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On hyperplane sections of reduced irreducible varieties of low codimension

By

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1. Introduction

Let X be an arithmetically Cohen-Macaulay variety (subscheme) of codimension 2 in $\boldsymbol{P}^n=\boldsymbol{P}^n(k)$, where k is an algebraically closed field. Let $I=I(X)$ denote the defining ideal of X in the polynomial ring $R=k[x_0, \cdots, x_n]$. By the Hilbert-Burch theorem we may assume that I is minimally generated by the maximal minors of an $r-1$ by r matrix (g_{ij}) of homogeneous elements of R. Let $a_1, \cdots, a_r$ be the degree of these generators. Then I has a minimal free resolution of the form

$$0 \longrightarrow \bigoplus_{i=1}^{r-1} R(-b_i) \xrightarrow{(g_{ij})} \bigoplus_{j=1}^{r} R(-a_j) \longrightarrow I \longrightarrow 0,$$

where $b_1, \cdots, b_{r-1}$ are positive integers with $\sum b_i=\sum a_j$. Put $u_{ij}=b_i-a_j$. We have $\deg g_{ij}=u_{ij}$, if $u_{ij}>0$ and $g_{ij}=0$ if $u_{ij}\leq 0$. Under the assumptions $a_1\leq\cdots\leq a_r$ and $b_1\leq\cdots\leq b_{r-1}$, the matrix (u_{ij}) is uniquely determined by X, and it carries all the numerical data about X. One calls (u_{ij}) the *degree matrix* of X [5].

In [24] Sauer proved that an arithmetically Cohen-Macaulay curve in $\boldsymbol{P}^3$ is smoothable if and only if $u_{ii+2}\geq 0$ for $i=1, \cdots, r-2$. At a first glance Sauer's result is surprising in so far as smoothability should solely depend on the Hilbert function of the curve (which of course is determined by the degree matrix but not vice versa). However, as observed by Geramita and Migliore [13], this numerical condition of the degree matrix can indeed be expressed in terms of the Hilbert function of C.

On the other hand, as noted in [13], Sauer ([24]) proved, though not explicitly stated, that a matrix of integers $u_{ij}=b_i-a_i$, where $a_1\leq\cdots\leq a_r$ and $b_1\leq\cdots\leq b_{r-1}$ are two sequences of positive integers with $\sum a_i=\sum b_j$, is the degree matrix of a smooth arithmetically Cohen-Macaulay curve in $\boldsymbol{P}^3$ if and only if $u_{ii+2}>0$ for $i=1, \cdots, r-2$. Here the reference to the stronger numerical invariant, the degree matrix, is indispensible, since the Hilbert function

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only recognizes smoothability.

Inspired by this observation, Chiantini and Orecchia [7] have shown that a collection of points in $\boldsymbol{P}^2$ is a hyperplane section of a smooth arithmetically Cohen-Macaulay curve in $\boldsymbol{P}^3$ if and only if its degree matrix satisfies the condition $u_{ii+2}>0$ for $i=1, \cdots, r-2$. This numerical condition is for instance satisfied if the points are in uniform position; see the discussion before Cor. 3.7. Thus the result of Chiantini and Orecchia is a certain converse to a theorem of Harris which says that a generic hyperplane section of an irreducible curve is a set of points in uniform position (see [15] and [16]).

In this paper we shall extend these results as follows.

Theorem 1.1. *Let $X\subset\boldsymbol{P}^n$ be a reduced arithmetically Cohen-Macaulay variety of codimension 2 with degree matrix (u_{ij}). Then X is a hyperplane section of a reduced irreducible arithmetically Cohen-Macaulay normal variety $Y\subset\boldsymbol{P}^{n+1}$ of codimension 2 if and only if $u_{ii+2}>0$ for $i=1, \cdots, r-2$.*

For example, the numerical conditions of the theorem are satisfied if one of the hypersurfaces of least possible degree passing through X is irreducible. This will be shown in 3.5.

Our method of proving 1.1 is different from the one of [24] and [13] (where linkage theory is employed). First we consider generic $r-1$ by r matrices (x_{ij}) whose entries are either zero or indeterminates and satisfy the condition $x_{ij}\neq 0$ for $i\geq j-2$. Then we specialize such a matrix to obtain a reduced irreducible arithmetically Cohen-Macaulay normal variety Z in a space $\boldsymbol{P}^m$, $m>n$, whose section with an n-space is X. Finally we descend from Z by a Bertini type theorem to a reduced irreducible arithmetically Cohen-Macaulay normal variety $Y\subset\boldsymbol{P}^{n+1}$ such that $X=Y\cap\boldsymbol{P}^n$. This method has the advantage to work as well in the Gorenstein case.

A result similar to 1.1 has been obtained by Mei-Chu Chang [4] for $n\leq 4$ when X is projectively Cohen-Macaulay.

Now let $X\subset\boldsymbol{P}^n$ be an arithmetically Gorenstein variety of codimension 3. By [2] and [29], the defining ideal $I=I(X)$ of X is minimally generated by the $2r$-pfaffians of a skew-symmetric $2r+1$ by $2r+1$ matrix (g_{ij}) of homogeneous forms of R. Let $a_1\leq\cdots\leq a_{2r+1}$ be the degree of these generators of I. Then I has a minimal free resolution of the form

$$0\longrightarrow R(-c)\longrightarrow\oplus_{i=1}^{2r+1}R(-b_i)\xrightarrow{(g_{ij})}\oplus_{i=1}^{2r+1}R(-a_j)\longrightarrow I\longrightarrow 0\,,$$

where $c=\frac{1}{r}(a_1+\cdots+a_{2r+1})$, $b_i=c-a_i$. If we put $u_{ij}=c-a_i-a_j$, we have deg $g_{ij}=u_{ij}$, if $u_{ij}>0$ and $g_{ij}=0$ if $u_{ij}\leq 0$. As before, the integer matrix (u_{ij}) will be called the *degree matrix* of X. With the method described above we obtain the following results which characterize the degree matrix of smooth arithmetically Gorenstein curves in $\boldsymbol{P}^4$ and hyperplane sections of reduced irreducible arithmetically Gorenstein varieties of codimension 3.

Theorem 1.2. *A matrix of* (u_{ij}) *of integers as above is the degree matrix of a smooth arithmetically Gorenstein curve in* $\boldsymbol{P}^4$ *if and only if* $u_{ij}>0$ *for all* i, j *with* $i+j=2r+4$.

Theorem 1.3. *Let* $X\subset\boldsymbol{P}^n$ *be a reduced arithmetically Gorenstein variety of codimension* 3 *with degree matrix* (u_{ij}). *Then* X *is a hyperplane section of a reduced irreducible arithmetically Gorenstein normal variety* $Y\subset\boldsymbol{P}^{n+1}$ *of codimension* 3 *if and only if* $u_{ij}>0$ *for all* i, j *with* $i+j=2r+4$.

Similarly as in case of codimension 2 varieties we show (Cor. 5.1) that the equivalent conditions of the theorem are satisfied if one of the hypersurfaces of least possible degree passing through X is irreducible.

The proofs of 1.1 resp. 1.2 and 1.3 will be found in Section 3 resp. Section 5. In Section 2 we deal with generic height 2 perfect prime ideals and their specializations. In Section 4 we list, for a fixed integer r> 2, all integers which occur as the degree of reduced irreducible arithmetically Cohen-Macaulay schemes of codimension 2 whose defining ideals are minimally generated by r elements. Finally in Section 6 we compute a Gröbner basis for generic height 3 Gorenstein ideals and deduce from this a formula for the degree of arithmetically Gorenstein varieties of codimension 3. We also compute the minimal free resolution of the ideal generated by the leading terms of a generic height 3 Gorenstein ideal.

2. Generic height 2 perfect prime ideals and specializations

Let us first explain why the degree matrix (u_{ij}) of an arithmetically Cohen-Macaulay reduced irreducible variety of codimension 2 satisfies the condition $u_{ii+2}>0$ for all i.

We prefer to use the algebraic language, and hence have to consider homogeneous perfect ideals $I\subset R=k[x_0, \cdots, x_n]$ of height 2. Their degree matrix is defined as in the introduction. Note that the assumptions $a_1\leq\cdots\leq a_r$ and $b_1\leq\cdots\leq b_{r-1}$ imply that $u_{ij}\leq u_{st}$ for all $i\leq s$ and $t\leq j$.

Lemma 2.1 (cf. [13], p. 3142). *Let* $I\subset R=k[x_0, \cdots, x_n]$ *be a height* 2 *perfect homogeneous ideal with degree matrix* (u_{ij}). *Suppose that* I *contains two forms of degree* a_1 *and* a_2 *(the least possible degrees) having no common factor. Then* $u_{ii+2}>0$ *for all* i.

Proof. Without restriction we may assume that the two forms of least possible degree are the elements f_1 and f_2 of a minimal homogeneous basis $f_1, \cdots, f_r$ of I, and that f_i is the maximal minor of an $r-1$ by r matrix (g_{ij}) of homogeneous forms obtained by deleting the i-th column, $i=1, \cdots, r$. Let (u_{ij}) be the degree matrix of (g_{ij}). If $u_{tt+2}\leq 0$ for some $t=1, \cdots, r-2$, we have $u_{ij}\leq 0$, and therefore $g_{ij}=0$ for all $i\leq t$ and $j\geq t+2$. Thus, the minor of the last $r-t-1$ rows and columns of (g_{ij}) is a factor of both maximal minors f_1

and f_2, a contradiction.

It is obvious that any height 2 homogeneous prime ideal contains two forms of least possible degree having no common factor (any form of least possible degree of a prime ideal is irreducible). Therefore, the degree matrix of any arithmetically Cohen-Macaulay reduced irreducible variety of codimension 2 and of all of its proper hyperplane sections (they have the same degree matrix) satisfies the condition $u_{ii+2}>0$ for all i.

One can obtain arithmetically Cohen-Macaulay varieties of codimension 2 with a given degree matrix (u_{ij}) which satisfies the condition $u_{ii+2}>0$ for $i=1, \cdots, r-2$ by specializing the generic cases. A generic case is given by an $r-1$ by r matrix (x_{ij}) whose non-zero entries are indeterminates and which satisfies the condition $x_{ij}\neq 0$ for $i\geq j-2$. We shall use induction on r to show that the ideal generated by the maximal minors of such a matrix is a perfect prime ideal. In order to make the induction hypothesis accessible we have to modify these cases a little bit as follows (cf. [1, Lemma 2] for a similar argument).

Lemma 2.2. *Let A be a Cohen-Macaulay normal domain. Let (x_{ij}) be an $r-1$ by r matrix such that $X=\{x_{ij}: i\geq j-2\}$ is a set of algebraically independent elements over A and $x_{ij}\in A$ if $i<j-2$. Let I be the ideal generated by the maximal minors of (x_{ij}). Then height $I=2$ and $A[X]/I$ is a Cohen-Macaulay normal domain.*

Proof. We prove the assertion by induction on r. If $r\leq 3$, the statement is trivial because then the entries of (x_{ij}) consist of algebraically independent elements. If $r>3$, we start with a general observation. Let x_{st} be an arbitrary element of X with $s\geq t-2$, and consider the matrix (x'_{ij}) whose entries belong to the ring $A[X, x_{st}^{-1}]$, and are given by

$$x'_{ij}=\begin{cases} x_{sj} & \text{if } i=s\,, \\ x_{ij}x_{st}-x_{it}x_{sj} & \text{if } i\neq s\,. \end{cases}$$

It is obvious that the ideal $IA[X, x_{st}^{-1}]$ is generated by the maximal minors of the new matrix (x'_{ij}). Since $x'_{it}=0$ for all $i\neq s$, and since $x'_{st}=x_{st}$ is a unit in $A[X, x_{st}^{-1}]$, the ideal $IA[X, x_{st}^{-1}]$ is as well generated by the maximal minors of the $r-2$ by $r-1$ submatrix (y_{ij}) of (x'_{ij}) obtained by deleting row s and column t.

Now we choose $x_{st}\in Z=\{x_{11}, x_{12}, x_{13}, x_{24}\}$. Let X' be the set of all elements of X in row s and column t of (x_{ij}), and denote by B the ring $A[X', x_{st}^{-1}]$. Then B is a Cohen-Macaulay domain, $A[X, x_{st}^{-1}]=B[Y]$ and $IA[X, x_{st}^{-1}]=IB[Y]$ where $Y=\{y_{ij}|i\geq j-2\}$. Note that Y is again a set of algebraically independent elements over B, and that $y_{ij}\in B$ for $i<j-2$. Thus we may apply the induction hypothesis, and conclude that $A[X, x_{st}^{-1}]/IA[X, x_{st}^{-1}]\cong B[Y]/IB[Y]$ is a Cohen-Macaulay normal domain, and that $IA[X, x_{st}^{-1}]$ is a

prime ideal of height 2. Moreover, $IA[X, x_{st}^{-1}]$ does not contain any element of X'.

We use these informations to deduce hat I has the required properties: Since I is the ideal of maximal minors of an $r-1$ by r matrix, its height is ≤ 2. Suppose it is less than 2. Then there exists a prime ideal $\mathfrak{P}$ containing I with height $\mathfrak{P}<2$, and so $\mathfrak{P}A[X, x_{st}^{-1}]=A[X, x_{st}^{-1}]$ since height $IA[X, x_{st}^{-1}]=2$. This is true for any $x_{st}\in Z$. Therefore it follows that $\mathfrak{P}$ contains the ideal (Z) which is absurd since height$(Z)=4$. (For this part of the proof it would have sufficed that Z contains two elements.) We conclude that height $I=2$, and hence by [19] the ring $A[X]/I$ is Cohen-Macaulay, and I is an unmixed ideal.

Next we claim that I is a prime ideal. Indeed, suppose there exist two different minimal prime ideals $\mathfrak{P}_1$ and $\mathfrak{P}_2$ of I. Since $IA[X, x_{st}^{-1}]$ is prime ideal, it follows that for any $x_{st}\in Z$ we have $x_{st}\in\mathfrak{P}_1$ or $x_{st}\in\mathfrak{P}_2$. None of the two prime ideals can contain all $x_{ts}\in Z$ since their height is two. Thus we may assume that $x_{11}\in\mathfrak{P}_1$ and $x_{12}\notin\mathfrak{P}_1$. But then $x_{11}\in\mathfrak{P}_1A[X, x_{12}^{-1}]=IA[X, x_{12}^{-1}]$, a contradiction.

It remains to show that $A[X]/I$ is normal. For this it suffices to prove that $A[X]/I$ satisfies the Serre condition R_1. Let $\mathfrak{P}\supset I$ be any prime ideal with height$(\mathfrak{P}/I)=1$. Then, since height $\mathfrak{P}=$height $I+1=3<$height(Z), there is an element $x_{st}\in Z$ not belonging to $\mathfrak{P}$, and hence $(A[X]/I)_{\mathfrak{P}}$ may be considered as the localization of the normal ring $A[X, x_{st}^{-1}]/IA[X, x_{st}^{-1}]$. Therefore it is regular, as desired.

Following [16] we call a homogeneous ideal I of a polynomial ring $\boldsymbol{Z}[X]$ over the ring $\boldsymbol{Z}$ of integers a *generically* (*perfect*, resp. *Gorenstein*, resp. *normal*) *prime ideal* if height$I$=height$(IA[X])$ and $A[X]/IA[X]$ is a (Cohen-Macaulay, resp. Gorenstein, resp. normal) domain for any (Cohen-Macaulay, resp. Gorenstein, resp. normal) domain A.

With this notation, 2.2 has the following consequence.

Corollary 2.3. *Let* (x_{ij}) *be an* $r-1$ *by* r *matrix such that the non-zero entries form a set* X *of algebraically independent elements over* $\boldsymbol{Z}$ *and* $x_{ij}\neq 0$ *for* $i\geq j-2$. *Let* I *be the ideal of* $\boldsymbol{Z}[X]$ *generated by the maximal minors of* (x_{ij}). *Then* I *is a height* 2 *generically perfect normal prime ideal.*

The following lemma describes the transfer of generic properties. We refer the reader to a survey on the widespread literature on this subject in [3], Chapter 3. The proof of the next lemma follows the patterns of [27].

Lemma 2.4. *Let* I *be a generically* (*perfect, resp. Gorenstein, resp. normal*) *homogeneous prime ideal of a polynomial ring* $\boldsymbol{Z}[X]=\boldsymbol{Z}[x_1, \cdots, x_n]$. *Let* S *be a Noetherian commutative ring with unity whose Jacobson radical contains a regular sequence* $a_1, \cdots, a_n$ *such that* $S/(a_1, \cdots, a_n)$ *is a* (*Cohen-Macaulay, resp. Gorenstein, resp. normal*) *domain. Let* J *denote the ideal generated by the elements* $\varphi(f)$, $f\in I$, *where* φ *is the ring homomorphism from*

$\mathbf{Z}[X]$ to S induced by the map $x_i \mapsto a_i$, $i=1, \cdots, n$. Then S/J is also a (Cohen-Macaulay, resp. Gorenstein, resp. normal) domain.

Proof. Since height(I)=height$(I\mathbf{Z}_p[X])$ for any prime number p, the ideal I does not contain any non-zero element of $\mathbf{Z}$; therefore $I \subseteq (x_1, \cdots, x_n)$. Put $\mathfrak{P}=(a_1, \cdots, a_n)$. Then $J \subseteq \mathfrak{P}$. Since $\mathfrak{P}$ is contained in the Jacobson radical of S, it suffices to show that the associated graded ring $\mathrm{gr}_{\mathfrak{P}/J}(S/J)$ is a (Cohen-Macaulay, resp. Gorenstein, resp. normal) domain (see e.g. [21, (2.1D)], [20, Theorem 4.11], and [6, Section 3]). Let A be the ring $S/\mathfrak{P}$. By the assumption on $a_1, \cdots, a_n$, $\mathrm{gr}_{\mathfrak{P}}(S) \cong A[X]$ and A is a (Cohen-Macaulay, resp. Gorenstein, resp. normal) domain. In the following we will identify $\mathrm{gr}_{\mathfrak{P}}(S)$ with $A[X]$. Let I^* denote the ideal of $A[X]$ generated by the leading forms of J. Then $\mathrm{gr}_{\mathfrak{P}}(S/J) \cong A[X]/I^*$. By the generic property of I we need only to show that $I^*=IA[X]$. Since I is a homogeneous ideal, this follows once we know that height(I^*)=height$(IA[X])$. To prove the latter equality, it is sufficient to show that height$(I^*) \leq$ height$(IA[X])$. Let K denote the quotient field of A. We have

$$\mathrm{height}(I^*) \leq \mathrm{height}(I^*K[X])$$

Since $\mathfrak{P}^n$ is a primary ideal for all $n \geq 0$, the order of any element of I with respect to the $\mathfrak{P}$-adic filtration remains the same when passing to the $\mathfrak{P}S_{\mathfrak{P}}$-adic filtration. Hence $I^*K[X]$ is the ideal of the leading forms of the elements of $JS_{\mathfrak{P}}$ in $\mathrm{gr}_{\mathfrak{P}S_{\mathfrak{P}}}(S_{\mathfrak{P}}) \cong K[X]$. This implies

$$\mathrm{height}(I^*K[X]) = \mathrm{height}(JS_{\mathfrak{P}})$$

(see [12, Kap. II, § 3]). But

$$\mathrm{height}(JS_{\mathfrak{P}}) \leq \mathrm{height}(I) = \mathrm{height}(IA[X])$$

by the superheight theorem of Hochster [17, (7.1)], and hence we obtain height $(I^*) \leq$ height$(IA[X])$.

Corollary 2.5. *Let (g_{ij}) be an $r-1$ by r matrix of homogeneous elements of a polynomial ring S over a field k which satisfies the following conditions:*
(i) *$g_{ij} \neq 0$ for $i \geq j-2$,*
(ii) *The elements $g_{ij} \neq 0$ form an S-regular sequence,*
(iii) *The factor ring $S/\mathfrak{P}$ is a (normal) Cohen-Macaulay domain, where $\mathfrak{P}$ denotes the ideal generated by the elements g_{ij}.*
Suppose moreover that the ideal J generated by the maximal minors of (g_{ij}) is homogeneous. Then S/J is a (normal) Cohen-Macaulay domain.

Proof. Let $\mathfrak{m}$ denote the maximal graded ideal of S. By 2.3 and 2.4, the local ring $(S/J)_{\mathfrak{m}}$ is a (normal) Cohen-Macaulay domain. Since J is a homogeneous ideal, this is equivalent to saying that S/J is a (normal) Cohen-Macaulay domain (see [20] or [6, Section I]).

Remark 2.6. The condition $x_{ij}\neq 0$ (resp. $g_{ij}\neq 0$) for $i\geq j-2$ of 2.3 (resp. 2.5) can not be weakened. For instance, the ideal generated by the 2-minors of the matrix

$$\begin{pmatrix} x & y & 0 \\ u & v & w \end{pmatrix}$$

is not a prime ideal.

3. Lifting height 2 perfect ideals

There is an easy way to construct, for a given arithmetically Cohen-Macaulay variety X of codimension 2 in $\boldsymbol{P}^n$ whose degree matrix satisfies the condition $u_{ii+2}>0$, a reduced irreducible arithmetically Cohen-Macaulay variety Y of codimension 2 in a larger projective space $\boldsymbol{P}^m\supset\boldsymbol{P}^n$ such that $X=Y\cap\boldsymbol{P}^n$.

To see this let us introduce the following terminology. We say that a homogeneous ideal $I\subset R=k[x_0,\cdots,x_n]$ can be *lifted* to an ideal $J\subset S=k[x_0,\cdots,x_m]$, $m\geq n$, if there exist linear forms $y_1,\cdots,y_r$ of S, $r=m-n$, such that $R\cong S/(y_1,\cdots,y_r)$ and $I\cong(J,y_1,\cdots,y_r)/(y_1,\cdots,y_r)$. Geometrically, this means that the variety defined by I in $\boldsymbol{P}^n$ is the intersection of the variety defined by $\boldsymbol{J}$ in $\boldsymbol{P}^m$ with a n-space.

Lemma 3.1. *Any height* 2 *perfect ideal* $I\subset R=k[x_0,\cdots,x_n]$ *whose degree matrix* (u_{ij}) *satisfies the condition* $u_{ii+2}>0$ *can be lifted to a height* 2 *perfect prime ideal* $J\subset S=k[x_0,\cdots,x_m]$ *for some integer* $m>n$ *such that* S/J *is a normal domain.*

Proof. Let (g_{ij}) be a Hilbert-Burch matrix of I having the degree matrix (u_{ij}). For every $u_{ij}>0$, we introduce new indeterminates $x_{ij}, y_{ij}, a_{ij}, b_{ij}, c_{ij}$ and put

$$G_{ij}=x_{ij}a_{ij}^{u_{ij}-1}+y_{ij}b_{ij}^{u_{ij}-1}+c_{ij}^{u_{ij}}+g_{ij}$$

For $u_{ij}\leq 0$ we put $G_{ij}=0$. Since $u_{ii+2}>0$, $G_{ij}\neq 0$ for all $i\geq j-2$. Let S be the polynomial ring over R in all indeterminates $x_{ij}, y_{ij}, a_{ij}, b_{ij}, c_{ij}$. It is obvious that the elements G_{ij} form a regular sequence of S. From the fact that the normality of a ring A is transfered to all rings of the form $A[x,y]/(ax+by+c)$, where x, y are indeterminates and a, b, c is a regular sequence of A ([26, Korollar 4.4]; see also [18] for a homogeneous version of this result), we can successively deduce that $S/\mathfrak{P}$ is a normal Cohen-Macaulay domain, where $\mathfrak{P}$ denotes the ideal generated by all G_{ij}. Let J be the height 2 homogeneous ideal of S generated by the maximal minors of the matrix (G_{ij}). Then I can be lifted to J since the G_{ij} specialize to g_{ij} for all i and j; moreover, by 2.5, S/J is a normal Cohen-Macaulay domain.

One can easily derive from 3.1 Sauer's characterization of the degree matrix of smooth arithmetically Cohen-Macaulay curves in $\boldsymbol{P}^3$.

Corollary 3.2 ([13], Theorem 4.1). *Let $a_1 \leq \cdots a_r$ and $b_1 \leq \cdots \leq b_{r-1}$ be two sequences of positive integers and put $u_{ij} = b_i - a_j$ for all i and j. The matrix (u_{ij}) is the degree matrix of a smooth arithmetically Cohen-Macaulay curve in $\boldsymbol{P}^3$ if and only if $u_{ii+2} > 0$ for $i = 1, \cdots, r-2$.*

Proof. By 2.1, we only need to prove the sufficient part of the statement. Let I be the height 2 perfect ideal generated by the maximal minors of the matrix (g_{ij}) with

$$g_{ij} = \begin{cases} x_{ij}^{u_{ij}} & \text{if } u_{ij} > 0\,, \\ 0 & \text{if } u_{ij} \leq 0\,. \end{cases}$$

By 3.1 we can lift I to a height 2 perfect prime ideal J in a polynomial ring $S = k[x_0, \cdots, x_m]$ such that S/J is a normal domain. Let $Y \subset \boldsymbol{P}^m$ be the arithmetically Cohen-Macaulay normal variety defined by J. There is a Bertini type theorem on hyperplane sections of normal varieties [11, Theorem 5.2] according to which there is a linear subspace $\boldsymbol{P}^3$ of $\boldsymbol{P}^m$ such that $X = Y \cap \boldsymbol{P}^3$ is a smooth arithmetically Cohen-Macaulay curve. Of course, X has the same degree matrix as Y, namely (u_{ij}).

To prove 1.1 we need a Bertini type theorem dealing with hyperplane sections passing through a fixed linear space.

Lemma 3.3. *Let J be a perfect homogeneous prime ideal in a polynomial ring S over k such that S/J is a normal domain. Assume that there are $r \geq 2$ linear forms $x_1, \cdots, x_r$ of S such that $I = (J, x_1, \cdots, x_r)$ is a reduced ideal with* height I = height $J + r$. *For a general linear form x in $(x_1, \cdots, x_r)$, the ideal (J, x) is a perfect prime ideal and $S/(J, x)$ is normal domain.*

Proof. We only need to show that $S/(J, x)$ satisfies Serre condition R_1. Let $\mathfrak{P} \supseteq (J, x)$ be an arbitrary prime ideal of S which corresponds to a singular point of Spec$(S/(J, x))$. We have to show that height $\mathfrak{P}/(J, x) \geq 2$. By Bertini's theorem on singularities (see [30] for char$(k) = 0$ and, e.g., [28, Lemma 3.5 (ii)] for char$(k) \neq 0$) we know that $\mathfrak{P} \supseteq I \cap \mathfrak{Q}$, where $\mathfrak{Q}$ is the defining ideal of the singular locus of J. If $\mathfrak{P} \supseteq I$, there is an associated prime ideal $\mathfrak{P}'$ of I such that $\mathfrak{P} \supseteq \mathfrak{P}'$. Since I is a reduced ideal with height I = height $J + r$, the local ring $(S/J)_{\mathfrak{P}'}$ is regular. From this it follows that $(S/(J, x))_{\mathfrak{P}'}$ is also regular. Thus, $\mathfrak{P}' \neq \mathfrak{P}$, and we obtain height $\mathfrak{P}/(J, x)$ = height $\mathfrak{P}/I$ + height $I/(J, x) \geq$ height $\mathfrak{P}/\mathfrak{P}' + r - 1 \geq 2$. If $\mathfrak{P} \supseteq \mathfrak{Q}$, we have

$$\text{height}\, \mathfrak{P}/(J, x) \geq \text{height}\, (\mathfrak{Q}, x)/(J, x) = \text{height}\, (\mathfrak{Q}, x)/\mathfrak{Q} + \text{height}\, \mathfrak{Q}/J - 1\,.$$

Since S/J satisfies the condition R_1, height $\mathfrak{Q}/J\geq 2$. We distinguish two cases: in the first case there exists no associated prime ideal of $\mathfrak{Q}$ of minimal height containing $(x_1, \cdots, x_n)$. Then, by the general choice of x, height$(\mathfrak{Q}, x)/\mathfrak{Q}>0$, and thus height $\mathfrak{P}/(J, x)\geq 2$. In the second case such an associated prime ideal $\mathfrak{Q}'$ exists. Then, since $\mathfrak{Q}'\supseteq I$, it follows as above for $\mathfrak{P}$ that height $\mathfrak{Q}'/(J, x)\geq 2$, and this implies that height $\mathfrak{Q}'/J\geq 3$. Hence height $\mathfrak{P}/(J, x)\geq$height $\mathfrak{Q}/J-1=$height $\mathfrak{Q}'/J-1\geq 2$, as required.

Remark 3.4. The assumption that $(J, x_1, \cdots, x_r)$ is a reduced ideal can not be removed in 3.3. For instance, let J be any height 2 homogeneous perfect prime ideal in $R=k[x_0, x_1, x_2, x_3]$ such that R/J is normal and $e(R/J)>1$. We may assume that height$(J, x_1, x_2)=$height $J+2=4$. Since $e(R/P)=1$ for any homogeneous prime ideal P of R with height $P=3$, the ideal (J, x) is never prime for any general linear form x in (x_1, x_2).

Now we are able to prove the first main result of this paper.

Theorem 3.5. *Let I be a height 2 perfect reduced homogeneous ideal I in the polynomial ring $R=k[x_0, \cdots, x_n]$. Assume I is minimally generated by forms of degree $a_1, \cdots, a_r$, $a_1\leq a_2\leq\cdots\leq a_r$, and has degree matrix (u_{ij}). Then the following conditions are equivalent:*

(i) $u_{ii+2}>0$ *for* $i=1, \cdots, r-2$.

(ii) *I can be lifted to a height 2 perfect prime ideal J in $S=k[x_0, \cdots, x_{n+1}]$ such that S/J is a normal domain.*

(iii) *I can be lifted to a height 2 perfect prime ideal.*

(iv) *I can be lifted to a height 2 perfect ideal which contains an irreducible form of degree a_1.*

(v) *I can be lifted to a height 2 perfect ideal which contains a form of degree a_1 and a form of degree a_2 with no common factor.*

Proof. The implications (ii)⇒(iii)⇒(iv)⇒(v) are obvious, (v)⇒(i) follows from 2.1 and 3.1. It remains to show that (i)⇒(ii). First, by 3.1, I can be lifted to a height 2 perfect prime ideal $J'\subset S'=k[x_0, \cdots, x_m]$ for some integer $m\geq n$ such that S'/J' is a normal domain. We may assume that $r=m-n\geq 2$, $R\cong S'/(x_{n+1}, \cdots, x_m)$ and $I\cong(J', x_{n+1}, \cdots, x_m)/(x_{n+1}, \cdots, x_m)$. Then $I'=(J', x_{n+1}, \cdots, x_m)$ is a reduced ideal with height $I'=$height $J'+r$. Applying 3.3 successively, we can find $r-1$ linear forms, say $x_{n+2}, \cdots, x_m$, such that $(J', x_{n+2}, \cdots, x_m)$ is a perfect prime ideal and $S'/(J', x_{n+2}, \cdots, x_m)$ is a normal domain. Let J be the ideal $(J', x_{n+2}, \cdots, x_m)/(x_{n+2}, \cdots, x_m)$ of the ring $S=k[x_0, \cdots, x_{n+1}]=S'/(x_{n+2}, \cdots, x_m)$. Then J is a height 2 perfect prime ideal of S and S/J is normal domain. It is obvious that I can be lifted to J.

Theorem 1.1 is just the equivalence (i)⟺(ii) of 3.5. In particular, we obtain the following result of Chiantini and Orecchia.

Corollary 3.6 (cf. [7]).*A collection X of points in $\boldsymbol{P}^2$ is a hyperplane section* *of a projectively normal curve in $\boldsymbol{P}^3$ if and only if the degree matrix (u_{ij}) of X satisfies the condition $u_{ii+2}>0$.*

Conditions (iv) and (v) of 3.5 can be easier checked than condition (i) in certain situations. Recall that a set X of points in $\boldsymbol{P}^2$ is in uniform position if all subsets of X with the same cardinality have the same Hilbert function, and that the general hyperplane section of any reduced irreducible curve in $\boldsymbol{P}^3$ is a collection of points in uniform position [15]. Since the defining ideal of any collection of points in uniform position in $\boldsymbol{P}^2$ has an irreducible form of the least possible degree [22, Remark 1.2], the equivalence (ii)$\Longleftrightarrow$(iv) implies

Corollary 3.7 (cf. [7]). *Every set of points in uniform position in $\boldsymbol{P}^2$ arises as a hyperplane section of a projectively normal curve in $\boldsymbol{P}^3$.*

Remark 3.8. (1) In spite of 2.1 and the equivalence (i)$\Longleftrightarrow$(v) of 3.5, one may ask whether a height 2 perfect reduced ideal I whose degree matrix satisfies the condition $u_{ii+2}>0$ always contains two forms of least possible degree having no common factor. The answer is negative. Consider, for example, the ideal $I=(x, yv-zu)\cap(y-u, z-v)$ which is generated by the 2-minors of the matrix

$$\begin{pmatrix} x & y & z \\ x & u & v \end{pmatrix}$$

(2) Without the assumption I being a reduced ideal, conditions (i), (iii), (iv), (v) of 3.5 are still equivalent. We do not know whether this assumption can be removed in 1.1 and 3.5.

4. The Hilbert function and multiplicity of height 2 perfect homogeneous prime ideals

In the following we will determine all possible degrees of reduced and irreducible arithmetically Cohen-Macaulay varieties of codimension 2 and their hyperplane sections.

Let $I\subset R=k[x_0, \cdots, x_n]$ be a height 2 perfect homogeneous ideal which is minimally generated by r elements. Let $A=(u_{ij})$ be the degree matrix of I. For convenience we set

$$u_i=u_{ii} \quad \text{and} \quad v_i=u_{ii+1}$$

for all i. By [5, Prop. 1] the multiplicity of R/I is given by:

$$e(R/I)=\sum_{i=1}^{r-1} u_i(v_i+\cdots+v_{r-1}).$$

Note that the integers u_i and v_i completely determine the matrix (u_{ij}), since

for all i, j, s and t we have
(a) $u_{ij}+u_{st}=u_{it}+u_{sj}$.

Conversely, suppose we are given positive integers u_i, v_i for $i=1, \cdots, r-1$, satisfying the conditions
(b) $u_i \geq v_i$,
(c) $u_{i+1} \geq v_i$.
Then these integers determine the degree matrix $A=(u_{ij})$ of a height 2 perfect homogeneous ideal, where we set $u_{ii}=u_i$ and $u_{ii+1}=v_i$ for $i=1, \cdots, r-1$, and where the other coefficients of A are defined via (a). In view of this fact we set

$$e(A)=\sum_{i=1}^{r-1} u_i(v_i+\cdots+v_{r-1}),$$

and

$$i(A)=a_1=v_1+\cdots+v_{r-1}.$$

It is now clear that the possible multiplicities $e(R/I)$ range over all integers $e(A)$ where A is an $r-1$ by r matrix arising from the u_i and v_j described as above. In particular, any integer $d \geq \binom{r}{2}$ occurs as a multiplicity of a height 2 perfect homogeneous ideal which is minimally generated by r elements. To see this one just chooses the $r-1$ by r degree matrix

$$\begin{pmatrix} 1 & 1 & & & \\ & \cdot & \cdot & & \\ & & 1 & 1 & \\ & & & u & 1 \end{pmatrix}$$

where $u=d-\binom{r}{2}+1$. On the other hand it is clear from the formula for $e(A)$ that we always have

$$e(R/I) \geq \binom{r}{2}.$$

The lower bound for $e(R/I)$ becomes sharper if we take into account the initial degree $a_1=v_1+\cdots+v_{r-1}$ of I; cf. [9] and [5].

Lemma 4.1. $e(R/I) \geq \binom{r}{2}+(a_1-r+1)(r+1)$.

Proof. If $r=2$, I is a complete intersection, and hence $e(R/I)=a_1a_2 \geq 1+(a_1-1)3$. If $r>2$, we denote by B the $r-2$ by $r-1$ matrix obtained from A by deleting the first row and column. We have

$$e(R/I)=e(A)=u_1a_1+e(B).$$

By induction we may assume that

$$e(B)\geq\binom{r-1}{2}+(a_1-v_1-r+2)r\,.$$

Since $u_1\geq v_1$, we obtain

$$\begin{aligned}e(A)&\geq v_1a_1+\binom{r-1}{2}+(a_1-v_1-r+2)r\\&=v_1a_1+\binom{r}{2}-r+1+(a_1-r+2)r-v_1r\\&\geq\binom{r}{2}+(a_1-r+1)(r+1)\end{aligned}$$

There is a further constraint for the degree matrix of a height 2 perfect homogeneous *prime* ideal. By Corollary 3.2, we have to add

(d) $v_i+v_{i+1}-u_{i+1}>0$

to the conditions (a)−(c).

Theorem 4.2. *Let E_r denote the set of all positive integers which occur as the multiplicity of height 2 perfect homogeneous prime ideals with r generators. Then*

$$E_3=\{n\in\boldsymbol{N}\mid n\geq 3\}\setminus\{4,6\}\,,$$

and

$$E_r=\left\{n\in\boldsymbol{N}\mid n\geq\binom{r}{2}\right\}\setminus\left\{\binom{r}{2}+1,\cdots,\binom{r}{2}+r-2,\binom{r}{2}+r,\binom{r}{2}+2r\right\},$$

for $r>3$.

The theorem shows that the additional requirement that I be prime gives, for $r>3$, the extra gaps $\binom{r}{2}+1,\cdots,\binom{r}{2}+r-2$, $\binom{r}{2}+r$, $\binom{r}{2}+2r$ in the sequence of possible multiplicities.

Proof of 4.2. Let C denote the set of all $r-1$ by r degree matrices $A=(u_{ij})$ whose elements u_i and v_i satisfy the conditions (b), (c) and (d).

We have $E_r=\{e(A)\colon A\in C\}$. If $i(A)=r-1$, then $v_1=\cdots=v_{r-1}=1$. Using (b), (c) and (d) it is easy to check that A has the following form

$$A(u)=\begin{pmatrix} u & 1 & & & \\ & 1 & 1 & & \\ & & \cdot & \cdot & \\ & & & 1 & 1 \end{pmatrix}$$

for some integer $u>0$. Similarly, if $i(A)=r$, we deduce that A must be one of the following matrices:

$$B_1(u)=\begin{pmatrix} u & 1 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & 2 & 2 \end{pmatrix}$$

$$B_i(u)=\begin{pmatrix} u & 1 & & & & & \\ & \cdot & \cdot & & & & \\ & & 2 & 2 & & & \\ & & & 2 & \cdot & & \\ & & & & \cdot & \cdot & \\ & & & & & 1 & 1 \end{pmatrix}$$

$$B_{r-1}(u)=\begin{pmatrix} u+1 & 2 & & & \\ & 2 & \cdot & & \\ & & \cdot & \cdot & \\ & & & 1 & 1 \end{pmatrix}$$

for some positive integer u, where the dots stand for 1, and where the first number 2 in the matrix $B_i(u)$ $(i=2, \cdots, r-2)$ appears in the $(r-i)$-th row. We have

(1) $e(A(u))=\binom{r}{2}+(u-1)(r-1)$

(2) $e(B_i(u))=\binom{r}{2}+ur+i$, $i=1, \cdots, r-1$.

By 4.1, $i(A)=r-1$ or r if $e(A)<\binom{r}{2}+2(r+1)$. Hence using (1) and (2) we can compute all numbers $\leq\binom{r}{2}+2r$ of E_r and obtain the following values $\binom{r}{2}$, $\binom{r}{2}+r-1$, $\binom{r}{2}+r+1, \cdots, \binom{r}{2}+2r-1$ and, if $r=3$, $\binom{r}{2}+2r=\binom{r}{2}+3(r-1)=9$. Moreover, from (2) we know that E_r contains the numbers $\binom{r}{2}+2r+1, \cdots, \binom{r}{2}+3r-1$. Hence, to complete the proof, we only need to show that E_r contains all integers $\geq\binom{r}{2}+3r$. For this we give a list of degree matrices $A\in C$ such that $e(A)$ covers all arithmetical progressions $a(r+1)+b$ of the integers $\geq\binom{r}{2}+3r$. In the following list, the first column gives the degree matrix A, the second column the number $e(A)$ and all possible numbers u:

$r=3$:

$$\begin{pmatrix} u & 2 & \\ & 2 & 2 \end{pmatrix} \qquad 4u+4, \quad u>1$$

$$\begin{pmatrix} u & 1 & \\ & 3 & 3 \end{pmatrix} \qquad 4u+9, \quad u>0$$

$$\begin{pmatrix} u & 2 & \\ & 3 & 2 \end{pmatrix} \qquad 4u+6, \quad u>1$$

$$\begin{pmatrix} u & 3 & \\ & 3 & 1 \end{pmatrix} \qquad 4u+3, \quad u>2$$

$r=4$:

$$\begin{pmatrix} u & 2 & & \\ & 2 & 1 & \\ & & 2 & 2 \end{pmatrix} \qquad 5u+10, \quad u>1$$

$$\begin{pmatrix} u & 2 & & \\ & 3 & 2 & \\ & & 2 & 1 \end{pmatrix} \qquad 5u+11, \quad u>1$$

$$\begin{pmatrix} u & 1 & & \\ & 2 & 2 & \\ & & 2 & 2 \end{pmatrix} \qquad 5u+12, \quad u>0$$

$$\begin{pmatrix} u & 2 & & \\ & 2 & 2 & \\ & & 2 & 1 \end{pmatrix} \qquad 5u+8, \quad u>1$$

$$\begin{pmatrix} u & 1 & & \\ & 2 & 2 & \\ & & 3 & 2 \end{pmatrix} \qquad 5u+14, \quad u>0$$

$r>4$:

$$\begin{pmatrix} u & 2 & & & \\ & 2 & 2 & & \\ & & 2 & \cdot & \\ & & & \cdot & \cdot \end{pmatrix} \qquad u(r+1)+\binom{r}{2}+r-2, \quad u>1$$

$$\begin{pmatrix} u & 1 & & & & \\ & 2 & 2 & & & \\ & & 2 & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & 2 & 2 \end{pmatrix} \qquad u(r+1)+\binom{r}{2}+2r\,, \quad u>0$$

$$\begin{pmatrix} u & 2 & & & \\ & 2 & \cdot & & \\ & & \cdot & \cdot & \\ & & & 2 & 2 \end{pmatrix} \qquad u(r+1)+\binom{r}{2}+r\,, \quad u>1$$

$$\begin{pmatrix} u & 2 & & & & \\ & 2 & \cdot & & & \\ & & \cdot & \cdot & & \\ & & & 2 & 2 & \\ & & & & 2 & 1 \end{pmatrix} \qquad u(r+1)+\binom{r}{2}+r+1\,, \quad u>1$$

$$\begin{pmatrix} u & 1 & & & & & \\ & \cdot & \cdot & & & & \\ & & 2 & 2 & & & \\ & & & 2 & 2 & & \\ & & & & 2 & \cdot & \\ & & & & & \cdot & \cdot \end{pmatrix} \qquad u(r+1)+\binom{r}{2}+2r-i\,, \quad u>0$$

Here the first number 2 appears in the i-th row, $2\leq i\leq r-3$

$$\begin{pmatrix} u & 1 & & & & \\ & \cdot & \cdot & & & \\ & & \cdot & \cdot & & \\ & & & 2 & 2 & \\ & & & & 2 & 2 \end{pmatrix} \qquad u(r+1)+\binom{r}{2}+r+2\,, \quad u>0$$

The proof of 4.2 is now complete.

As a by-product of the above proof we obtain the following description of the case when I contains a form of degree $r-1$ (the least possible initial degree).

Lemma 4.3. *Let I be a height 2 perfect homogeneous prime ideal in a polynomial ring R over k which is minimally generated by $r\geq 3$ elements. Suppose that I contains a form of degree $r-1$. Then*

$$e(R/I)=\binom{r}{2}+u(r-1)$$

for some integer $u\geq 0$, *and* I *is generated by one form of degree* $r-1$, *and* $r-1$ *forms of degree* $u+r-1$.

From 4.2 we immediately obtain the following upper bound for the minimal number of generators, $\nu(I)$ of I.

Corollary 4.4. *Let* I *be a height 2 perfect homogeneous prime ideal in a polynomial ring* R *over* k *with* $e(R/I)=d$. *Suppose that* $d=\binom{r}{2}+i$, $r\geq 2$ *and* $0\leq i\leq r-1$. *Then* $\nu(I)\leq r-1$ *if* $i\neq 0$, $r-1$ *and* $\nu(I)\leq r$ *if* $i=0$, $r-1$. *Moreover, these bounds are sharp.*

Remark 4.5. For the larger class of height 2 perfect homogeneous (not necessarily prime) ideals of multiplicity $d=\binom{r}{2}+i$, $r\geq 2$ and $0\leq i\leq r-1$, the bound is always $\nu(I)\leq r$. (Compare this result with the main Theorem in [8].

5. The degree matrix of height 3 Gorenstein ideals

Let $I\subset R=k[x_0,\cdots,x_n]$ be a height 3 Gorenstein homogeneous ideal. By the structure theorem of Buchsbaum and Eisenbud [2], there exists an integer $r\geq 1$ such that I is minimally generated by the $2r$-pfaffians of an $2r+1$ by $2r+1$ skew-symmetric matrix (g_{ij}) with homogeneous entries. We denote by p_i the pfaffian of the skew-symmetric matrix which is obtained from (g_{ij}) by deleting the i-th row and i-th column. Then $I=(p_1,\cdots,p_{2r+1})$. Let $a_1,\cdots,a_{2r+1}$ be the degrees of these pfaffians. Then R/I, since it is Gorenstein, has a self-dual free homogeneous R-resolution

$$0\longrightarrow R(-c)\longrightarrow\bigoplus_{i=1}^{2r+1}R(-b_i)\xrightarrow{(g_{ij})}\bigoplus_{j}^{2r+1}R(-a_j)\longrightarrow R\longrightarrow R/I\longrightarrow 0\,.$$

We may assume that

$$a_1\leq a_2\leq\cdots\leq a_{2r+1}\,,$$

and, since the resolution is self-dual, that

$$b_i=c-a_i \quad\text{for}\quad i=1,\cdots,2r+1\,.$$

The matrix (u_{ij}) with $u_{ij}=b_i-a_j=c-a_i-a_j$ for $i,j=1,\cdots,2r+1$ is called the *degree matrix* of I. It is clear that (u_{ij}) is symmetric and that $\deg g_{ij}=u_{ij}$ for all i,j, ($g_{ii}=0$, and so may have any degree, say u_{ii}).

Note that (u_{ij}) satisfies the following conditions:

(a) $u_{ij}\geq u_{st}$ for all $i\leq s$ and $j\leq t$,

(b) $u_{ij}+u_{st}=u_{it}+u_{sj}$,

(c) $u_{ij}>0$ for all i, j with $i+j=2r+3$.

The first two conditions are obvious. Concerning (c), suppose that $u_{i2r+3-i}\leq 0$ for some i. Then, by (a), $u_{st}\leq 0$ for all $s\geq i$ and $t\geq 2r+3-i$, and hence $g_{st}=0$ for the same indices. From this it follows then easily that p_1 is zero, a contradiction.

Conversely, any symmetric matrix (u_{ij}) of integers which satisfies the conditions (a), (b) and (c), is the degree matrix of a height 3 Gorenstein ideal. To simplify notation we set

(1) $\quad u_i=u_{i2r+1-i}\,,\quad i=1, \cdots, r$

(2) $\quad v_i=u_{i2r+2-i}\,,\quad i=1, \cdots, r$

(3) $\quad w_i=u_{i+12r+2-i}\,,\quad i=1, \cdots, r\,.$

Note that the integers u_i and v_i determine all other coefficients of the degree martix (u_{ij}), and that the integers u_i, v_i and w_i are all positive, by (c).

Now let (g_{ij}) be the skew-symmetric matrix with $g_{i2r+1-i}=x^{u_i}$, $g_{i2r+2-i}=y^{v_i}$, and $g_{i+12r+2-i}=u^{w_i}$ for $i=1, \cdots, r$, while $g_{ij}=0$ for all other i, j with $i<j$. Here x, y, z are indeterminates. The ideal I of the $2r$-pfaffians of (g_{ij}) is (x, y, z)-primary. Indeed, $p_1=z^{w_1+\cdots+w_r}$, $p_{r+1}=y^{v_1+\cdots+v_r}+\cdots$ and $p_{2r+1}=x^{u_1+\cdots+u_r}$. By [2], I is a height 3 graded Gorenstein ideal with relation matrix (g_{ij}). Thus I has degree matrix (u_{ij}), and condition (a) implies that $a_1\leq\cdots\leq a_{2r+1}$, where $a_i=\deg p_i$ for $i=1, \cdots, 2r+1$.

Note that

$$a_i=\sum_{j=1}^{i-1}v_j+\sum_{j=1}^{r}w_j\,,\qquad i=1, \cdots, r+1\,,$$

$$a_{r+i}=\sum_{j=1}^{r-i+1}v_j+\sum_{j=r-i+2}^{r}u_j\,,\qquad i=2, \cdots, r+1\,,$$

and that

$$c=\frac{1}{r}\sum_{i=1}^{2r+1}a_i\,.$$

To prove the last equation, we observe that $v_i=c-a_i-a_{2r+2-i}$ for $i=1, \cdots, r$. Adding up these equations, we obtain

$$a_{r+1}=\sum_{i=1}^{r}v_i=rc-\sum_{i=1}^{r}a_i-\sum_{i=r+2}^{2r+1}a_i\,,$$

and the assertion follows.

The aim of this section is to characterize the degree matrix of homogeneous height 3 Gorenstein *prime* ideals; see 1.2 and 1.3.

Lemma 5.1. *Let $I\subset R=k[x_0, \cdots, x_n]$ be a height 3 Gorenstein homogeneous ideal with degree matrix (u_{ij}). Suppose that I contains two forms of degree a_1 and a_2 (the least possible degrees) having no common factor. Then $u_{ij}>0$ for all i, j with $i+j=2r+4$, $i=3, \cdots, r+1$.*

Proof. Without restriction we may assume that the forms are the pfaffians p_1 and p_2 of the skew-symmetric matrix (g_{ij}). If $u_{t,2r+4-t}\leq 0$ for some $t=3, \cdots, r+2$, we have $u_{ij}\leq 0$ and therefore $g_{ij}=0$ for all $i\geq t$ and $j\geq 2r+4-t$. Thus, the $2(r+2-t)$-pfaffian obtained from (g_{ij}) by deleting the first $t-1$ and the last $t-2$ rows and columns is a common factor of p_1 and p_2, a contradiction.

Proof of the necessary part of 1.2 and 1.3. It is obvious that any height 3 homogeneous prime ideal contains two forms of least possible degree having no common factor. Therefore, the degree matrix of any arithmetically Gorenstein reduced irreducible variety of codimension 3 and of all of its hyperplane sections (they have the same degree matrix) satisfies the condition $u_{ij}>0$ for $i+j=2r+4$, $i=3, \cdots, r+1$.

To prove the sufficient part of 1.2 and 1.3 we follow the approach of Section 2 and Section 3.

Lemma 5.2. *Let A be a Gorenstein normal domain. Let (x_{ij}) be a $2r+1$ by $2r+1$ skew-symmetric matrix such that $X=\{x_{ij};\ i+j\leq 2r+4,\ i<j\}$ is a set of indeterminates over A and $x_{ij}\in A$ for $i+j>2r+4$, $i<j$. Let I be the ideal generated by the $2r$-pfaffians of (x_{ij}). Then* height$I=3$, *and $A[X]/I$ is a normal Gorenstein domain.*

Proof. We prove the assertion by induction on r. If $r=1$, the statement is trivial because then the entries of (x_{ij}) consist of algebraically independent elements. If $r>1$, we start with a general observation. Let x_{st} be an arbitrary element of X with $s+t\leq 2r+4$ and $s<t$, and consider the matrix (x'_{ij}) whose entries belong to the ring $A[X, x_{st}^{-1}]$, and are given by

$$x'_{ij}=\begin{cases} x_{st}x_{ij} & \text{if } i=t \text{ or } j=t\ , \\ x_{st}x_{ij}-x_{it}x_{sj}-x_{jt}x_{is} & \text{if both } i, j\neq t\ . \end{cases}$$

It is obvious that the new matrix (x'_{ij}) is skew-symmetric and that its $2r$-pfaffians generate the ideal $IA[X, x_{st}^{-1}]$. Let (y_{ij}) be the $2r-1$ by $2r-1$ submatrix of (x'_{ij}) obtained by deleting the rows s, t and columns s, t. Then (y_{ij}) is also a skew-symmetric matrix. Since the entries of row s and column s of (x'_{ij}) are zero except $x'_{st}=x'_{ts}=x_{st}^2$, every non-vanishing $2r$-pfaffian of (x'_{ij}) is the product of x_{st}^2 with a $2(r-1)$-pfaffian of (y_{ij}). Therefore the $2(r-1)$-pfaffians of (y_{ij}) also generate the ideal $IA[X, x_{st}^{-1}]$. Now we choose x_{st} in the set

$$Z=\begin{cases}\{x_{14}, x_{15}, x_{24}, x_{25}, x_{35}\} & \text{if } r=2\,,\\ \{x_{12r+1}, x_{22r+1}, x_{32r+1}, x_{42r}, x_{42r-1}\} & \text{if } r=3\,,\\ \{x_{12r+1}, x_{22r+1}, x_{32r+1}, x_{42r}, x_{52r-1}\} & \text{if } r>3\,.\end{cases}$$

Let X' denote the set of all elements of X in the rows s and t, and denote by B the ring $A[X', x_{st}^{-1}]$. Then B is a Gorenstein normal domain, $A[X, x_{st}^{-1}]=B[Y]$ and $IA[X, x_{st}^{-1}]=IB[Y]$ where $Y=\{y_{ij}|i+j\leq 2r+4, i<j\}$. Note that Y is a set of algebraically independent elements over B, and that $y_{ij}\in B$ for $i+j>2r+4$, $i<j$. These two facts (with r replaced by $r-1$) remain true for the matrix (y_{ij}) with respect to the indexing of its entries according to their row and column position (which differs from the given indexing which is induced by that of (x'_{ij})). Thus we may apply the induction hypothesis, and conclude that $A[X, x_{st}^{-1}]/IA[X, x_{st}^{-1}]\cong B[Y]/IB[Y]$ is a Gorenstein normal domain, and that $IA[X, x_{st}^{-1}]$ is a prime ideal of height 3. Now the proof follows exactly the line of arguments of the proof of 2.2, hence we ommit it.

In Section 2 we introduced generic properties of ideals. We refer to this notion in the following corollary which is a consequence of 5.2.

Corollary 5.3. *Let (x_{ij}) be an $2r+1$ by $2r+1$ skew-symmetric matrix such that the non-zero entries x_{ij}, $i<j$, form a set X of algebraically independent elements over $\boldsymbol{Z}$ and $x_{ij}\neq 0$ for $i+j\leq 2r+4$. Let I be the ideal of $\boldsymbol{Z}[X]$ generated by the $2r$-pfaffians of (x_{ij}). Then I is a height 3 generically Gorenstein normal prime ideal.*

Combining 5.3 with 2.4 we obtain the following result on specializations of generic height 3 Gorenstein prime ideals.

Corollary 5.4. *Let (g_{ij}) be a $2r+1$ by $2r+1$ skew-symmetric matrix of homogeneous elements of a polynomial ring S over a field k which satisfy the following conditions:*

(i) $g_{ij}\neq 0$ *for* $i+j\leq 2r+4$, $i<j$,
(ii) *The elements* $g_{ij}\neq 0$ *form a regular sequence of* S,
(iii) *The factor ring* $S/\mathfrak{P}$ *is a (normal) Gorenstein domain, where* $\mathfrak{P}$ *is the ideal generated by the elements* g_{ij}.

Suppose that the ideal J generated by the $2r$-pfaffians of (g_{ij}) is homogeneous. Then S/J is a (normal) Gorenstein domain.

From 5.4 we deduce, similarly as in the proof of 3.1, the following result which allows to construct, for a given arithmetically Gorenstein variety $X\subset \boldsymbol{P}^n$ of codimension 3 with the condition $u_{ij}>0$ for $i+j\leq 2r+4$, $i<j$, an arithmetically Gorenstein normal variety Y of codimension 3 in a larger projective space such that X is the intersection of Y with a linear space.

Lemma 5.5. *Let I be a height 3 Gorenstein ideal in $R=k[x_0, \cdots, x_n]$.*

If $r=1$ or $r\geq 2$ and its degree matrix (u_{ij}) satisfies the condition $u_{ij}>0$ for $i+j=2r+4$, $i=3, \cdots, r+1$, then I can be lifted to a height 3 Gorenstein prime ideal $J\subset S=k[x_0, \cdots, x_m]$ for some integer $m>n$ such that S/J is a normal domain.

Now we are able to complete the proofs of 1.2 and 1.3.

Proof of the sufficient part of 1.2. Let (u_{ij}) be the degree matrix a height 3 Gorenstein homogeneous ideal in some polynomial ring R over k. Suppose that $u_{ij}>0$ for $i+j=2r+4$, $i=3, \cdots, r+1$. By 5.5 we can lift I to a height 3 Gorenstein prime ideal J in a polynomial ring $S=k[x_0, \cdots, x_m]$ such that S/J is a normal domain. Let $Y\subset \boldsymbol{P}^m$ be the arithmetically Cohen-Macaulay normal variety defined by J. Using Bertini's theorem on hyperplane sections of normal varieties [11, Theorem 5.2] we can find a linear subspace $\boldsymbol{P}^4$ of $\boldsymbol{P}^m$ such that $X=Y\cap \boldsymbol{P}^4$ is a smooth arithmetically Gorenstein curve. Of course, X has the same degree matrix (u_{ij}) as Y.

Proof of the sufficient part of 1.3. Let $I\subset R=k[x_0, \cdots, x_n]$ be the reduced defining ideal of X. Suppose that the degree matrix of X satisfies the condition $u_{ij}>0$ for $i+j=2r+4$. By 5.5, I can be lifted to a height 3 Gorenstein prime ideal $J'\subset S'=k[x_0, \cdots, x_m]$ for some integer $m\geq n$ such that S'/J' is a normal domain. We may assume that $r=m-n\geq 2$, $R\cong S'/(x_{n+1}, \cdots, x_m)$ and $I\cong(J', x_{n+1}, \cdots, x_m)/(x_{n+1}, \cdots, x_m)$. Then $I'=(J', x_{n+1}, \cdots, x_m)$ is a reduced ideal with heightI'=height$J'+r$. Applying 3.5 successively, we can find $r-1$ linear forms, say $x_{n+2}, \cdots, x_m$, such that $(J', x_{n+2}, \cdots, x_m)$ is a Gorenstein prime ideal and $S'/(J', x_{n+2}, \cdots, x_m)$ is a normal domain. Let J be the ideal $(J', x_{n+2}, \cdots, x_m)/(x_{n+2}, \cdots, x_m)$ of the ring $S=k[x_0, \cdots, x_{n+1}]=S'/(x_{n+2}, \cdots, x_m)$. Then J is a height 3 Gorenstein prime ideal and S/J is a normal domain. Let $Y\subset \boldsymbol{P}^{n+1}$ be the arithmetically Gorenstein normal variety defined by the ideal $J$and H the hyperplane $x_{n+1}=0$. It is obvious that $X=Y\cap H$.

From Theorem 1.3 and Lemma 5.1 we immediately obtain the following result.

Corollary 5.6. Let $X\subset \boldsymbol{P}^n$ be a reduced arithmetically Gorenstein variety of codimension 3. Let H be a hypersurface of least possible degree containing X. If H is irreducible, then X is the hyperplane section of a reduced irreducible arithmetically Gorenstein normal variety of codimension 3 in $\boldsymbol{P}^n$.

6. The multiplicity of height 3 perfect homogeneous Gorenstein ideals

The purpose of this section is to give a compact formula for the multiplicity of a height 3 homogeneous Gorenstein ideal in terms of its degree matrix. We begin with the generic situation: let k be a field, R the polynomial ring over k in the indeterminates x_{ij}, $1 \leq i < j \leq 2r+1$, and let $A = (g_{ij})$ be an $2r+1$ by $2r+1$ skew-symmetic matrix with $g_{ij} = x_{ij}$ for all $1 \leq i < j \leq 2r+1$. Given $i_1 < i_2 < \cdots < i_j$, $j \leq 2r+1$, we let $A_{i_1 i_2 \cdots i_j}$ be the skew-symmetric matrix which is obtained from A by deleting the i_1-th, i_2-th, $\cdots$, i_j-th column and row of A, and denote by $p_{i_1 i_2 \cdots i_j}$ the pfaffian of $A_{i_1 i_2 \cdots i_j}$.

We intend to compute a Gröbner basis of $I = (p_1, \cdots, p_{2r+1})$ with respect to a suitable order of the monomials. First we order the indeterminates according to their position in A row by row from the right to the left:

$$x_{12r+1} > x_{12r} > \cdots > x_{12} > x_{22r+1} > \cdots > x_{23} > x_{32r+1} > \cdots ,$$

and extend this order to all monomials by the reverse lexicographical order.

Theorem 6.1. *The generators $p_1, \cdots, p_{2r+1}$ form a Gröbner basis of I.*

Proof. We denote by f^* the leading form of a polynomial f. Let us first compute p_i^*. Let $i \neq 1, 2r+1$, then

$$p_i = \sum_{j=2}^{i-1} (-1)^j x_{1j} p_{1ij} + \sum_{j=i+1}^{2r+1} (-1)^{j+1} x_{1j} p_{1ij} .$$

Since non of the p_{1ij} contains the variable x_{12r+1} which is the largest in the given order, it follows from this expansion that

$$p_i^* = (\pm x_{12r+1} p_{1i2r+1})^* = \pm x_{12r+1} p^*_{1i2r+1} ,$$

and hence

$$(p_2^*, \cdots, p^*_{2r}) = x_{12r+1}(q_2^*, \cdots, q^*_{2r}) ,$$

where $q_i = p_{1i2r+1}$ for $i = 2, \cdots, r$.

For $i=1$, we have $p_1 = \sum_{i=3}^{2r+1} (-1)^i x_{2i} p_{12i}$, and so $p_1^* = \pm x_{22r+1} p^*_{22r+1}$. It follows by induction that

$$p_1^* = \pm x_{22r+1} x_{32r} \cdots x_{i2r+3-i} \cdots x_{rr+2} .$$

Similarly

$$p^*_{2r+1} = \pm x_{12r} x_{22r-1} \cdots x_{i2r+1-i} \cdots x_{rr+1} .$$

Let $J = (p_1^*, \cdots, p^*_{2r+1})$. We conclude that

$$\text{(1)} \qquad J = \Big(\prod_{i=1}^{r} x_{i+12r+2-i}, \prod_{i=1}^{r} x_{i2r+1-i}, x_{12r+1}\Big) \cap (q_2^*, \cdots, q^*_{2r}) .$$

We want to prove that $J = I^*$. The inclusion $J \subset I^*$ is obvious. To prove the other inclusion notice that J is reduced and equidimensional (as may be seen by induction), so that it suffices to show that $e(R/J) = e(R/I^*)$.

In order to compute $e(R/I^*)$ we observe that $e(R/I^*) = e(S)$, where $S=$

R/I. We reduce S modulo a sequence of 1-forms, and obtain an Artinian ring, $\bar{S}$. Then $e(S)=e(\bar{S})=l(\bar{S})$, the length of $\bar{S}$. The degree of the socle of the Gorenstein ring $\bar{S}$ is $2r-2$, and the defining equations of $\bar{S}$ are of degree r. From this and the symmetry of the Hilbert function it follows easily that

$$H_{\bar{S}}(i)=\begin{cases}\binom{i+2}{2} & \text{if } 0\leq i\leq r-1\\ \binom{2r-i}{2} & \text{if } r\leq i\leq 2r-2\,.\end{cases}$$

Therefore $e(S)=2\sum_{i=0}^{r-2}\binom{i+2}{2}+\binom{r+1}{2}=\sum_{i=1}^{r}i^2$.

On the other hand it follows from the presentation of J that

$$e(R/J)=r^2+e(R/(q_2^*,\cdots,q_{2r}^*))\,.$$

By induction we may assume that

$$(q_2^*,\cdots,q_{2r}^*)=(q_2,\cdots,q_{2r})^*\,,$$

and so

$$e(R/(q_2^*,\cdots,q_{2r}^*))=e(R/(q_2,\cdots,q_{2r}))=\sum_{i=1}^{r=1}i^2\,.$$

Thus, indeed, $e(R/I^*)=e(R/J)$.

Quite generally, for an ideal I, one has height$I$=heightI^*; but depthR/I^* may be less than depthR/I. In other words, if I is perfect, I^* needs not to be perfect. In our case we have

Proposition 6.2. *R/I^* has the minimal free homogeneous*

$$0\longrightarrow\bigoplus_{i=r+2}^{2r+1}R(-i)\longrightarrow\bigoplus_{i=r+2}^{2r}R(-i)\oplus R(-r-1)^{2r+1}$$
$$\longrightarrow R(-r)^{2r+1}\longrightarrow R\longrightarrow R/I^*\longrightarrow 0\,.$$

In particular, I^* is perfect, and R/I^* is Gorenstein if and only if $r=1$.

Proof. We proceed by induction on r. The case $r=1$ is trivial. Now let I_r denote the ideal of the pfaffians $p_1,\cdots,p_{2r+1}$ of A, and by I_{r-1} the corresponding ideal for A_{12r+1}. The proof will be based on equation (1) in the proof of 6.1:

$$I_r^*=K\cap I_{r-1}^*\,,$$

where K is generated by a regular sequence m_1, m_2, m_3 with $\deg m_1=\deg m_2=r$ and $\deg m_3=1$. Moreover we have $K+I_{r-1}^*=(I_{r-1}^*, m_3)$. Therefore we get the exact sequence

$$0 \longrightarrow R/I_r{}^* \longrightarrow R/K \oplus R/I^*_{r-1} \longrightarrow R/(I^*_{r-1}, m_3) \longrightarrow 0$$

which yields the exact sequence

$$0 \longrightarrow \mathrm{Tor}_4(k, R/(I^*_{r-1}, m_3)) \longrightarrow \mathrm{Tor}_3(k, R/I_r{}^*) \xrightarrow{(\alpha_1, \alpha_2)}$$

$$\mathrm{Tor}_3(k, R/K) \oplus \mathrm{Tor}_3(k, R/I^*_{r-1}) \xrightarrow{\binom{\beta_1}{\beta_2}} \mathrm{Tor}_3(k, R/(I^*_{r-1}, m_3))$$

We observe that β_2 is injective: indeed, the exact sequence

$$0 \longrightarrow R/I^*_{r-1} \xrightarrow{m_3} R/I^*_{r-1} \longrightarrow R/(I^*_{r-1}, m_3) \longrightarrow 0$$

(m_3 is R/I^*_{r-1}-regular) gives the exact sequence

$$\mathrm{Tor}_3(k, R/I^*_{r-1}) \xrightarrow{m_3} \mathrm{Tor}_3(k, R/I^*_{r-1}) \xrightarrow{\beta_2} \mathrm{Tor}_3(k, R/(I^*_{r-1}, m_3)),$$

and multiplication by m_3 on $\mathrm{Tor}(k, \_)$ is the zero-map.

Next observe that $\beta_1 = 0$: we have

$$\mathrm{Tor}_3(k, R/K)_i = \begin{cases} 0 & \text{for } i \neq 2r+1, \\ k & \text{for } i = 2r+1. \end{cases}$$

On the other hand, using the induction hypothesis, we see that $\mathrm{Tor}_3(k, R/(I^*_{r-1}, m_3))_{2r+1} = 0$, and thus the conclusion follows.

Now $0 = \beta_1\alpha_1 + \beta_2\alpha_2 = \beta_2\alpha_2$, and since β_2 is injective, we see that $\alpha_2 = 0$. Therefore we obtain the exact sequence

$$0 \longrightarrow \mathrm{Tor}_4(k, R/(I^*_{r-1}, m_3)) \longrightarrow \mathrm{Tor}_3(k, R/I_r{}^*) \xrightarrow{\alpha_1} \mathrm{Tor}_3(k, R/K) \longrightarrow 0.$$

Notice that $\mathrm{Tor}_4(k, R/(I^*_{r-1}, m_3)) \cong \mathrm{Tor}_3(k, R/I^*_{r-1})(-1)$; therefore, the induction hypothesis implies

$$\mathrm{Tor}_3(k, R/I_r{}^*)_i = \begin{cases} k & \text{for } i = r+2, \cdots, 2r+1 \\ 0 & \text{otherwise}. \end{cases}$$

Finally, to compute the shifts in the second step of the resolution we note that R/I_r and $R/I_r{}^*$ have the same Hilbert series. In both cases we use their resolutions to compute it:

$$H_{R/I_r}(t) = (1-(2r+1)t^r + (2r+1)t^{r+1} - t^{2r+1})/(1-t)^3,$$

and

$$H_{R/I_r^*}(t)=(1-(2r+1)t^r+X(t)-\sum_{i=r+2}^{2r+1}t^i)/(1-t)^3 .$$

Here $X(t)$ stands for the polynomial corresponding to the yet unknown part of the resolution of R/I_r^*.

The comparison shows that

$$X(t)=(2r+1)t^{r+1}+\sum_{i=r+2}^{2r}t^i .$$

This completes the proof.

Now we come to the main application of Theorem 6.1.

Proposition 6.3. *With the notations of* (1) *in Section* 5 *we have*

$$e(R/I)=\sum_{i=1}^{r}v_i(\sum_{j=i}^{r}u_j)(\sum_{j=i}^{r}w_j) .$$

Proof. The multiplicity $e(R/I)$ is a polynomial function in the entries u_{ij} of the degree matrix of I. This can be seen if one uses the minimal R-free resolution of R/I in order to compute $e(R/I)$. Thus, if we give a (polynomial) formula for $e(R/I)$ in terms of the u_{ij} or (u_i, v_i, w_i) under the assumption that all $u_{ij}>0$, this formula is also valid without this assumption. Without loss of generality we may therefore assume that all $u_{ij}>0$.

Consider the (flat) homomorphism φ: $k[X]\rightarrow k[X]$, $\varphi(x_{ij})=x_{ij}^{u_{ij}}$, $X=\{x_{ij}: 1\leq i<j\leq 2r+1\}$. Let $B=(x_{ij}^{u_{ij}})$ be the image of A under φ, and let p_i be the pfaffian of A_i and q_i the pfaffian of B_i. Then $q_i=\varphi(p_i)$ for $i=1,\cdots,2r+1$.

We claim that $q_1,\cdots,q_{2r+1}$ is a Gröbner basis of $I=(q_1,\cdots,q_{2r+1})$. In order to prove the claim we employ the following well-known criterion: let $\mathfrak{R}$ be a generating set of homoeneous relations of $q_1^*,\cdots,q_{2r+1}^*$. Then $q_1,\cdots,q_{2r+1}$ is a Gröbner basis of I, if any element of $\mathfrak{R}$ can be lifted to a relation of $q_1,\cdots,q_{2r+1}$.

A relation $\sum_i a_i q_i^*$ is called *homogeneous* if

(a) a_i is a monomial for all i,
(b) $\deg a_i q_i^*=\deg a_j q_j^*$ for all i,j with $a_i, a_j\neq 0$.

Here the degree of a monomial is its exponent. The common degree in (b) is called the *degree of the relation* $a=(a_1,\cdots,a_{2r+1})$. We say that the relation $a=(a_1,\cdots,a_{2r+1})$ of $q_1^*,\cdots,q_{2r+1}^*$ can be lifted to relation of $(q_1,\cdots,q_{2r+1})$ if there exist $h_i\in R$, $i=1,\cdots,2r+1$, such that:

(1) $\sum_i h_i q_i=0$,

(2) $h_i^*=a_i$ for all i with $a_i\neq 0$,

(3) $\deg h_i^* \deg q_i^* > \deg a$ for all i with $a_i = 0$ ($>$in the reverse lexicographical order).

Note that for any $f \in R$ which is homogeneous in the usual sense we have $\varphi(f)^* = \varphi(f^*)$; in particular, $q_i^* = \varphi(p_i^*)$ for $i = 1, \cdots, 2r+1$.

Since φ is flat we conclude that there exists a generating set $\mathfrak{R}$ of homogeneous relations of $q_1^*, \cdots, q_{2r+1}^*$ such that for each $(a_i) \in \mathfrak{R}$ there exists a homogeneous relation (b_i) of $p_1^*, \cdots, p_{2r+1}^*$ with $\varphi(b_i) = a_i$. Since $(p_1^*, \cdots, p_{2r+1}^*) = (p_1, \cdots, p_{2r+1})^*$, we can lift each (b_i) to (h_i) satisfying (1), (2) and (3). Then $(\varphi(h_1), \cdots, \varphi(h_{2r+1}))$ is the required lifting for $(a_1, \cdots, a_{2r+1})$. This proves the claim.

Now for I^* we have a decompostion corresponding to the one of J in the proof of 6.1:

$$I_r^* = \Big(\prod_{i=1}^{r} x_{i+1\,2r+2-i}^{w_i}, \prod_{i=1}^{r} x_{i\,2r+1-i}^{u_i}, x_{1\,2r+1}^{v_1}\Big) \cap I_{r-1}^* .$$

Both components have codim 3, and so

$$e(R/I) = e(R/I_r^*) = v_1\Big(\sum_{i=1}^{r} u_i\Big)\Big(\sum_{i=1}^{r} w_i\Big) + e(R/I_{r-1}^*) .$$

Hence the formula follows by induction on r.

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