Regularity, closedness and spectral dimensions of the Dirichlet forms on P.C.F. self-similar sets

By

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§ 1. Introduction

Today, there has been many works about the self-similar sets. One well known framework for describing them is due to Hutchinson ([2]): Let \( F_i (1 \leq i \leq N) \) be contractions on a complete metric space \( X \). Then there exists a unique compact set \( K \) which satisfies \( K = \bigcup_{i=1}^{N} F_i(K) \). \( K \) is called a self-similar set with respect to \( \{F_i\}_{i=1}^{N} \).

\( K \) is said to be finitely ramified if \( \#(\bigcup_{i,j}(F_i(K) \cap F_j(K))) < \infty \). The Sierpinski gasket (Example 2.1) and nested fractals (e.g. Example 2.3) introduced by Lindström ([8]) are the examples of finitely ramified fractals.

In [3], Kigami defined and studied Laplace operators on the finitely ramified fractals systematically. There he defined post critically finite (P.C.F. for short) self-similar sets, which are almost the same concept as finitely ramified fractals. He introduced the notion of the “harmonic structure” and the “regularity of the harmonic structure” on them to define qua-

Example 2.1

Sierpinski gasket

Communicated by Prof. S. Watanabe, March 27, 1992
Example 2.2

Hata's tree-like set

Example 2.3

Lindström's snowflake

dratic forms and Laplace operators as a limit of difference operators. He proved that this quadratic form is closed and is a Dirichlet form if associated with a regular harmonic structure. In his study, he used the boundedness of the reproducing kernel.

On the other hand, spectral analysis of the Laplace operators was studied on nested fractals which are a subset of the class of P.C.F. self-similar sets ([1], [8]). There, spectral dimensions, which express the asymptotic frequency of the eigenvalues of the corresponding Laplace operators were identified.

In this paper, we will focus our attention on the P.C.F. self-similar sets. First, we construct another method to prove the closedness of the quadratic form. This method works even if the harmonic structure is not regular, when the reproducing kernel is unbounded. The key inequality (Poincare type inequality), originally obtained by Kusuoka ([6]), is Lemma 3.2, which
concludes that the form is closed under a natural self-similar measure. We then compute the spectral dimension of the Laplace operator with respect to the self-similar measure by extending the methods employed in the spectral analysis on nested fractals. We also analyze the regularity condition in the probabilistic way. The ideas of our methods come from Fukushima ([1]), Kigami([4]) and Kusuoka([7]).

In § 2, we briefly summarize Kigami's results on P.C.F. self-similar sets. In § 3, we prove the closedness of the quadratic form under the self-similar measure. As a result, we see that the regularity condition is not necessary to construct the Dirichlet form. In § 4, we determine the spectral dimensions. It is surprising that the spectral dimensions are not less than 2 in general. In § 5, we indicate a sufficient condition for the harmonic structure to be regular. In § 6, we give one-parameter family of the harmonic structures on the Sierpinski gasket which contains the Brownian motion as a special case. In § 7, we indicate a sufficient condition for the existence of the harmonic structure.

This work is originally from the author's Master thesis at Kyoto University (1991). In the thesis, the author restricted the class of fractals for some technical reason. After then, he was influenced by Kigami ([4]) to complete the research.

The author would like to express his sincere gratitude to Prof. S. Watanabe for his constant encouragement. Also he is very grateful to Prof. S. Kusuoka and Dr. J. Kigami for their fruitful advice, Prof. M. T. Barlow for correcting the English in the manuscript.

§ 2. P. C. F. self-similar sets

First, we introduce the one-sided shift space and give some notation.

Notation. 1) Let \( S = \{1, 2, \ldots, N\} \). The one-sided shift space \( \Omega \) is defined by

\[ \Omega = S^N. \]

Also, let

\[ \Omega_n = S^n. \]

2) For \( \omega \in \Omega \), we denote the \( i \)-th symbol by \( \omega_i \) and write \( \omega = \omega_1 \omega_2 \omega_3 \cdots. \)

3) If \( \omega \in \Omega_n \), we define \( |\omega| = n. \)

4) We denote \( i = iii \cdots \) for \( i \in S. \)

5) Let \( \sigma : \Omega \to \Omega \) be the shift map. I.e. \( \sigma \omega = \omega_2 \omega_3 \cdots \) if \( \omega = \omega_1 \omega_2 \cdots. \)

Define \( \delta_s : \Omega \to \Omega \) as \( \delta_s \omega = s \omega \) for \( s \in S. \)

We introduce the notion of self-similar structures and P. C. F. self-similar sets.
Definition 2.1. Let $K$ be a compact connected metric space and for each $s \in S$, $F_s : K \to K$ be a continuous injection. Then, $(K, N, \{F_s\}_{s \in S})$ is said to be a self-similar structure on $K$ if there exists a continuous surjection $\pi : \mathcal{Q} \to K$ such that $\pi \circ \delta_s = F_s \circ \pi$ for every $s \in S$. Further, for $\omega \in \mathcal{Q}_n$, we denote $F_\omega = F_{\omega_1} \circ F_{\omega_2} \cdots \circ F_{\omega_n}$ and $K_\omega = F_\omega(K)$. In particular, $K_s = F_s(K)$ for $s \in S$.

Remark that Hutchinson’s self-similar set is a self-similar structure in the sense of Definition 2.1 by taking $\pi(\omega) = \bigcap_{n \geq 1} F_{\omega_1} \circ F_{\omega_2} \cdots \circ F_{\omega_n}(K)$.

Definition 2.2. Let $\mathcal{L} = (K, S, \{F_s\}_{s \in S})$ be a self-similar structure on $K$. Then the critical set of $\mathcal{L}$ is defined by
\begin{equation}
C(\mathcal{L}) = \pi^{-1}(\bigcup_{s, t \in S, s \neq t}(K_s \cap K_t))
\end{equation}
and the post critical set of $\mathcal{L}$ is defined by
\begin{equation}
P(\mathcal{L}) = \bigcup_{n \geq 1} \sigma^n(C(\mathcal{L})).
\end{equation}
$\mathcal{L}$ is called post critically finite, or P.C.F. for short, if $P = P(\mathcal{L})$ is a finite set.

In the following, we consider a P. C. F. self-similar set $(K, S, \{F_s\}_{s \in S})$. In order to make our framework clear, we give some examples.

Example 2.1: Sierpinski Gasket
\[ S = \{1, 2, 3\}. \]
\[ \pi(C) = \{q_1, q_2, q_3\}; \quad \pi^{-1}(q_1) = \{23, 3\}, \quad \pi^{-1}(q_2) = \{13, 31\}, \]
\[ \pi^{-1}(q_3) = \{12, 21\}. \]
\[ \pi(P) = \{p_1, p_2, p_3\}; \quad \pi^{-1}(p_i) = \{i\} \quad \text{for} \quad i = 1, 2, 3. \]

Example 2.2: Hata’s Tree-like Set
\[ S = \{1, 2\}. \]
\[ \pi(C) = \{q\}; \quad \pi^{-1}(q_1) = \{12, 21\}, \]
\[ \pi(P) = \{p_1, p_2, p_3\}; \quad \pi^{-1}(p_i) = \{i\} \quad \text{for} \quad i \in S, \quad \pi^{-1}(p_3) = \{12\}. \]

Example 2.3: Lindstrøm’s snowflake
\[ S = \{1, 2, 3, 4, 5, 6, 7\}. \]
\[ \pi(C) = \{q_i : 1 \leq i \leq 12\}; \]
\[ \pi^{-1}(q_i) = \{i + \frac{1}{2}, i + 1\} \quad \text{for} \quad 1 \leq i \leq 6, \]
\[ \pi^{-1}(q_i) = \{7i, 7i + 3\} \quad \text{for} \quad 7 \leq i \leq 12. \]
Example 2.4

\[ \langle i \rangle = i \text{ if } i \leq 6 \text{ and } \langle i \rangle = i - 6 \text{ if } 7 \leq i \].

\[ \pi(P) = \{ p_i : 1 \leq i \leq 6 \}; \quad \pi^{-1}(p) = \{ \langle i \rangle \} \text{ for } 1 \leq i \leq 6. \]

**Example 2.4: (Introduced by Kigami)**

\[ S = \{1, 2, 3, 4\} . \]

\[ \pi(C) = \{ q_i : 1 \leq i \leq 6 \}; \quad \pi^{-1}(q_1) = \{23, 32\}, \quad \pi^{-1}(q_2) = \{13, 31\}, \]
\[ \quad \pi^{-1}(q_3) = \{12, 21\}, \quad \pi^{-1}(q_4) = \{41, 123, 132\}, \]
\[ \quad \pi^{-1}(q_5) = \{42, 213, 231\}, \quad \pi^{-1}(q_6) = \{43, 312, 321\}. \]

\[ \pi(P) = \{ p_i : 1 \leq i \leq 6 \}; \quad \pi^{-1}(p) = \{ \langle i \rangle \} \text{ for } i = 1, 2, 3, \quad \pi^{-1}(p_4) = \{23, 32\}, \]
\[ \pi^{-1}(p_5) = \{13, 31\}, \quad \pi^{-1}(p_6) = \{12, 21\}. \]

**Definition 2.3.** Given positive numbers \( r_1, r_2, \ldots, r_N \), define \( a = \sum_{i=1}^{N} r_i^{-1} \), \( \theta_i = (ar_i)^{-1} \). Let \( \tilde{\nu} \) be a Bernoulli measure on \( \Omega \) such that \( \tilde{\nu}(\omega_k = i) = \theta_i \) for all \( k \geq 1, 1 \leq i \leq N \). Let \( \nu \) be the induced measure of \( \tilde{\nu} \) on \( K \) by \( \pi \). \( \nu \) is the self-similar measure in the sense of Hutchinson ([2]). We sometimes identify \( \nu \) and \( \tilde{\nu} \), \( L^2(K, \nu) \) and \( L^2(\Omega, \tilde{\nu}) \) because \( \pi \) is almost surely one to one.

**Notation.** 1) For \( m \geq 0 \), let

\[ P^{(m)} = \bigcup_{\omega \in \Omega_m} \omega P \], \( V_m = \pi(P^{(m)}) \) and \( \hat{V}_m = V_m - V_0 \).

Moreover, \( B_\omega = F_\omega(\pi(P)) \) for \( \omega \in \Omega_m \) for any \( m \geq 0 \).

2) For the finite set \( V, V' \), we define

\[ l(V) = \{ f : V \to \mathbb{R} \} , \]
\[ L(V, V') = \{ A | A : l(V) \to l(V') \text{ and } A \text{ is linear} \}. \]

\[ L(V) = L(V, V). \]

**Definition 2.4.** A pair \((D, r) \in L(V_0) \times l(S)\) is called a quasi-harmonic structure on \(K\) if it satisfies the following.

1) \( r_s > 0 \) for each \( s \in S \).
2) \( D = 'D \).
3) \( D \) is irreducible.
4) \( D_{pq} < 0, \sum_{q \in V_0} D_{pq} = 0 \) for each \( p \in V_0 \).
5) \( D_{pq} \geq 0 \) if \( p \neq q \).

From the quasi-harmonic structure \((D, r)\), we have a difference operator \(H_m\) on \(V_m\).

**Definition 2.5.** A difference operator \(H_m \in L(V_m)\) is defined by

\[ H_m = \sum_{\omega \in D_m} r_{\omega}^{-1} D\omega R_{\omega}, \]

where \(R_{\omega} : l(V_m) \to l(V_0)\) is defined by \(R_{\omega}(u) = u \circ F_{\omega}\) and \(r_{\omega} = r_{\omega_1} \cdots r_{\omega_m}\).

In the following, we decompose \(H_m\) into

\[ H_m f = \begin{pmatrix} T_m & J_m \\ \ast \end{pmatrix} \begin{pmatrix} f |_{V_0} \\ f |_{V_m} \end{pmatrix}, \]

where \(T_m \in L(V_0), J_m \in L(V_0, \hat{V}_m), X_m \in L(\hat{V}_m)\).

Especially we write \(T = T_1, J = J_1, X = X_1\).

Next, we give the notion of the harmonic structure. With this structure, harmonic functions on \(V_m\) w.r.t. \(H_m\) automatically become harmonic functions on \(V_{m-1}\) w.r.t. \(H_{m-1}\).

**Definition 2.6.** A quasi-harmonic structure \((D, r)\) is called a harmonic structure if there exists \(\lambda > 0\) such that

\[ T - 'JX^{-1}J = \lambda^{-1}D. \]

Further, a harmonic structure \((D, r)\) is said to be regular if \(r_s < \lambda\) for each \(s \in S\).

Regularity of the harmonic structure is a condition for the reproducing kernel of the following quadratic form \((\mathcal{F}, \mathcal{E})\) to be bounded continuous.

We introduce some results derived from the definition of the harmonic structure.

**Definition 2.7.** \(f \in C(K) = \{ f : f \text{ is a continuous function on } K \}\) is said to be harmonic if and only if \(H_m f |_{V_0} = 0\) for all \(m \geq 1\).
Theorem 2.8 ([3], Theorem 4.12). Let \((D, r)\) be a harmonic structure. Then for any \(\rho \in l(V_0)\), there exists a unique harmonic function \(f\) with \(f|_{V_0} = \rho\).

Definition 2.9. For \(f \in l(\bigcup_{n \geq 0} V_n)\), \(P_m f\) is a continuous function on \(K\) satisfying the following.
1) \(P_m f|_{V_m} = f|_{V_m}\),
2) \(H_n(P_m f)|_{V_{m+1} \setminus V_m} = 0\) for all \(n > m\).

By Theorem 2.8, if \((D, r)\) is a harmonic structure, \(P_m f\) exists uniquely for all \(f \in l(\bigcup_{n \geq 0} V_n), m \geq 1\).

For \(u, v \in l(V_m)\), define
\[
\mathcal{E}_m(u, v) = -\lambda_m^n \langle uH_m v \rangle.
\]

By Corollary 6.14 in [3], we see that \(\mathcal{E}_m(u, u) \leq \mathcal{E}_{m+1}(u, u)\) for \(u \in l(V_{m+1})\) (equality holds if and only if \(P_m u|_{V_{m+1}} = u\)). Thus we can define a symmetric form \(\mathcal{E}\) as follows. For \(f \in l(\bigcup_{n \geq 0} V_n)\), let
\[
\mathcal{E}(f, f) = \lim_{m \to \infty} \mathcal{E}_m(f|_{V_m}, f|_{V_m}) \quad \text{and}
\]
\[
\mathcal{F} = \{ f \in l(\bigcup_{n \geq 0} V_n), \mathcal{E}(f, f) < \infty \}.
\]

For \(u, v \in \mathcal{F}\), define
\[
\mathcal{E}(u, v) = \lim_{m \to \infty} \mathcal{E}_m(u|_{V_m}, v|_{V_m}).
\]

Now we give the main theorem by Kigami ([3]).

Theorem 2.10. Let \((D, r)\) be a regular harmonic structure on \(K\). Then, the following holds.
1) \(\mathcal{F} \subset C(K)\).
2) \((\mathcal{F}, \mathcal{E})\) is a regular local Dirichlet form on \(L^2(K, \nu)\).
3) Let \(\mathcal{E}_{\langle \nu \rangle}(f, g) = \mathcal{E}(f, g) + a(f, g)\nu(K, \nu)\) for \(a > 0\). Then \((\mathcal{F}, \mathcal{E}_{\langle \nu \rangle})\) admits a continuous bounded reproducing kernel.

Remark. 1) In fact, Kigami discussed the quadratic form \((\mathcal{F}, \mathcal{E})\) on \(L^2(K, \mu)\) for any everywhere dense probability measure.
2) If we denote \(A\) the corresponding self-adjoint operator of \((\mathcal{F}, \mathcal{E})\), by the above theorem we know \(-A\) has a compact resolvent. Therefore spectrums of \(-A\) consist of eigenvalues and we can express them using the min-max principle.

§ 3. Closedness of the quadratic form

In this section, we prove that the quadratic form \((\mathcal{F}, \mathcal{E})\) in § 2 is closed on \(L^2(K, \nu)\) and has a compact resolvent even if \((D, r)\) is not regular, following
the method of Kusuoka [6]. As a result, $(\mathcal{D}, \mathcal{E})$ is a regular local Dirichlet form and we can analyze the asymptotic behavior of eigenvalues using the min-max principle. First, we give some examples of quasi-harmonic structures which are not regular.

**Example 3.1**

$S=\{1, 2, 3, 4\}$.

$\pi(C)=\{q_1, q_2, q_3\}; \quad \pi^{-1}(q_1)=\{23, 32, 41\}, \quad \pi^{-1}(q_2)=\{13, 31, 43\}$,

$\pi^{-1}(q_3)=\{12, 21, 42\}$.

$\pi(P)=\{p_1, p_2, p_3\}; \quad \pi^{-1}(p_i)=\{i\} \text{ for } i=1, 2, 3$.

$D=\begin{pmatrix}
-2 & 1 & 1 \\
1 & -2 & 1 \\
1 & 1 & -2
\end{pmatrix}, \quad r=(1, 1, 1, r^{-1})$.

Then, (2.5) holds if and only if $\lambda=\frac{3r+5}{2r+3}$. Thus $(D, r)$ is a harmonic structure but not regular if $r\leq\frac{-1+\sqrt{5}}{2}$.

**Example 3.2**

$S=\{1, 2, 3, 4, 5\}$.

$\pi(C)=\{q\}; \quad \pi^{-1}(q)=\{12, 21, 31, 41, 51\}$.

$\pi(P)=\{p_1, p_2\}; \quad \pi^{-1}(p_1)=\{1\}, \quad \pi^{-1}(p_2)=\{2\}$.

$D=\begin{pmatrix}
-1 & 1 \\
1 & -1
\end{pmatrix}, \quad r=(1, 1, r^{-1}, r^{-1}, r^{-1})$.

Then, (2.5) holds if and only if $\lambda=2$. Thus $(D, r)$ is a harmonic structure but not regular if $r\leq\frac{1}{2}$.
Let \#V_0 = l. As \#P < \infty, all the elements of P are preperiodic.

Noting this, we define a map \( \rho: \{1, 2, \ldots, l\} \to \pi^{-1}(V_0) \) such that \( \rho(i) = \rho(i)q(i) \). Here \( |\rho(i)| = s_i \) and \( |q(i)| = t_i \) for some \( s_i, t_i \in N \). \( q(i) \) is a periodic part of \( \rho(i) \). We select representatives of \( \pi^{-1}(V_0) \) adequately so that \( \rho \) satisfies \( \pi(\rho(\{1, 2, \ldots, l\})) = V_0 \).

**Lemma 3.1.** There is a positive constant \( C < \infty \) such that

\[
\int_a |g(\pi(\omega)) - g(\pi([\omega, \rho(j)]_m))|^2 \psi(d\omega) \leq C(a\lambda)^{-m} E(g, g)
\]

for any \( g \in C(K) \) such that \( g|_{\cup_{n \geq 0} V_n} \in \mathcal{F} \).

Here \( m \geq 0, 1 \leq j \leq l \) and \( [\omega, \omega']_m = \omega_1\omega_2 \cdots \omega_m\omega_{m+1}\omega_{m+2} \cdots \).

**Proof.** Our proof is essentially the same as the proof of (4.12) in [6]. First, remark that, from Theorem 4.15 in [3] we have

\[
(3.1) \quad a\lambda = \sum_{i=1}^N \frac{\lambda}{r_i} > 1.
\]

By the irreducibility of \( D \) and by the definition of \( H_m \), we have

\[
\sup_m (a\lambda)^m \int_a \sup_{1 \leq j \leq l} |g(\pi([\omega, q(i)]_m)) - g(\pi([\omega, q(j)]_m))|^2 \psi(d\omega) \leq C E(g, g).
\]

Thus, by changing the constant \( C \),

\[
\sup_m (a\lambda)^m \int_{|\omega| = \sum_{1 \leq j \leq l} \sup_{|\rho(j)| + 1 \leq j \leq l} |g(\pi([\omega, \omega'q(j)]_m)) - g(\pi([\omega, \omega'q(j)]_m))|^2 \psi(d\omega)
\]

\[
\leq C E(g, g).
\]

As \( K \) is connected, we see

\[
\int_a |g(\pi([\omega, \rho(i)]_{m+1})) - g(\pi([\omega, \rho(i)]_m))|^2 \psi(d\omega) \leq C(a\lambda)^{-m} E(g, g).
\]

Summing this up and using (3.1), we obtain the result.

For each \( f \in \mathcal{F} \), consider \( P_m f \in C(K) \). Noting that \( P_m f|_{V_m} = f|_{V_m} \) and \( E(P_m f, P_m f) = E_m(f, f) \), then by Lemma 3.1, we see that \( \{P_m f\} \) is a Cauchy sequence in \( L^2(K, \nu) \). Let the limit be \( i(f) \). We have the map \( i: \mathcal{F} \to L^2(K, \nu) \). We will show this map is injective. Before that, we give the extension of Lemma 3.1, which is an easy consequence of the above results.

**Lemma 3.2.** There is a positive constant \( C < \infty \) such that
\[
(3.2) \quad \int_{\mathcal{F}} |i(g)\pi(\omega) - g(\pi([\omega, \rho(j)]_m)|^2 \tilde{\nu}(d\omega) \leq C(a\lambda)^{-m} \mathcal{E}(g, g)
\]
for any \(g \in \mathcal{F}, m \geq 0, 1 \leq j \leq l\).

**Proposition 3.3.** \(i: \mathcal{F} \to L^2(K, \nu)\) is injective.

**Proof.** Let \(|\rho(j)| = s_j, |q(j)| = t_j\) for \(1 \leq j \leq l\). By the definition of \(P, q(j) \in P\). Thus, from Lemma 4.15.3 in [3],

\[
(3.3) \quad (a\lambda)^{t_j} \theta_{q(j)} = \frac{\lambda^{t_j}}{r_{q(j)}} > 1.
\]

Also, from the above lemma, there is a positive constant \(C\) such that

\[
C(a\lambda)^{-m} \mathcal{E}(g, g) \geq \int_{\mathcal{F}} |i(g)\pi(\omega) - g(\pi([\omega, q(j)]_m)|^2 \tilde{\nu}(d\omega)
\]

\[
\geq \int_{\rho(j)[\rho(j)-q(j)]} |i(g)\pi(\omega) - g(\pi([\omega, q(j)]_m)|^2 \tilde{\nu}(d\omega)
\]

\[
= \theta_{p(j)} \theta_{q(j)} \int_{\mathcal{F}} |i(g)\pi(\rho(j)]_m) - g(\pi(\rho(j))]\tilde{\nu}(d\omega).
\]

(Here \(m\) is chosen as \(s_j + nt_j = m\).) Therefore, using the triangle inequality, we have

\[
(3.4) \quad |g(\pi(\rho(j))] - \left(\int_{\mathcal{F}} |i(g)\pi(\rho(j)]_m)\tilde{\nu}(d\omega)\right)^{\frac{1}{2}}
\]

\[
\leq \left(\int_{\mathcal{F}} |g(\pi(\rho(j))] - i(g)\pi(\rho(j))]_m)\tilde{\nu}(d\omega)\right)^{\frac{1}{2}}
\]

\[
\leq \{C'((a\lambda)^{t_j} \theta_{q(j)})^{-n} \mathcal{E}(g, g)\}^{\frac{1}{2}}.
\]

Now, suppose that \(g \in \mathcal{F}, i(g) = 0\) (\(L^2\) sense). By the above, we have

\[
|g(\pi(\rho(j))]| \leq \{C'((a\lambda)^{t_j} \theta_{q(j)})^{-n} \mathcal{E}(g, g)\}^{\frac{1}{2}}.
\]

From (3.3), by letting \(n \to \infty\), we see \(g(\pi(\rho(j))] = 0\). In the same way, we know \(g(\pi([\omega, \rho(j)]_m]) = 0\) for \(1 \leq j \leq l, \omega \in \Omega, m \geq 0\). Thus, we have \(g = 0\).

In the following, we consider \(\mathcal{F} \subset L^2(K, \nu)\).

**Proposition 3.4.** \((\mathcal{F}, \mathcal{E})\) is closed in \(L^2(K, \nu)\).

**Proof.** First, remark that by an easy consequence of Lemma 3.2, we have,
(3.5) \( |g(\pi([\omega, \rho(j)]_m))| \leq \{\theta_{\omega_1} \cdots \theta_{\omega_m}\}^{-\frac{1}{2}}[\|g\|_2^2 + \{C(a\lambda)^{-m} \mathcal{E}(g, g)\}^{\frac{1}{2}}] \),

(3.6) \( \|g\|_2 \leq g(\pi(\rho(j))) + \{C \mathcal{E}(g, g)\}^{\frac{1}{2}} \),

for \( 1 \leq j \leq l \), \( \omega \in \Omega \), \( m \geq 0 \), \( g \in \mathcal{F} \).

Now, suppose that \( g_n \in \mathcal{F} \) and \( \mathcal{E}_{(\cdot)}(g_n - g_n', g_n - g_n') \to 0 \) as \( n, n' \to \infty \) (remember that \( \mathcal{E}_{(\cdot)}(\cdot, \cdot) = \mathcal{E}(\cdot, \cdot) + (\cdot, \cdot)_{L^2} \)). If we substitute \( g_n - g_n' \) for \( g \) in (3.5), we see \( \{g_n(\pi([\omega, \rho(j)]_m))\} \) is a Cauchy sequence because of the assumption of \( \{g_n\} \).

Denote the limit as \( n \to \infty \) by \( g(\pi([\omega, \rho(j)]_m)) \). Then we have the limit \( g \in \mathcal{L}(\bigcup_{n \geq 0} V_n) \). For each \( k, n \geq 0 \),

\[
\mathcal{E}_k(g_n - g, g_n - g) = \lim_{m \to \infty} \mathcal{E}_k(g_n - g_m, g_n - g_m) 
\leq \sup_{m \geq n} \mathcal{E}(g_n - g_m, g_n - g_m).
\]

Therefore, we have

(3.7) \( \mathcal{E}(g_n - g, g_n - g) \leq \sup_{m \geq n} \mathcal{E}(g_n - g_m, g_n - g_m) \).

Thus \( g \in \mathcal{F} \) and \( \mathcal{E}(g_n - g, g_n - g) \to 0 \) as \( n \to \infty \) because the right hand side of (3.7) goes to 0. Also, we see that \( g_n \to g \) in \( L^2 \) by (3.6).

In [3], Kigami used the bounded reproducing kernel to prove the closedness of the quadratic form. But the reproducing kernel is not bounded when the harmonic structure is not regular. Here, we have proved closedness even when the harmonic structure is not regular. We can check Markov property e.t.c. just the same way as [3], and see that the quadratic form \( (\mathcal{F}, \mathcal{E}) \) is a regular local Dirichlet form on \( L^2(K, \nu) \).

Next we show that this Dirichlet form has a compact resolvent.

**Proposition 3.5.** \((\mathcal{F}, \mathcal{E})\) has a compact \( a \)-order resolvent for each \( a > 0 \).

**Proof.** Define \( T_m g(\pi(\omega)) = g(\pi([\omega, \rho(1)]_m)) \). By (3.2), we have

\[
\|g - T_m g\|_2 \leq C(a\lambda)^{-m} \mathcal{E}(g, g) \leq C(a\lambda)^{-m} \mathcal{E}_{<\alpha>(g, g)}.
\]

Thus if we let \( I : (\mathcal{F}, \mathcal{E}_{<\alpha>}) \to L^2(K, \nu) \) be the inclusion map, we have

\[
\|I - T_m\|_2 \|g\|_2 \leq \{C(a\lambda)^{-m}\}^{\frac{1}{2}}.
\]

Therefore, \( T_m \to I \) in the operator norm. As \( T_m \) is a compact operator, we see \( I \) is a compact operator. Thus \( I^* \) is also a compact operator. As \( I^* \) equals to the \( a \)-order resolvent operator, the \( a \)-order resolvent is compact.
Remark 3.6. Given positive numbers $\mu_1, \ldots, \mu_N$ such that $\sum_{i=1}^{N} \mu_i = 1$, let $\bar{\mu}$ be a Bernoulli measure on $\Omega$ such that $\bar{\mu}(\omega_i = i) = \mu_i$ for all $k \geq 1, 1 \leq i \leq N$. (\bar{\nu} satisfies this condition. Also, if $(D, r)$ is regular, all the everywhere dense Bernoulli measures satisfy this condition.) Let $\mu$ be the induced measure of $\bar{\mu}$ on $K$ by $\pi$. Then we can prove $\mathcal{F} \subset L^2(K, \mu)$ in the same way. Proposition 3.4, 3.5 hold when the base measure is $\mu$. (To generalize Proposition 3.4, we need a scaling property of the Dirichlet form with general Bernoulli measure, which is an extension of Lemma 4.2.)

If we denote $A$ the corresponding self-adjoint operator of $(\mathcal{F}, \mathcal{E})$, by the above proposition we know $-A$ has a compact resolvent. Therefore spectrums of $-A$ consist of eigenvalues and we can express them using the min-max principle.

§ 4. Computation of the spectral dimensions

In this section, we will identify the asymptotic frequency of the eigenvalues of the self-adjoint operator. In the followibg, we fix the harmonic structure $(D, r)$ and the self-similar measure $\nu$ introduced in Definition 2.3.

Definition 4.1. For $f \in l(K)$, define $\sigma: l(K) \to l(K) (1 \leq i \leq N)$ by

$$\sigma f(\omega) = f \circ F_i(\omega) \quad \text{for } \omega \in K.$$ 

By definition, we see

$$R_{\omega}(f|_{\nu_{m_i}}) = R_{\omega}(\sigma f|_{\nu_m}) \quad \text{for } \omega \in \Omega_m.$$ 

Lemma 4.2. (1) If $f, g \in \mathcal{F}$, then

$$\mathcal{E}(f, g) = a\lambda \sum_{i=1}^{N} \mathcal{E}(\sigma_i f, \sigma_i g) \theta_i.$$ 

(2) If $f \in L^1(K, \nu)$, then

$$\int_K \sigma f d\nu = \frac{1}{\theta_i} \int_{K_i} f d\nu.$$ 

Proof. By definition and (4.1),

$$\mathcal{E}_{m+1}(f|_{\nu_{m+1}}, g|_{\nu_{m+1}}) = -\lambda^{m+1} (f|_{\nu_{m+1}}) H_{m+1}(g|_{\nu_{m+1}})$$

$$= -\lambda^{m+1} \sum_{|\omega|=m+1} r^{-1}_\omega f(R_\omega(f|_{\nu_{m+1}})) D(R_\omega(g|_{\nu_{m+1}}))$$

$$= -\lambda^{m+1} \sum_{i=1}^{N} \sum_{|\omega|=m} r_i^{-1} r^{-1}_\omega f(R_\omega(\sigma_i f|_{\nu_m})) D(R_\omega(\sigma_i g|_{\nu_m}))$$

$$-\lambda^{m+1} \sum_{i=1}^{N} \sum_{|\omega|=m} r_i^{-1} r^{-1}_\omega f(R_\omega(\sigma_i f|_{\nu_m})) D(R_\omega(\sigma_i g|_{\nu_m}))$$

$$= -\lambda^{m+1} \sum_{i=1}^{N} \sum_{|\omega|=m} r_i^{-1} r^{-1}_\omega f(R_\omega(\sigma_i f|_{\nu_m})) D(R_\omega(\sigma_i g|_{\nu_m}))$$
\[
E_m(f, g) = \frac{\lambda}{r_i} E_m(\sigma f, \sigma g).
\]

2) We define a subspace \( F_i \subseteq l(F_i(\cup_{n \geq 0} V_n)) \) by

\[
F_i = \{ f | f \in l(F_i(\cup_{n \geq 0} V_n)), \quad \lim_{m \to -\infty} E_m^i(f, f) < \infty \},
\]

and a symmetric form \( E_i \) on \( F_i \) by

\[
E_i^i(f, g) = \lim_{m \to -\infty} E_m^i(f, g).
\]

By definition,

\[
\sigma_i F_i = F_i.
\]

Therefore we can easily prove \((F_i, E_i)\) is a regular local Dirichlet form on \( L^2(K_i, \nu) \). Let \(-\Delta_i\) be the self-adjoint operator of this form.

**Definition 4.4.** We define

\[
F_i^0 = \{ f \in F | f(\omega) = 0 \quad \text{if} \quad \omega \in V_0 \}
\]

\[
F_i^0 = \{ f \in F | f(\omega) = 0 \quad \text{if} \quad \omega \in V_1 \}.
\]

Also, let \(-\Delta_i^0\) be the self-adjoint operator of \((F_i^0, E_i)\) on \( L^2(K_i, \nu) \).

We have the following relations between eigenvalues of \(-\Delta\) and \(-\Delta_i\).

**Proposition 4.5.** The following are equivalent.

1) \( k \) is the eigenvalue of \(-\Delta\) (resp. \(-\Delta^0\)) with the eigenfunction \( \sigma g \).
2) \( a\lambda k \) is the eigenvalue of \(-\Delta_i\) (resp. \(-\Delta_i^0\)) with the eigenfunction \( g \).

**Proof.** (2) is equivalent to

\[
E_i^i(g, h) = a\lambda k(g, h)_{L^2(K_i, \nu)} \quad \text{for all} \quad h \in F_i (\text{resp.} \ F_i^0).
\]
From Lemma 4.2, this is equivalent to

$$\frac{\lambda}{F_i} \mathcal{E}(\sigma, g, \sigma, h) = a\lambda k\theta_i(\sigma, g, \sigma, h)\nu^i_{(K, \nu)}.$$  

This is equivalent to (1) because (4.4) holds.

**Remark.** From Proposition 4.5, we see that eigenvalues of $-\Delta_i$ (resp. $-\Delta_i^0$) are exactly the same for each $i$.

**Proposition 4.6.**

(4.5) $\#\{\lambda|\lambda \text{ is an eigenvalue of } -\Delta, \lambda \leq x\} \leq N\#\{\lambda|\lambda \text{ is an eigenvalue of } -\Delta_i, \lambda \leq x\}$,

(4.6) $\#\{\lambda|\lambda \text{ is an eigenvalue of } -\Delta_i^0, \lambda \leq x\} \geq N\#\{\lambda|\lambda \text{ is an eigenvalue of } -\Delta_i^0, \lambda \leq x\}$.

**Proof.** First, let $F_i$ be the totality of the function $u \in l(\cup_{n \geq 0} V_n)$ such that for any $1 \leq i \leq N$, there exists $u_i \in F_i$ whose restriction to $F_i(\cup_{n \geq 0} V_n \setminus V_0)$ is equal to $u|_{F_i(\cup_{n \geq 0} V_n \setminus V_0)}$. If we define

$$\tilde{\mathcal{E}}(u, v) = \sum_{i=1}^{N} \mathcal{E}^i(u_i, v_i) \quad \text{for } u, v \in \mathcal{F},$$

then $(\mathcal{F}, \tilde{\mathcal{E}})$ is a local Dirichlet form on $L^2(K, \nu)$.

By the definition of $\mathcal{E}^i$ and Lemma 4.2, $\tilde{\mathcal{E}} = \mathcal{E}$ on $\mathcal{F}$ (note that $\mathcal{F} \subset \mathcal{F}$). Therefore if we denote the n-th eigenvalue of an operator $A$ by $\nu_n(A)$ in general, we have $\nu_n(-\Delta) \geq \nu_n(-\tilde{A})$. (Remark that we can also apply the min-max principle to compute eigenvalues of $-\tilde{A}$.) It is clear that the right hand side is equal to $N \cdot \nu_n(-\Delta')$. Hence we obtain (4.5).

Let $F^0_i$ be as follows:

$$F^0_i = \{f \in F^0 | f = 0 \text{ on } V_i\}.$$  

Then, (4.6) is proved in the same way.

**Notation 4.7.** Let $\rho(x)$ and $\rho^0(x)$ be defined by

$$\rho(x) = \#\{\lambda|\lambda \text{ is an eigenvalue of } -\Delta, \lambda \leq x\},$$

$$\rho^0(x) = \#\{\lambda|\lambda \text{ is an eigenvalue of } -\Delta_i^0, \lambda \leq x\}.$$  

Now we arrive at the main theorem.

**Theorem 4.8.** If we define $d_s = 2\log N / \log a\lambda$, then we have

$$0 < \liminf_{x \to +\infty} \frac{\rho(x)}{x^\frac{s}{2}} \leq \limsup_{x \to +\infty} \frac{\rho(x)}{x^\frac{s}{2}} < +\infty.$$  

Therefore, the spectral dimension of $-\Delta$ on $L^2(K, \nu)$ is $d_s$. 
Proof. From Proposition 4.5 and 4.6, we have
\[ \frac{1}{N} \rho(x) \leq \#(\lambda | \lambda \text{ is an eigenvalue of } -\Delta, \lambda \leq x) = \rho\left( \frac{x}{a \lambda} \right) \]
and
\[ \frac{1}{N} \rho^0(x) \geq \#(\lambda | \lambda \text{ is an eigenvalue of } -\Delta^0, \lambda \leq x) = \rho^0\left( \frac{x}{a \lambda} \right). \]

Thus, if we let \( N_i(x) = N^{-i} \rho((a \lambda)^{i} x) \) and \( N_i^0(x) = N^{-i} \rho^0((a \lambda)^{i} x) \), then \( N_i \) is non-increasing and \( N_i^0 \) is non-decreasing when \( l \) increases. Denote by \( N_0 \) and \( N_0^0 \) their limits as \( l \to \infty \), respectively. If we take \( \lambda_0 \) sufficiently large, we have
\[ 0 < N_0^0(\lambda_0) \leq N_0(\lambda_0) < \infty. \]

For each \( x \), choose \( l \) so that \((a \lambda)^{i-1} \lambda_0 < x < (a \lambda)^i \lambda_0 \). Then, \( N^{i-1} \lambda_0^{d_k} \leq x \leq N^{i} \lambda_0^{d_k} \). Therefore, we have
\[ \frac{\rho((a \lambda)^{i-1} \lambda_0)}{N^{i-1}} \cdot (N \lambda_0^{d_k})^{-1} \leq \frac{\rho(x)}{x^{d_k}} \leq \frac{\rho((a \lambda)^i \lambda_0)}{N^{i}} \cdot N \lambda_0^{d_k}. \]

Taking \( x \to \infty \) (thus \( l \to \infty \)), we finally obtain
\[ 0 < (N \lambda_0^{d_k})^{-1} \cdot N_0(\lambda_0) \leq \lim \inf \frac{\rho(x)}{x^{d_k}} \leq \lim \sup \frac{\rho(x)}{x^{d_k}} \leq N \lambda_0^{d_k} \cdot N_0(\lambda_0) < + \infty. \]

Remark. 1) When \((D, r)\) is a regular harmonic structure, \(a \lambda > N\). Thus the spectral dimension of \(-\Delta\) on \(L^2(K, \nu)\) is less than 2 in this case.
2) In the case of Example 3.2, \(a \lambda = 4 + 6r\). Therefore the spectral dimension is greater than 2 when \(r < \frac{1}{6}\).
3) [Relations to Kusuoka's Dirichlet forms.] In [6], Kusuoka constructed Dirichlet forms on quotient spaces of the one-sided shift spaces. Here we briefly mention some relations between Kigami's Dirichlet forms and Kusuoka's ones.

Let \((D, r)\) be a harmonic structure. If we let \(A_s = R_s \left( \begin{array}{cc} I & 0 \\ -X^{-1} J \end{array} \right)\), then
\[ \sum_{s=1}^{N} w_s^t A_s D A_s = \sum_{s=1}^{N} w_s (I - t X^{-1})^t R_s D R_s \left( \begin{array}{cc} I & 0 \\ -X^{-1} J \end{array} \right) \]
Thus, the equation (4.2) in [6] is satisfied if we let \( w_s = \theta_s, \ Q_0 = DP, \ \lambda = \frac{1}{a\lambda}. \)

Further, \( S_n f = P_n f, \ M = l \) and \( D = k \) in his paper.

By the same proof, we see that the spectral dimension of the generator of the Kusuoka’s Dirichlet form is \( \frac{2\log N}{\log \lambda}. \)

4) Recently, Kigami-Lapidus ([5]) extends our methods and obtains the following: If \( (D, r) \) is a regular harmonic structure and \( \mu \) is an arbitrary everywhere dense Bernoulli probability measure with \( \mu(\omega_k = i) = \mu_i, \) and if \( d_s \) is the unique positive number such that \( \Sigma_{i=1}^{N} \left( \frac{\mu_i r_i}{\lambda} \right)^{d_s/2} = 1, \) then the spectral dimension of the Dirichlet form \((\mathcal{F}, \mathcal{E})\) on \( L^2(K, \mu) \) is \( d_s. \)

Combined their result and Remark 3.6, we see that if \( \mu_i \) satisfies \( \frac{\mu_i r_i}{\lambda} < 1, \) then their result holds even if \( (D, r) \) is not regular.

§ 5. Regularity of the P.C.F self-similar sets

In this section, we will consider a regularity condition of the harmonic structure in a probabilistic way and give a sufficient condition for the harmonic structure to be regular under some assumption. Let \#S = N, \#\( V_0 = l \) and \( 2 \leq k \leq \min\{N, l\}. \) Throughout this chapter, we assume the following.

**Assumption 5.1.** Let \( V_0 = \{s_1, s_2, \ldots, s_l\}. \) Then \( s_i = \pi(i) \) for \( 1 \leq i \leq k. \)

Remark that \( \pi^{-1}(s_i) = i \) for \( 1 \leq i \leq k \) because \#\( P < \infty. \)

Now, let \( \bar{\delta}_{v_0} = \inf\{j > 0: X(j) \in V_0\} \) where \( X \) is a random walk on \( V_1 \) corresponding to \( H_1, \ c_i = P_s[ X(\bar{\delta}_{v_0}) = s_i] \) \( (1 \leq i \leq k). \)

Then, the following theorem holds.

**Theorem 5.2.** \[
\frac{1 - c_1}{r_1} = \frac{1 - c_2}{r_2} = \cdots = \frac{1 - c_k}{r_k} = \frac{1}{\lambda}. \]

**Proof.** Let \( \mu_p = -D_{pp} \) \( (D_{pp} \) is a \( (p, p) \)-element of \( D, \ 1 \leq p \leq l), \)

\( D_{pq} = D_{pq} \mu_q^{-1} \)

and
If we let $P^{(0)} = D + I$, then $P^{(0)}$ is a transition probability of the random walk on $V_0$ corresponding to $D$. (I.e. $P_{xy}^{(0)}$ is a transition probability from $y$ to $x$.) We have

$$D = (P^{(0)} - I)U.$$ 

Let

$$P^{(1)} = (\sum_{s \in s'_{rs}} \sum_{s \in s'_{rs}} \sum_{s' \in s'_{rs}} (T_{rs} U_{rs})(\sum_{s \in s'_{rs}} \sum_{s' \in s'_{rs}} (T_{rs} U_{rs})^{-1} = \begin{pmatrix} A & B \\ C & K \end{pmatrix}.$$ 

$\sum_{s \in s'_{rs}} \sum_{s' \in s'_{rs}} (T_{rs} U_{rs})$ is a diagonal matrix and $P^{(1)}$ gives a transition probability of the random walk on $V_1$ corresponding to $H_1$. If we denote

$$\sum_{s \in s'_{rs}} \sum_{s' \in s'_{rs}} (T_{rs} U_{rs}) = \begin{pmatrix} T_{U} & 0 \\ 0 & X_{U} \end{pmatrix},$$

then we have

$$P^{(1)} = H_1 \begin{pmatrix} T_{U}^{-1} & 0 \\ 0 & X_{U}^{-1} \end{pmatrix} + \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} I + TT_{U}^{-1} & 'JX_{U}^{-1} \\ JT_{U}^{-1} & I + XX_{U}^{-1} \end{pmatrix} \begin{pmatrix} A & B \\ C & K \end{pmatrix}.$$ 

Let $Q_n$ be the matrix such that

$$Q_n = P_{xy} = P_{\bar{\sigma} \bar{\nu}_0} = P_{\bar{x} \bar{y}}(\bar{\sigma} \bar{\nu}_0) = x, \bar{\nu}_0 = n.$$ 

Then

$$Q_n = (I 0) \begin{pmatrix} A & B \\ C & K \end{pmatrix}^{n-1} \begin{pmatrix} A & B \\ C & K \end{pmatrix}$$

$$= (I 0) \begin{pmatrix} 0 & BK^{n-2} \\ 0 & K^{n-1} \end{pmatrix} \begin{pmatrix} A & B \\ C & K \end{pmatrix}.$$ 

Thus
\[(5.3) \quad \sum_{n=1}^{\infty} Q_n = (I - K) A + B(I+K+K^2+\cdots) \begin{pmatrix} A & B \\ C & K \end{pmatrix} = (I - K) A + B(I-K)^{-1} \begin{pmatrix} A & B \\ C & K \end{pmatrix} = (A + B(I-K)^{-1}C, B + B(I-K)^{-1}K).\]

By (5.1), we know \(I - K\) is invertible. Thus it is easy to check \((I-K)^{-1}=I+K+K^2+\cdots\). From (5.2) and (5.3), we have

\[(5.4) \quad (A + B(I-K)^{-1}C)_{2y} = p_1[X(\delta_{y_0}) = x] \text{ for } x, y \in V_0.\]

On the other hand, by (2.5) we see

\[I - S + TT_v^{-1} - JX^{-1}JT_v^{-1} = \frac{1}{\lambda} DT_v^{-1} + I - S.\]

where \(S = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_l \end{pmatrix}\).

Substituting (5.1) for this equation, we have

\[(5.5) \quad A + B(I-K)^{-1}C - S = \frac{1}{\lambda} DT_v^{-1} + I - S.\]

From (5.4), we see that the diagonal of the left hand side of (5.5) is 0. By the Assumption 5.1 and by the definition of \(T_v\),

\[(5.6) \quad (T_v)_{ij} = 0 \text{ for } 1 \leq i \neq j \leq l, \quad (T_v)_{ii} = \frac{H_i}{r_i} \text{ for } 1 \leq i \leq k.\]

Combined (5.5) and (5.6), we obtain

\[\frac{1-c_1}{r_1} = \frac{1-c_2}{r_2} = \cdots = \frac{1-c_k}{r_k} = \frac{1}{\lambda}.\]

**Corollary 5.3.** All the harmonic structures are regular if \(k=N\leq l\).

**Proof.** If

\[(5.7) \quad \#(V_0 \cap F_s(V_0)) \leq 1 \text{ for each } 1 \leq s \leq N,\]

then it is clear that \(c_i \neq 0\) for \(1 \leq i \leq l\). Thus the result is an easy consequence of the above theorem.

In the general case, we change the self-similar structure \(\mathcal{L}\) by \(\mathcal{L}_m\) as in the
proof of Theorem 4.12 in [3]. By choosing $m$ so that $\mathcal{L}_m$ can satisfy (5.7), we obtain the result in the same way.

**Remark.** The corresponding result is already proved in [3] Lemma 4.15.3. We have proved it in a probabilistic way.

From this corollary, we know all the harmonic structures are regular for the Sierpinski gasket and Hata's tree like set. But this is not true in general if $t < N$.

§ 6. **An example on the Sierpinski gasket**

In this section, we construct one-parameter family of regular local Dirichlet forms on the Sierpinski gasket. Let

$$P^{(0)} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1-p \\ p & 1-p & 0 \end{pmatrix}$$

be the transition probability on $V_0$. Let

$$D = \begin{pmatrix} -2p & p & p \\ p & -1 & 1-p \\ p & 1-p & -1 \end{pmatrix} = (P^{(0)} - I) \begin{pmatrix} 2p & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $r = (s^{-1}, 1, 1)$. By easy calculations, the equation (2.5) is equivalent to the following.

(6.1) $$(p-1)(p-3)s^2 - 2(p-1)^2s + p(p-2) = 0 ,$$

(6.2) $$\lambda = \frac{2+s-p}{s(2-p)} .$$

As (6.1) has the unique positive solution $s$ for each $0 < p < 1$, we know that there exists a unique harmonic structure for each $0 < p < 1$. From the result obtained in § 5, this harmonic structure is regular. Thus we have a regular local Dirichlet form for each $0 < p < 1$. In the case of $p = \frac{1}{2}$, it is the Brownian motion. If we let $p \to 1$, then $a\lambda \to \infty$ and the spectral dimension with respect to the self-similar measure goes to 0. If we let $p \to 0$, then $a\lambda \to \frac{16}{3}$ and the spectral dimension goes to $\frac{2\log 3}{\log 16 - \log 3}$.

§ 7. **Weakly symmetric P.C.F. self-similar sets**

In this section, we introduce a class of P.C.F. self-similar sets which has the
Definition 7.1. A P.C.F. self-similar set \((K, S, \{F_s\}_{s \in S})\) is called weakly symmetric if the following holds.

1) \(#S(=N) \geq \#V_0(=l)\).
2) There exist a \(l\)-order cycle \(\tau \in \mathbb{G}_l\) and a continuous surjection \(\tilde{\tau}: K \to K\) with order \(l\) such that

\[
\pi \circ \tilde{\tau} = \tilde{\tau} \circ \pi.
\]

(7.1)

Here, \(\tilde{\tau}: S^N \to S^N\) is the mapping which operate \(\tau\) on each coordinate.

Remark 7.2. It is easy to check the following.

1) \(\tilde{\tau}\) is a homeomorphism.
2) \(\tilde{\tau}(V_m) = V_m\).

Without loss of generality, we let \(\tau = (1, 2, \ldots, l)\) and line up \(V_0\) as \[
\begin{pmatrix}
S_1 \\
S_2 \\
\vdots \\
S_l
\end{pmatrix}
\]

where \(\tilde{\tau}(s_i) = s_{i+1}\) for \(1 \leq i \leq l - 1\) and \(\tilde{\tau}(s_1) = s_l\).

Definition 7.3. \(D \in L(V_0)\) is said to be a weakly symmetric structure if it satisfies the following:

There exist non-negative constants \(p_i(1 \leq i \leq l - 1)\) such that

\[
\sum_{i=1}^{l-1} p_i = 1, \quad p_i = p_{l-i}(1 \leq i \leq l - 1)
\]

and if we denote \(u = (-1, p_1, p_2, \ldots, p_{l-1})\), then the \(i\)-th row of \(D\) is \(\tau^{1-i}u\).

Let

\[
\mathcal{H}(V_0) = \{D \in L(V_0) | D\text{ is a harmonic structure.}\}\text{ and} \mathcal{H}_w(V_0) = \{D \in L(V_0) | D\text{ is a weakly symmetric structure.}\}\.
\]

We consider the space of \(l \times l\)-matrices as a Banach space with the norm \(\|A\| = \max_{i,j}|A_{ij}|\). Then the next two lemmas are clear.

Lemma 7.4. \(\mathcal{H}_w(V_0) \subset \mathcal{H}(V_0)\) if \(l\) is a prime number.

Lemma 7.5. \(\mathcal{H}_w(V_0)\) is a convex closed compact subset of \(L(V_0)\).

Now we prove the key lemma.
Lemma 7.6. If we let $r=(1,1,\ldots,1,r_{i+1},\ldots,r_N)$, then $T^{-1}J^{-1}J$ is a cyclic matrix.

Proof. For $f \in l(V_o)$, let $f^\sharp = f \circ \tilde{r}$. Then, it is easy to check the following relations.

\begin{align}
(7.2) & \quad Df^\sharp = (Df)^\sharp \\
(7.3) & \quad Rwf^\sharp = (Rr(\omega)f)^\sharp \\
(7.4) & \quad \tilde{r}Rwf^\sharp = (\tilde{r}Rr(\omega)f)^\sharp
\end{align}

where we consider $Rwf \in l(V_o)$.

Using this, we obtain $H_1f^\sharp = (H_1f)^\sharp$. Thus, we have

\[(T^{-1}J^{-1}J)(f|_{V_0})^\sharp = ((T^{-1}J^{-1}J)f|_{V_0})^\sharp.\]

Let $H = \{D|D$ is a $l \times l$-matrix with $D_{11} \neq 0\}$ and $g: H \rightarrow H$ be $g(D) = -\frac{1}{D_{11}}D$. Then we have the following proposition.

Proposition 7.7. Let $r=(1,1,\ldots,1,r_{i+1},\ldots,r_N)$, $l$ be a prime number and $D$ be a weakly symmetric structure. Then, $g \circ \mathcal{S}_r$ maps $\mathcal{H}_w(V_0)$ to $\mathcal{H}_w(V_0)$, where $\mathcal{S}_r(D) = T^{-1}J^{-1}J$.

Proof. If we use Proposition 4.3 in [3] and the above lemmas, this is clear.

Now, by Schauder’s fixed point theorem, if $(D, r)$ and $l$ satisfy the assumptions of Proposition 7.7, there exists $D \in \mathcal{H}_w(V_0)$ such that $g \circ \mathcal{S}_r(D) = D$. Therefore $\mathcal{S}_r(D) = \frac{1}{\lambda}D$ for some $\lambda > 0$. Thus we obtain the existence of the harmonic structure in this case. We give one typical example which is weakly symmetric. By the above results, we have non-degenerate Dirichlet forms on this fractal.

Example 7.1

\[S = \{1, 2, 3, 4, 5, 6\}.\]
\[\pi(C) = \{q_i : 1 \leq i \leq 10\};\]
\[\pi^{-1}(q_i) = \{i, i+1\}, \langle i+1 \rangle^i \text{ for } 1 \leq i \leq 5,\]
Example 7.1

\begin{align*}
\pi^{-1}(q_i) &= \{ \langle i \rangle, \langle i \rangle \rangle, \ldots \} \text{ for } 6 \leq i \leq 10. \\
Here \langle i \rangle &= i \text{ if } i \leq 5 \text{ and } \langle i \rangle &= i - 5 \text{ if } 6 \leq i.
\end{align*}

When $l$ is not a prime number, the fixed points can be degenerate. (I.e. the matrix $D$ is not irreducible.) In this case, the author does not know whether there exist non-degenerate fixed points or not.

\section*{References}


