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On numerical invariants of Noetherian local rings of characteristic p

By

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1. Introduction

Throughout this paper, all rings are commutative with identity. Let R be a Noetherian ring of characteristic p, where p is a prime number. For an ideal I of R, we denote by I^* the tight closure of I (see Definition 3.1). R is called weakly F-regular when every ideal I of R is tightly closed, that is $I^*=I$. The concept of tight closure and their fundamental properties were given by M. Hochster and C. Huneke in [4] and [5]. They proved that regular rings are weakly F-regular and that weakly F-regular rings are normal (cf. [5, § 4, § 5]).

Now, we introduce the following two invariants for a local ring $(R, \mathfrak{m})$ of characteristic p.

$$t(R):=\sup l_R(I^*/I)\,, \qquad \text{where } I \text{ runs all } \mathfrak{m}\text{-primary ideals}.$$

$$t_0(R):=\sup l_R(Q^*/Q)\,, \qquad \text{where } Q \text{ runs all parameter ideals of } R\,.$$

In this article, we will discuss the following:

Problems. (1) Estimate the values $t(R)$ and $t_0(R)$.

(2) When $t(R)$ (respectively $t_0(R)$) is finite, what can one say about the ring R?

Hochster and Huneck proved that $t(R)=0$ if and only if R is weakly F-regular (cf. [5, (4.16) Proposition]) and that a Gorenstein local ring with $t_0(R)=0$ is weakly F-regular (cf. [4, Theorem 5.1]). A local ring R with $t_0(R)=0$ is called F-rational (cf. [2]). They also proved that $I\subset I^*\subset \overline{I}$ and $\overline{I^n}\subset I^*$ if I is generated by n-elements (the Briançon-Skoda theorem) in [5], where $\overline{I}$ is the integral closure of I. We also introduce the following two invariants, which will be useful to inverstigate $t(R)$ and $t_0(R)$.

$$i(R):=\sup l_R(\overline{I}/I)\,, \qquad \text{where } I \text{ runs all } \mathfrak{m}\text{-primary ideals}.$$

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$$i_0(R) := \sup l_R(\bar{Q}/Q)\,,\quad \text{where } Q \text{ runs all parameter ideals of } R\,.$$

First of all, in section 2 we shall investigate these values $i(R)$ and $i_0(R)$, and we shall give

Theorem 1.1. *Let $(R, \mathfrak{m})$ be a d-dimensional Noetherian local ring (which is not necessarily of characteristic p). We set $N=\sqrt{(0)}$, $A=R/N$ and denote by $\bar{A}$ the integral closure of A in its total quotient ring. Then we have*

$$i(R)=i_0(R)=\begin{cases} l_R(R)-1 & d=0 \\ l_A(\bar{A}/A)+l_R(N) & d=1 \\ \infty & d\geq 2 \end{cases}$$

In section 3, we shall prove that $t(R)=t_0(R)=i_0(R)$ when $\dim R\leq 1$ (see Proposition 3.3). When $\dim R\geq 2$, however, the behavior of $t(R)$ and $t_0(R)$ is rather complicated. We shall give several examples in these cases. Rings in these examples are not normal with $t_0(R)<\infty$. But, if we put a restriction on the depth$R$, we get the following theorem, which will be proved in section 4.

Theorem 1.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of characteristic p. If* depth$R\geq 2$ *and $t_0(R)<\infty$, then R is normal.*

Section 4 is also devoted to the study of the compatibility of taking tight closure with localization. It is shown in [7] that any localization of F-rational Cohen-Macaulay local ring is again F-rational. We shall generalize this result to the case that R has F.L.C. (see Theorem 4.1). Furthermore, we shall prove the following:

Theorem 1.3. *Let $(R, \mathfrak{m})$ be a complete local ring of characteristic p. If R is equi-dimnsional and $t_0(R)<\infty$, then*

(1) *R has F.L.C.*

(2) *$t_0(R_{\mathfrak{p}})=0$ for any $\mathfrak{p}\in \mathrm{Spec}R\backslash\{\mathfrak{m}\}$.*

Finally in section 5, we shall treat the tight closure in polynomial extensions.

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2. Results on the integral closure

Throughout this section, $(R, \mathfrak{m})$ means a Noetherian local ring. We denote by N the nilradical of R and by A the factor ring R/N. In this section we have no restriction on the characteristic of R. The integral closure of an ideal I and the integral closure of a ring R in its total quotient ring will be denoted by $\bar{I}$ and $\bar{R}$, respectively. Our purpose is to investigate the behavior of length $l_R(\bar{I}/I)$, when I runs all $\mathfrak{m}$-primary ideals or all parameter ideals of R. From now on we denote by $\mathcal{I}(R)$ the set of all $\mathfrak{m}$-pramary ideals and by $\mathcal{I}_0(R)$ the set of all parameter ideals.

Definition 2.1. *We set* $i(R)=\sup_{I\in\mathcal{I}(R)} l_R(\bar{I}/I)$ *and* $i_0(R)=\sup_{Q\in\mathcal{I}_0(R)} l_R(\bar{Q}/Q)$.

We begin with an easy but useful

Lemma 2.2. (1) $l_R(\bar{I}/I)=l_A(\overline{IA}/IA)+l_R(N/I\cap N)$ *for any* $I\in\mathcal{I}(R)$.

(2) $i_0(A)\leq i_0(R)$ *and* $i(A)\leq i(R)$.

Proof. (1) Take $I\in\mathcal{I}(R)$ and set $J=IA$. We have that $\bar{I}\supset N$ and $\bar{J}=\bar{I}/N$. Hence we get the following cmmutative diagram with exact rows.

$$\begin{array}{ccccccccc} 0 & \to & I\cap N & \to & I & \to & J & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & N & \to & \bar{I} & \to & \bar{J} & \to & 0\,. \end{array}$$

By the snake lemma, we get the assertion.

(2) For any $J\in\mathcal{I}_0(A)$, there exists $I\in\mathcal{I}_0(R)$ such that $J=IA$. By (1), we have $l_R(\bar{I}/I)\geq l_A(\bar{J}/J)$ and $i_0(A)\leq i_0(R)$. Similary we have $i(A)\leq i(R)$.

When $\dim R=1$, we can calculate $i(R)$ and $i_0(R)$ as follows:

Lemma 2.3. *If* $\dim R=1$, *then* $i(R)=i_0(R)=l_A(\bar{A}/A)+l_R(N)$.

Proof. (Step. 1) The case where $l_A(\bar{A}/A)+l_R(N)$ is infinite. First we assume $l_R(N)=\infty$. Take $a\in R$ such that $(a)\in\mathcal{I}_0(R)$. Then $(a^n)\cap N=a^nN$, since a^n is an A-regular element for any $n>0$. Hence $a^{n+1}N\neq a^nN$ for all $n>0$. By Lemma 2.2, we get

$$l_R(\overline{(a^n)}/(a^n))\geq l_R(N/(a^n)\cap N)=l_R(N/a^nN)\geq n\,.$$

Therefore we have $i(R)\geq i_0(R)=\infty$.

Next we assume $l_A(\bar{A}/A)=\infty$. Then $\bar{A}$ is not finitely generated as an A-algebra. We can choose $a_1, a_2, \cdots, a_n, \cdots\in\bar{A}$ such that $A_n\neq A_{n+1}$ for all $n\geq 0$, where $A_n=A[a_1, a_2, \cdots, a_n]$. Then there exists $c_n\in[A:_AA_n]$ such that $(c_n)\in\mathcal{I}_0(A)$ for each n. Since $\overline{(c_n)}=c_n\bar{A}\cap A\supset c_nA_n\supset(c_n)$ and since c_n is A_n-regular,

$$l_A(\overline{(c_n)}/(c_n))\geq l_A(c_nA_n/(c_n))=l_A(A_n/A)\geq n \text{ for all } n\geq 0\,.$$

Hence we get $i(R)\geq i_0(R)\geq i_0(A)=\infty$.

(Step. 2) The case where both $l_R(N)$ and $l_A(\bar{A}/A)$ are finite. We can choose $a\in R$ such that $(a)\in\mathcal{I}_0(R)$, $aN=(0)$ and $a\in[A:_R\bar{A}]$. Setting $b=a$ mod N, b is an A-regular element. Hence we have $\overline{(b)}=b\bar{A}\cap A=b\bar{A}$ and

$l_A(\overline{(b)}/(b))=l_A(\bar{A}/A)$. Since $(a)\cap N=aN=(0)$ and by Lemma 2.2, we get

$$l_R(\overline{(a)}/(a))=l_A(\overline{(b)}/(b))+l_R(N/(a)\cap N)=l_A(\bar{A}/A)+l_R(N)\,.$$

Therefore $i_0(R)\geq l_A(\bar{A}/A)+l_R(N)$.

Next we show the opposite inequality. Set $S=R[X]_{\mathfrak{m}R[X]}$ and $B=S/N'$, where X is an indeterminate over R and N' is the nilradical of S. By Lemma 2.4 stated below, it is sufficient to prove that $i(S)\leq l_B(\bar{B}/B)+l_S(N')$. Hence we may assum $|R/\mathfrak{m}|=\infty$. Take a minimal reduction (a) of I for $I\in\mathfrak{I}(R)$. Then $\bar{I}=\overline{(a)}$ and $b:=a \bmod N$ is an A-regular element. Hence

$$l_R(\bar{I}/I)\leq l_R(\overline{(a)}/(a))=l_A(\overline{(b)}/(b))+l_R(N/(a)\cap N)\leq l_A(\bar{A}/A)+l_R(N)\,.$$

Thus we have $i(R)\leq l_A(\bar{A}/A)+l_R(N)$.

Let X be an indeterminate over a local ring $(R,\mathfrak{m})$. Set $S=R[X]_{\mathfrak{m}R[X]}$ and $A=R/N$, where $N=\sqrt{(0)}$. Then NS is the nilradical of S. We set $B=S/NS$. Then $B\cong A[X]_{\mathfrak{n}A[X]}$, where $\mathfrak{n}=\mathfrak{m}A$. Under the situation above we have the following:

Lemma 2.4. (1) $l_R(N)=l_S(NS)$.

(2) $l_A(\bar{A}/A)=l_B(\bar{B}/B)$.

(3) $i(R)\leq i(S)$.

Proof. By [1, Chapter 5, § 1, 3°, Proposition 13], $\bar{A}[X]\cong\overline{A[X]}$ as $A[X]$-algebras. Set $T=A[X]\backslash\mathfrak{n}A[X]$. Because $A[X]$ and $T^{-1}A[X]$ have the same total quotient ring, $T^{-1}\overline{A[X]}$ coincides with $\overline{T^{-1}A[X]}$ in the total quotient ring. Hence $\bar{B}\cong\bar{A}\otimes_A B$ and $\bar{B}/B\cong(\bar{A}/A)\otimes_A B$. The canonical ring homomorphisms $R\to S$ and $A\to B$ are faithfully flat and their closed fibres are fields. Then we get (1) and (2). For any $I\in\mathfrak{I}(R)$, $IS\in\mathfrak{I}(S)$ and $\bar{I}S\subset\overline{IS}$ hold. Since $l_R(\bar{I}/I)\leq l_S(\overline{IS}/IS)$, we have $i(R)\leq i(S)$.

Now we complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Assume $\dim R=0$. Then $\bar{I}=\mathfrak{m}$ for any ideal I of R. Hence $l_R(\bar{I}/I)\leq l_R(\mathfrak{m})=l_R(R)-1$. In particular, take $(0)\in\mathfrak{I}_0(R)$. Then $l_R(\overline{(0)}/(0))=l_R(\mathfrak{m}/(0))=l_R(R)-1$. Thus we get $i(R)=i_0(R)=l_R(R)-1$.

Next we assumed $\dim R=d\geq 2$. Take $Q\in\mathfrak{I}_0(R)$. Set $Q=(a_1, a_2, \cdots, a_d)R$ and $Q_n=(a_1{}^n, a_2{}^n, \cdots, a_d{}^n)R$ for each $n>0$. Then we can check $\overline{Q_n}\supset Q^n$ and we have

$$l_R(\overline{Q_n}/Q_n)\geq l_R(Q^n/Q_n)\geq l_R(Q^n/\mathfrak{m}Q^n+Q_n)\,.$$

On the other hand, $\mathfrak{m}Q_n\subset\mathfrak{m}Q^n\cap Q_n$ and the following sequence is exact.

$$Q_n/\mathfrak{m}Q_n\to Q^n/\mathfrak{m}Q^n\to Q^n/\mathfrak{m}Q^n+Q_n\to 0\,.$$

Since $a_1, a_2, \cdots, a_d$ are analytically independent,

$$l_R(Q^n/\mathfrak{m}Q^n+Q_n)\geq \mu_R(Q^n)-\mu_R(Q_n)$$
$$=\binom{n+d-1}{d-1}-d\,,$$

where $\mu_R(\cdot)$ denotes the number of minimal generators of an R-module. This implies $i_0(R)=\infty$, because $d\geq 2$.

Finally in this section, we consider whether there exists a local ring R with $i(R)$ $(=i_0(R))=n$ for any given non-negative integer n when $\dim R\leq 1$.

Now we set $C=k[[t]]$, which is a formal power series ring over a field k. As an example of dimension 0, we have

Example 2.5. $i(C/t^{n+1}C)=l_R(C/t^{n+1}C)-1=n$.

Furthermore, we set $A=k[[t^i|n+1\leq i\leq 2n+1]]\subset C$. Then A is a local ring with the maximal ideal $\mathfrak{n}$. Since $\bar{A}=C$, we have $l_A(\bar{A}/A)=n$. On the other hand, let m be any non-negative integer and set $R=A\times(A/\mathfrak{n})^m$ (the idealization). Then we have

Example 2.6. R is a 1-dimensional Noetherian local ring with $i(R)=n+m$.

Proof. Because the nilradical N of R is $(A/\mathfrak{n})^m$, R/N and A are isomorphic as A-algebras. It is easy to check that R is a Noeteran local ring with the maximal ideal $\{(a, b)|a\in\mathfrak{n}, b\in(A/\mathfrak{n})^m\}$ and that $\dim R=\dim A=1$. Therefore we have $i(R)=l_A(\bar{A}/A)+l_R(N)=n+m$.

There are two cases where $i(R)=\infty$. One is the case where $l_A(\bar{A}/A)=\infty$ and another is the case where $l_R(N)=\infty$. Both cases can occur. For the first case, there exists a Noetherian local domain with $\dim R=1$ whose integral closure is not module-finite over R(cf. [8, Appendix, Example 3]). For the second case, let S be a Noetherian local ring of dimension 1 and set $R=S\times S$ (the idealization). Then R is a Noetherian local ring of dimension 1 and $l_R(N)=l_S(N)=l_S(S)=\infty$.

3. Calculations of $t(R)$ and $t_0(R)$

In this section we return to the study of tight closure. From now on we assume that all rings are of characteristic p. R° denotes the complement of the union of the minimal primes of R, i.e., $R^\circ=R\setminus\bigcup_{\mathfrak{p}\in\operatorname{Min}R}\mathfrak{p}$. For an ideal I and an integer $t>0$, we denote by $I^{[t]}$ the ideal generated by $\{a^t|a\in I\}$. The tight closure I^* of I is defined as follows:

Definition 3.1. Let R be a commutative ring of characteristic p and let I be an ideal of R. For $x \in R$, we say that $x \in I^*$ if there exists $c \in R^\circ$ such that $cx^{p^e} \in I^{[p^e]}$ for all sufficiently large integer e.

I^* is an ideal of R and we always have $I \subset I^* \subset \overline{I}$ (cf. [5, (5.2) Theorem]). In particular, by the Briançon-Skoda theorem, $(x)^* = \overline{(x)}$ holds for $x \in R^\circ$ (cf. [5, (5.8) Corollary]). It means that * and $^{-}$ are the same operator on $\mathcal{F}_0(R)$ when R is a local ring of dimension 1.

In section 1, we defined $t(R)$ and $t_0(R)$. Here we recall their definitions.

Definition 3.2. Let $(R, \mathfrak{m})$ be a Noetherian local ring of characteristic p. We set

$$t(R) = \sup_{I \in \mathcal{F}(R)} l_R(I^*/I) \quad \text{and} \quad t_0(R) = \sup_{Q \in \mathcal{F}_0(R)} l_R(Q^*/Q)$$

Obviously, $t_0(R) \leq i_0(R)$ and $t(R) \leq i(R)$, because $I^* \subset \overline{I}$. The following proposition is a corollary of Theorem 1.1.

Proposition 3.3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of characteristic p. If $\dim R \leq 1$, then we have $t(R) = t_0(R) = i_0(R) = i(R)$.*

Proof. If $\dim R = 0$, then R° consists only of the units of R. Since $I^* = \overline{I} = \mathfrak{m}$ for any ideal I of R, we get $t_0(R) = t(R) = i(R) = l_R(R) - 1$.

If $\dim R = 1$, then $Q^* = \overline{Q}$ for any $Q \in \mathcal{F}_0(R)$. Thus $t_0(R) = i_0(R) = i(R)$. Since $t_0(R) \leq t(R) \leq i(R)$ in general, we get the conclusion.

By Theorem 1.1, $i(R)$ is always infinite when $\dim R \geq 2$. But $t(R)$ and $t_0(R)$ are zero for a regular ring R.

Remark 3.4. Now assume that $\dim R = 1$ and $|R/\mathfrak{m}| = \infty$. Then * and $^{-}$ are the same operator on $\mathcal{F}(R)$. Indeed, for given $I \in \mathcal{F}(R)$, we can take a minimal reduction (a) of I. Then $\overline{I} = \overline{(a)}$ and $(a) \in \mathcal{F}_0(R)$. Hence

$$\overline{(a)} = \overline{I} \supset I^* \supset (a)^* = \overline{(a)}\,.$$

Next we shall try to calculate $t(R)$ and $t_0(R)$. The following is useful for this purpose.

Lemma 3.5. *Let $(R, \mathfrak{m})$ be a d-dimensional Noetherian local ring of characteristic p $(d > 0)$. Assume that*

(1) *$\overline{R}$ is a weakly F-regular ring, i.e., $J = J^*$ for any ideal J of $\overline{R}$,*

(2) *$\overline{R}$ is module-finite over R and $\mathfrak{m} \subset [R :_R \overline{R}]$.*

Then

$I^* = I\overline{R}$ and $l_R(I^*/I) = \mu_R(I\overline{R}) - \mu_R(I)$ for any ideal I of R, and $t_0(R)$

$\leq d(\mu_R(\bar{R})-1)$.

Proof. Take $c\in\mathfrak{m}\cap R^{\circ}$. Then for any $x\in I$, for any $y\in\bar{R}$ and for any integer $e>0$,

$$c(xy)^{pe}=x^{pe}(cy^{pe})\in I^{[pe]}.$$

Hence $xy\in I^*$ and $I\bar{R}\subset I^*$. Since $\bar{R}$ is weakly F-regular, $\sqrt{(0)}\subset(0)^*=(0)$. Thus $\bar{R}$ is reduced and so is R. Because R° consists of non zero divisors of R, we have $R^{\circ}\subset(\bar{R})^{\circ}$. Then $I^*\bar{R}\subset(I\bar{R})^*$ and we get

$$I^*\subset I^*\bar{R}\subset(I\bar{R})^*=I\bar{R}\subset I^*.$$

By the condition (2), $\mathfrak{m}\bar{R}=\mathfrak{m}$ and $\mathfrak{m}I^*=I(\mathfrak{m}\bar{R})=\mathfrak{m}I$. Therefore

$$l_R(I^*/I)=\mu_R(I^*)-\mu_R(I)=\mu_R(I\bar{R})-\mu_R(I).$$

In particular, for any $Q\in\mathfrak{F}_0(R)$, we have

$$l_R(Q^*/Q)=\mu_R(Q\bar{R})-\mu_R(Q)\leq d\cdot\mu_R(\bar{R})-d.$$

Thus we have $t_0(R)\leq d\cdot\mu_R(\bar{R})-d$.

In the following examples 3.6, 3.7 and 3.8, we assume that k is a field of characteristic p.

Example 3.6. *Let $S=k[[X_1, X_2, \cdots, X_d]]$ be a formal power series ring in d-variables $(d>0)$ over k. We denote by $\mathfrak{n}$ the maximal ideal of S. For $\mathfrak{a}\in\mathfrak{F}(S)$, we set $R=k+\mathfrak{a}\subset S$ and $\mathfrak{m}=\mathfrak{a}$. Then*

(1) *$(R,\mathfrak{m})$ is a Noetherian local ring of dimension d.*

(2) *$t_0(R)\leq d\cdot l_R(S/R)$.*

(3) *If $d\geq 2$ and $\mathfrak{a}\neq\mathfrak{n}$, then $t(R)=\infty$.*

Proof. (1), (2) Because $l_R(S/R)<l_S(S/\mathfrak{a})<\infty$, R is Noetherian by Eakin-Nagata's theorem. Since $S=\bar{R}$, R is a local ring of dimension d. Hence R satisfies the conditions of Lemma 3.5, and we have

$$t_0(R)\leq d(\mu_R(S)-1)=d(l_R(S/\mathfrak{a})-1)=d\cdot l_R(S/R).$$

(3) We choose $\lambda>0$ such that $\mathfrak{n}^{\lambda}\subset\mathfrak{a}$ and set $Q=(X_1^{\lambda}, X_2^{\lambda}, \cdots, X_d^{\lambda})R$. Then $Q\in\mathfrak{F}_0(R)$ and $QS\in\mathfrak{F}_0(S)$. By Lemma 3.5, we have

$$\begin{aligned} l_R((Q^n)^*/Q^n)&=\mu_R(Q^nS)-\mu_R(Q^n)\\ &=\mu_S(Q^nS/\mathfrak{a}Q^nS)-\binom{d+n-1}{d-1}. \end{aligned}$$

On the other hand, since $Q\subset\mathfrak{a}$, we have isomorphisms of graded $S/\mathfrak{a}$-algebras:

$$\oplus_{n\geq 0}Q^nS/\mathfrak{a}Q^nS\cong(S/\mathfrak{a})\otimes_S\mathrm{Gr}_S(QS)$$
$$\cong(S/\mathfrak{a})[Z_1, Z_2, \cdots, Z_d]$$

where $Z_1, Z_2, \cdots, Z_d$ are indeterminates over S.

Thus we have

$$l_R((Q^n)^*/Q^n)=l_S(S/\mathfrak{a})\binom{d+n-1}{d-1}-\binom{d+n-1}{d-1}.$$

Consequently, we have $t(R)=\infty$, because $\mathfrak{a}\neq\mathfrak{n}$ and $d\geq 2$.

Example 3.7. Let $S=k[[X_1, X_2, \cdots, X_m, Y_1, Y_2, \cdots, Y_d]]$ be a formal power series ring in $(m+d)$-variables over k, where d and m are integers with $0<m\leq d$. Set $R=S/(X_1, X_2, \cdots, X_m)\cap(Y_1, Y_2, \cdots, Y_d)$. Then

(1) $t(R)=\begin{cases}1 & \text{if } m=1\\ \infty & \text{if } m\geq 2\end{cases}$

(2) $m\leq t_0(R)\leq d$. In particular, if $m=1$, then $t_0(R)=1$.

Proof. Let x_i, y_j be respectibely the images of X_i, Y_j in R $(1\leq i\leq m, 1\leq j\leq d)$. We set $P_1=(x_1, x_2, \cdots, x_m)R$ and $P_2=(y_1, y_2, \cdots, y_d)R$. Because $\mathrm{Min}\, R=\{P_1, P_2\}$, we have $\bar{R}\cong R/P_1\times R/P_2$ and the following exact sequence

$$0\to R\to R/P_1\times R/P_2\to R/P_1+P_2\to 0\,.$$

Now $\mathfrak{m}=P_1+P_2$ is the maximal ideal of R and $(R, \mathfrak{m})$ satisfies the conditions of Lemma 3.5. Hence for any ideal I of R,

$$l_R(I^*/I)=\mu_R(I\bar{R})-\mu_R(I)$$
$$=\mu_R(I+P_1/P_1)+\mu_R(I+P_2/P_2)-\mu_R(I)\,.$$

Suppose $m=1$. Then R/P_2 is a D.V.R. Hence $\mu_R(I+P_2/P_2)\leq 1$. Bcause $\mu_R(I+P_1/P_1)\leq\mu_R(I)$, we have $l_R(I^*/I)\leq 1$ and $t(R)\leq 1$. But R is normal whenever $t_0(R)=0$ (cf. [5, § 5]). Therefore $t(R)=t_0(R)=1$.

Set $Q=(x_1+y_1, x_2+y_2, \cdots, x_m+y_m, y_{m+1}, \cdots, y_d)R$. Then $Q\in\mathcal{F}_0(R)$ and $Q+P_i/P_i=\mathfrak{m}/P_i\in\mathcal{F}_0(R/P_i)$ for $i=1, 2$, and we have:

$$l_R((Q^n)^*/Q^n)=\mu_R((\mathfrak{m}/P_1)^n)+\mu_R((\mathfrak{m}/P_2)^n)-\mu_R(Q^n)$$
$$=\binom{d+n-1}{d-1}+\binom{m+n-1}{m-1}-\binom{d+n-1}{d-1}$$

$$=\binom{m+n-1}{m-1}.$$

Thus $t(R)=\infty$, if $m>1$.

Finally we shall prove $m\leq t_0(R)\leq d$. Take $Q\in\mathfrak{F}_0(R)$. Then $\mu_R(Q+P_2/P_2)\leq\mu_R(Q)=d$. On the other hand, $\mu_R(Q+P_2/P_2)\geq m$, because $\mathrm{ht}(Q+P_2/P_2)=\dim R/P_2=m$. Similarly we get $\mu_R(Q+P_1/P_1)=d$. Hence we see $m\leq l_R(Q^*/Q)\leq d$ and $m\leq t_0(R)\leq d$.

Example 3.7 shows that there exists a local ring R with $t_0(R)=d$ for any given integer d. But we do not know whether there exists a local ring whose $t(R)$ is different from 0, 1 or ∞.

M. Nagata constructed a non-regular local ring $(A,\mathfrak{m})$ with $e(A)=1$ (cf. [8, Appendix, Example 2]). For this local ring we have

Example 3.8. $t(A)=t_0(A)=1$.

Proof. Recall that the local ring A is constructed as follows: The integral closure B of A is regular and module-finite over A. B has only two maximal ideals M and N such that $\dim B_M=2$, $\dim B_N=1$ and that $B/M=B/N=k$. Further $A=k+M\cap N$. Hence $(A,\mathfrak{m})$, where $\mathfrak{m}=M\cap N$, is a Noetherian local ring and satisfies the conditions of Lemma 3.5. Thus for any ideal I of A, we have

$$l_A(I^*/I)=\mu_A(IB)-\mu_A(I).$$

Now $\mu_A(IB)=l_A(IB/\mathfrak{m}IB)=l_B(IB\otimes_B(B/\mathfrak{m}))$ and $IB\otimes_B(B/\mathfrak{m})\cong(IB\otimes_B B/M)\oplus(IB\otimes_B B/N)$ as B-modules. Hence $\mu_A(IB)=\mu_{B_M}(IB_M)+\mu_{B_N}(IB_N)$. Thus we get $l_A(I^*/I)\leq 1$, because B_N is a D.V.R. and because $\mu_{B_M}(IB_M)\leq\mu_A(I)$. Finally we have $t(A)=t_0(A)=1$, because A is not normal.

4. Finiteness of $t_0(R)$

Here we study the case where $t_0(R)$ is finite. Recall that, if $t_0(R)=0$, then R is normal. However, examples in the previous section are of $t_0(R)<\infty$, but not normal. Now we shall prove Theorem 1.2 which asserts that R is normal when depth $R\geq 2$ and $t_0(R)<\infty$.

Proof of Theorem 1.2. Take an R-regular element $a\in R$. Moreover, choose $a_2, a_3, \cdots, a_d\in R$ so that $a, a_2, \cdots, a_d$ form a system of parameters of R, where $d=\dim R$. Put $I=(a)$ and $\lambda=t_0(R)$. Then we have

$$\mathfrak{m}^\lambda[(I+(a_2{}^n, a_3{}^n, \cdots, a_d{}^n))^*/I+(a_2{}^n, a_3{}^n, \cdots, a_d{}^n)]=0.$$

for all $n>0$. Therefore

$$\mathfrak{m}^{\lambda}I^{*}\subset\mathfrak{m}^{\lambda}(I+(a_2^{n}, a_3^{n}, \cdots, a_d^{n}))^{*}\subset I+(a_2^{n}, a_3^{n}, \cdots, a_d^{n})$$

and we get $\mathfrak{m}^{\lambda}I^{*}\subset I$. Thus $I^{*}/I\subset H_{\mathfrak{m}}^{0}(R/I)=(0)$, where $H_{\mathfrak{m}}^{i}(\cdot)$ means the i-th local cohomology module. Consequently we have $(a)=(a)^{*}=\overline{(a)}$, by the Briançon-Skoda therem. On the other hand, it is well known that R is normal if and only if $\overline{(a)}=(a)$ holds for every non zero divisor a of R. Hence we get the conclusion.

By Theorem 1.2 we can easily see that, if R is a Cohen-Macaulay local ring of $\dim R\geq 2$ and is not a domain (e.g., $R=k[[X, Y, Z]]/(XYZ)$, where k is a field), then $t_0(R)=\infty$.

Now we recall the concept of F.L.C. Let $(R, \mathfrak{m})$ be a local ring of dimension d. We say that R has F.L.C. when $H_{\mathfrak{m}}^{i}(R)$ is of finite length for any $i\neq d$. It is well known that $(R, \mathfrak{m})$ has F.L.C. if and only if there exsts an integer $t>0$ such that

$$\mathfrak{m}^{t}[(a_1, a_2, \cdots, a_{d-1}):_R a_d]\subset(a_1, a_2, \cdots, a_{d-1}),$$

for any system of parameters $a_1, a_2, \cdots, a_d$ of R (cf. [3, (37.10) Theorem]).

Theorem 4.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of characteristic p. If R has F.L.C. and $t_0(R)<\infty$, then $t_0(R_{\mathfrak{p}})=0$ for any $\mathfrak{p}\in\operatorname{Spec}R\backslash\{\mathfrak{m}\}$.*

A local ring R is called F-rational when $t_0(R)=0$ (cf. [2]). Any localization of R is known to be F-rational when R is an F-rational Chen-Macaulay local ring (cf. [7]). Theorem 4.1 is a generalization of this fact. In order to prove Theorem 4.1, we need some preparations.

Lemma 4.2. *Let R be a Noetherian local ring and $\mathfrak{p}\in\operatorname{Spec}R$ with $\operatorname{ht}\mathfrak{p}=r$. If $J\in\mathcal{F}_0(R_{\mathfrak{p}})$, then there exists a subsystem of parameters $a_1, a_2, \cdots, a_r$ of R such that $J=(a_1, a_2, \cdots, a_r)R_{\mathfrak{p}}$.*

Proof. We shall prove the assertion by induction on r. When $r=0$, we have nothing to show. Suppose $r>0$. Choose $f_1, f_2, \cdots, f_r\in R$ such that $J=(f_1, f_2, \cdots, f_r)R_{\mathfrak{p}}$ and put $I=(f_1, f_2, \cdots, f_r)R$. Moreover we put $\mathcal{F}=\{P\in\operatorname{Min}R\mid P\not\supset I\}$ and $\mathcal{G}=\{P\in\operatorname{Min}R\mid P\supset I\}$. Because $\mathcal{F}\neq\phi$, there exists $d\in\bigcap_{P\in\mathcal{F}}P\backslash\bigcup_{P\in\mathcal{G}}P$. Then the image $d/1$ in $R_{\mathfrak{p}}$ is a nilpotent, so we can assume $d/1=0$ in $R_{\mathfrak{p}}$. Because $I\not\subset\bigcup_{P\in\mathcal{F}}P$, we can choose $z\in(f_2, f_3, \cdots, f_r)R$ such that $f_1+z\notin\bigcup_{P\in\mathcal{F}}P$ (cf. [6, Theorem 124]). Now we put $a=d+f_1+z$. Then $J=(a, f_2, \cdots, f_r)R_{\mathfrak{p}}$, $a\notin\bigcup_{P\in\operatorname{Min}R}P$ and $\operatorname{ht}\mathfrak{p}/(a)=r-1$. By applying the hypothesis of induction to $R/(a)$, $\mathfrak{p}/(a)$ and $J/(a)$, we get the assertion.

Lemma 4.3 ([7, Lemma (2.2)]). *Let R be a Noetherin ring of characteristic p. Suppose that an ideal I of R satisfies $\operatorname{Ass}_R R/I\subset\operatorname{Max}R$. Then $I^{*}R_{\mathfrak{p}}=(IR_{\mathfrak{p}})^{*}$ for any $\mathfrak{p}\in\operatorname{Spec}R$.*

We also need the following proposition to prove Theorem 4.1.

Proposition 4.4. Let R be a d-dimensional Noetherian local ring of characteristic p $(d>0)$ and let I be an ideal generated by a subsystem of parameters $f_1, f_2, \cdots, f_{d-1}$ of R. If R has F.L.C., then $I^*R_{\mathfrak{p}}=(IR_{\mathfrak{p}})^*$ for any $\mathfrak{p} \in \mathrm{Spec}R$.

Proof. Choose $f_d \in R$ so that $f_1, \cdots, f_{d-1}, f_d$ form a system of parameters of R and put $S=R[1/f_d]$. We first claim that $I^*S=(IS)^*$ holds. Indeed, $I^*S \subset (IS)^*$ holds in general. Suppose $x/1 \in (IS)^*$, where $x \in R$. Then there exists $c/1 \in S^{\circ}$ such that $(c/1)(x/1)^{p^e} \in I^{[p^e]}S$ for all $e \gg 0$. We may choose $c \in R^{\circ}$ (see [5, (4.14) Proposition]). Hence

$$f_d^j c x^{p^e} \in I^{[p^e]}=(f_1^{p^e}, f_2^{p^e}, \cdots, f_{d-1}^{p^e}),$$

for some $j>0$, but j may be depend on the integer e. Since R has F.L.C., there exists a positive integer t, which is independent of e and j, such that

$$\mathfrak{m}^t[(f_1^{p^e}, f_2^{p^e}, \cdots, f_{d-1}^{p^e}):_R f_d^j] \subset (f_1^{p^e}, f_2^{p^e}, \cdots, f_{d-1}^{p^e}).$$

Thus $c^{t+1}x^{p^e} \in I^{[p^e]}$, and $x \in I^*$. Hene $I^*S=(IS)^*$.

Now $\mathrm{Ass}_S S/IS \subset \mathrm{Max}S$, so by Lemma 4.3 we get $(IS)^*S_{\mathfrak{q}}=(IS_{\mathfrak{q}})^*$, where $\mathfrak{q}=\mathfrak{p}S$. Therefore we have

$$I^*R_{\mathfrak{p}}=I^*S_{\mathfrak{q}}=(IS)^*S_{\mathfrak{q}}=(IS_{\mathfrak{q}})^*=(IR_{\mathfrak{p}})^*.$$

Proof of Theorem 4.1. We can assume $d=\dim R>0$. Take $\mathfrak{p} \in \mathrm{Spec}R \setminus \{\mathfrak{m}\}$ and $J \in \mathcal{F}_0(R_{\mathfrak{p}})$. We shall prove $J^*=J$ by induction on $\dim R/\mathfrak{p}$.

Suppose $\dim R/\mathfrak{p}=1$. Then $\mathrm{ht}\,\mathfrak{p}=d-1$ (cf. [3, (37.6) Corollary]). By Lemma 4.2, we can choose $a_1, \cdots, a_{d-1}, a_d \in R$, which form a system of parameters of R, such that $J=(a_1, a_2, \cdots, a_{d-1})R_{\mathfrak{p}}$. Putting $\lambda=t_0(R)$ and $I=(a_1, a_2, \cdots, a_{d-1})R$, we get $\mathfrak{m}^{\lambda}I^* \subset I$ by the same argument as in the proof of Theorem 1.2. Hence $I^*R_{\mathfrak{p}}=IR_{\mathfrak{p}}=J$. By Proposition 4.4, $I^*R_{\mathfrak{p}}=(IR_{\mathfrak{p}})^*=J^*$. Therefore we have $J=J^*$.

Suppose $\dim R/\mathfrak{p} \geq 2$. Take $P \in \mathrm{Spec}R$ such that $\mathfrak{m} \neq P \supset \mathfrak{p}$ and $\dim R/P=1$. By applying the hypothesis of induction to R_P and $\mathfrak{p}R_P$, we have $t_0(R_{\mathfrak{p}})=t_0((R_P)_{\mathfrak{p}R_P})=0$.

Finally we prove Theorem 1.3. But, for its proof, we need the following:

Theorem 4.5 ([5, (4.8) Theorem]). *Let R be an equi-dimensional Noetherian complete local ring of characteristic p and let $a_1, a_2, \cdots a_d$ be a system of parameters of R. Then*

$$[(a_1, a_2, \cdots, a_{d-1}):_R a_d] \subset (a_1, a_2, \cdots, a_{d-1})^*$$

Proof. Let A be a regular local subring of R, where R is module-finite over A and $a_1, a_2, \cdots, a_d \in A$. Then there is an A-free submodule F of R such that R/F is a torsion A-module. Hence there exists a non zero element c of A, which satisfies $cR \subset F$. Since R is equi-dimensional, $P \cap A = (0)$ for any $P \in \mathrm{Min}R = \mathrm{Assh}R$. Thus $c \in R^\circ$.

Let $x \in [(a_1, a_2, \cdots, a_{d-1}) :_R a_d]$. Then by the same argument as in [5, (4.8) Theorem], we can check that $cx^{p^e} \in (a_1{}^{p^e}, a_2{}^{p^e}, \cdots, a_{d-1}^{p^e})$ for any $e \geq 0$. Therefore $x \in (a_1, a_2, \cdots, a_{d-1})^*$.

Proof of Theorem 1.3. Let $a_1, a_2, \cdots, a_d$ be a system of parameters of R and $\lambda = t_0(R)$. Then by Thorem 4.5, we get

$$[(a_1, a_2, \cdots, a_{d-1}) :_R a_d] \subset (a_1, a_2, \cdots, a_{d-1})^* .$$

Hence

$$\mathfrak{m}^\lambda [(a_1, a_2, \cdots, a_{d-1}) :_R a_d] \subset (a_1, a_2, \cdots, a_{d-1}) .$$

This means that R has F.L.C. Thus applying Theorem 4.1, we get the desired conclusion.

5. Polynomial extensions of F-rational rings

Here we shall study polynomial extensions of an F-rational ring.

Theorem 5.1. *Let R be a Cohen-Macaulay local ring of characteristic p and let S be a polynomial ring $R[X_1, X_2, \cdots X_d]$ in d-variables over R. If $R_\mathfrak{m}$ is an F-rational ring for any $\mathfrak{m} \in \mathrm{Max}R$ (hence for any $\mathfrak{p} \in \mathrm{Spec}R$), then S_P is an F-rational ring for any $P \in \mathrm{Spec}S$.*

If R is a Gorenstein local ring, then weakly F-regularity is equivalent to F-rationality (cf. [4, Theorem 5.1]). Hence we get

Corollary 5.2. *If R is a weakly F-regular Gorenstein ring of characterstic p, then so is the polynomal ring $R[X_1, X_2, \cdots, X_d]$.*

R. Fedder and K. Watanabe proved the following:

Proposition 5.3 ([2, Proposition (2.2)]). *Let R be a Cohen-Macaulay local ring of characteristic p and assume that $Q^* = Q$ for some $Q \in \mathfrak{F}_0(R)$. Then R is an F-rational ring.*

Proof of Theorem 5.1. We may assume $S = R[X]$ and $P \in \mathrm{Max}S$. Put $\mathfrak{m} = P \cap R$. By localizing R at $\mathfrak{m}$ we may assume $(R, \mathfrak{m})$ is a Cohen-Macaulay local domain such that $P \cap R = \mathfrak{m}$. If $\dim R = 0$, then the assertion is obvious. Suppose $\dim R = d > 0$. By Proposition 5.3 it is sufficient to show that there exists $J \in \mathfrak{F}_0(S_P)$ such that $J = J^*$.

Let $a_1, a_2, \cdots, a_d$ be a system of parameters of R and $\mathfrak{q}=(a_1, a_2, \cdots, a_d)R$. Since $S/\mathfrak{m}S=(R/\mathfrak{m})[X]$ is a principal ideal domain, there exists a monic polynomial $f\in S$ such that $(\bar{f})=P/\mathfrak{m}S\subset S/\mathfrak{m}S$. We put $I=\mathfrak{q}S+(f)$. Then I is a P-primary ideal of S, and we have only to prove $I=I^*$ (see Lemma 4.3).

Suppose $I^*\neq I$. We choose $\varphi\in I^*\backslash I$ such that $\deg\varphi$ is minimal. Then $\deg\varphi<\deg f$. Since $\varphi\in I^*$, there exists $\xi\in S^\circ$ such that

$$\xi\varphi^{p^e}\in I^{[p^e]}=\mathfrak{q}^{[p^e]}S+(f^{p^e}), \qquad \text{for all} \quad e\gg 0 .$$

If $0\neq\bar{\xi}\bar{\varphi}^{p^e}\in S/\mathfrak{q}^{[p^e]}S$, then

$$\deg\xi+p^e\deg\varphi\geq\deg\bar{\xi}\bar{\varphi}^{p^e}\geq\deg f^{p^e}=p^e\deg f .$$

Hence $p^e(\deg f-\deg\varphi)\leq\deg\xi$. Therefore there exists $e_1\geq 0$ such that $\xi\varphi^{p^e}\in\mathfrak{q}^{[p^e]}S$ for all $e\geq e_1$. Let cX^n be the leading term of ξ and aX^m be the leading term of φ. Then $ca^{p^e}\in\mathfrak{q}^{[p^e]}$ for all $e\geq e_1$ and $c\in R^\circ$. Hence $a\in\mathfrak{q}^*=\mathfrak{q}$ and $aX^m\in I$. This contradicts to the minimality of $\deg\varphi$.

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