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The analytic continuation of the scattering kernel associated with the Schrödinger operator with a penetrable wall interaction

Dedicated to the Professor Yōjirō Hasegawa on his 60th birthday

By

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§ 0. Introduction

A penetrable wall interaction (δ sphere interaction) is described by the Schrödinger operator formally given by

$$H = -\Delta + q(x)\delta(|x|-a) \quad \text{in} \quad L_2(\boldsymbol{R}^3), \tag{0.1}$$

where $q(x)$ is real and smooth on $S_a = \{x; |x| = a\}$ $(a>0)$ and δ denotes the one-dimensional delta function. This has a long history mainly in nuclear physics (Petzold [8], Nussenzveig [7], Antoine-Gesztesy-Shabani [1] and references therein).

The first problem one meets is that of the selfadjoint realization of the formal expression (0.1). As a rigorous selfadjoint operator H corresponding to (0.1), we adopt one uniquely determined by the following quadratic form h:

$$h[u, v] = (\nabla u, \nabla v) + \langle q\gamma u, \gamma v\rangle, \quad \mathrm{Dom}[h] = H^1(\boldsymbol{R}^3), \tag{0.2}$$

where γ is the trace operator from $H^1(\boldsymbol{R}^3)$ to $L_2(S_a)$, $\mathrm{Dom}[h]$ denotes the form domain of h, $(\ ,\ )$ means the $L_2(\boldsymbol{R}^3)$ inner product, $\langle\ ,\ \rangle$ the $L_2(S_a)$ inner product, and $H^m(\boldsymbol{G})$ the Sobolev space of order m over $\boldsymbol{G}$. It is seen that H is characterized as follows (Ikebe-Shimada [4, Theorem 1.7]):

$$\mathrm{Dom}(H) = \left\{u \in H^2(\boldsymbol{R}^3 \backslash S_a) \cap H^1(\boldsymbol{R}^3); \left(\frac{\partial u}{\partial r}\right)_+ - \left(\frac{\partial u}{\partial r}\right)_- = q\gamma u\right\}, \tag{0.3}$$

$$Hu = -\Delta u \quad \text{in} \quad \boldsymbol{R}^3 \backslash S_a \quad \text{for} \quad u \in \mathrm{Dom}(H),$$

where $\left(\frac{\partial u}{\partial r}\right)_\pm$ means the trace of the radial derivative $\frac{\partial u}{\partial r}$ to $L_2(S_a)$ from $\{|x| > a\}$ and $\{|x| < a\}$, respectively. Another way of selfadjoint realization of

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(0.1) may be found in Antoine et al. [1] and also in Shimada [10] for the approximation problem. Another type of penetrable wall interaction (δ' sphere interaction) has been discussed in Antone et al. [1] and Ikebe [3]. If we put $q(x)\equiv 0$ in (0.2), the resulting operator H_0 is seen to be the free Hamiltonian:

$$(0.4)\qquad H_0=-\Delta,\ \mathrm{Dom}(H_0)=H^2(\boldsymbol{R}^3)\,.$$

The scattering matrix $S_r(r>0)$ associated with the pair H and H_0 is defined by

$$(0.5)\qquad (S_r u)(\omega)=u(\omega)-ir\int_{S_1}F(r,\omega,\omega')u(\omega')\mathrm{d}\omega' \quad \text{for} \quad u\in L_2(S_1)$$

whre $F(r, \omega, \omega')$ which is called th scattering kernel (amplitude) for the scattering from the initial direction ω' to the final direction ω at energy r^2, can be represented as

$$(0.6)\qquad F(r,\omega,\omega')=\frac{1}{8\pi^2}\langle(1-\tilde{T}_r)^{-1}(e^{ir\omega'\cdot x}),\ q(x)e^{ir\omega\cdot x}\rangle\,,$$

(Shimada [11, Theorem 1.4, Lemma 1.7]). Here $\tilde{T}_\kappa$ is an integral operator with a complex parameter κ defined by

$$(0.7)\qquad \tilde{T}_\kappa u(x)=\frac{-1}{4\pi}\int_{S_a}\frac{e^{i\kappa|x-y|}}{|x-y|}q(y)u(y)\mathrm{d}S_y \quad \text{for} \quad u\in L_2(S_a)\,,$$

which is seen to be a compact operator on $L_2(S_a)$ for all $\kappa\in \boldsymbol{C}$ (Ikebe-Shimada [4, Lemma 2.4]).

In the present paper, we shall study the analytic poperties of the scattering kernel $F(r, \omega, \omega')$ with respect to r. Dolph-McLeod-Thoe [2] investigated this problem for Schrödinger operators with exponentially decaying potentials and further, yet at the formal level, dealt with the case that $q(x)\equiv const.$ Our procedure can be carried out rigorously including non-spherically symmetric potentials. In § 1 we obtain the analytic continuation of $F(r, \omega, \omega')$ to the whole complex plain as a meromorphic function of r. In § 2 we discuss the poles of $F(r, \omega, \omega')$ in $\mathcal{I}r\geq 0$ ($\mathcal{I}r$=imaginary part of r). In $\mathcal{I}r\geq 0$, the poles can appear only on the imaginary axis. In $\mathcal{I}r>0$, they correspond to the negative eigenvalues of H, and on $\mathcal{I}r=0$, they produce zero resonance if any. In § 3 we examine the region in which no poles can appear and deal with the case that $q(x)\equiv const.$ in § 4.

1. Analytic continuation of the scattering kernel

As mentioned in § 0, the scattering kernel $F(r, \omega, \omega')$ has the form:

$$F(r, \omega, \omega')=\frac{1}{8\pi^2}\langle(1-\tilde{T}_r)^{-1}(e^{ir\omega'\cdot x}), q(x)e^{i\bar{r}\omega\cdot x}\rangle \tag{1.1}$$

$$\text{for} \quad (r, \omega, \omega')\in(0, \infty)\times S_1\times S_1 .$$

We shall show that it can be continued analytically to the whole complex plain $\boldsymbol{C}$.

Theorem 1.1. *The scattering kernel $F(r, \omega, \omega')$ can be continued to the whole complex plain $\boldsymbol{C}$ as a $C(S_1\times S_1)$-valued meromorphic function in r, where $C(\boldsymbol{G})$ denotes the Banach space of continuous functions on $\boldsymbol{G}$ with maximum norm.*

Proof. We have only to show that

$$\kappa\mapsto\langle(1-\tilde{T}_\kappa)^{-1}(e^{i\kappa\omega'\cdot x}), q(x)e^{i\bar{\kappa}\omega\cdot x}\rangle$$

is a $C(S_1\times S_1)$-valued meromorophic function on $\boldsymbol{C}$. First we note that for each $\omega\in S_1$, $\kappa\mapsto e^{i\kappa\omega\cdot x}$ is an $L_2(S_a)$-valued entire function and $\kappa\mapsto q(x)e^{i\bar{\kappa}\omega\cdot x}$ an $L_2(S_a)$-valued anti-analytic entire function:

$$e^{i\kappa\omega'\cdot x}=e^{i\kappa_0\omega'\cdot x}e^{i(\kappa-\kappa_0)\omega'\cdot x} \tag{1.2}$$

$$=\sum_{l=0}^{\infty}\frac{e^{i\kappa_0\omega'\cdot x}}{l!}i^l(\omega'\cdot x)^l(\kappa-\kappa_0)^l \quad \text{in} \quad L_2(S_a) ,$$

$$q(x)e^{i\bar{\kappa}\omega\cdot x}=q(x)e^{i\bar{\kappa}_0\omega\cdot x}e^{i\overline{(\kappa-\kappa_0)}\omega\cdot x} \tag{1.3}$$

$$=\sum_{m=0}^{\infty}\frac{q(x)e^{i\bar{\kappa}_0\omega\cdot x}}{m!}i^m(\omega\cdot x)^m\overline{(\kappa-\kappa_0)}^m \quad \text{in} \quad L_2(S_a) ,$$

$$\text{for} \quad (\kappa, \kappa_0, \omega, \omega')\in \boldsymbol{C}\times\boldsymbol{C}\times S_1\times S_1 .$$

Since it is seen that $\kappa\mapsto\tilde{T}_\kappa$ is a $\boldsymbol{B}(L_2(S_a))$-valued entire function such that $\tilde{T}_\kappa$ is compact for each $\kappa\in\boldsymbol{C}$, $\kappa\mapsto(1-\tilde{T}_\kappa)^{-1}$ turn out to be a $\boldsymbol{B}(L_2(S_a))$-valued meromorophic in $\boldsymbol{C}$ by the analytic Fredholm theorem (e.g. Reed-Simon [9, Theorem VI. 14]). Here $\boldsymbol{B}(\boldsymbol{X}, \boldsymbol{Y})$ denotes the Banach space of linear bounded operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$ ($\boldsymbol{B}(\boldsymbol{X}, \boldsymbol{X})=\boldsymbol{B}(\boldsymbol{X})$). Thus for each $\kappa_0\in\boldsymbol{C}$, there exists $\rho_0>0$ such that for $0<|\kappa-\kappa_0|<\rho_0$

$$(1-\tilde{T}_\kappa)^{-1}=\sum_{j=-n}^{\infty}A_j(\kappa-\kappa_0)^j \quad \text{in} \quad \boldsymbol{B}(L_2(S_a)) , \tag{1.4}$$

where $A_j\in\boldsymbol{B}(L_2(S_a))$ an n is a nonnegative integer. From (1.2), (1.3) and (1.4) it follows that for each $(\omega, \omega')\in S_1\times S_1$ $\langle(1-\tilde{T}_\kappa)^{-1}(e^{i\kappa\omega'\cdot x}), q(x)e^{i\bar{\kappa}\omega\cdot x}\rangle$ has the following Laurent expansion on $0<|\kappa-\kappa_0|<\rho_0$:

$$\langle(1-\tilde{T}_\kappa)^{-1}(e^{i\kappa\omega'\cdot x}), q(x)e^{i\bar{\kappa}\omega\cdot x}\rangle \tag{1.5}$$

$$=\sum_{j=-n,l=0,m=0}^{\infty}\langle A_j e^{i\kappa_0\omega'\cdot x}(\omega'\cdot x)^l,\ q(x)e^{i\overline{\kappa_0}\omega\cdot x}(\omega\cdot x)^m\rangle$$

$$\times\frac{i^{l-m}(\kappa-\kappa_0)^{l+m+j}}{l!m!}\,.$$

It is easily seen that each term of (1.5) is continuous in $(\omega, \omega')\in S_1\times S_1$ and (1.5) converges in the $C(S_1\times S_1)$-topology, if we note that

$$\|e^{i\kappa\omega\cdot x}(\omega\cdot x)^l\|_{L_2(S_a)}\leq a^l e^{a|\mathcal{I}\kappa|}\sqrt{\frac{4\pi}{2l+1}}$$

and Cauchy's inequality:

$$\|A_j\|_{B(L_2(S_a))}\leq \max_{|\kappa-\kappa_0|=\rho}\|(1-\tilde{T}_\kappa)^{-1}\|_{B(L_2(S_a))}\rho^{-j}\,.$$

Hereafter by $F(\kappa, \omega, \omega')$ we denote the continued scattering kernel. Thus (1.1) holds for $(\kappa, \omega, \omega')\in \boldsymbol{C}\times S_1\times S_1$. We should remark that the poles of $F(\kappa, \omega, \omega')$ are necessarily those of $(1-\tilde{T}_\kappa)^{-1}$, while the converse is not true in general (see Theorem 2.3 and a remark there). Further the analytic Fredholm theorem asserts that κ is a pole of $(1-\tilde{T}_\kappa)^{-1}$ if and only if $(1-\tilde{T}_\kappa)u=0$ has a nonzero solution.

Lemma 1.2. *For each* $(\omega, \omega')\in S_1\times S_1$ *we have*

$$\overline{F(\kappa, \omega, \omega')}=F(-\bar{\kappa}, \omega, \omega')\,. \tag{1.6}$$

In particular, the poles are symmetrically placed with respect to the imaginary axis, if they exist.

Proof. Assume that κ_0 is not a pole of $(1-\tilde{T}_\kappa)^{-1}$. Then since it holds that

$$\overline{\tilde{T}_\kappa u(x)}=\tilde{T}_{-\bar{\kappa}}\overline{u(x)}\,, \tag{1.7}$$

$(1-\tilde{T}_{-\overline{\kappa_0}})u=0$ implies $(1-\tilde{T}_{\kappa_0})\bar{u}=0$ i.e. $u=0$. Thus it follows that $-\overline{\kappa_0}$ is not a pole of $(1-\tilde{T}_\kappa)^{-1}$. Again using (1.7) we have

$$\overline{(1-\tilde{T}_{\kappa_0})^{-1}u}=(1-\tilde{T}_{-\overline{\kappa_0}})^{-1}\bar{u}\,, \tag{1.8}$$

from which (1.6) holds in this case.

Assume that κ_0 is a pole of $(1-\tilde{T}_\kappa)^{-1}$. Then there exists a sequence $\{\kappa_j\}_{j=0}^{\infty}$ which converges to κ_0 such that each κ_j is ot a pole o $(1-\tilde{T}_\kappa)^{-1}$. Since (1.6) holds for each $\kappa=\kappa_j$, letting j tend to ∞, (1.6) holds for $\kappa=\kappa_0$, where κ_0 is a pole of $F(\kappa, \omega, \omega')$ or not.

2. Poles of the scattering kernel in $\mathscr{I}\kappa \geq 0$

We shall discuss the poles of $F(\kappa, \omega, \omega')$ in the region $\mathscr{I}\kappa \geq 0$. First in $\mathscr{I}\kappa > 0$ we have

Theorem 2.1. *In the region $\mathscr{I}\kappa > 0$, there are no poles o $F(\kappa, \omega, \omega')$ except on the imaginary axis, and on the imaginary axis at most a finite number of poles can occur. Further if $i\lambda(\lambda > 0)$ is a pole of $F(\kappa, \omega, \omega')$, $-\lambda^2$ is a negative eigenvalue of H.*

Imitating the proof of the theorem, we can show that $i\lambda(\lambda > 0)$ is a pole of $(1-\tilde{T}_\kappa)^{-1}$ if and only if $-\lambda^2$ is a negative eigenvalue of H. We also remark that $F(\kappa, \omega, \omega')$ actually has a pole in $\mathscr{I}\kappa > 0$ where $q(x) \equiv V_0(const.) < -1/a$ (see § 4).

Proof of Theorem 2.1. Assume that κ_0 is a pole of $F(\kappa, \omega, \omega')$. Then since κ_0 is a pole of $(1-\tilde{T}_\kappa)^{-1}$, there exists a nonzero vector u in $L_2(S_a)$ such that $(1-\tilde{T}_{\kappa_0})u = 0$ by the analytic Fredholm theorem. Thus from Lemma 2.12 in Ikebe-Shimada [4] it follows that κ_0^2 is an eigenvalue of H, which implies $\kappa_0^2 < 0$ because H has no positive eigenvalue by Theorem 5,2 in Ikebe-Shimada [4]. Therefore κ_0 must be pure imaginary. Since H is bounded from below (Ikebe-Shimada [4, Theorem 1.5]), we have $\mathscr{I}\kappa_0 \leq A(-A^2\ (A > 0)$ is a lower bound of $H)$. Thus it is seen that only a finite number of poles can appear on the positive imaginary axis.

Theorem 2.2. *In the region $\mathscr{I}\kappa > 0$, the poles of $F(\kappa, \omega, \omega')$ are simple.*

Proof. We have only to show that the poles of $(1-\tilde{T}_\kappa)^{-1}$ are simple. Let us recall the following identity (Ikebe-Shimada [4, (7.2)]): for κ such that $\mathscr{I}\kappa > 0$ and $\kappa^2 \in \rho(H)$

$$(2.1) \qquad (H-\kappa^2)^{-1} = (H_0-\kappa^2)^{-1} + T_\kappa(1-\tilde{T}_\kappa)^{-1}\gamma(H_0-\kappa^2)^{-1},$$

where $\rho(H)$ denotes the resolvent set of H, T_κ the integral operator with a complex parameter κ defined by

$$(2.2) \qquad T_\kappa u(x) = \frac{-1}{4\pi}\int_{S_a}\frac{e^{i\kappa|x-y|}}{|x-y|}q(y)u(y)dS_y \quad (x \in \boldsymbol{R}^3)$$

$$\text{for} \quad u \in L_2(S_a),$$

which is a bounded operator from $L_2(S_a)$ to $H^1(\boldsymbol{R}^3)$ if $\mathscr{I}\kappa > 0$ (Ikebe-Shimada [4, Lemma 2.6]). After operating γ from left on the both sides of (2.1) and using $\tilde{T}_\kappa = \gamma T_\kappa$ (Ikebe-Shimada [4, Lemma 2.7]), we have by operating $(H_0-\kappa^2)$ from right

$$(2.3) \qquad (1-\tilde{T}_\kappa)^{-1}\gamma = \gamma(H-\kappa^2)^{-1}(H_0-\kappa^2) \quad \text{on} \quad H^2(\boldsymbol{R}^3).$$

Assume that $\kappa_0(\mathcal{I}\kappa_0>0)$ is a pole of $(1-\tilde{T}_\kappa)^{-1}$. Then since κ_0^2 is an eigenvalue of H by Theorem 2.1, $(H-\kappa^2)^{-1}$ has the following Laurent expansion near κ_0:

$$(2.4)\qquad (H-\kappa^2)^{-1}=\frac{E(\{\kappa_0^2\})}{\kappa_0^2-\kappa^2}+A(\kappa),$$

where $E(\cdot)$ denotes the spectral measure associated with H and $A(\kappa)$ is an operator valued analytic funtion near κ_0. Noting that Range$(H-\kappa^2)^{-1}$ $\subset H^1(\boldsymbol{R}^3)$ and Range$(E(\{\kappa_0^2\}))\subset H^1(\boldsymbol{R}^3)$, we have Range$(A(\kappa))\subset H^1(\boldsymbol{R}^3)$, so that

$$(2.5)\qquad \gamma(H-\kappa^2)^{-1}=\frac{\gamma E(\{\kappa_0^2\})}{\kappa_0^2-\kappa^2}+\gamma A(\kappa).$$

On the other hand, $(1-\tilde{T}_\kappa)^{-1}$ has the following form near κ_0

$$(2.6)\qquad (1-\tilde{T}_\kappa)^{-1}=\frac{A_{-n}}{(\kappa_0-\kappa)^n}+\frac{A_{-n+1}}{(\kappa_0-\kappa)^{n-1}}+\cdots,$$

where $A_j\in\boldsymbol{B}(L_2(S_a))$ $(j=-n,\ -n+1,\cdots)$. Compared with (2.3), (2.5) and (2.6), we obtain

$$(2.7)\qquad A_{-n}\gamma=0 \quad \text{on} \quad H^2(\boldsymbol{R}^3) \quad \text{if} \quad n\geq 2,$$

Since $\gamma H^2(\boldsymbol{R}^3)$ is dense in $L_2(S_a)$ and A_{-n} bouned on $L_2(S_a)$, (2.7) implies

$$(2.8)\qquad A_{-n}=0 \quad \text{on} \quad L_2(S_a) \quad \text{if} \quad n\geq 2,$$

from which the assertion follows.

Theorem 2.3. *In the region* $\mathcal{I}\kappa=0$ (*real axis*), $F(\kappa,\omega,\omega')$ *may or may not have a pole, which is necessarily simple, only at the origin. Further* 0 *is a pole of* $F(\kappa,\omega,\omega')$ *if and only if it is a zero resonance of H.*

For the zero resonance, see Jensen-Kato [5] and Shimada [11].

Proof. If $\kappa>0$ is a pole of $F(\kappa,\omega,\omega')$, by the analytic Fredholm theorem there exists a nonzero vector u such that

$$(2.9)\qquad (1-\tilde{T}_\kappa)u=0.$$

But (2.9) implies that $u=0$ by Lemma 7.8 in Ikebe-Shimada [4], which is a contradiction. Thus noting Lemma 1.2, it turns out that $F(\kappa,\omega,\omega')$ has no pole on the real axis except for the origin. The rest of the statement follows from Theorems 5.1-5.4 in Shimada [11].

Taking into account Theorems 5.1-5.4 in Shimada [11], we can show that when H has zero resonance (in fact, it occurs e.g. if $q(x)\equiv -1/a$), $F(\kappa,\omega,\omega')$ has the following Laurent expansion at $\kappa=0$:

(2.10) $$F(\kappa, \omega, \omega')=\frac{-i}{2\pi\kappa}+\cdots.$$

On the other hand, in general $(1-\tilde{T}_\kappa)^{-1}$ has the following form at $\kappa=0$:

(2.11) $$(1-\tilde{T}_\kappa)^{-1}=\frac{C_{-2}}{\kappa^2}+\frac{C_{-1}}{\kappa}+\cdots.$$

In this case, it is seen that $C_{-2}\neq 0$ in $\boldsymbol{B}(L_2(S_a))$ is if and only if H has a zero eigenvalue.

3. Poles of the scattering kernel in $\mathcal{I}\kappa<0$

In this section we shall examine the region in which no pole exists. If $1-\tilde{T}_\kappa$ has the inverse in some region $D\subset \boldsymbol{C}$, $F(\kappa, \omega, \omega')$ has no pole in D by (1.1) and the analytic Fredholm theorem. Therefore we shall seek the region on which κ satisfy $\|\tilde{T}_\kappa{}^2\|_{H.S.}<1$, where $\|\cdot\|_{H.S.}$ denotes the Hilbert-Schmidt norm. It is known that for each $\kappa\in\boldsymbol{C}$, $\tilde{T}_\kappa{}^2$ belongs to the Hilbert-Schmidt class (Ikebe-Shimada [4, Lemma 2.8]). The same idea has been used to prove the exponential decay of the solution for the wave equation in Mochizuki [6]. We shall prove the next

Lemma 3.1. *For $(\kappa, \epsilon)\in\boldsymbol{C}\times(0, 1/2)$, we have*

(3.1) $$\|\tilde{T}_\kappa{}^2\|_{H.S.}^2\leq Ce^{8a|\mathcal{I}\kappa|}\left(\epsilon^2|\log\epsilon|^2+\frac{|\log\epsilon|}{\epsilon^2|\kappa|^2}\right),$$

where C is a constant which is independent of (κ, ϵ)

Our main theorem of his section is

Theorem 3.2. Let $\kappa=x+iy$ and let $0<\delta<1$. Then there exists a constant $C_\delta>0$ such that in the region:

$$y>-\frac{1-\delta}{8a}\log|x|+C_\delta$$

$$|x|>4, \quad y<0,$$

$F(\kappa, \omega, \omega')$ has no pole.

Proof of Theorem 3.2. If $\|\tilde{T}_\kappa{}^2\|_{H.S.}<1$, since $\|\tilde{T}_\kappa{}^2\|\leq\|T_\kappa{}^2\|_{H.S.}<1$ ($\|T\|$ is the operator norm of T), it is seen that $(1-\tilde{T}_\kappa{}^2)^{-1}$ exists and belongs to $\boldsymbol{B}(L_2(S_a))$ using the Neumann series. This assures us that $(1-\tilde{T}_\kappa)^{-1}$ also exists by the relation

(3.2) $$(1-\tilde{T}_\kappa)^{-1}=(1-\tilde{T}_\kappa{}^2)^{-1}(1+\tilde{T}_\kappa),$$

which implies that κ is not a pole of $F(\kappa, \omega, \omega')$ as mentioned above.

Now let $\kappa=x+iy$ ($|x|>4$, $y<0$) and let $0<\delta<1$. Since $\epsilon^{\delta}|\log\epsilon|$ is bounded for $\epsilon\in(0,1/2)$, we have taking $\epsilon=|x|^{-1/2}$ in Lemma 3.1

$$\|\tilde{T}_{\kappa}^{2}\|_{H.S.}^{2}\leq Ce^{8a|\mathfrak{I}\kappa|}\left(\epsilon^{2(1-\delta)}+\frac{1}{|x|^{2}\epsilon^{2(1+\delta)}}\right) \tag{3.3}$$

$$=2Ce^{-8ay}|x|^{-(1-\delta)},$$

where C is independent of (κ, ϵ), however, may depend on δ. Since the R.H.S. of (3.3)<1 if and only if

$$y>-\frac{1-\delta}{8a}\log|x|+\frac{\log(2C)-1}{8a}, \tag{3.4}$$

$F(\kappa, \omega, \omega')$ has no pole in the region on which $\kappa=x+iy$ stisfies $|x|>4$, $y<0$ and (3.4).

We will devote the rest of this section to prove Lemma 3.1. We write the integral kernel $K(x, y)$ of $\tilde{T}_{\kappa}^{2}$ as

$$K(x, y)=\left(\frac{1}{4\pi}\right)^{2}\int_{S_a}\mathrm{d}S_z\frac{e^{i\kappa|x-z|}}{|x-z|}q(z)\frac{e^{i\kappa|z-y|}}{|z-y|}q(y) \tag{3.5}$$

$$=\left(\frac{1}{4\pi}\right)^{2}\int_{S_1}\mathrm{d}\omega\frac{e^{ia\kappa(|\omega_x-\omega|+|\omega-\omega_y|)}}{|\omega_x-\omega||\omega-\omega_y|}q(a\omega)q(a\omega_y),$$

where $x=a\omega_x$, $y=a\omega_y$ and $=a\omega(\omega_x, \omega_y, \omega\in S_1)$. Thus we have

$$\|\tilde{T}_{\kappa}^{2}\|_{H.S.}^{2}=a^{4}\int_{S_1\times S_1}\mathrm{d}\omega_1\mathrm{d}\omega_2|K(a\omega_1, a\omega_2)|^{2}, \tag{3.6}$$

$$\leq\left(\frac{a}{4\pi}\right)^{4}\max_{x\in S_a}|q(x)|^{2}\int_{S_1\times S_1}\mathrm{d}\omega_1\mathrm{d}\omega_2|I(\omega_1, \omega_2)|^{2},$$

where

$$I(\omega_1, \omega_2)=\int_{S_1}\mathrm{d}\omega\frac{e^{ia\kappa(|\omega_1-\omega|+|\omega-\omega_2|)}}{|\omega_1-\omega||\omega-\omega_2|}q(a\omega). \tag{3.7}$$

We will show Lemma 3.1 by proving a series of lemmas. First we have

Lemma 3.3. *Let $0<\epsilon<1$. Then we have for $\kappa\in\boldsymbol{C}$*

$$\int_{D_1}d\omega_1 d\omega_2|I(\omega_1, \omega_2)|^{2}\leq Ce^{8a|\mathfrak{I}\kappa|}\epsilon^{2}|\log\epsilon|^{2}, \tag{3.8}$$

where C is a constant which is independent of $(\kappa, \epsilon)\in\boldsymbol{C}\times(0,1)$ and $D_1=D_1(\epsilon)$ denotes

$$D_1=D_1(\epsilon)=\{(\omega_1, \omega_2)\in S_1\times S_1; |\omega_1-\omega_2|\leq 2\epsilon\} \tag{3.9}$$

$$\cup\{(\omega_1, \omega_2)\in S_1\times S_1;\ |\omega_1-\omega_2|\geq 2(1-\epsilon^2)\}\ .$$

Proof. Here and in the sequel, we use the same letter C to denote a constant, which may be different but independent of $(\kappa, \epsilon)\in \boldsymbol{C}\times(0, 1/2)$. First we note by (3.7)

$$(3.10)\qquad |I(\omega_1, \omega_2)|\leq Ce^{4a|\mathfrak{I}\kappa|}\int_{S_1}d\omega\frac{1}{|\omega_1-\omega||\omega-\omega_2|}$$
$$\leq Ce^{4a|\mathfrak{I}\kappa|}(1+|\log|\omega_1-\omega||)\ .$$

Let us introduce the polar coordinates (r, θ, ϕ) such that $|\omega_1-\omega_2|=2\sin\frac{\theta}{2}$ i.e. the z-axis is taken as the ω_1-direction and θ denotes the angle between ω_1 and ω_2. Let θ_1 and θ_2 be such that $\sin\frac{\theta_1}{2}=\epsilon$ and $\sin\frac{\theta_2}{2}=1-\epsilon^2$, respectively. Then we have by (3.10) and Fubini's theorem

$$(3.11)\qquad \int_{D_1}d\omega_1 d\omega_2|I(\omega_1, \omega_2)|^2$$
$$\leq Ce^{8a|\mathfrak{I}\kappa|}\int_{S_1}d\omega_1\Big(\int_0^{\theta_1}+\int_{\theta_2}^{\pi}\Big)d\theta\sin\theta\Big(1+\Big|\log\Big(2\sin\frac{\theta}{2}\Big)\Big|\Big)^2\ .$$

A simple computation shows that

$$(3.12)\qquad \int_0^{\theta_1}d\theta\sin\theta\Big(1+\Big|\log\Big(2\sin\frac{\theta}{2}\Big)\Big|\Big)^2\leq C\epsilon^2|\log\epsilon|^2\ ,$$

$$(3.13)\qquad \int_{\theta_2}^{\pi}d\theta\sin\theta\Big(1+\Big|\log\Big(2\sin\frac{\theta}{2}\Big)\Big|\Big)^2\leq C\epsilon^2\ .$$

The assertion follows from (3.11), (3.12) and (3.13).

We shall proceed to the case $2\epsilon<|\omega_1-\omega_2|<2(1-\epsilon^2)$. Let $D_2=D_2(\epsilon)$ be a subset of $S_1\times S_1$ defined by

$$(3.14)\qquad D_2=\{(\omega_1, \omega_2)\in S_1\times S_1;\ 2\epsilon<|\omega_1-\omega_2|<2(1-\epsilon^2)\}\ .$$

Lemma 3.4. *Let* $(\omega_1, \omega_2)\in D_2$ *and let* $\epsilon\in(0, 1/2)$. *Then*

$$(3.15)\qquad \int_{D_2}\frac{1}{|\omega_1-\omega_2|^2}d\omega_1 d\omega_2\leq C|\log\epsilon|\ .$$

Proof. Let θ_1 and θ_2 be such that $\sin\frac{\theta_1}{2}=\epsilon$ and $\sin\frac{\theta_2}{2}=1-\epsilon^2$, respectively. Then using the polar coordinates (r, θ, ϕ) introduced in the proof of Lemma 3.3, we have by Fubini's theorem

(3.16) $$\int_{D_2}\frac{1}{|\omega_1-\omega_2|^2}d\omega_1 d\omega_2=\int_{S_1}d\omega_1 2\pi\int_{\theta_1}^{\theta_2}d\theta\sin\theta\frac{1}{4\sin^2\frac{\theta}{2}}$$
$$=\int_{S_1}d\omega_1 2\pi(\log(1-\epsilon^2)-\log\epsilon)$$
$$\leq C|\log\epsilon|$$

Let $2\varphi_1\left(0<\varphi_1<\frac{\pi}{2}\right)$ be the angle between ω_1 and ω_2 i.e.

(3.17) $$|\omega_1-\omega_2|=2\sin\varphi_1\,.$$

We introduce the new coordinate system, under which ω_1 and ω_2 are represented as $\omega_1=(\cos\varphi_1, \sin\varphi_1, 0)$ and $\omega_2=(\cos\varphi_1, -\sin\varphi_1, 0)$, respectively. Under this we write $\omega\in S_1$, using the polar coordinates, as

(3.18) $$\omega=(\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)\,,\quad 0\leq\vartheta\leq\pi\,,\quad -\pi\leq\varphi\leq\pi\,.$$

Let $\gamma_i(i=1, 2)$ be the angle between ω_i and ω. Then we have

(3.19) $$\cos\gamma_1=\omega\cdot\omega_1=\sin\vartheta\cos(\varphi-\varphi_1)$$

(3.20) $$\cos\gamma_2=\omega\cdot\omega_2=\sin\vartheta\cos(\varphi+\varphi_1)$$

and by the change of variables: $\omega\mapsto(\vartheta, \varphi)$

(3.21) $$I(\omega_1, \omega_2)=\int_{-\pi}^{\pi}d\varphi\int_0^{\pi}d\vartheta\frac{e^{2ia\kappa\left(\sin\frac{\gamma_1}{2}+\sin\frac{\gamma_2}{2}\right)}}{4\sin\frac{\gamma_1}{2}\sin\frac{\gamma_2}{2}}\sin\vartheta q(\vartheta, \varphi)$$
$$\equiv\int_{-\pi}^{\pi}d\varphi\int_0^{\pi}d\vartheta f(\vartheta, \varphi)\,.$$

Let $\chi_i(i=1, 2, 3, 4)$ be smooh functions on $\boldsymbol{R}$ such that

(3.22) $$\operatorname{supp}\chi_1(\vartheta)\subset\left[\epsilon, \frac{\pi}{2}-\epsilon\right]\cup\left[\frac{\pi}{2}+\epsilon, \pi-\epsilon\right],\quad 0\leq\chi_1(\vartheta)\leq1\,,$$
$$\chi_1(\vartheta)=1\quad\text{on}\quad\left[2\epsilon, \frac{\pi}{2}-2\epsilon\right]\cup\left[\frac{\pi}{2}+2\epsilon, \pi-2\epsilon\right],$$
$$|\chi_1'(\vartheta)|\leq\frac{C}{\epsilon}\quad\text{on}\quad[\epsilon, 2\epsilon]\cup[\pi-2\epsilon, \pi-\epsilon]\,,$$

(3.23) $$\operatorname{supp}\chi_3(\varphi)\subset\left[-\pi+\epsilon, \pi-\epsilon\right],\quad 0\leq\chi_3(\varphi)\leq1\,,$$
$$\chi_3(\varphi)=1\quad\text{on}\quad[-\pi+2\epsilon, \pi-2\epsilon]\,,$$

$$|\chi_3'(\varphi)| \leq \frac{C}{\epsilon} \quad \text{on} \quad [-\pi+\epsilon, -\pi+2\epsilon] \cup [\pi-2\epsilon, \pi-\epsilon],$$

and

(3.24) $$\chi_2(\vartheta) = 1 - \chi_1(\vartheta), \quad \chi_4(\varphi) = 1 - \chi_3(\varphi).$$

Then we have by (3.21) and (3.24)

(3.25) $$\begin{aligned} I(\omega_1, \omega_2) &= \int_{-\pi}^{\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\vartheta (\chi_1 + \chi_2)(\chi_3 + \chi_4) f \\ &= \int_{-\pi}^{\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\vartheta \chi_1 \chi_3 f + \int_{-\pi}^{\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\vartheta (\chi_2 \chi_3 + \chi_4) f \\ &\equiv I_1(\omega_1, \omega_2) + I_2(\omega_1, \omega_2). \end{aligned}$$

Lemma 3.5. *Let* $\epsilon \in (0, 1/2)$ *and* $\kappa \in \boldsymbol{C}$. *Then*

(3.26) $$\int_{D_2} |I_2(\omega_1, \omega_2)|^2 d\omega_1 d\omega_2 \leq C e^{8a|\mathfrak{I}\kappa|} \epsilon^2 |\log \epsilon|.$$

Proof. Let $E_0 = \operatorname{supp}(\chi_2 \chi_3 + \chi_4)$ and let $E_\pm$ be subsets defined by

(3.27) $$E_+(E_-) = \{(\vartheta, \varphi) \in E_0;\ \varphi \geq 0 (\varphi \leq 0)\}.$$

By (3.22)-(3.24), we note that

(3.28) $$m(E_0) \leq C_\epsilon,$$

where $m(\cdot)$ denotes the Lebesgue measure on $\boldsymbol{R}^2$. Since $2\sin\frac{\gamma_2}{2} = |\omega - \omega_2| \geq \frac{1}{2}|\omega_1 - \omega_2|$ on E_+ and $2\sin\frac{\gamma_1}{2} = |\omega - \omega_1| \geq \frac{1}{2}|\omega_1 - \omega_2|$ on E_-, we have

(3.29) $$\begin{aligned} |I_2(\omega_1, \omega_2)| &\leq \int_{E_0} \mathrm{d}\vartheta d\varphi |f| \\ &\leq C e^{4a|\mathfrak{I}\kappa|} |\omega_1 - \omega_2|^{-1} \left(\int_{E_+} \frac{\sin\vartheta}{\sin\frac{\gamma_1}{2}} \mathrm{d}\vartheta d\varphi + \int_{E_-} \frac{\sin\vartheta}{\sin\frac{\gamma_2}{2}} \mathrm{d}\vartheta \mathrm{d}\varphi \right). \end{aligned}$$

By the change of variables Ω; $(\vartheta, \varphi) \mapsto \omega \in S_1$, we have

(3.30) $$\int_{E_+} \frac{\sin\vartheta}{\sin\frac{\gamma_1}{2}} \mathrm{d}\vartheta \mathrm{d}\varphi = \int_{\Omega(E_+)} \frac{2}{|\omega - \omega_1|} \mathrm{d}\omega.$$

Again, using the polar coordinates introduced in the proof of Lemma 3.3, we obtain

$$(3.31)\qquad \int_{\Omega(E_+)}\frac{1}{|\omega-\omega_1|}\mathrm{d}\omega=\int_{\tilde{E}_+}\frac{\sin\theta}{2\sin\frac{\theta}{2}}\mathrm{d}\theta\mathrm{d}\phi$$

$$=\int_{\tilde{E}_+}\cos\frac{\theta}{2}\mathrm{d}\theta\mathrm{d}\phi\leq C\epsilon\,,$$

where $\tilde{E}_+$ denotes the region corresponding to $\Omega(E_+)$ by the above transformation, so that $m(\tilde{E}_+)\leq C\epsilon$ by (3.2). Similarly we have

$$(3.32)\qquad \int_{E_-}\frac{\sin\vartheta}{\sin\frac{\gamma_2}{2}}\mathrm{d}\vartheta d\varphi\leq C\epsilon\,,$$

By (3.29)-(3.32), we have for $(\omega_1, \omega_2)\in D_2$

$$(3.33)\qquad |I_2(\omega_1, \omega_2)|\leq C\epsilon e^{4a|\mathfrak{I}\kappa|}|\omega_1-\omega_2|^{-1}\,.$$

The assertion follows from Lemma 3.4 and (3.33) immediately.

Let

$$u=2\left(\sin\frac{\gamma_1}{2}+\sin\frac{\gamma_2}{2}\right)$$

and

$$v=2\left(\sin\frac{\gamma_1}{2}-\sin\frac{\gamma_2}{2}\right).$$

Consider the map F: $(\vartheta, \varphi)\mapsto(u, v)$. Then we can easily check the next.

Lemma 3.6. *The map F: $(\vartheta, \varphi)\mapsto(u, v)$ is injective on $(0, \pi/2)\times(-\pi, \pi)$ and $(\pi/2, \pi)\times(-\pi, \pi)$, respectively.*

Lemma 3.7. *Let J be the Jacobian of F. Then*

$$(3.34)\qquad J=\frac{\sin(2\varphi_1)\sin\vartheta\cos\vartheta}{2\sin\frac{\gamma_1}{2}\sin\frac{\gamma_2}{2}}\,.$$

Proof. By (3.19) and (3.20) we have

$$(3.35)\qquad -\sin\gamma_1\frac{\partial\gamma_1}{\partial\vartheta}=\cos\vartheta\cos(\varphi-\varphi_1)\,,$$

$$(3.36)\qquad -\sin\gamma_2\frac{\partial\gamma_2}{\partial\vartheta}=\cos\vartheta\cos(\varphi+\varphi_1)\,,$$

and hence

(3.37)
$$\frac{\partial u}{\partial\vartheta}=\cos\frac{\gamma_1}{2}\frac{\partial\gamma_1}{\partial\vartheta}+\cos\frac{\gamma_2}{2}\frac{\partial\gamma_2}{\partial\vartheta}$$
$$=\frac{-\cos\vartheta}{2\sin\frac{\gamma_1}{2}\sin\frac{\gamma_2}{2}}\left\{\sin\frac{\gamma_2}{2}\cos(\varphi-\varphi_1)+\sin\frac{\gamma_1}{2}\cos(\varphi+\varphi_1)\right\}.$$

Similarly we have

(3.38)
$$\frac{\partial u}{\partial\varphi}=\frac{\sin\vartheta}{2\sin\frac{\gamma_1}{2}\sin\frac{\gamma_2}{2}}\left\{\sin\frac{\gamma_2}{2}\sin(\varphi-\varphi_1)+\sin\frac{\gamma_1}{2}\sin(\varphi+\varphi_1)\right\},$$

(3.39)
$$\frac{\partial v}{\partial\vartheta}=\frac{-\cos\vartheta}{2\sin\frac{\gamma_1}{2}\sin\frac{\gamma_2}{2}}\left\{\sin\frac{\gamma_2}{2}\cos(\varphi-\varphi_1)-\sin\frac{\gamma_1}{2}\cos(\varphi+\varphi_1)\right\},$$

(3.40)
$$\frac{\partial v}{\partial\varphi}=\frac{\sin\vartheta}{2\sin\frac{\gamma_1}{2}\sin\frac{\gamma_2}{2}}\left\{\sin\frac{\gamma_2}{2}\sin(\varphi-\varphi_1)-\sin\frac{\gamma_1}{2}\sin(\varphi+\varphi_1)\right\},$$

From (3.37)-(3.40), (3.34) follows immediately.

Lemma 3.8. *Let* $(\omega_1, \omega_2)\in D_2$. *Then*

(3.41)
$$\int_{-\pi}^{\pi}\frac{|\sin(\varphi-\varphi_1)|}{\sin\frac{\gamma_1}{2}}d\varphi\leq 8\,,$$

(3.42)
$$\int_{-\pi}^{\pi}\frac{|\sin(\varphi+\varphi_1)|}{\sin\frac{\gamma_2}{2}}d\varphi\leq 8\,.$$

Proof. In view of (3.19), we have

(3.43)
$$\int_{-\pi}^{\pi}\frac{|\sin(\varphi-\varphi_1)|}{\sin\frac{\gamma_1}{2}}\mathrm{d}\varphi$$
$$=\left(\int_{-\pi}^{-\pi+\varphi_1}-\int_{-\pi+\varphi_1}^{\varphi_1}+\int_{\varphi_1}^{\pi}\right)\frac{\sqrt{2}\sin(\varphi-\varphi_1)}{\sqrt{1-\sin\vartheta\cos(\varphi-\varphi_1)}}\mathrm{d}\varphi$$
$$=\frac{8\sqrt{2}}{\sqrt{1+\sin\vartheta}+\sqrt{1-\sin\vartheta}}\leq 8\,,$$

which implies (3.41). (3.42) is similarly obtained.

Lemma 3.9. *Let* $(\omega_1, \omega_2)\in D_2$ *and let* $\epsilon\in(0, 1/2)$. *Then*

$$(3.44)\qquad \left(\int_0^{\frac{\pi}{2}-\epsilon}+\int_{\frac{\pi}{2}+\epsilon}^{\pi}\right)d\vartheta\left(\frac{|\cos(\varphi-\varphi_1)|}{\sin\frac{\gamma_1}{2}}+\frac{|\cos(\varphi+\varphi_1)|}{\sin\frac{\gamma_2}{2}}\right)\leq\frac{C}{\epsilon},$$

where C is a constant which is independent of φ, φ_1 *and* ϵ.

Proof. We shall only show that

$$(3.45)\qquad \int_0^{\frac{\pi}{2}-\epsilon}d\vartheta\frac{|\cos(\varphi-\varphi_1)|}{\sin\frac{\gamma_1}{2}}\leq\frac{C}{\epsilon}.$$

The other cases can be dealt with similarly. Since $\cos\vartheta\geq\sin\epsilon$ on $\left[0,\frac{\pi}{2}-\epsilon\right]$, we have by (3.19)

$$(3.46)\qquad \int_0^{\frac{\pi}{2}-\epsilon}d\vartheta\frac{|\cos(\varphi-\varphi_1)|}{\sin\frac{\gamma_1}{2}}$$

$$\leq\frac{1}{\sin\epsilon}\int_0^{\frac{\pi}{2}-\epsilon}\frac{\sqrt{2}\cos\vartheta|\cos(\varphi-\varphi_1)|}{\sqrt{1-\sin\vartheta\cos(\varphi-\varphi_1)}}$$

$$=\frac{\sqrt{2}|\cos(\varphi-\varphi_1)|(1-\sqrt{1-\cos\epsilon\cos(\varphi-\varphi_1)})}{\sin\epsilon\cos(\varphi-\varphi_1)}$$

$$\leq\frac{C}{\sin\epsilon}\leq\frac{C}{\epsilon},$$

which implies (3.45).

Lemma 3.10. *Let* $(\omega_1, \omega_2)\in D_2$ *and* $\epsilon\in(0, 1/2)$. *Then we have for* $\kappa\in\boldsymbol{C}$

$$(3.47)\qquad |I_1(\omega_1, \omega_2)|\leq\frac{Ce^{4a|\Im\kappa|}}{\epsilon|\kappa|\sin(2\varphi_1)}.$$

Proof. We write $I_1(\omega_1, \omega_2)$ as

$$(3.48)\qquad I_1(\omega_1, \omega_2)=\int_{E_1}+\int_{E_2}\equiv I_{11}+I_{12},$$

where $E_1=\left\{(\vartheta, \varphi); 0<\vartheta<\frac{\pi}{2}, -\pi<\varphi<\pi\right\}$ and $E_2=\left\{(\vartheta, \varphi); \frac{\pi}{2}<\vartheta<\pi, -\pi<\varphi<\pi\right\}$. We shall show (3.47) for I_{11}. I_{12} can be estimated similarly. In view of Lemmas 3.6, 3.7, integration by parts with respect to u gives

$$(3.49)\qquad I_{11}(\omega_1, \omega_{22})=\frac{1}{2\sin(2\varphi_1)}\int_{F(E_1)}e^{ia\kappa u}\frac{q\chi_1\chi_3}{\cos\vartheta}dudv$$

$$=\frac{-1}{2ia\kappa\sin(2\varphi_1)}\int_{F(E_1)}e^{ia\kappa u}\frac{\partial}{\partial u}\left(\frac{q\chi_1\chi_3}{\cos\vartheta}\right)\mathrm{d}u\mathrm{d}v$$

$$=\frac{-1}{2ia\kappa\sin(2\varphi_1)}\int_{E_1}e^{2ia\kappa\left(\sin\frac{\gamma_1}{2}+\sin\frac{\gamma_2}{2}\right)}v_\varphi\frac{\partial}{\partial\vartheta}\left(\frac{q\chi_1\chi_3}{\cos\vartheta}\right)\mathrm{d}\vartheta\mathrm{d}\varphi$$

$$+\frac{1}{2ia\kappa\sin(2\varphi_1)}\int_{E_1}e^{2ia\kappa\left(\sin\frac{\gamma_1}{2}+\sin\frac{\gamma_2}{2}\right)}v_\vartheta\frac{\partial}{\partial\varphi}\left(\frac{q\chi_1\chi_3}{\cos\vartheta}\right)\mathrm{d}\vartheta\mathrm{d}\varphi$$

$$\equiv J_1+J_2,$$

where the abbreviation $v_\vartheta=\frac{\partial v}{\vartheta}$ is used. Using (3.22), (3.23), (3.40) and Lemma 3.8, we obtain by Fubini's theorem

$$|J_1|\leq\frac{Ce^{4a|\Im\kappa|}}{|\kappa|\sin(2\varphi_1)}\int_{E_1}\left\{\frac{|\sin(\varphi-\varphi_1)|}{\sin\frac{\gamma_1}{2}}+\frac{|\sin(\varphi+\varphi_1)|}{\sin\frac{\gamma_2}{2}}\right\} \tag{3.50}$$

$$\times\left\{\frac{|\chi_1(\vartheta)|}{\cos\vartheta}+\frac{|\chi_1'(\vartheta)|}{\cos\vartheta}+\frac{|\chi_1(\vartheta)|\sin\vartheta}{\cos^2\vartheta}\right\}\sin\vartheta\mathrm{d}\vartheta\mathrm{d}\varphi$$

$$\leq\frac{Ce^{4a|\Im\kappa|}}{|\kappa|\sin(2\varphi_1)}\left\{\int_\epsilon^{\frac{\pi}{2}-\epsilon}\mathrm{d}\vartheta\left(\frac{\sin\vartheta}{\cos\vartheta}+\frac{\sin^2\vartheta}{\cos^2\vartheta}\right)\right.$$

$$\left.+\frac{1}{\epsilon}\left(\int_\epsilon^{2\epsilon}+\int_{\frac{\pi}{2}-2\epsilon}^{\frac{\pi}{2}-\epsilon}\right)\mathrm{d}\vartheta\frac{\sin\vartheta}{\cos\vartheta}\right\}$$

$$\leq\frac{Ce^{4a|\Im\kappa|}}{\epsilon|\kappa|\sin(2\varphi_1)}.$$

Similarly, we have by (3.22), (3.23), (3.39) and Lemma 3.9

$$|J_2|\leq\frac{Ce^{4a|\Im\kappa|}}{|\kappa|\sin(2\varphi_1)}\int_{E_1}\left\{\frac{|\cos(\varphi-\varphi_1)|}{\sin\frac{\gamma_1}{2}}+\frac{|\cos(\varphi+\varphi_1)|}{\sin\frac{\gamma_2}{2}}\right\} \tag{3.51}$$

$$\times(|\chi_3(\varphi)|+|\chi_3'(\varphi)|)|\chi_1(\vartheta)|\mathrm{d}\vartheta\mathrm{d}\varphi$$

$$\leq\frac{Ce^{4a|\Im\kappa|}}{\epsilon|\kappa|\sin(2\varphi_1)}\left\{\int_{-\pi+\epsilon}^{\pi-\epsilon}\mathrm{d}\varphi+\frac{1}{\epsilon}\left(\int_{-\pi+\epsilon}^{-\pi+2\epsilon}+\int_{\pi-2\epsilon}^{\pi-\epsilon}\right)\mathrm{d}\varphi\right\}$$

$$\leq\frac{Ce^{4a|\Im\kappa|}}{\epsilon|\kappa|\sin(2\varphi_1)}.$$

From (3.49), (3.50) and (3.51) follows (3.47) for $I_{11}(\omega_1,\omega_2)$.

Lemma 3.11. *Let* $\epsilon\in(0,1/2)$. *Then we have for* $\kappa\in\boldsymbol{C}$

$$\int_{D_2}|I_1(\omega_1,\omega_2)|^2d\omega_1d\omega_2\leq Ce^{8a|\Im\kappa|}\frac{|\log\epsilon|}{\epsilon^2|\kappa|^2}. \tag{3.52}$$

Proof. By Lemma 3.10 we have only to show that

$$\int_{D_2} \frac{1}{\sin^2(2\varphi_1)} d\omega_1 d\omega_2 \leq C|\log\epsilon|. \tag{3.53}$$

Since $\sin^2(2\varphi_1) = |\omega_1 - \omega_2|^2\left(1 - \frac{1}{4}|\omega_1 - \omega_2|^2\right)$ by (3.17), we have using the polar coordinates introduced in the proof of Lemma 3.3 $\left(\sin\frac{\theta_1}{2} = \epsilon,\ \sin\frac{\theta_2}{2} = 1 - \epsilon^2 \text{ and } |\omega_1 - \omega_2| = 2\sin\frac{\theta}{2}\right)$

$$\begin{aligned} \int_{D_2} \frac{1}{\sin^2(2\varphi_1)} d\omega_1 d\omega_2 &= \int_{D_2} \frac{d\omega_1 d\omega_2}{|\omega_1 - \omega_2|^2\left(1 - \frac{1}{4}|\omega_1 - \omega_2|^2\right)} \\ &= \int_{S_1} d\omega_1 \int_{\theta_1}^{\theta_2} d\theta 2\pi \sin\theta \frac{1}{4\sin^2\frac{\theta}{2}\left(1 - \sin^2\frac{\theta}{2}\right)} \\ &= 2\pi \int_{S_1} d\omega_1 \int_{\theta_1}^{\theta_2} d\theta \frac{1}{\sin\theta} \\ &= 8\pi^2\left(\left|\log\tan\frac{\theta_2}{2}\right| - \left|\log\tan\frac{\theta_1}{2}\right|\right) \\ &\leq C|\log\epsilon|. \end{aligned} \tag{3.54}$$

Now we are in a position to prove Lemma 3.1.

Proof of Lemma 3.1. (3.1) follows from Lemmas 3.3, 3.5 and 3.11 immediately.

4. Poles of the scattering kernel with constant density

Throughout this section, we assume that $q(x) \equiv V_0$ (const. $\neq 0$). Then for $(r, \omega, \omega') \in (0, \infty) \times S_1 \times S_1$, $F(r, \omega, \omega')$ has an expansion of the form:

$$F(r, \omega, \omega') = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} F_l(r) Y_l^m(\omega) \overline{Y_l^m(\omega')}, \tag{4.1}$$

which converges absolutely and in $L_2(S_1 \times S_1)$. Here

$$F_l(r) = \frac{2l+1}{2\pi} \frac{V_0 a^2 j_l^2(ar)}{1 + iV_0 a^2 r j_l(ar) h_l^{(1)}(ar)} \tag{4.2}$$

and $\{Y_l^m\}$ ($l = 0, 1, 2, \cdots, m = -l, -l+1, \cdots, l$) denotes the spherical harmonics which provides an orthonormal basis for $L_2(S_1)$, j_l ($l = 0, 1, 2, \cdots$) the spherical Bessel functions and $h_l^{(1)}$ ($l = 0, 1, 2, \cdots$) the spherical Hankel functions of the first kind. F_l corresponds to the scattering amplitude of the scattered wave with the angular momentum l (Shimada [11, Lemma 7.5]).

Since $\{Y_l^m\}$ is an orthonormal basis for $L_2(S_1)$, integrating over $S_1\times S_1$ with respect to (ω, ω') after multiplying the both sides of (4.1) by $\overline{Y_l^m(\omega)}$ $Y_l^m(\omega')$ gives

$$(4.3)\qquad (F(r, \omega, \omega'), Y_l^m(\omega)\overline{Y_l^m(\omega')})_{L_2(S_1\times S_1)}=\frac{4\pi}{2l+1}F_l(r)\,.$$

In view of the unicity theorem and Theorem 1.1, (4.3) holds for r such that $F(\kappa, \omega, \omega')$ and $F_l(\kappa)$ are analytic near r. Thus the poles of $F_l(\kappa)$ ($l=0, 1, 2, \cdots$) are necessarily those of $F(\kappa, \omega, \omega')$. We shall study the poles of $F_l(\kappa)$. By (4.2) it is seen that the solutions except 0 of the equation:

$$(4.4)\qquad e^{2ia\kappa}=1-\frac{2ia\kappa}{V_0 a},$$

are the poles of $F_0(\kappa)$. (4.4) has been obtained by Petzold [8] and Dolph et al. [2]. A straightforward computation shows that (4.4) has a solution $i\lambda(\lambda>0)$ if $V_0<-1/a$ and $i\lambda(\lambda<0)$ if $-1/a<V_0<0$, so that $F(\kappa, \omega, \omega')$ has a pole $i\lambda$. Further we have

Theorem 4.1. *For large $n\in \boldsymbol{N}$, in the region $\frac{\pi n}{a}<\Re\kappa<\frac{\pi(n+1)}{a}$ there exists exactly one pole κ_n of $F_0(\kappa)$ such that*

$$(4.5)\qquad \kappa_n=\frac{\pi}{2a}\left(2n+1+\frac{1}{2}\operatorname{sgn}V_0\right)-\frac{i}{2a}\log\frac{\pi}{2a}\left(2n+1+\frac{1}{2}\operatorname{sgn}V_0\right)$$
$$-\frac{i}{2a}\log\frac{2}{|V_0|}+o(1)\quad as\quad n\to\infty\,,$$

where $\operatorname{sgn}V_0$ *denotes the signature of* V_0.

The theorem tells us that the pole $\kappa=x+iy$ of $F_0(\kappa)$ asymptotically approaches to a curve $y=\frac{-1}{2a}\log|x|-\frac{1}{2a}\log\frac{2}{|V_0|}$ as $|\Re\kappa|\to\infty$. For $V_0<0$, Petzold [8] obtained this result. Here we give an elementary proof.

Proof. Let $\kappa=x+iy$ $(x>0, y<0)$ and let $2ia\kappa=p+iq(p=-2ay, q=2ax)$. Then (4.4) is written as

$$(4.6)\qquad e^p e^{iq}=(1+Ap)+iAp\quad\left(A=\frac{-1}{V_0 a}\right).$$

For $n=1, 2, \cdots$, consider a segment $L(p, n)$ and a circle $C(p)$ in $\boldsymbol{C}$ defined by

$$L(p, n)=\{(1+Ap)+iAq;\, 2\pi n\leq q\leq 2\pi(n+1)\}\,,$$

$$C(p)=\{e^{p+iq};\ 2\pi n\leq q\leq 2\pi(n+1)\}\,.$$

First we seek an interval on which there exists an intersection of $L(p, n)$ and $C(p)$. Let $f(p)=e^{2p}-(1+Ap)^2$. Then it is seen that $f(p)$ monotonouly increases to infinity for large p. We can take a strictly increasing sequence $\{c_n\}$ such that c_n goes to infinity and $f(c_n)=(2\pi An)^2$. For large n, it is seen that $L(p, n)$ and $C(p)$ can intersect only for $p\in[c_n, c_{n+1}]$ at a unique intersection point $1+Ap+i\,\mathrm{sgn}A\sqrt{f(p)}$. Thus we can take $q_1(p)$ and $q_2(p)$, which are uniquely determined for $p\in[c_n, c_{n+1}]$, such that

$$(4.7)\qquad e^p e^{iq_1(p)}=(1+Ap)+iAq_2(p)\,,$$

$$2\pi n\leq q_1(p),\ q_2(p)\leq 2\pi(n+1)\,,\quad \left(q_2(p)=\frac{\sqrt{f(p)}}{|A|}\right).$$

We shall show that there exists exactly one solution $p\in[c_n, c_{n+1}]$ such that $q_1(p)=q_2(p)$ i.e.

$$(4.8)\qquad \arctan\left(\frac{\mathrm{sgn}A\sqrt{f(p)}}{1+Ap}\right)+\left(2n+\frac{1}{2}-\frac{1}{2}\mathrm{sgn}A\right)\pi=\frac{\sqrt{f(p)}}{|A|}\,,$$

where $\arctan\theta\in\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$. Let

$$g(p)=\arctan\left(\frac{\mathrm{sgn}A\sqrt{f(p)}}{1+Ap}\right)+\left(2n+\frac{1}{2}-\frac{1}{2}\mathrm{sgn}A\right)\pi-\frac{\sqrt{f(p)}}{|A|}\,.$$

Then it is easily seen that $g(c_n)>0$, $g(c_{n+1})<0$ and $g(p)$ is monotone for large p, which implis that there exists a unique solution p_n of (4.8) for large n by the intermediate value theorem. If we put $q_n=q_0(p_n)$, $\frac{1}{2a}(q_n-ip_n)$ turns out to be a unique solution of (4.4) for large n. By (4.8) we have as $n\to\infty$

$$(4.9)\qquad q_n=\arctan\left(\frac{\mathrm{sgn}A\sqrt{f(p)}}{1+Ap}\right)+\left(2n+\frac{1}{2}-\frac{1}{2}\mathrm{sgn}A\right)\pi$$

$$=\left(2n+\frac{1}{2}-\frac{1}{2}\mathrm{sgn}A\right)\pi+\frac{\pi}{2}+o(1)$$

$$=\left(2n+1+\frac{1}{2}\mathrm{sgn}\,V_0\right)\pi+o(1)\,.$$

Taking the imaginary part of the both sides of (4.6), we have by (4.9)

$$e^{p_n}(-\mathrm{sgn}\,V_0+o(1))=\frac{-1}{V_0a}\left(2n+\frac{1}{2}+\frac{1}{2}\mathrm{sgn}\,V_0\right)\pi+o(1)$$

and hence

(4.10) $$p_n=\log\frac{2}{|V_0|}+\log\frac{\pi}{2a}\left(2n+1+\frac{1}{2}\operatorname{sgn}V_0\right)+o(1)\,.$$

From (4.9) and (4.10), (4.5) follows immediately.

Lemma 4.2. *Let $V_0<0$ and let l be a positive integer. For large n, in the region $2\pi n\leq s\leq 2\pi(n+1)$ $(r+is\in \boldsymbol{C})$, there exists a unique solution r_n+is_n of the equation:*

(4.11) $$e^{r+is}=(-1)^l\{(1+Ar)+iAs\}\,,\quad \left(A=\frac{-1}{V_0a}\right),$$

which behaves, when $n\to\infty$, as

(4.12) $$r_n=\log\left\{2n+1+\frac{1}{2}(-1)^{l-1}\right\}\pi A+o(1)\,,$$

$$s_n=\left\{2n+1+\frac{1}{2}(-1)^{l-1}\right\}\pi+o(1)\,.$$

Proof. The assertion is obtained in a way similar to the proof of Theorem 4.1.

We remark that $\frac{1}{2a}(s_n-ir_n)$ asymptotically approaches to the curve $y=\frac{-1}{2a}\log|x|-\frac{1}{2a}\log\frac{2}{|V_0|}$ $(x+iy\in \boldsymbol{C})$ as $n\to\infty$. For the pole of $F_l(\kappa)$ $(l\geq 1)$, we have the next

Theorem 4.3. *Let $V_0<0$ and let $\epsilon_n=\frac{\epsilon}{\log n}$ $(\epsilon>0)$. Then for large n, in the regin $\frac{\pi n}{a}<\Re\kappa<\frac{\pi(n+1)}{a}$ there exists exactly one pole $\kappa_n=x_n-iy_n$ of $F_l(\kappa)$ such that*

(4.13) $$\frac{1}{2a}(1-\epsilon_n)r_n\leq y_n\leq\frac{1}{2a}(1+\epsilon_n)r_n\,.$$

Since $\epsilon_n r_n=\epsilon(1+o(1))$ as $n\to\infty$, the poles of $F_l(\kappa)(l\geq 1)$ only appear near the curve $y=\frac{-1}{2a}\log|x|-\frac{1}{2a}\log\frac{2}{|V_0|}$ $(x+iy\in \boldsymbol{C})$, if $|\Re\kappa|$ is sufficiently large.

Proof. Let us recall (4.2). A straightforward computation shows that

(4.14) $$\kappa\{1+iV_0a^2\kappa j_l(a\kappa)h_l^{(1)}(a\kappa)\}$$

$$=\frac{-iV_0}{2}\left\{(-1)^l e^{2ia\kappa}-1+\frac{2ia\kappa}{V_0a}+R(\kappa)\right\},$$

where $R(\kappa)$ is analytic on $\boldsymbol{C}\backslash\{0\}$ such that

$$(4.15)\qquad |R(\kappa)| \leq C\frac{e^{2a|\mathcal{I}\kappa|}}{|\kappa|}.$$

Using (4.14), (4.15), Lemma 4.2 and Rouche's theorem, we have the conclusion. We omit the details.

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