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On a property of Fourier coefficients of cusp forms of half-integral weight

By

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Introduction

Since Shimura's epock-making paper [S1] appeared, modular forms of half-integral weight have been recognized to be an important object comparable to those of integral weight. This short note is to show a property of Fourier coefficients of cusp forms of half-integral weight. Before we state our result, we should remind ourself of a property of Fourier coefficients of cusp forms of integral weight. Let $F(z)=\sum_{n=1}^{\infty}A_n e^{2\pi inz}$ be a primitive form in $S_k(N,\chi)$, then the following fact is well known ([M2], (4.6.17)):

$$A_n=\chi(n)\overline{A_n} \qquad \textit{if} \quad (n,N)=1 .$$

Though this is quite simple and easy to prove, it sometimes plays an important role. The present note is to show that a parallel (but somewhat weaker) relation holds for cusp forms of half-integral weight by using this relation and the main theorem of Waldspurger ([W]). We denote by $S_k(N,\chi)$ the space of cusp forms of weight k with level N and character χ, and for an odd integer k we denote by $\mathfrak{S}_{k/2}(N,\chi)$ the space of cusp forms of weight $k/2$ (half-integral weight) with level N and character χ.

Theorem. *Let $f(z)=\sum_{n=1}^{\infty}a_n e^{2\pi inz}\in\mathfrak{S}_{k/2}(N,\chi)$ be a common eigen function of Hecke operators $\mathfrak{T}(p^2)$ for all prime numbers p prime to N, and F the primitive cusp form in $S_{k-1}(N,\chi^2)$ corresponding to f. Let m and n be square-free positive integers relatively prime to N satisfying*

$$m/n\in(\boldsymbol{Q}_p^{\times})^2$$

for all prime numbers p dividing N. If $L((1-k)/2,F,\overline{\chi}\psi_{m,k})$ and $L((1-k)/2,F,\overline{\chi}\psi_{n,k})$ are not 0, then

$$a_m\overline{a_n}\chi(n)=\overline{a_m}a_n\chi(m) .$$

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Here $L(s, F, \overline{\chi}\psi_{m,k})$ resp. $L(s, F, \overline{\chi}\psi_{n,k})$ is a Dirichlet series attached to F with character $\overline{\chi}\psi_{m,k}$ resp. $\overline{\chi}\psi_{n,k}$. For the precise definitions of them and characters $\psi_{m,k}$, $\psi_{n,k}$ see the text. A similar result is also found in [S2]. Since Shimura pointed out in [S1] the importance of Fourier coefficients a_n of cusp forms of half-integral weight, especially for square-free n, several authors obtained interesting formulas expressing a_n (or we rather say a_n^2 or $|a_n|^2$) (see [K-Z], [K], [W], [S2]). Our result is simple but useful to abridge some of those results. In fact, this result is based on a question arizen from a communication with Shimura, to whom authors wish to express their hearty gratitude. In the end of the article, we attach a table of Fourier coefficients of cusp forms of half-integral weight calculated by M.Yamauchi using a table of M.Ueda([U]) by their courtesy.

1 For a positive integer N, we denote by $\Gamma_0(N)$ the congruence modular group

$$\Gamma_0(N)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL_2(\boldsymbol{Z})\,|\, c\equiv 0 \pmod N\right\} .$$

For an elemnt $\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL_2(\boldsymbol{Z})$, we put

$$\gamma(z)=\frac{az+b}{cz+d}\ , \qquad j(\gamma, z)=cz+d\ .$$

For an odd positive integer k and for $\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in \Gamma_0(4)$, we put

$$J_{k/2}(\gamma, z)=\varepsilon_d^{-1}\left(\frac{c}{d}\right)(cz+d)^{1/2}(cz+d)^{(k-1)/2}\ .$$

Here $\varepsilon_d = 1$ or i according as $d\equiv 1$ or $3 \pmod 4$, and $(cz+d)^{1/2}$ is defined as usual. Let f be a function on the upper half plane $\boldsymbol{H}$. For an element $\gamma\in SL_2(\boldsymbol{Z})$ and an integer k, we put

$$(f|_k\gamma)(z)=j(\gamma, z)^{-k}f(\gamma(z))\ .$$

We also define, for a positive odd integer k and $\gamma\in\Gamma_0(4)$,

$$(f|_{k/2}\gamma)(z)=J_{k/2}(\gamma, z)^{-1}f(\gamma(z))\ .$$

For a positive integer N, we denote by χ a Dirichlet character defined modulo N. We put

$$\chi(\gamma)=\chi(d) \qquad \textit{for} \quad \gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

For N, χ and a positive integer k, we denote by $S_k(N, \chi)$ the space of holomorphic functions $f(z)$ on $\boldsymbol{H}$ satisfying

$$(f|_k\gamma)(z)=\chi(\gamma)f(z)$$

and the usual cuspidal condition. Now assume that N is divisible by 4. For N, χ and an odd integer k, we denote by $\mathfrak{S}_{k/2}(N, \chi)$ the space of holomorphic functions $f(z)$ on $\boldsymbol{H}$ satisfying

$$(f|_{k/2}\gamma)(z)=\chi(\gamma)f(z) \ .$$

and the usual cuspidal condition. Those spaces are called the spaces of cusp forms of integral weight or half-integral weight, respectively. For prime numbers p not dividing N, we denote by $T(p)$ the Hecke oprators acting on $S_k(N, \chi)$ and by $\mathfrak{T}(p^2)$ the Hecke oprators acting on $\mathfrak{S}_{k/2}(N, \chi)$, respectvely (for the definitions of Hecke operators, see [M2] and [S1]).

For a cusp form $f(z)$ of either integral or half-integral weight, we express its Fourier expansion as

$$f(z)=\sum_{n=1}^{\infty}a(n,f)e^{2\pi inz} \ .$$

We also define a function f_ρ by

$$f_\rho(z)=\sum_{n=1}^{\infty}\overline{a(n,f)}e^{2\pi inz} \ .$$

Then we see easily that

$$f_\rho(z)=\overline{f(-\bar{z})} \ .$$

Lemma 1. (1) *If $F(z)$ belongs to $S_k(N, \chi)$, then $F_\rho(z)$ belongs to $S_k(N, \bar{\chi})$. Moreover, if $F|_kT(p)=\lambda F$ then $F_\rho|_kT(p)=\bar{\lambda}F_\rho$.*
(2) *Let k be an odd positive integer. If $f(z)$ belongs to $\mathfrak{S}_{k/2}(N, \chi)$, then $f_\rho(z)$ belongs to $\mathfrak{S}_{k/2}(N, \bar{\chi})$. Moreover, if $f|_{k/2}\mathfrak{T}(p^2)=\lambda f$ then $f_\rho|_{k/2}\mathfrak{T}(p^2)=\bar{\lambda}f_\rho$.*

Proof. The first part of (1) is well known ([M2], Lemma 4.3.2). Let us prove the first part of (2). For $\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL_2(\boldsymbol{Z})$, we put

$$\gamma'=\begin{pmatrix} a & -b \\ -c & d \end{pmatrix} .$$

If $\gamma\in\Gamma_0(N)$, then we see that

$$\overline{J_{k/2}(\gamma', -\bar{z})}=\overline{\varepsilon_d^{-1}\left(\frac{-c}{d}\right)(c\bar{z}+d)^{1/2}(c\bar{z}+d)^{(k-1)/2}}$$

$$=\varepsilon_d^{-1}\left(\frac{c}{d}\right)(cz+d)^{1/2}(cz+\mathrm{d})^{(k-1)/2}=J_{k/2}(\gamma, z) \ .$$

Therefore we obtain that

$$f_\rho\left(\frac{az+d}{cz+d}\right)=\overline{f\left(-\overline{\frac{az+b}{cz+d}}\right)}$$
$$=\overline{f\left(\frac{a(-\bar{z})-b}{-c(-\bar{z})+d}\right)}$$
$$=\overline{\chi(\gamma)J_{k/2}(\gamma', -\bar{z})f(-\bar{z})}$$
$$=\overline{\chi(\gamma)}J_{k/2}(\gamma, z)f_\rho(z) \ .$$

Since $f_\rho(\gamma(z))=\overline{f(\gamma'(-\bar{z}))}$, the cuspidal condition is obviously satisfied by f_ρ. The second parts of both (1) and (2) can be directly proved using the Fourier expansion and the definition of Hecke operators. Since we omit the definition of Hecke operators, we also omit the proof. q.e.d.

From now on, we denote by k a positive odd integer $k\geq 3$ and by N a positive integer divisible by 4 and χ a Dirichlet character defined modulo N. Let $f\in\mathfrak{S}_{k/2}(N, \chi)$ be a common eigen function of Hecke operators $\mathfrak{T}(p^2)$ for all primes $p\,(p\nmid N)$ and $f|_{k/2}\mathfrak{T}(p^2)=\lambda_p f$. Then there exists a unique primitive form $F\in S_{k-1}(N, \chi^2)$ such that $F|_{k-1}T(p)=\lambda_p F$. We call F the primitive form corresponding to f. For a primitive form $F(z)$ in $S_{k-1}(N, \chi^2)$ of conductor N, we put

$$\mathfrak{S}_{k/2}(N, \chi, F)=\left\{f\in\mathfrak{S}_{k/2}(N, \chi)\,\middle|\, \begin{array}{l} f|_{k/2}\mathfrak{T}(p^2)=a(p, F)f \\ \text{for almost all } p\nmid N \end{array}\right\} .$$

For a modular form f and a Dirichlet character ψ, we define a Dirichlet series $L(s, f, \psi)$ attached to f and ψ by

$$L(s, f, \psi)=\sum_{n=1}^{\infty}a(n, f)\psi(n)n^{-s} \ .$$

For a square-free positive integer n and an odd integer k, we denote by ψ_n the Dirichlet character corresponding to the quadratic extension $\boldsymbol{Q}(\sqrt{n})$ and $\psi_{n,k}$ the product of ψ_n with $\left(\frac{-1}{\ }\right)^{(k-1)/2}$. Now one of the main results of Waldspurger ([W]) is as follows:

Lemma 2. ([W], Corollaire 2). *The notation and the assumptions being as above, let $f\in\mathfrak{S}_{k/2}(N, \chi, F)$ and m, n square-free positive integers prime to N satisfying $m/n\in(\boldsymbol{Q}_p^\times)^2$ for all prime numbers p dividing N. Then*

$$a(m,f)^2L((1\text{-}k)/2,F,\overline{\chi}\psi_{n,k})\chi(n/m)n^{k/2-1}$$
$$=a(n,f)^2L((1\text{-}k)/2,F,\overline{\chi}\psi_{m,k})\mathrm{m}^{k/2-1}\ .$$

Corollary 3. *Let the notation and the assumptions be the same as in Lemma 2. Assume neither $L((1-k)/2,F,\overline{\chi}\psi_{m,k})$ nor $L((1-k)/2,F,\overline{\chi}\psi_{n,k})$ vanishes. Then for any two element $f,g\in\mathfrak{S}_{k/2}(N,\chi,F)$,*

$$a(m,f)a(n,g)=a(n,f)a(m,g)\ .$$

Proof. By Lemma 2, we easily see that

$$a(m,f)^2a(n,g)^2=a(n,f)^2a(m,g)^2$$

for any two elements f and g in $\mathfrak{S}_{k/2}(N,\chi,F)$. If $a(m,f)a(n,f)=0$, then $a(m,f)a(n,g)=a(n,f)a(m,g)$ is clear. Assume $a(m,f)a(n,f)\neq 0$. Since f and g are arbitrary, it also holds if we take $f+g$ in place of g. This implies the equality

$$a(m,f)^2(a(n,f)+a(n,g))^2=a(n,f)^2(a(m,f)+a(m,g))^2\ .$$

Therefore we obtain

$$2a(m,f)a(n,f)\,(a(m,f)a(n,g)-a(n,f)a(m,g))=0\ .$$

By our assumption we see $a(m,f)a(n,g)=a(n,f)a(m,g)$. q.e.d.

Proof of the theorem. Let f be an element in $\mathfrak{S}_{k/2}(N,\chi)$ which is an eigen function of Hecke operators $\mathfrak{T}(p^2)$ for all prime numbers p prime to N. Denote by λ_p the eigenvalue of $\mathfrak{T}(p^2)$ for f or

$$f|_{k/2}\mathfrak{T}(p^2)=\lambda_p f$$

and denote by F the corresponding primitive form in $S_{k-1}(N,\chi^2)$. Then we see that

$$F|_{k-1}T(p)=\lambda_p F\ .$$

Now we put

$$f'(z)=\sum_{n=1}^{\infty}\chi(n)\overline{a(n,f)}e^{2\pi inz}\ .$$

and let F' the primitive cusp form in $S_{k-1}(N,\chi^2)$ corresponding to f'. Then by [S1], Lemma 3.6, and the theory of primitive forms ([M2], §4.6), we see easily that

$$f'|_{k/2}\mathfrak{T}(p^2)=\chi(p)^2\overline{\lambda_p}f'$$

and also

$$F'|_{k-1}T(p)=\chi(p)^2\overline{\lambda_p}F' .$$

Since λ_p is the p-th Fourier coefficient of F and $\chi(p)^2\overline{\lambda_p}$ is the p-th Fourier coefficients of F', we see by the property of coefficients of cusp forms of integral weight mentioned in the introduction that

$$\lambda_p=\chi(p)^2\overline{\lambda_p} .$$

This implies that eigen values of $T(p)$ for F and F' are the same for all prime numbers p not dividing N and therefore F and F' must coincide by the theory of primitive forms ([M1], [M2]). Since f and f' correspond to the same primitive form F, we can apply Corollary 3 and obtain our result by taking f' as g there. q.e.d.

Table

The following is a table of Fourier coefficients $a(n)$ of cusp forms f_A, f_B, f_C, and f_D in $\mathfrak{S}_{5/2}(28, \chi)$ with character χ of order 3 defined modulo 7. We note that $\dim_{\mathbf{C}}\mathfrak{S}_{5/2}(28, \chi)=4$ and $\{f_A, f_B, f_C, f_D\}$ is a basis of $\mathfrak{S}_{5/2}(28, \chi)$ consisting of common eigen functions of Hecke operators $\mathfrak{T}(p^2)$ satisfying $(p, 28)=1$. In the table, ω implies $1^{1/3}$. We note f_A belongs to $\mathfrak{S}_{5/2}(28, \chi, F_A)$ and f_B belongs to $\mathfrak{S}_{5/2}(28, \chi, F_B)$ with primitive forms $F_A, F_B\in S_4(14, \overline{\chi})$ and f_C and f_D belong to $\mathfrak{S}_{5/2}(28, \chi, G)$ with a primitive form $G\in S_4(7, \overline{\chi})$. In the table, we omit the bracket in the numerator. So, for example, $-3+3\omega/2$ should be read as $(-3+3\omega)/2$.

$=f_A=$

n	mod 8	mod 7	$(n/7)$	$\chi(n)$	$a(n)$	$\chi(n)\bar{a}(n)/a(n)$
(1)	1	1	+	1	1	1
(2)	2	2	+	ω^2	$\omega/2$	1
(3)	3	3	−	ω	$-3+3\omega/2$	−1
(4)	4	4	+	ω	$2+2\omega$	1
(5)	5	5	−	ω^2	0	
(6)	6	6	−	1	$-5-10\omega/2$	−1
(7)	7	0		0	$-3-8\omega/2$	0
(8)	0	1	+	1	$-1-2\omega$	−1
(9)	1	2	+	ω^2	-4ω	1
(10)	2	3	−	ω	$-1+\omega/2$	−1
(11)	3	4	+	ω	$13+13\omega/2$	1
(12)	4	5	−	ω^2	$-6-3\omega$	−1
(13)	5	6	−	1	0	
(14)	6	0		0	$24+15\omega/2$	0

(15)	7	1	+	1	$-17/2$	1
(16)	0	2	+	ω^2	4ω	1
(17)	1	3	−	ω	$-5+5\omega$	-1
(18)	2	4	+	ω	$-1-\omega$	1
(19)	3	5	−	ω^2	$26+13\omega/2$	-1
(20)	4	6	−	1	0	
(21)	5	0		0	0	
(22)	6	1	+	1	$13/2$	1
(23)	7	2	+	ω^2	$-37\omega/2$	1
(24)	0	3	−	ω	$5-5\omega$	-1
(25)	1	4	+	ω	$-2-2\omega$	1
(26)	2	5	−	ω^2	$-28-14\omega$	-1
(27)	3	6	−	1	$3+6\omega/2$	-1
(28)	4	0		0	$5-3\omega$	0
(29)	5	1	+	1	0	
(30)	6	2	+	ω^2	$51\omega/2$	1
(31)	7	3	−	ω	$-13+13\omega/2$	-1
(32)	0	4	+	ω	$-2-2\omega$	1
(33)	1	5	−	ω^2	$14+7\omega$	-1
(34)	2	6	−	1	$-9-18\omega/2$	-1
(35)	3	0		0	$-56-35\omega/2$	0
(36)	4	1	+	1	8	1
(37)	5	2	+	ω^2	0	
(38)	6	3	−	ω	$31-31\omega/2$	-1
(39)	7	4	+	ω	$3+3\omega$	1
(40)	0	5	−	ω^2	$-2-\omega$	-1
(41)	1	6	−	1	$6+12\omega$	-1
(42)	2	0		0	$21+56\omega/2$	0
(43)	3	1	+	1	2	1
(44)	4	2	+	ω^2	13ω	1
(45)	5	3	−	ω	0	
(46)	6	4	+	ω	$-29-29\omega/2$	1
(47)	7	5	−	ω^2	$70+35\omega/2$	-1
(48)	0	6	−	1	$-6-12\omega$	-1
(49)	1	0		0	$-19-18\omega$	0
(50)	2	1	+	1	6	1

$=f_B=$

n	mod 8	mod 7	$(n/7)$	$\chi(n)$	$a(n)$	$\chi(n)\bar{a}(n)/a(n)$
(1)	1	1	+	1	0	
(2)	2	2	+	ω^2	1	ω^2
(3)	3	3	−	ω	$-2-\omega/3$	$-\omega^2$

(4)	4	4	+	ω	0	
(5)	5	5	−	ω^2	$2+4\omega/3$	$-\omega^2$
(6)	6	6	−	1	$-1+\omega$	$-\omega^2$
(7)	7	0		0	$-1-5\omega/3$	0
(8)	0	1	+	1	$-2-2\omega$	ω^2
(9)	1	2	+	ω^2	0	
(10)	2	3	−	ω	$-2-\omega/3$	$-\omega^2$
(11)	3	4	+	ω	ω	ω^2
(12)	4	5	−	ω^2	$2+4\omega/3$	$-\omega^2$
(13)	5	6	−	1	$4-4\omega/3$	$-\omega^2$
(14)	6	0		0	$5+4\omega/3$	0
(15)	7	1	+	1	$3+3\omega$	ω^2
(16)	0	2	+	ω^2	0	
(17)	1	3	−	ω	0	
(18)	2	4	+	ω	-2ω	ω^2
(19)	3	5	−	ω^2	$-5-10\omega/3$	$-\omega^2$
(20)	4	6	−	1	$4-4\omega/3$	$-\omega^2$
(21)	5	0		0	$-8+2\omega/3$	0
(22)	6	1	+	1	$3+3\omega$	ω^2
(23)	7	2	+	ω^2	-3	ω^2
(24)	0	3	−	ω	$4+2\omega$	$-\omega^2$
(25)	1	4	+	ω	0	
(26)	2	5	−	ω^2	$-8-16\omega/3$	$-\omega^2$
(27)	3	6	−	1	$-5+5\omega/3$	$-\omega^2$
(28)	4	0		0	$-8+2\omega/3$	0
(29)	5	1	+	1	$-4-4\omega$	ω^2
(30)	6	2	+	ω^2	-5	ω^2
(31)	7	3	−	ω	$10+5\omega/3$	$-\omega^2$
(32)	0	4	+	ω	4ω	ω^2
(33)	1	5	−	ω^2	0	
(34)	2	6	−	1	$1-\omega/3$	$-\omega^2$
(35)	3	0		0	$25+20\omega/3$	0
(36)	4	1	+	1	0	
(37)	5	2	+	ω^2	2	ω^2
(38)	6	3	−	ω	$-10-5\omega/3$	$-\omega^2$
(39)	7	4	+	ω	-6ω	ω^2
(40)	0	5	−	ω^2	$2+4\omega/3$	$-\omega^2$
(41)	1	6	−	1	0	
(42)	2	0		0	$-1-5\omega$	0
(43)	3	1	+	1	$-4-4\omega$	ω^2
(44)	4	2	+	ω^2	2	ω^2
(45)	5	3	−	ω	$8+4\omega/3$	$-\omega^2$
(46)	6	4	+	ω	ω	ω^2
(47)	7	5	−	ω^2	$5+10\omega/3$	$-\omega^2$

(48)	0	6	$-$	1	$4-4\omega/3$	$-\omega^2$
(49)	1	0		0	0	
(50)	2	1	$+$	1	$4+4\omega$	ω^2

$=f_C=$

n	mod 8	mod 7	$(n/7)$	$\chi(n)$	$a(n)$	$\chi(n)\bar{a}(n)/a(n)$
(1)	1	1	$+$	1	1	1
(2)	2	2	$+$	ω^2	$-\omega/2$	1
(3)	3	3	$-$	ω	$-1+\omega/2$	-1
(4)	4	4	$+$	ω	$2+2\omega$	1
(5)	5	5	$-$	ω^2	$-4-2\omega$	-1
(6)	6	6	$-$	1	$-3-6\omega/2$	-1
(7)	7	0		0	$7/2$	0
(8)	0	1	$+$	1	-3	1
(9)	1	2	$+$	ω^2	4ω	1
(10)	2	3	$-$	ω	$-7+7\omega/2$	-1
(11)	3	4	$+$	ω	$7+7\omega/2$	1
(12)	4	5	$-$	ω^2	$6+3\omega$	-1
(13)	5	6	$-$	1	$-4-8\omega$	-1
(14)	6	0		0	$-7\omega/2$	0
(15)	7	1	$+$	1	$-3/2$	1
(16)	0	2	$+$	ω^2	-12ω	1
(17)	1	3	$-$	ω	$3-3\omega$	-1
(18)	2	4	$+$	ω	$5+5\omega$	1
(19)	3	5	$-$	ω^2	$-18-9\omega/2$	-1
(20)	4	6	$-$	1	$4+8\omega$	-1
(21)	5	0		0	$14+14\omega$	0
(22)	6	1	$+$	1	$3/2$	1
(23)	7	2	$+$	ω^2	$9\omega/2$	1
(24)	0	3	$-$	ω	$-9+9\omega$	-1
(25)	1	4	$+$	ω	$-2-2\omega$	1
(26)	2	5	$-$	ω^2	$4+2\omega$	-1
(27)	3	6	$-$	1	$-7-14\omega/2$	-1
(28)	4	0		0	$-21-21\omega$	0
(29)	5	1	$+$	1	4	1
(30)	6	2	$+$	ω^2	$-19\omega/2$	1
(31)	7	3	$-$	ω	$9-9\omega/2$	-1
(32)	0	4	$+$	ω	$2+2\omega$	1
(33)	1	5	$-$	ω^2	$-18-9\omega$	-1
(34)	2	6	$-$	1	$17+34\omega/2$	-1
(35)	3	0		0	$7\omega/2$	0
(36)	4	1	$+$	1	-8	1

(37)	5	2	+	ω^2	30ω	1
(38)	6	3	−	ω	$1-\omega/2$	−1
(39)	7	4	+	ω	$-11-11\omega$	1
(40)	0	5	−	ω^2	$42+21\omega$	−1
(41)	1	6	−	1	$-2-4\omega$	−1
(42)	2	0		0	$-21/2$	0
(43)	3	1	+	1	14	1
(44)	4	2	+	ω^2	-21ω	1
(45)	5	3	−	ω	$20-20\omega$	−1
(46)	6	4	+	ω	$-27-27\omega/2$	1
(47)	7	5	−	ω^2	$-14-7\omega$	−1
(48)	0	6	−	1	$-2-4\omega$	−1
(49)	1	0		0	$21+14\omega$	0
(50)	2	1	+	1	-6	1

$=f_D=$

n	mod 8	mod 7	$(n/7)$	$\chi(n)$	$a(n)$	$\chi(n)\bar{a}(n)/a(n)$
(1)	1	1	+	1	1	1
(2)	2	2	+	ω^2	$3\omega/2$	1
(3)	3	3	−	ω	$3-3\omega/2$	−1
(4)	4	4	+	ω	$-6-6\omega$	1
(5)	5	5	−	ω^2	$4+2\omega$	−1
(6)	6	6	−	1	$9+18\omega/2$	−1
(7)	7	0		0	$-21/2$	0
(8)	0	1	+	1	1	1
(9)	1	2	+	ω^2	4ω	1
(10)	2	3	−	ω	$21-21\omega/2$	−1
(11)	3	4	+	ω	$-21-21\omega/2$	1
(12)	4	5	−	ω^2	$-2-\omega$	−1
(13)	5	6	−	1	$4+8\omega$	−1
(14)	6	0		0	$21\omega/2$	0
(15)	7	1	+	1	$9/2$	1
(16)	0	2	+	ω^2	4ω	1
(17)	1	3	−	ω	$3-3\omega$	−1
(18)	2	4	+	ω	$-15-15\omega$	1
(19)	3	5	−	ω^2	$54+27\omega/2$	−1
(20)	4	6	−	1	$4+8\omega$	−1
(21)	5	0		0	$-14-14\omega$	0
(22)	6	1	+	1	$-9/2$	1
(23)	7	2	+	ω^2	$-27\omega/2$	1
(24)	0	3	−	ω	$3-3\omega$	−1
(25)	1	4	+	ω	$-2-2\omega$	1

(26)	2	5	−	ω^2	$-12-6\omega$	−1
(27)	3	6	−	1	$21+42\omega/2$	−1
(28)	4	0		0	$7+7\omega$	0
(29)	5	1	+	1	-4	1
(30)	6	2	+	ω^2	$57\omega/2$	1
(31)	7	3	−	ω	$-27+27\omega/2$	−1
(32)	0	4	+	ω	$10+10\omega$	1
(33)	1	5	−	ω^2	$-18-9\omega$	−1
(34)	2	6	−	1	$-51-102\omega/2$	−1
(35)	3	0		0	$-21\omega/2$	0
(36)	4	1	+	1	24	1
(37)	5	2	+	ω^2	-30ω	1
(38)	6	3	−	ω	$-3+3\omega/2$	−1
(39)	7	4	+	ω	$33+33\omega$	1
(40)	0	5	−	ω^2	$-14-7\omega$	−1
(41)	1	6	−	1	$-2-4\omega$	−1
(42)	2	0		0	$63/2$	0
(43)	3	1	+	1	-42	1
(44)	4	2	+	ω^2	7ω	1
(45)	5	3	−	ω	$-20+20\omega$	−1
(46)	6	4	+	ω	$81+81\omega/2$	1
(47)	7	5	−	ω^2	$42+21\omega/2$	−1
(48)	0	6	−	1	$-10-20\omega$	−1
(49)	1	0		0	$21+14\omega$	0
(50)	2	1	+	1	18	1

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