frac{1}{s}\right]^{\dagger\dagger}+\tau_\Omega(\nu)\cdot\hat{\kappa}_K(C)+O(s)$$

$$=\left[-2^{n-2}R_F\langle \Phi_\omega, \boldsymbol{d}_\Delta 1\rangle\cdot\frac{1}{s}\right]^\dagger+\sqrt{d_F}\langle \Phi_\omega, \boldsymbol{d}_\Delta f\rangle$$

$$-\left[2^{n-2}R_F\sum_g\left\langle \Phi_\omega, \boldsymbol{d}_\Delta\log\frac{y_g}{|(\omega_g-\omega^\iota{}_g)/\rho_g|}\right\rangle\right]^\dagger$$

$$-[2^{n-2}R_F\log(4^{e_0}N_K\mathfrak{A}_1)\cdot\langle \Phi_\omega, \boldsymbol{d}_\Delta 1\rangle]^\dagger\,,$$

where $\rho_g=1$ or 2 according as $g\in\Omega_0{}^r$ or $g\in\Omega_0{}^c$. (6.6) shows (6.3) in the case where $\mathfrak{f}\neq I_K$.

Assume that $\mathfrak{f}=I_K$. Comparing the coefficients of s^{-1} on both sides of (6.6), we have

$$\text{(6.7)}\qquad 2^{n-2}R_F\langle \Phi_\omega, \boldsymbol{d}_\Delta 1\rangle=\begin{cases}\dfrac{R_K}{W_K} & (\text{if } \Omega=\phi,\text{ i.e., } e_0=0)\\ 0 & (\text{if } \Omega\neq 0,\text{ i.e., } e_0>0)\,,\end{cases}$$

which we can also obtain by a direct elementary calculation of $\langle \Phi_\omega, \boldsymbol{d}_\Delta 1\rangle$. Further, $\left\langle \Phi_\omega, \boldsymbol{d}_\Delta\log\frac{y_g}{|(\omega_g-\omega^\iota{}_g)/\rho_g|}\right\rangle$ vanishes in the following cases: (i) $g\in\Omega_0{}^c$; (ii) $e_0\geqq 2$; (iii) $\Omega_0=\{e\}$ $(e_0=1)$ and $g\neq e$. If it does not vanish, we can use the following equation, i.e.,

$$\text{(6.8)}\qquad \frac{y_r}{\omega_r-\omega^\iota{}_r}=\frac{|(z_r-\omega_r)(z_r-\omega^\iota{}_r)|}{(\omega_r-\omega^\iota{}_r)^2}\cdot\cos\theta_r\,,$$

where θ is the element of I^a derived from $\varphi_\omega{}^{-1}(z)$ in § 5. Note that θ associated with $z\in Z$ depends only on Z. (6.8) is obtained from the latter equation for $r\in\Omega_0{}^r$ in (5.7), see [Si1]. Hence we obtain (6.3) in the case where $\mathfrak{f}=I_K$ and our proof is completed.

Corollary 1. *Let the assumption on Ω be as in Theorem* 1. *For $C\in H_K(\Omega)$, let $\mathfrak{a}$, $\mathfrak{b}$ be ideals of F and ω an element of $K-F$ such that $\mathfrak{d}_{K/F}{}^{-1}C\ni\mathfrak{A}=\mathfrak{a}+\mathfrak{b}\omega$. Put $f(z)=f_{\mathfrak{a}\times\mathfrak{b}}(z)$ and $\Phi_\omega=\Phi_\omega(\Gamma_{\mathfrak{a}\times\mathfrak{b}})$. Then, we have*

(6.9) $$(-1)^{e_0}\cdot \tilde{\kappa}_K(C)=\sqrt{d_F}\langle \Phi_\omega, \boldsymbol{d}_\Delta f\rangle-2^{n-2}R_F\cdot T\ ,$$

where

(6.10) $$T=\begin{cases} R_{K/F}\log N_K(\mathfrak{A}\mathfrak{a}^{-1})+\sum_{e'\in\Omega_0'}\left\langle \Phi_\omega, \boldsymbol{d}_\Delta\log\left(\dfrac{\mathfrak{T}(z_{e'})}{\omega_{e'}-\omega^{\iota}_{e'}}\right)\right\rangle \\ \qquad (\text{if } \Omega_0=\phi) \\ \dfrac{1}{2}\langle \Phi_\omega, \boldsymbol{d}_\Delta\log(z_e-\omega_e)(z_e-\omega^{\iota}_e)\rangle \\ \qquad (\text{if } \Omega_0=\{e\}) \\ 0 \qquad (\text{if } e_0\geqq 2)\,. \end{cases}$$

Remark. In the case where $\mathfrak{f}=I_K$ and $\Omega=\phi$, each term of $\sum_{e'}$ in (6.3) and (6.10) can be rewritten by (6.8) as $\left\langle Z, \boldsymbol{d}_\Delta\left|\dfrac{(z_{e'}-\omega_{e'})(z_{e'}-\omega^{\iota}_{e'})}{(\omega_{e'}-\omega^{\iota}_{e'})^2}\right|\right\rangle$, where $Z=Z(\Gamma, t_{e'}, D)\in\Phi_\omega$ with $t_{e'}$ such that $-(t_{e'}-\omega_{e'})^{-1}(t_{e'}-\omega^{\iota}_{e'})\in i\boldsymbol{R}^{\times}_{+}$. This expression corresponds to Heck's, see [Si1].

Proof. Take $\lambda\in(\mathfrak{A}\mathfrak{d}_{K/F})^{-1}$ such that $\lambda_l>0$ $(l\in\Omega)$ $(\lambda\neq 0$ if $\Omega=\phi)$, and put $\nu=\lambda^{-1}(\omega-\omega^{\iota})^{-1}$, $\mu=\omega^{\iota}\nu$. Then, we have $(\lambda)\mathfrak{A}\mathfrak{d}_{K/F}\in C$, $((\lambda)\mathfrak{A}\mathfrak{d}_{K/F})^{-1}=\mathfrak{a}^{-1}\mu+\mathfrak{b}^{-1}\nu$ and $\omega^{\iota}=\mu/\nu$. Considering $((\lambda)\mathfrak{A}\mathfrak{d}_{K/F})^{-1}$ as $\mathfrak{A}$ in Theorem 1, $\mathfrak{A}_1$ in the theorem becomes $I_F+\mathfrak{a}^{-1}\mathfrak{b}\omega^{\iota}$, whose norm is equal to that of $\mathfrak{A}\mathfrak{a}^{-1}=I_F+\mathfrak{a}^{-1}\mathfrak{b}\omega$. Since $\tau_\Omega(\nu)=\tau_\Omega(\omega-\omega^{\iota})=(-1)^{e_0}$ and $\langle\Phi_\omega, \boldsymbol{d}_\Delta f\rangle=\langle\Phi_{\omega^{\iota}}, \boldsymbol{d}_\Delta f\rangle$, we obtain our corollary from Theorem 1.

In particular, if K is a CM-field, Theorem 1 and Corollary 1 are restated as follows:

Corollary 2. *Let K be a CM-field. For $C\in H_K(\mathfrak{f})$, let notations and assumptions be as in Theorem* 1. *Then, we have*

(611) $$\kappa_K(C)=\frac{|E_K\colon E_K(\mathfrak{f})|}{|E_K\colon E_F|}\cdot\{\sqrt{d_F}f_{\mathfrak{a}\times\mathfrak{b}}(a, b; \omega)-[2^{n-2}R_F\log N_K\mathfrak{A}_1]^{\dagger}\}\,.$$

Corollary 3. Let K be a CM-field. For $C\in H_K(I_K)$, let notations and assumptions be as in Corollary 1. Then, we have

(6.12) $$\kappa_K(C)=\frac{1}{|E_K\colon E_F|}\cdot\{\sqrt{d_F}f_{\mathfrak{a}\times\mathfrak{b}}(\omega)-2^{n-2}R_F\log N_K(\mathfrak{A}\mathfrak{a}^{-1})\}\,.$$

Remark. Corollary 2 includes the Katayama's result in [Ka] on the case of CM-fields as our special case (i.e. the case $n=2$), and Corollary 3 corresponds to the Konno's in [Ko]. In particular, if $n=1$, i.e., $F=\boldsymbol{Q}$ and K is an imaginary quadratic field, (6.11) and (6.12) are rewritten as

(6.13) $$\kappa_K(C)=-\frac{1}{W_K(\mathfrak{f})}\log\|\phi(a, b; \omega)\|\quad\text{if } \mathfrak{f}\neq I_K\,,$$

$$\kappa_K(C) = -\frac{1}{W_K}\log\|2\pi\sqrt{N_K([1, \omega])}\,\eta^2(\omega)\| \quad \text{if } \mathfrak{f}=I_K\,, \tag{6.14}$$

which are classical results. Note that the values in $\|\ \|$ on the right hand side of (6.13) and (6.14) give rise to units in a class field of K which is related to $\mathfrak{f}$, see [R], [K-L].

If K is a totally real and $\Omega=\Omega^a$, it is known that $\kappa_K(C)\in \boldsymbol{Q}$ by Klingen [Kl], Siegel [Si2], [Si3] and Shintani [Sh1]. Then, we have

Corollary 4. *For* $\mathfrak{F}^{\sigma}_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)$ *defined in* (4.18), *let* $\boldsymbol{d}\mathfrak{F}=\mathfrak{F}^{\sigma}_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)\wedge_g dz_g$. *If* $K=F(\omega)$ *is a totally real quadratic extension of* F, $\mathfrak{a}+\mathfrak{b}\omega$ *is an ideal of* K, *and* $\sigma=+$, *then,* $\langle \Phi_\omega, \boldsymbol{d}\mathfrak{F}\rangle \in \sqrt{d_F}\,\boldsymbol{Q}$.

Remark. Corollary 4 holds also for an arbitrary $\sigma\in\mathfrak{S}^a$ with a suitable change of the definition of Φ_ω. Probably the second assumption in the corollary seems not to be required.

In this case, Goldstein carried out the same calculation of $\kappa_K(C)$ as ours in [G] for the purpose of obtaining the rationality of $\kappa_K(C)$ and an explicit formula for it. He asserted in § 5 of [G] that $\langle \Phi_\omega, \boldsymbol{d}_\Delta f\rangle \left(=\int_Z \partial_{\Omega_0{}^a} f dz_{\Omega_0{}^a}\right)$ was described by sums and differences of finitely many special values of f at some points associated with Z. However, it has a mistake in general except the case $n=1$. In the case $n=1$, this integral may be expressed as a difference of two special values of f, which is one of elementary properties of holomorphic functions. Of course, it is possible also in the cases where $\Omega_0'=\phi$ and $\#(\Omega_0)=1$.

Now, we assume that $\Omega_0'=\phi$ and $\Omega_0=\{e\}$. In this case, K has two real and $(n-1)$-imaginary places, and the archimedean conductor Ω consists of the two real places of K.

For $C\in H_K(\mathfrak{f}\Omega)$, let $\mathfrak{a}$, $\mathfrak{b}$, (a, b) and ω be as in Theorem 1, Put $E_{K/F}=\{\varepsilon\in E_K|\varepsilon\varepsilon^{\iota}=1, \varepsilon\gg 0\}$ and $E_{K/F}(\mathfrak{f})=\{\varepsilon\in E_{K/F}|\varepsilon\equiv 1(\mathrm{mod}\ \mathfrak{f})\}$. Note that their ranks as abelian groups are 1. Let ϵ be the generator of $E_{K/F}(\mathfrak{f})$ such that $\epsilon_e>1$ (where e should be considered as an element of Ω as in § 5), and put $A=\psi_\omega{}^{-1}(\epsilon)$ where ψ_ω is as in § 5. Take $t_0\in\mathfrak{H}$ and put $t_1=A_e t_0$. For ω, let ω_0 and ω_1 be the elements of $\mathfrak{H}^a$ defined by

$$(\omega_0)_g=\begin{cases} t_0 & (\text{if } g=e) \\ \omega_c & (\text{if } g=c\in\Omega_0{}^c)\,, \end{cases} \qquad (\omega_1)_g=\begin{cases} t_1 & (\text{if } g=e) \\ \omega_c & (\text{if } g=c\in\Omega_0{}^c)\,. \end{cases} \tag{6.15}$$

We can take a path in $\mathfrak{H}^a$ from ω_0 to ω_1 as $Z\in\Phi_\omega$ in Theorem 1.

Corollary 5. *For* $C\in H_K(\mathfrak{f}\Omega)$, *let notations and assumptions be as in Theorem* 1 *and as above. Put* $f(z)=f^{\Omega_0}_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)$. *Then, we have*

$$\kappa_K(C)=\tau_\Omega(\nu)\frac{|E_K\colon E_K(\mathfrak{f}\Omega)|}{|E_K\colon E_F E_{K/F}(\mathfrak{f})|}\cdot\left\{\frac{\sqrt{d_F}}{2\pi i}(f(\omega_0)-f(\omega_1))-[2^{n-2}R_F T]^{\dagger}\right\}, \tag{6.16}$$

where

$$T=\frac{1}{4\pi i}\log\frac{(t_0-\omega_e)(t_0-\omega^{\iota}{}_e)}{(t_1-\omega_e)(t_1-\omega^{\iota}{}_e)}. \tag{6.17}$$

Here the branch of $\log z$ *is taken in such a way as* $0<arg(z)<2\pi$.

Note that the right hand side of (6.16) is independent of the choice of $t_0\in \mathfrak{H}$. In particular, we can take t_0 and t_1 as

$$t_0=\frac{(\epsilon_e\omega^{\iota}{}_e-\epsilon^{\iota}{}_e\omega_e)+(\omega_e-\omega^{\iota}{}_e)i}{\epsilon_e-\epsilon^{\iota}{}_e},\quad t_1=\frac{(\epsilon_e\omega_e-\epsilon^{\iota}{}_e\omega^{\iota}{}_e)+(\omega_e-\omega^{\iota}{}_e)i}{\epsilon_e-\epsilon^{\iota}{}_e}. \tag{6.18}$$

For such t_0 and t_1, we have $T=\frac{1}{4}$, and

Corollary 6. *For* $C\in H_K(\Omega)$, *let notations and assumptions be as in Corollary* 1 *and as above (i.e.,* $\Omega_0'=\phi$ *and* $\Omega_0=\{e\}$*). Let* ω_0, ω_1 *be as in* (6.15) *associated with* ω *in the corollary and with* t_0, t_1 *in* (6.18). *Then, we have*

$$\kappa_K(C)=\frac{|E_K\colon E_K(\Omega)|}{|E_K\colon E_F E_{K/F}|}\cdot\left\{\frac{\sqrt{d_F}}{2\pi i}(f^{\Omega_0}_{\mathfrak{a}\times\mathfrak{b}}(\omega_1)-f^{\Omega_0}_{\mathfrak{a}\times\mathfrak{b}}(\omega_0))+2^{n-4}R_F\right\}. \tag{6.19}$$

Remark. Stark conjectured in [St1] and [St2] that for an algebraic number field k, if the value $e_1'+r_2$ in (2.1) is equal to $1, \kappa_k(C)$ (for $\kappa_k(C)\in H_k(\mathfrak{f}\Omega)$) would be expressed as a form $q\log|\varepsilon|$, where $q\in \boldsymbol{Q}^\times$ and ε is a unit in the class field corresponding to the unit element of $H_k(\mathfrak{f}\Omega)$. (In fact, the conjecture holds if $k=\boldsymbol{Q}$ and k is an imaginary quadratic field, the former of which is the classical result and the latter the result in [St2]). Corollaries 5 and 6 in the case $n=2$ shows that a difference of two special values of $f^{\Omega_0}_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)$ would express a value of the form $\frac{2\pi i}{\sqrt{d_F}}\log|\varepsilon|$, and hence $f^{\Omega_0}_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)$ seems to describe class fields of K, where K is quartic with two real and one imaginary places, i.e., the case where $e_1=2$ ($e_0=1$), e_1' ($=e_0'$)$=0$ and r_2 ($=c_0$) $=1$. The case where $e_1'=1$ and $r_2=0$ is treated by Shintani in [Sh2], [Sh3] and [Sh4] besides the above papers. He has shown in them that $\kappa_k(C)$ can be expressed by special values of multiple gamma functions in this case. We may consider that our function $f^{\Omega_0}_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)$ takes the place of a multiple gamma function in Shintani's case. In general, for arithmeticity of $\kappa_k(C)$, Stark-Shintani conjecture predicts that $\kappa_k(C)$ would be expressed as a homogeneous polynomial in the logarithm of several units which belong to a suitable class field related to C's, where the polynomial is of degree $e_1'+r_2$, and with $\boldsymbol{Q}$-coefficients. (For example, (6.13), (6.14) and Corollary 3 are such cases as Stark-Shintani conjecture holds, see [Sh5], [St2], [T].) Here it is conjectured that the difference of two special values of $f^{\Omega_0}_{\mathfrak{a}\times\mathfrak{b}}(a, b; z)$ in Corollaries 5 and 6 for an arbitrary n would be expressed as a homogeneous polynomial in the logarithm of several units with above conditions.

§ 7. Numerical examples

In this section, we shall give several numerical examples.

We consider the Artin L-function $L_{L/K}(s, \chi)$ with an abelian extension L/K of finite degree, where K is a quadratic extension of a real quadratic field F, (i.e., the case $=2$ in Theorem 1), L is the maximal unramified (with respect to all finite places) extension of K (and so $\mathfrak{f}=I_K$), and χ is a non-trivial character of $\mathrm{Gal}(L/K)$ of degree 1 satisfying the condition that the subfield of L corresponding to χ is totally imaginary (and so $\Omega_0'=\phi$). This case is classified into three cases in view of archimedean places of K. Now we consider the following two cases: (1) K has two real and one imaginary places; (2) K is a CM-field. The cases (1) and (2) correspond to Corollary 6 and 3 respectively, and in each case, the coefficient of the leading term in the Taylor expansion of $L_{L/K}(s, \chi)$ at $s=0$ (simply say 'the leading term' below) is calculated by each corollary. If $L_{L/K}(s, \chi)$ is expressed as a product of Hecke L-functions on $\boldsymbol{Q}$ or on imaginary quadratic fields, the leading term is calculated also by means of classical results. In such a case, we can compare two computational datas by the different methods and make sure our formula for $\kappa_K(C)$ holds in the numerical sense. The formulas for $L_{L/K}(1, \chi)$ in the case (2) have already appeared in [Ko], [Ka], however, they have some numerical mistakes. Furthermore, to the author's knowledge, the case (1) has not appeared in the literature yet. Thus it seems to be meaningful that such numerical examples are given here.

We take $\boldsymbol{Q}(\sqrt{3})$ as F in the case (1), and $\boldsymbol{Q}(\sqrt{2})$ in the case (2). In boh case, we have $h_F=1$, and so we can take $f(z)=f_{I_F\times I_F}(z)$ as $f_{\mathfrak{a}\times\mathfrak{b}}(z)$ in the corollaries. We denote by $\tilde{f}(z)$ the function which is obtained from $f(z)$ by eliminating the third term in (4.5), and by $[\mu, \nu]$ the ideal $I_F\mu+I_F\nu$ of K. In the following, we denote by C_0 (or $\tilde{C}_0$) and by C_1 (or $\tilde{C}_1$) the unit and a generator of an ideal class group when it is cyclic. We note that a representative of $C_1^j\in H_K(I_K)$ (where $C_1^0=C_0$) can be given in the form $[1, \omega_j]$ with $\omega_j \in K-F$.

The case (1): Assume that $F=\boldsymbol{Q}(\sqrt{3})$. Corollary 6 says that

$$\kappa_K(C)=\frac{|E_K\colon E_K(\Omega)|}{|E_K\colon E_F E_{K/F}|}\left\{\frac{\sqrt{12}}{2\pi i}(f^{\Omega_0}_{\mathfrak{a}\times\mathfrak{b}}(\omega_1)-f^{\Omega_0}_{\mathfrak{a}\times\mathfrak{b}}(\omega_0))+\frac{1}{4}\log(2+\sqrt{3})\right\}, \tag{7.1}$$

with the notation in the corollary. Note that we can substitute an arbitrary subgroup U of $E_{K/F}$ with finite index for $E_{K/F}$.

Now we consider, as K, $K_1=F(\sqrt{1+\sqrt{3}})$ and $K_2=F(\sqrt{-1-\sqrt{3}})$. In this case, $h_{K_1}=h_{K_2}=1$ and $L=K_1(\sqrt{-1})=K_2(\sqrt{-1})$, which is biquadratic over F, and intermediate fields of L/F are K_1, K_2 and $F(\sqrt{-1})$. Here we have

$$(7.2)\qquad \begin{cases} L_{L/K_1}(s,\chi_1)=\zeta_{K_2}(s)\zeta_F(s)^{-1}\cdot\zeta_{\boldsymbol{Q}(\sqrt{-1})}(s)\zeta_{\boldsymbol{Q}(\sqrt{-3})}(s)\zeta(s)^{-2} \\ L_{L/K_2}(s,\chi_2)=\zeta_{K_1}(s)\zeta_F(s)^{-1}\cdot\zeta_{\boldsymbol{Q}(\sqrt{-1})}(s)\zeta_{\boldsymbol{Q}(\sqrt{-3})}(s)\zeta(s)^{-2}\,. \end{cases}$$

Put $\varepsilon_0=2+\sqrt{3}$, $\varepsilon_1=\sqrt{3}+\sqrt{1+\sqrt{3}}$, $\varepsilon_2=1+\sqrt{-1+\sqrt{3}}$, $\varepsilon_3=\varepsilon_1{}^2\varepsilon_0$ and $\varepsilon_4=\varepsilon_2{}^2\varepsilon_0$, which are all totally positive. E_F, E_{K_1} and E_{K_2} are generated by $\{\varepsilon_0, -1\}$, $\{\varepsilon_1, \varepsilon_0, -1\}$, and $\{\varepsilon_2, \varepsilon_0, -1\}$ respectively. Let U_{K_1}, U_{K_2} be the subgroups of E_{K_1}, E_{K_2} generated by ε_3, ε_4 respectively. We have $|E_{K_1}: E_{K_1}^+|=|E_K: E_FU_{K_1}|=2$ and $|E_{K_2}: E_{K_2}^+|=|E_K: E_FU_{K_2}|=2$. Let $H_{K_j}(I_{K_j}\Omega_j)=\{C_{j0}, C_{j1}\}$ ($j=1$ or 2).For $\omega_{K_1}=\sqrt{1+\sqrt{3}}$ and $\omega_{K_2}=\sqrt{-1+\sqrt{3}}$, we have $I_{K_j}=[1, \omega_{K_j}]$, $\mathfrak{d}_{K/F}=(\omega_{K_j}-\omega_{K_j}{}^{\iota})\in C_{j1}$, and $\kappa_{K_j}(C_{j1})=-\kappa_{K_j}(C_{j0})$ by Lemma 1. Then the leading terms ξ_1, ξ_2 on the left hand side of (7.2) are given by $-2\kappa_{K_1}(C_{11})$, $-2\kappa_{K_2}(C_{21})$ respectively. Applying ω_{K_1} and ε_3, or ω_{K_2} and ε_4 to ω and ϵ in (6.18), we calculate $\kappa_{K_1}(C_{11})$, $\kappa_{K2}(C_{21})$ by (7.1) as

$$\begin{cases} \kappa_{K_1}(C_{11}) \\ \quad=-0.21278114473005757006425652733586971262375634596622 81\cdots \\ \kappa_{K_2}(C_{21}) \\ \quad=-0.31296930925401556657118984848190188154782967771147 64\cdots\,, \end{cases}$$

and hence

$$(7.3)\qquad \begin{cases} \xi_1=0.4255622894601151401285130546717394252475126919324562\cdots \\ \xi_2=0.6259386185080311331423796969638037630956593554229528\cdots\,. \end{cases}$$

On the other hand, we have $R_{Kj}/R_F=\log\varepsilon_0+2\log\varepsilon_j$ ($j=1, 2$). By (2.5), we obtain the leading terms on the right hand side of (7.2), i.e.,

$$(7.4)\qquad \begin{cases} \frac{1}{6}\log(\varepsilon_0\varepsilon_2{}^2) \\ \quad=0.42556228946011514012851305467173942524751269193245 5\cdots \\ \frac{1}{6}\log(\varepsilon_0\varepsilon_1{}^2) \\ \quad=0.62593861850803113314237969696380376309565935542295 2\cdots\,. \end{cases}$$

Comparing (7.3) with (7.4), we can see the coincidence of the leading terms on both sides of (7.2) up to 10^{-50}.

The case (2): Assume that $F=\boldsymbol{Q}(\sqrt{2})$. Corollary 3 says that if $E_K=E_F$, for $\kappa_K{}^*(C)=\kappa_K(C)-\sqrt{2}\kappa_F(I_F)$, we have

$$(7.5)\qquad \kappa_K{}^*(C)=2\sqrt{2}\,\hat{f}(\omega)-\log(1+\sqrt{2})\log(N(\mathfrak{J}(\omega)))\,,$$

where $N(\Im(\omega))=\prod_g|\Im(\omega_g)|$. Then, the leading term of $L_{L/K}(s, \chi)$ can be obtained by calculating a linear combination of $\kappa_K{}^*(C)$'s. In the following examples (a), (b), (c) and (d), $E_K=E_F$ always holds.

(a) $K=F(\sqrt{-5})$. In this case, $h_K=2$ and $L=K(\sqrt{-1})$, which is an abelian extension over $\boldsymbol{Q}$ with $\mathrm{Gal}(L/Q)\cong \boldsymbol{Z}/2\boldsymbol{Z}\times\boldsymbol{Z}/2\boldsymbol{Z}\times\boldsymbol{Z}/2\boldsymbol{Z}$. For the non-trivial character χ of $\mathrm{Gal}(L/K)$, we have

$$L_{L/K}(s, \chi)=\zeta_{\boldsymbol{Q}(\sqrt{-1})}(s)\zeta_{\boldsymbol{Q}(\sqrt{-2})}(s)\zeta_{\boldsymbol{Q}(\sqrt{5})}(s)\zeta_{\boldsymbol{Q}(\sqrt{10})}(s)\zeta(s)^{-4}\,. \tag{7.6}$$

The leading term ξ on the left hand sie is given by $\kappa_K{}^*(C_0)-\kappa_K{}^*(C_1)$ where $H_K(I_K)=\{C_0, C_1\}$. Applying $\omega_0=(\sqrt{2}+\sqrt{-10})/2$, $\omega_1=(1+\sqrt{2}+\sqrt{-5})/2$ to ω in (7.5), we have

$$\begin{cases}\kappa_K{}^*(C_0)=3.92788272032221983102718644131337264653504802979680 95\cdots\\ \kappa_K{}^*(C_1)=3.05282478090198317831631871602600271354202512784950 7\cdots\,,\end{cases}$$

and hence

$$\xi=0.87505793942023665271086772528736993299302290194730 24\cdots\,. \tag{7.7}$$

On the other hand, as for the right hand side of (7.6), the leading term is calculated by (2.5), i.e.,

$$\begin{aligned}&\log\Big(\frac{1+\sqrt{5}}{2}\Big)\log(3+\sqrt{10})\\ &\quad=0.87505793942023665271086772528736993299302290194730 5\cdots\,.\end{aligned} \tag{7.8}$$

Comparing (7.7) with (7.8), the leading terms on both sides of (7.6) coincide up to 10^{-50}.

(b) $K=F(\sqrt{-7})$. In this case, $h_K=2$ and $L=K(\sqrt{-1+2\sqrt{2}})$, which is a cyclic extension of degree 4 over $M=\boldsymbol{Q}(\sqrt{-14})$. For the non-trivial character χ of $\mathrm{Gal}(L/K)$ and the two characters χ_1, χ_2 of order 4 of $\mathrm{Gal}(L/M)$, we have

$$L_{L/K}(s, \chi)=L_{L/M}(s, \chi_1)L_{L/M}(s, \chi_2)\,. \tag{7.9}$$

Let ξ, ξ_1, ξ_2 be the leading term of $L_{L/K}(s, \chi)$, $L_{L/M}(s, \chi_1)$, $L_{L/M}(s, \chi_2)$, respectively. The value ξ is given by $\kappa_K{}^*(C_0)-\kappa_K{}^*(C_1)$ where $H_K(I_K)=\{C_0, C_1\}$. Applying $\omega_0=(1+\sqrt{-7})/2$ and $\omega_1=(3+2\sqrt{2}+\sqrt{-7})/6$ to ω in (7.5), we have

$$\begin{cases}\kappa_K{}^*(C_0)=3.38597236328174849867103680666023054291345727257091 38\cdots\\ \kappa_K{}^*(C_1)=2.98531587451384558680599145038954373506392983864730 64\cdots\,,\end{cases}$$

and hence

$$\xi=0.40065648876790291186504535627068680784952743392360 73\cdots\,. \tag{7.10}$$

On the other hand, for $H_M(I_M)=\{\tilde{C}_0, \tilde{C}_1, \tilde{C}_1^2, \tilde{C}_1^3\}$, we have $\kappa_M(\tilde{C}_1)=\kappa_M(\tilde{C}_1^3)$, and $\kappa_M(\tilde{C}_0)$, $\kappa_M(\tilde{C}_1^2)$ are calculated by (6.14) as

$$\begin{cases} \kappa_M(\tilde{C}_0) \\ \quad =0.12125016013712343600662413378181330035165166384594 67\cdots \\ \kappa_M(\tilde{C}_1^2) \\ \quad =-0.51172415906382401414155911229837262972120603384266 09\cdots . \end{cases}$$

Since $\xi_1=\xi_2=\kappa_M(\tilde{C}_0)-\kappa_M(\tilde{C}_1^2)$, we have

$$\xi_1=\xi_2=0.6329743192009474501481832460801859300728576976886077\cdots ,$$

and hence

(7.11)
$$\xi_1\xi_2=0.4006564887679029118650453562706868078495274339236077\cdots .$$

Comparing (7.10) with (7.11), we see that $\xi=\xi_1\xi_2$, i.e., the leading terms on both sides of (7.9) coincide up to the order of 10^{-50}.

(c) $K=F(\sqrt{-23})$. In this case $h_K=6$ and $L=\boldsymbol{Q}(X^4-2X^3+3X^2-2X-1, X^3-X-1)$, i.e., the minimal splitting field of $(X^4-2X^3+3X^2-2X-1)(X^3-X-1)$ over $\boldsymbol{Q}$. Put $L_0=\boldsymbol{Q}(X^4-2X^3+3X^2-2X-1)$. L_0 is a quadratic extension of K, and also the maximal unramified extension over $M=\boldsymbol{Q}(\sqrt{-46})$. Conditions for L_0, K and M are the same as those in (b) considering L_0 in place of L in (b), and so $\tilde{C}_j$'s, χ_1, χ_2, χ and ξ, ξ_1, ξ_2 are defined similarly. Here

$$L_{L_0/K}(s, \chi)=L_{L_0/M}(s, \chi_1)L_{L_0/M}(s, \chi_2) . \tag{7.12}$$

The value ξ is given by $\kappa_K{}^*(C_0)-2\kappa_K{}^*(C_1)+2\kappa_K{}^*(C_1^2)-\kappa_K{}^*(C_1^3)$ where $H_K(I_K)=\{C_0, C_1, C_1^2, C_1^3, C_1^4, C_1^5\}$. Here note that $\kappa_K{}^*(C_1)=\kappa_K{}^*(C_1^5)$, $\kappa_K{}^*(C_1^2)=\kappa_K{}^*(C_1^4)$. Applying $\omega_0=(1+\sqrt{-23})/2$, $\omega_1=(5-4\sqrt{2}+\sqrt{-23})/20$, $\omega_2=(-1+\sqrt{-23})/4$ and $\omega_3=(5-4\sqrt{2}+\sqrt{-23})/10$ to ω in (7.5), we have

$$\begin{cases} \kappa_K{}^*(C_0)=6.979272403433464444627104763733417306462380602500969 8\cdots \\ \kappa_K{}^*(C_1)=4.246219690926266204531787248760116780327289883740879 8\cdots \\ \kappa_K{}^*(C_1^2)=3.212639079665091983464779192431858707361104972652026 5\cdots \\ \kappa_K{}^*(C_1^2)=2.869966588418536553322449299872525730575614987842913 3\cdots , \end{cases}$$

and hence

$$\xi=2.0421445924925794491706393512043754299543957924803498\cdots . \tag{7.13}$$

On the other hand, by (6.14), we have

$$\begin{cases} \kappa_M(\tilde{C}_0) \\ \quad =1.71334260845867597196290713917923623025815621594039 44\cdots \\ \kappa_M(\tilde{C}_1{}^2) \\ \quad =0.28430636241990615997273028089308826341740118424120 8\cdots, \end{cases}$$

and so,

$$\xi_1=\xi_2=1.429036246038769811990176858286147966840755031699186 4\cdots.$$

Hence

(7.14)
$$\xi_1\xi_2=2.04214459249257944917063935120437542995439579248035 98\cdots.$$

Comparing (7.13) with (7.14), it can be seen that $\xi=\xi_1\xi_2$ up to 10^{-50}.

(d) $K=F(\sqrt{-19})$. In this case, $h_K=3$ and $L=K(X^3-X^2-2X-2)$ which is also the maximal unramified extension over $M=\boldsymbol{Q}(\sqrt{-38})$ ($h_M=6$). For a character χ of order 3 of $\mathrm{Gal}(L/K)$ and χ_6, χ_3 of order 6, 3 of $\mathrm{Gal}(L/M)$, we have

$$L_{L/K}(s, \chi)=L_{L/M}(s, \chi_6)L_{L/M}(s, \chi_3). \tag{7.15}$$

Let ξ, ξ_6, ξ_3 be the leading terms of $L_{L/K}(s, \chi)$, $L_{L/M}(s, \chi_6)$, $L_{L/M}(s, \chi_3)$, respectively. The value ξ is given by $\kappa_K{}^*(C_0)-\kappa_K{}^*(C_1)$ where $H_K(I_K)=\{C_0, C_1, C_1{}^2\}$. Here note that $\kappa_K{}^*(C_1)=\kappa_K{}^*(C_1{}^2)$. Applying $\omega_0=(1+\sqrt{-19})/2$ and $\omega_1=(3+2\sqrt{2}+\sqrt{-19})/6$ to ω in (7.5), we have

$$\begin{cases} \kappa_K{}^*(C_0) \\ \quad =5.98472171238984709537601925740644159610791120132001 25\cdots \\ \kappa_K{}^*(C_1) \\ \quad =2.87352633591885806474267882582340584283073292369623 79\cdots, \end{cases}$$

and hence

$$\xi=3.11119537647098903063334043158303575327717827762377 46\cdots. \tag{7.16}$$

On the other hand, for $H_M(I_M)=\{\tilde{C}_0, \tilde{C}_1, \tilde{C}_1{}^2, \tilde{C}_1{}^3, \tilde{C}_1{}^4, \tilde{C}_1{}^5\}$, we have $\kappa_M(\tilde{C}_1)=\kappa_M(\tilde{C}_1{}^5)$, $\kappa_M(\tilde{C}_1{}^2)=\kappa_M(\tilde{C}_1{}^4)$ and $\xi_6=\kappa_M(\tilde{C}_0)+\kappa_M(\tilde{C}_1)-\kappa_M(\tilde{C}_1{}^2)-\kappa_M(\tilde{C}_1{}^3)$, $\xi_3=\kappa_M(\tilde{C}_0)-\kappa_M(\tilde{C}_1)-\kappa_M(\tilde{C}_1{}^2)+\kappa_M(\tilde{C}_1{}^3)$. By (6.14), we obtain

$$\begin{cases} \kappa_M(\tilde{C}_0) \\ \quad =1.38980255782621893014164885489616503586207472707716 27\cdots \\ \kappa_M(\tilde{C}_1) \\ \quad =-0.21268018542389649408210981310374179061857796405172 78\cdots \\ \kappa_M(\tilde{C}_1{}^2) \\ \quad =-0.40562649894777679074080580391903982860678284894163 81\cdots \\ \kappa_M(\tilde{C}_1{}^3) \\ \quad =0.12253634375891531159164754828577665085750462903570 65\cdots, \end{cases}$$

and so

$$\begin{cases} \xi_6=1.4602125275911839152086972974256864229927749829313664\cdots \\ \xi_3=2.1306455859568075265562120202047233059449401691062352\cdots. \end{cases}$$

Hence

(7.17)

$$\xi_6\xi_3=3.1111953764709890306333404315830357532771782776237756\cdots.$$

Comparing (7.16) with (7.17), it can be seen that $\xi=\xi_6\xi_3$, i.e., the leading terms on both sides of (7.15) coincide up to 10^{-50}.

§ 8. Appendix. Coincidence of L-functions

For a character χ of $H_K(\mathfrak{f}\Omega)$, the value $\lim_{s\to 0}\frac{L_K(s,\chi)}{s^{2e_0'+c_0}}$ can be expressed as a linear combination of (6.2) by Theorem 1, in which we need the assumption that Ω is stable under the action of $\mathrm{Gal}(K/F)$.

In this section, we consider the case where Ω is not necessarily stable under the action of $\mathrm{Gal}(K/F)$ and $\lim_{s\to 0}\frac{{}_K(s,\chi)}{s^{2e_0'+c_0}}$ has the same expression as above (Theorem 2). (Note that if χ is primitive, the value is the coefficient of the leading term of $L_K(s,\chi)$ in the Taylor expansion at $s=0$.) The ideas in this section are based on Shintani [Sh4].

Though the following lemma seems to be essentially contained in Ishii [I], we shall write down it in a suitable form for the proof of Theorem 2.

Lemma 4. *For a non-abelian finite group G with its center Z and its commutator D, the followings hold:*
(1) *If G has an abelian subgroup H with index 2, then $H\supset Z$.*
(2) *Assume that there exists H as in* (1) *satisfying $|H\colon Z|=2$. Then, there are exactly three abelian subgroups of G with index 2.*
(3) *Assume that there exist H as in* (1) *and an element $\sigma\in H-Z$ of order 2 in G. For $\iota\in G-H$, put $\sigma^\iota=\iota^{-1}\sigma\iota$ and $J=\{1,\sigma,\sigma^\iota,\sigma\sigma^\iota\}$. Then the following*

(i), (ii) are equivalent: (i) $J\supset D$; (ii) $|H: Z|=2$. If (i), (ii) hold, then $D=\{1, \sigma\sigma^{\iota}\}\subset Z$.

(4) *Assume that there exist H and σ as in* (3) *satisfying (i), (ii). For a character χ_0 of Z of degree* 1, *non-trivial on D, there exists a unique irreducible character Φ of G of degree* 2 *such that* $\mathrm{Res}_G^Z\Phi=2\chi_0$. *$\Phi$ is expressed as* $\mathrm{Ind}_H^G\chi$ *where χ is a character of H of degree* 1 *such that* $\mathrm{Res}_H^Z\chi=\chi_0$. $\mathrm{Ind}_H^G\chi$ *is independent of the choice of three H's and two χ's for each H.*

Proof. (1), (2), (3) are elementary. Here we only give the proof of (4).

Let χ_0 be a character of Z of degree 1, non-trivial on D, and χ a character of H of degree 1 such that $\mathrm{Res}_H^Z\chi=\chi_0$. For $\iota\in G-H$, define χ^{ι} as $\chi^{\iota}(h)=\chi(\iota h\iota^{-1})$ for $h\in H$. Then, χ^{ι} is also a character of H of degree 1 such that $\mathrm{Res}_H^Z\chi^{\iota}=\chi_0$. Since χ_0 is non-trivial on D and so $\chi(\sigma\sigma^{\iota})=\chi_0(\sigma\sigma^{\iota})\neq 1$ by (3), we have $\chi\neq\chi^{\iota}$. Put $\Phi=\mathrm{Ind}_H^G\chi$. Then, $\mathrm{Res}_G^H\Phi=\chi+\chi^{\iota}$, $\mathrm{Res}_G^Z\Phi=2\chi_0$, and we have

$$\langle\Phi, \Phi\rangle_G=\langle\mathrm{Ind}_H^G\chi, \Phi\rangle_G=\langle\chi, \mathrm{Res}_G^H\Phi\rangle_H=\langle\chi, \chi+\chi^{\iota}\rangle_H=1\,.$$

This shows that Φ is irreducible and it satisfies the condition in (4).

Next, assume that there exists an irreducible character Φ of G of degree 2 such that $\mathrm{Res}_G^Z\Phi=2\chi_0$. Let $\mathrm{Res}_G^H\Phi=\chi_1+\chi_2$ with two characters χ_1 and χ_2 of H of degree 1. From the calculation with respect to inner products of characters:

$$1\geqq\langle\Phi, \mathrm{Ind}_H^G\chi_j\rangle_G=\langle\mathrm{Res}_G^H\Phi, \chi_j\rangle_H=\langle\chi_1+\chi_2, \chi_j\rangle_H=1+\langle\chi_1, \chi_2\rangle_H\geqq 1\,,$$

we obtain $\langle\Phi, \mathrm{Ind}_H^G\chi_j\rangle_G=1$ $(j=1, 2)$, $\langle\chi_1, \chi_2\rangle_H=0$, and so $\Phi=\mathrm{Ind}_H^G\chi_1=\mathrm{Ind}_H^G\chi_2$, $\chi_1\neq\chi_2$. By the assumption $\mathrm{Res}_G^Z\Phi=2\chi_0$, we have $\mathrm{Res}_H^Z\chi_1=\mathrm{Res}_H^Z\chi_2=\chi_0$. It shows that the expression of Φ in (4) is possible and such a Φ as we consider associated with a given χ_0 is unique.

Let $\Omega_0{}^a$, $\Omega_0{}^r$, $\Omega_0{}^c$ and Ω^a, Ω^r, Ω^c be as § 5 and § 6, and we modify decomposition of $\Omega_0{}^r$ and Ω^r as follows. We decompose $\Omega_0{}^r$ into disjoint three subsets Ω_0', Ω_0 and $\Omega_0{}^s$. Let Ω', Ω be the set of all the places of K lying above Ω_0', Ω_0 respectively. Further, we decompose the set of real places of K lying above $\Omega_0{}^s$ into two subsets $\Omega^{s'}$, Ω^s in such a way as one of the two places lying above a place of $\Omega_0{}^s$ belongs to $\Omega^{s'}$ and the other belongs to Ω^s. Hence the archimedean places of F and K are described as follows:

$$\begin{array}{ccccc}
 & \overbrace{\cdots\boldsymbol{R}\quad\boldsymbol{R}\cdots}^{\Omega'} & \overbrace{\cdots\boldsymbol{R}\quad\boldsymbol{R}\cdots}^{\Omega} & \overbrace{\cdots\boldsymbol{R}\quad\boldsymbol{R}\cdots}^{\Omega^{s'}\cup\Omega^s} & \overbrace{\cdots\boldsymbol{C}\cdots}^{\Omega^c} \\
K: & & & & \\
| & \diagdown\diagup & \diagdown\diagup & \diagdown\diagup & | \\
F: & \underbrace{\cdots\ \boldsymbol{R}\ \cdots}_{\Omega_0'} & \underbrace{\cdots\ \boldsymbol{R}\ \cdots}_{\Omega_0} & \underbrace{\cdots\ \boldsymbol{R}\ \cdots}_{\Omega_0{}^s} & \underbrace{\cdots\boldsymbol{R}\cdots}_{\Omega_0{}^c}
\end{array}\,.$$

By the assumptions, we have $\#(\Omega^{s'})=\#(\Omega^s)=\#(\Omega_0{}^s)$. Put $\Omega^u=\Omega\cup\Omega^s$ and $\Omega^{u'}$

$=\Omega'\cup\Phi^{s'}$.

Let χ be a primitive character of $H_K(\mathfrak{f}\Omega^u)$, and L the class field corresponding to the kernel of χ. Note that the archimedean places of L above $\Omega^{u'}$ are real and the others imaginary. If $\Omega_0^s\neq\phi$, L/F is not a Galois extension. Let L_0/F be the maximal Galois extension in L/F, L_1 the Galois closure of L/F, and put $G=\mathrm{Gal}(L_1/F)$, $H=\mathrm{Gal}(L_1/K)$. Then, $L_1=LL^{\iota}$ and $L_0=L\cap L^{\iota}(\iota\in G-H)$. Also note that the archimedean places of L_0 above $\Omega_0'\cup\Omega_0^s$ and Ω_0^c are real and imaginary respectively.

For $\eta\in H$, let $\eta^{\iota}=\iota^{-1}\eta\iota\in H$. Here we shall describe Theorem 2.

Theorem 2. *Assume that $c_0\leqq 1$, $[L: L_0]=2$ and L_0/F is an abelian extension. Then, there exist a quadratic extension $\tilde{K}$ of F, an integral ideal $\tilde{\mathfrak{f}}$ of $\tilde{K}$, a subset $\tilde{\Omega}$ of real places of $\tilde{K}$ which is stable under the action of* Gal(K/F)*, and a primitive character $\tilde{\chi}$ of $H_{\tilde{K}}(\tilde{\mathfrak{f}}\tilde{\Omega})$, which satisfy $L_K(s,\chi)=L_{\tilde{K}}(s,\tilde{\chi})$.*

Proof. Regarding χ as a character of H through the Artin map, we consider $L_K(s,\chi)$ also as Artin L-function $L_{L_1/K}(s,\chi)$. By the assumption, L_1/L_0 is a biquadratic extension with intermediate fields L, L^{ι}. Let σ be a generator of Gal(L_1/L). Then, σ^{ι} is a generator of Gal(L_1/L^{ι}), and since L_0/F is abelian extension, $J=\mathrm{Gal}(L_1/L_0)=\{1,\sigma,\sigma^{\iota},\sigma\sigma^{\iota}\}$ contains the commutator D of G. Thus, we see that H satisfies the assumption (1) of Lemma 4, and J satisfies (ii) (and so (i)) in (3) of the lemma. By (2) of the lemma, there exist two abelian subgroups of G with index 2, besides H itself. Choose one of such two subgroups $\tilde{H}$. In the case $c_0=1$, choose it so that σ_c, $\sigma_c{}^{\iota}\notin\tilde{H}$, where σ_c is a Frobenius automorhism corresponding to the place in Ω_0^c. Let $\tilde{K}$ be the invariant field of $\tilde{H}$. By (4) of the lemma with the fact that $\chi(\sigma\sigma^{\iota})=-1$, there exists a character $\tilde{\chi}$ of $\tilde{H}$ of degree 1 such that $\mathrm{Ind}_H^G\chi=\mathrm{Ind}_{\tilde{H}}^G\tilde{\chi}$. By using properties of Artin L-functions, we see that

$$L_{L_1/K}(s,\chi)=L_{L_1/F}(s,\mathrm{Ind}_H^G\chi)=L_{L_1/F}(s,\mathrm{Ind}_{\tilde{H}}^G\tilde{\chi})=L_{L_1/\tilde{K}}(s,\tilde{\chi}).$$

Let $\tilde{\mathfrak{f}}\tilde{\Omega}$ be the conductor of the ideal group of K corresponding to L_1. Regarding $\tilde{\chi}$ as a character of $H_{\tilde{K}}(\tilde{\mathfrak{f}}\tilde{\Omega})$ through the Artin map, and we consider Artin L-function $L_{L_1/\tilde{K}}(s,\tilde{\chi})$ as Hecke L-function $L_{\tilde{K}}(s,\tilde{\chi})$ again. This shows the equation in Theorem 2.

Hereafter we consider whether an element of Ω_0^a is real or not on $\tilde{K}$ or L_1. We may assume that $\iota\notin\tilde{H}$. For $g\in\Omega_0^a$, let σ_g be the Frobenius automorphism of Gal(L_1/F) corresponding to g. Note that $\sigma_g=\sigma_g{}^{\iota}$ and $\sigma_g\in Z$ (=the center of G) are euivalent.

If $g\in\Omega_0'$, then the places of L_1 lying above g and so those of $\tilde{K}$ are all real since those of L and L^{ι} are all real.

If $g\in\Omega_0^c$, then our choice of $\tilde{H}$ shows that the places of $\tilde{K}$ lying above g and so those of L_1 are all imaginary.

Consider the case where $g\in\Omega_0^s$. Since each real place of L_0 lying above

Ω^s is imaginary on L and real on L^ι, each Frobenius automorphism with respect to L_1/K corresponding to a place in Ω^s or $\Omega^{s'}$ is all equal to σ^ι or σ respectively. (3) of Lemma 4 shows that $\sigma\sigma^\iota$ belongs to Z, and the fact $\sigma \neq \sigma^\iota$ shows that $\sigma, \sigma^\iota \in H - Z$. Since $H \cap \tilde{H} = Z$, we obtain $\sigma, \sigma^\iota \notin \tilde{H}$. Hence the places of $\tilde{K}$ lying above $\Omega_0{}^s$ are all imaginary.

Lastly, let $g \in \Omega_0$. Then, σ_g is non-trivial and so the places of L_1 lying above g are all imaginary. Assume that $\sigma_g \neq \sigma_g{}^\iota$. Since $\sigma, \sigma^\iota \in H$, we have $\sigma_g, \sigma_g{}^\iota \in H - Z$ and so $\sigma_g, \sigma_g{}^\iota \notin H$. This means that the places of $\tilde{K}$ lying above g are all imaginary. If $\sigma_g = \sigma_g{}^\iota$, it belongs to Z and so belongs to $\tilde{H}$. Then, the places of $\tilde{K}$ lying above g are all real.

By all the above consideration, we see that the archimedean part $\tilde{\Omega}$ of the conductor of the ideal group of $\tilde{K}$ corresponding to L_1 is stable under the action of Gal(K/F). On th other hand, since χ_0 is faithful on H by the choice of L, $\tilde{\chi}$ is also faithful on $\tilde{H}$ and so the archimedean part of the conductor of $\tilde{\chi}$ is equal to $\tilde{\Omega}$. This completes the proof.

Remark. The first part of the above proof is contained in [1]. The latter part can be regarded as a generalization of the proof of Proposition 5.1 in the paper.

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