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On Eakin-Nagata-Formanek Theorem

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The theorem of Eakin-Nagata ([1], [3], cf. [4]) was generalized by Formanek [2] and the main purpose of the present note is to give a new proof of the generalized result, which will be recalled below.

The writer likes to emphasize here that we avoided the use of Zorn Lemma in our new proof.

In this note, we mean by a *ring* a commutative ring with identity. If M is a module over a ring R, then a submodule of the form IM, with an ideal I of R, is called an *extended submodule* of M. Then the generalization can be stated as follows:

Eakin-Nagata-Formanek Theorem. *Let M be a finitely generated module over a ring R. If M satisfies the maximum condition on extended submodules, then M is a noetherian R-module, consequently, $R/(\text{Ann } M)$ is a noetherian ring, where* $\text{Ann } M=\{x\in R \mid xM=0\}$.

Before proving the theorem, we prove a lemma as follows:

Lemma. *Let M be a module over a ring R. For an $a\in R$, we denote by $0:a$ the ideal $\{x\in R \mid ax=0\}$. If* $\text{Ann } M=\{0\}$, *then we have*

$$\text{Ann}(M/(0:a)M)=0:a\,.$$

Proof. The inclusion $0:a\subseteq\text{Ann }(M/(0:a)M)$ is clear. As for the converse inclusion, $z\in\text{Ann}(M/(0:a)M)\Rightarrow zM\subseteq(0:a)M\Rightarrow az=0\Rightarrow z\in 0:a$. QED

Proof of the theorem. We use a double induction on the number of generators of M and the largeness of extended submodules of M. We may assume that M is generated by n elements and $\text{Ann } M=0$. Then our induction hypothesis is that (1) the assertion is true for R-modules generated by less than n elements and (2) if I is a non-zero ideal of R, then M/IM is a noetherian module. Note that if $n=1$, then the extended submodules are in one-one correspondence with ideals of R modulo Ann M (preserving inclusion relation). Thus the assertion is clear in this case.

(i) The case where there are non-zero elements a, b of R such that $ab=0$: By our induction hypothesis, $M/(0:a)M$ is a noetherian module and

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therefore $R/(0\colon a)$ is noetherian by the lemma above. Consider the R-module aM. aM is generated by n elements and is really an $R/(0:a)$-module. Thus aM is noetherian. Since M/aM is noetherian by our induction hypothesis, it follows that M is noetherian.

(ii) The other case: R is an integral domain. Let $u_1, \cdots, u_n$ be a set of generators of M. M/Ru_1 is an R-module generated by $n-1$ elements and therefore it is a noetherian module by our induction hypothesis. Set $I=\mathrm{Ann}(M/Ru_1)$. Then R/I is noetherian. If $I=\{0\}$, then R is noetherian. So, we assume that $I\neq\{0\}$. Take a non-zero element c of I. We have only to prove that any non-zero submodule N of M is finitely generated. Take a non-zero element w of N. Then cw is expressed as bu_1 with $b\in R$. M/bcM is a noetherian module by our induction hypothesis. bcM is generated by n elements and $bcM\subseteq Rbu_1\subseteq N$, which implies that N is finitely generated. QED

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