J. Math. Kyoto Univ. (JMKYAZ)
34-1 (1994) 41-46

Bicharacteristic curves and wellposedness for hyperbolic equations with non-involutive multiple characteristics

BY

Nobuhisa IWASAKI

Let us consider a third order hyperbolic operator as

(1.1) $$P(x, D)=P_3+P_2+P_1+P_0 .$$

We assume the principal part P_3 is the product of three first order operators a_i, where the symbols of a_i are given by

$$a_i=(\xi_0-\Lambda_i)$$

with $\Lambda_i(x, \xi')$ real valued pseudodifferential operators of order 1 in x'. We also assume the characteristics of a_i crosses each other non-involutively, that is,

$$\{a_i, a_j\}\neq 0 , \qquad \text{for} \quad i\neq j \quad \text{on} \quad a_i=a_j .$$

Then we can divide the characteristics of the hyperbolic operator P into two types in geometric view points. We indicate the types by sign(a_0, a_1, a_2) which is defined as

$$\text{sign}(a_0, a_1, a_2)=+1$$

if $\{a_0, a_1\}$, $\{a_1, a_2\}$, $\{a_2, a_0\}$ have the same sign $+$ or $-$, and

$$\text{sign}(a_0, a_1, a_2)=-1$$

otherwise.

Here we say these types correspond to the different types of operators for the well posed Cauchy problem. Ivriĭ-Petkov said the lower terms should be restricted in general. In our case it is said that the subprincipal symbol $p_2{}^s$ is spanned by a_i's, that is,

(1.2) $$p_2{}^s=b_0a_0+b_1a_1+b_2a_2 .$$

So we assume this condition, also. The result is that the case with the sign

Received April 1, 1992

$=-1$ is the extension type of effectively hyperbolic operators, and the other case with the sign $=+1$ requires some more restrictions to the subprincipal part $p_2{}^s$. Let us put for $p_2{}^s=\sum_{j=0}^2 b_j a_j$

$$c_0=b_0/\{a_1, a_2\}, \quad c_1=b_1/\{a_2, a_0\}, \quad c_2=b_2/\{a_0, a_1\}. \tag{1.3}$$

In general, we say a higher order hyperbolic operator P has triple characteristics crossing non-involutively with the sign $+1$ (or -1) at $X=(x, \xi)$ if its symbol is denoted as the above third order hyperbolic operator with the sign $+1$ (or -1) times an elliptic symbol mod smooth kernels at a conic neighborhood of $X=(x, \xi)$.

Theorem (main). *If the Cauchy problem for a higher order hyperbolic operator P were well posed, then on the triple characteristics $\{a_0=a_1=a_2=0\}$ crossing non-involutively with the sign $+1$, its subprincipal symbol $p_2{}^s$ should satisfy the condition* (1.2) (*Ivriĭ-Petkov*) *and that*

$$ic_j \in \boldsymbol{Z}+1/2 \quad \textit{for some } j \tag{1.4}$$

or

$$\operatorname{Im}\sum_{j=0}^{2} c_j=0, \tag{1.5}$$

where we defined the normalized subprincipal symbols c_j on the double characteristics by (1.3) *near the triple characteristics.*

At first we note the geometrical behavior of bicharacteristics for P of (1.1). In case of the sign -1, any bicharacteristic polygon goes out from the singular characteristics of P after turnning finite times at the double characteristics. On the other hand, in case of the sign $+1$, there exist bicharacteristic polygons which turn infinitely many times at the double characteristics and converge to (come out from) the triple charateristics, namely, we get the following lemma.

Lemma 1.1. *We consider the case of sign $+1$ for P of* (1.1).

1) *There exist bicharacteristics γ_j ($j \in \boldsymbol{N}$) of a_k ($k \in \{0, 1, 2\}$, $k=j$ mod 3) and cross points X_j on the double characteristics $\{a_k \neq a_l=a_m=0\}$ ($l, m=j+1, j+2$ mod 3; $l, m \in \{0, 1, 2\}$) such that*

$$\gamma_j \cap \gamma_{j+1}=\{X_j\}$$

and

$$\lim_{j\to\infty} X_j=X_\infty \in \{a_0=a_1=a_2=0\}.$$

2) *We may parametrize γ_j by x_0 because $\{a_k, x_0\}>0$. So, there exist increasing* (*or decreasing resp.*) *numbers τ_j converging to $\tau_\infty=\lim_{j\to\infty}\tau_j$ such that*

$$\gamma_j(\tau_j)=\gamma_{j+1}(\tau_j)=X_j\ ,$$

and $\gamma_j(x_0)$ *never touch the singular (double or triple) characteristics on* $x_0>\tau_j(x_0<\tau_j$, *resp.).*

3) *We define a bicharacteristic polygon* $\gamma(\tau)$ *as*

$$\gamma(\tau)=\gamma_{j+1}(\tau) \qquad for \qquad \tau_j<\tau<\tau_{j+1}\ . \quad (\tau_j>\tau>\tau_{j+1},\ resp.)$$

Then

$$\lim_{\tau\to\tau_\infty}\gamma(\tau)=X_\infty\in\{a_0=a_1=a_2=0\}\ .$$

For simplicity, we may assume the wellposedness means that there exists an open domain Ω such that for any $t\leq T(<+\infty)$ for any $f\in C_0^\infty(\Omega)$ there exists a solution $u\in C_0^\infty(\Omega)$ of

$$Pu=f \qquad \text{on} \qquad \Omega_t \qquad (\text{unique on } \Omega_t)\ ,$$

where

$$\Omega_t=\{x\in\Omega;\ x_0\leq t\}\ ,$$

so that, for any seminorm $\|\ \|_s$ of $C_0^\infty(\Omega_t)$ there exists another seminorm $\|\ \|_r$ (r and C_s are independent of t) such that

$$\|u\|_s\leq C_s\|f\|_r\ . \tag{1.6}$$

We know two following properties about the construction of parametrices for the heigher order hyperbolic operator P.

Let Y_0 be a point of the simple characteristics of P.

Lemma 1.2. 1) *There exists a parametrix by means of the Fourier integral operators in* (x',ξ') *with parameter* x_0 *for* $Pu=0$, *which satisfies a given initial asymptotic amplitude supported on a conic neighborhood of* $Y'=(y',\eta')$.

2) *For any fixed* x_0 *in a neighborhood of* y_0, *the parametrix has an asymptotic expansion supported on a conic neighborhood of the bicharacteristc curve through* Y_0.

3) *The leading term of asymptotic expansion at* x_0 *is the pull back of the leading term of that at* y_0 *time the non-degenerate symbol. So, if the given leading term at* Y_0 *is non-degenrate, then the leading term of parametrix is also non-degenarate.*

4) *The parametrix with the same initial asymptotic expansion is unique modulo smooth kernels.*

(See any book or papers about the Fourier integral operators.)

Let us consider the parametrix near the double characteristics consisted with two characteristic surfaces $a_0=0$, $a_1=0$ crossing non-involutively $\{a_0, a_1\}\neq 0$. We can say the following lemma from N. Iwasaki [3]. $Y_0\in\{a_0=a_1=0\}$. γ_0 and γ_1 are bicharacteristic curves of a_0 and a_1, resp., crossing at Y_0. We may assume that near Y_0, the operator P is denoted as

$$P=RQ \quad \text{mod smooth kernels},$$

where R is non degenerate at Y_0,

$$Q=a_0a_1+b,$$

and b is an operator of order 1.

We denote γ_j by $\gamma_{j-}(\gamma_{j+})$ at $x_0<y_0$ (at $x_0>y_0$). Let b^s be the subprincipal symbol of Q, and put

$$c^s=b^s/\{a_0, a_1\}.$$

Lemma 1.3. *Let us assume that*

$$ic^s\notin \boldsymbol{Z}+1/2.$$

1) *For given asymptotic expansion at $x_0<y_0$ supported on a small conic neighborhood of $\gamma_{0-}(x_0)$, which never touch γ_1, there exists a parametrix for $Pu=0$ modulo smooth kernel such that the parametrix is smooth on γ_{1-} outside of a small conic neighborhood of Y_0.*

2) *The parametrix has asymptotic expansion on $\gamma_{j+}(x_0>y_0)$ with supports on conic neighborhoods of $\gamma_{j+}(x_0)$. The leading terms are the pull backs of the given leading term on γ_{0-} time non-degenerate terms.*

3) *Let the order of the given asymptotic expansion on γ_{0-} be σ. Then the order of that on γ_{0+} is σ also and the order of that on γ_{1+} is $\sigma-Imc^s(Y_0)$,*

Now, we can prove Theorem. Let us fix the indices s, r of the estimate (1.6) from the wellposedness. We assume for (1.3)

$$\operatorname{Im}\sum_{j=0}^{2} c_j<0.$$

Then we can choose an integer N as large as

$$ic_j\notin \boldsymbol{Z}+1/2 \quad \text{for all} \quad j \tag{1.7}$$

$$-(\operatorname{Im}\sum_{j=0}^{2} c_j)N>LN>r-s, \tag{1.8}$$

on a conic neighborhood of Y_0 of triple characteristics. We can find the bicharacteristic polygon $\gamma(x_0)$ in a neighborhood of Y_0' as lemma 1.1 which starts at $x_0=\tau_0$ and turns
at $x_0=\tau_j(j=1, 2, \cdots)$. We may assume (1.7) and (1.8) hold near $\gamma(x_0)$. Let us give any asymptotic expansion with non-degenerate leading term at $\gamma(\tau_0)$ with

order $-r$, supported in a sufficiently small conic neighborhood of $\gamma(\tau_0)$. We can succeed the constructions of parametrices as on the simple characteristics from τ_0 to τ_1, near the bifurcated characteristics at $x_0=\tau_1$, on the simple characteristics from τ_1 to τ_2, and so on. Then we get the parametrix which has an asymptotic expansion on $\tau_3<x_0<\tau_4$ with the order $>-r+L$. After N rounds, we get the order of asymptotic expansion greater than

$$-r+LN>-s\,.$$

The singularity of the parametrix we construct is included in a propagation cone of P because the parametrix is smooth near $\gamma_j(x_0)$ for $x_0<\tau_j$ outside of a small conic neighborhood of γ. The existence of such a parametrix contrudict the estimate (1.6) from the wellposedness because we can give a series of solutions of $Pu_k=f_k$, which violetes the estimate (1.6), by using such a parametrix.

In case that

$$\mathrm{Im}\sum_{j=0}^{2}c_j>0\,,$$

we choose a polygon in $x_0>y_0$ and construct a parametrix starting from $x_0<\tau_{3N}<\tau_{3N-1}<\tau_{3N-2,\cdots})$ and coming to $x_0=\tau_0$.

Let us consider a special type such that the principal part of P is a product of first order operators a_j $(j=1,\cdots,m)$ with simple characteristics. We assume that da_j, $j=1,\cdots,m$ are linearly independent vectors. In this case, the Ivriĭ-Petkov result proves that the lower terms P_k (order k) have to be spanned by $(m-2k)$-th products of $\{a_j\}$. We say, here, such a lower term satisfies the natural limit conditions.

Corollary. *If P is wellposed for any lower term satisfying the natural limit conditions, then any triple set of $\{a_j\}$ has the sign -1.*

This means that we can introduce an order to the set $\{a_j\}$ such that

$$a_k\leq a_j \quad \text{if} \quad \{a_k,a_j\}\leq 0\,.$$

Moreover, when taking localizatios of P, it is equivalent to the condition that the propagation cones never touch the singular part of characteristics. T. Nishitani expects that the operators satisfying this condition might be the extensive type of effectively hyperbolic operators to the case of multiple characteristics.

DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY

References

[1] V. Ya Ivriĭ and Y. M. Petkov, Necessary conditions for the Cauchy problem for non-strictly hyperbolic equations to be well-posed, Uspehi Mat. Nauk, **29** (1974) 3-70.

[2] N. Iwasaki, The Cauchy problem for hyperbolic equations with double characteristcs, Publ. RIMS, Kyoto Univ., **19** (1983), 927-942.

[3] N. Iwasaki, A paramerix at noninvolutively crossing characteristic points, J. Math. Kyoto Univ. **31-4** (1991), 899-908.