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On the Cauchy problem for Schrödinger type equations and the regularity of solutions

By

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1. Introduction and main results

In this paper we consider evolution equations for second order differential operators with skew-symmetric principal parts

$$(1.1)\qquad \begin{cases} (\partial_t + a^w(x, D))u = f \text{ in } \mathcal{D}'((0, T)\times \mathbf{R}^d_x) \\ u(0, x) = u_0(x). \end{cases}$$

Here we assume

(A0) $a = ia_2 + a_1 + a_0$, $a_j \in S^j_{1,0}$ $(j=0, 1, 2)$ and a_2 is real.

Especially we have in mind the following simple equations:

$$(1.2)\qquad \begin{cases} \left(\partial_t + i\frac{1}{2}|D_x|^2 + \sum_{j=1}^{d} b_j(x)D_j + c(x)\right)u = f \text{ in } \mathcal{D}'((0, T)\times \mathbf{R}^d_x) \\ u(0, x) = u_0(x), \end{cases}$$

where $b_j(x), c(x) \in \mathcal{B}^\infty(\mathbf{R}^d)$.

The aim of this paper is to give a sufficient condition for the Cauchy problem (1.1), especially (1.2), to be H^s (or H^∞) well posed, and under that condition we will show the additional regularity of the solutions.

More precisely we consider the following conditions:

(A1) There exists $e \in S^1_{1,0}$ such that $e(x, \xi) \geq \delta\langle\xi\rangle$ with some $\delta > 0$ and that $\{e, a_2\} \in S^1_{1,0}$. Here $\langle\xi\rangle = (10 + |\xi|^2)^{\frac{1}{2}}$.

(A2) There exist $p \in S^0_{1,0}$ of real value and $C > 0$ such that

$$(1.3)\qquad H_{a_2}p + \operatorname{Re} a \geq -C.$$

(A3) There exist $p \in S(\log\langle\xi\rangle, |dx|^2 + \langle\xi\rangle^{-2}|d\xi|^2)$, of real value, $K > 0$ and $C > 0$ such that

$$(1.4)\qquad H_{a_2}p + \operatorname{Re} a_1 \geq -K\log\langle\xi\rangle - C.$$

Here $H_p q = \{p, q\} = \sum_{j=1}^{d} (\partial_{\xi_j}p\,\partial_{x_j}q - \partial_{x_j}p\,\partial_{\xi_j}q)$.

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Remark. If a_2 is uniformly elliptic or independent of x, then (A1) is satisfied with $e=(1+a_2^2)^{\frac{1}{4}}$ or $\langle\xi\rangle$ respectively.

Theorem 1.1. *Let $s\in\mathbf{R}$ and suppose (A0), (A1) and (A2). Then for any $u_0\in H^s$ and $f\in L^1([0,T];H^s)$ there exists a solution $u\in C([0,T];H^s)$ of (1.1) satisfying*

$$\|u(t)\|_s\leq C_1(\|u(0)\|_s+\int_0^t\|f(\tau)\|_s d\tau),\quad 0\leq t\leq T, \tag{1.5}$$

and it is unique in $C([0,T];H^{-\infty})$. Moreover if $f\in L^2([0,T];H^s)$, then $u\in L^2([0,T];X^s)$ and also satisfies

$$\int_0^t\|u(\tau)\|_{X^s}^2 d\tau\leq C_2(\|u(0)\|_s^2+\int_0^t\|f(\tau)\|_s^2 d\tau),\ 0\leq t\leq T. \tag{1.6}$$

Here X^s is a Hilbert space whose norm is defined by

$$\|u\|_{X^s}^2=((H_{a_2}p+\mathrm{Re}\ a_1)^w(x,D)\langle D\rangle^s u,\ \langle D\rangle^s u)+C_s\|u\|_s^2 \tag{1.7}$$

with a large constant $C_s>0$.

Theorem 1.2. *Let $s\in\mathbf{R}$. Suppose (A0), (A1) and (A3). Then for any $u_0\in H^s$ and $f\in L^1([0,T];H^s)$ there exists a solution $u\in C([0,T];H^{s-\gamma})$ of (1. 1) satisfying*

$$\|u(t)\|_{s-\gamma}\leq C_1(\|u(0)\|_s+\int_0^t\|f(\tau)\|_s d\tau),\ 0\leq t\leq T \tag{1.8}$$

and it is unique in $C([0,T];H^{-\infty})$. Here γ, $C>0$ and γ is independent of s.

Remark. The proofs of Lemmas 2.2 and 2.3 contain more information: if $u_0\in H^s$ and $\langle D\rangle^{-mt}f\in L^1([0,T];H^s)$, then $\langle D\rangle^{-M-mt}u\in C([0,T];H^s)$ and the following estimate holds with m, M, $C>0$;

$$\|u(t)\|_{s-M-mt}\leq C(\|u(0)\|_s+\int_0^t\|f(\tau)\|_{s-m\tau}d\tau),\ 0\leq t\leq T. \tag{1.9}$$

Here we can choose $M=M_2-M_1$ if $M_1\log\langle\xi\rangle+C_1\leq p(x,\xi)\leq M_2\log\langle\xi\rangle+C_2$ with C_1, $C_2\in\mathbf{R}$ and m as any real number satisfying $H_{a_2+\mathrm{Im}a_1}p+\mathrm{Re}\ a_1+m\log\langle\xi\rangle\geq -C$ with some $C>0$.

Corollary 1.3. *Suppose (A0), (A1) and (A3). Then for any $u_0\in H^\infty$ and $f\in L^1([0,T];H^\infty)$ there exists a uique solution $u\in C([0,T];H^\infty)$ of (1.1).*

Corollary 1.4. *Let $s\in\mathbf{R}$ and put* $\mathrm{Re}\ b(x)=(\mathrm{Re}\ b_1(x),\dots,\mathrm{Re}\ b_d(x))$. *Suppose $\lambda(t)$ is a positive non-increasing function in $C([0,\infty))\cap L^1(0,\infty))$. If*

(1.10) $|\mathrm{Re}\ b(x)|\leq\lambda(|x|)$,

then (1. 2) satisfies (A1) and (A2) with

(1.11) $\|u\|_{X^s}^2=(\lambda(|x|)D\rangle^{s+\frac{1}{2}}u,\ \langle D\rangle^{s+\frac{1}{2}}u)+C_s\|u\|_s^2.$

Therefore Theorem 1.1 is applicable.

Corollary 1.5. *Let $s\in\mathbf{R}$ and suppose $\lambda(t)$ is a positive non-increasing function in $C([0,\infty))$ satisfying $\int_0^t\lambda(\tau)\,d\tau\leq L\log(t+1)+C$ with $L, C>0$. If*

(1.10) $|\mathrm{Re}\ b(x)|\leq\lambda(|x|)$,

then (1. 2) satisfies (A1) and (A3). Therefore Theorem 1.2 is applicable.

Remark 1.6. We can take $\gamma=2L$ if $|\mathrm{Re}\ b(x)|\leq L\langle x\rangle^{-1}$ and $|\mathrm{Im}\ \partial_j b_k(x)|\leq C(\log\langle x\rangle)^{-1}$ with $C>0$. In this case p can be chosen as follows:

$$p(x,\xi)=\frac{Mx\cdot\xi}{\langle x\rangle\langle\xi\rangle}+Lf\left(\frac{x\cdot\xi}{\langle x\rangle\langle\xi\rangle}\right)\log\frac{|x\cdot\xi|}{\sqrt{1+|x|^2+|\xi|^2}}.$$

Here $M>0$ is a large constant and f is a real valued function in $C^\infty(\mathbf{R})$ such that $f(t)=0\ (|t|<1-2\varepsilon), =1\ (t>1-\varepsilon), -1\ (t<-1+\varepsilon)$ and $f'(t)\geq0$ with $0<\varepsilon\ll1$.

On the well-posedness of the Cauchy problem for Schrödinger type equations there seems to be a gap between the necessity and sufficiency.

For the necessity there are works such as [Mi 1, 2], [Ichi 3, 4] in the L^2 case, and [Ichi 2], [Ta 2], [Ha] in the H^∞ case. We quote the necessary condition from [Ichi 3] in a little changed form to make the comparison with (A2) easy.

Theorem (W. Ichinose). *Let $a_2=\sum_{j,k=1}^d a_{jk}(x)\xi_j\xi_k$, $a_1=\sum_{j=1}^d b_j(x)\xi_j$ and $a_0=c(x)$, where $a_{jk}, b_j, c\in\mathfrak{B}^\infty(\mathbf{R}^d)$, $a_{jk}=a_{kj}\in\mathbf{R}$ and $C_1|\xi|^2\leq|a_2(x,\xi)|\leq C_2|\xi|^2$ with $C_j>0$. If (1. 1) is L^2 well posed on $[0, T]$ (see [Ichi 3] for the precise definition), then*

$$(1.12)\qquad \inf_{(t,y,\eta)\in[0,T]\times\mathbf{R}^d\times\mathbf{R}^d}\mathrm{Re}\int_0^t a_1(X(\tau,y,\eta),\ \Xi(\tau,y\ \eta))\,d\tau>-\infty.$$

Here $(X(t,y,\eta),\ \Xi(t,y,\eta))$ are the integral curve of the Hamilton vector field $H_{a_2}=\sum_{j=1}^d(\partial_{\xi_j}a_2\partial_{x_j}-\partial_{x_j}a_2\partial_{\xi_j})$ passing through (y,η) at $t=0$.

The simplest condition to assure (1.12) is the existence of a bounded real valued function $p(x,\xi)$ of class C^1 such that $H_{a_2}p+\mathrm{Re}\ a_1\geq-C$ with some $C>0$, which is the origin of (A2). In general we can not hope that $p\in S_{1,0}^0$. Even so this is the case of (1.2) (see Corollary 1.4). Similarly (A3) originates in the necessary condition for the H^∞ well-posedness.

For the sufficiency of the well-posedness of (1.2) there are works such as [Mi 2], [Ichi 1], [Ta 3] and [Ba]. These works all rely on the method given in [Mi 2], based on the $S_{0,0}$ calculus. So they can not help to assume some conditions on Im $b(x)$ in addition to $b(x)\in\mathfrak{B}^\infty(\mathbf{R}^d)$. In contrast the author uses rather simple energy method, which is based on good symbol classes $S_{1,0}$ or $S(m, (\log\langle\xi\rangle)^2|dx|^2+\langle\xi\rangle^{-2}(\log\langle\xi\rangle)^2|d\xi|^2)$, and formulates the sufficient condition in a stable way. The defect of this approach is that he can not handle the delicate case treated in [Mi 2], [Ichi 1] and [Ta 3].

Acknowledgement. At the preliminarly stage of this paper, I proved Theorem 1.1 and Corollarly 1.4: however. I proved Corollarly 1.5 under the conditions $|\mathrm{Re}\, b_j(x)|\leq C\langle x\rangle^{-1}$, $|\mathrm{Im}\partial_k b_j(x)|\leq C(\log\langle x\rangle)^{-1}$, $1\leq j, k\leq d$ with $C>0$ (see Remark 1.6).

Soon after I informed Mr. Baba of this approach, he suggested that to add the linearly time-dependent term to p in (A3) might eliminate the second condition. I thank him for this advice.

Notation. For general notation, especially concerning the Weyl calculus, see [Hö, chapter 18].

$$\langle\xi\rangle=(10+|\xi|^2)^{\frac{1}{2}}\ (\xi\in\mathbf{R}^d).\quad L^2=L^2(\mathbf{R}^d),\ (\cdot,\cdot)=(\cdot,\cdot)_{L^2},\ \|\cdot\|=\|\cdot\|_{L^2}.$$

$$H^s=H^s(\mathbf{R}^d)=\{u\in S'(\mathbf{R}^d)\ :\ \langle\xi\rangle^s\hat{u}(\xi)\in L^2\},\ \|u\|_s=\|\langle\xi\rangle^s\hat{u}(\xi)\|.$$

$$H^\infty=\bigcap_{s\in\mathbf{R}}H^s,\ H^{-\infty}=\bigcup_{s\in\mathbf{R}}H^s,\ C([0,T];H^{-\infty})=\bigcup_{s\in\mathbf{R}}C([0,T];H^s).$$

$$\mathfrak{B}^\infty=\mathfrak{B}^\infty(\mathbf{R}^d)=\{f\in C^\infty(\mathbf{R}^d)\ :\ \partial^\alpha f\in L^\infty \text{ for all } \alpha\}.$$

For $S^m_{\rho,\delta}$ and $S(m,g)$, see [Hö, chapter 18]. $C([0,T];w-S(m,g))$ is the set of all $\phi\in C([0,T];C^\infty(\mathbf{R}^d\times\mathbf{R}^d))$ such that $\{\phi(t,\cdot,\cdot)\}_{0\leq t\leq T}$ is bounded in $S(m,g)$.

For $p\in S(m,g)$, $u\in S$,

$$(p^w(x,D)u)(x)=\frac{1}{(2\pi)^d}\iint e^{i(x-y)\cdot\xi}\,p\left(\frac{x+y}{2},\xi\right)u(y)\,dyd\xi;$$

$\sigma(p^w(x,D))=p(x,\xi)$.

2. Proofs

Theorems 1.1 and 1.2 follow the a priori estimates in Lemmas 2.4 and 2.3 respectively with standard argument (see, for examples, [Hö, the proof of theorem 23.1.2, p.387]).

In this section we abbreviate $S(m, |dx|^2+\langle\xi\rangle^{-2}|d\xi|^2)$ to $S(m)$ and $p^w(x,D)$ to $p(x,D)$.

Lemma 2.1. *Let $\phi\in S(\log\langle\xi\rangle)$ and suppose there exist real numbers m_1, m_2, C_1 and C_2 such that*

$$m_1\log\langle\xi\rangle+C_1\leq\phi(x,\xi)\leq m_2\log\langle\xi\rangle+C_2.$$

Then

(1) $e^{\phi}\in S(\langle\xi\rangle^{m_2}, g)$, $g=(\log\langle\xi\rangle)^2|dx|^2+\langle\xi\rangle^{-2}(\log\langle\xi\rangle)^2|d\xi|^2$.

(2) *There exists $q\in S(\langle\xi\rangle^{-2}(\log\langle\xi\rangle)^4)$ such that*

$$\begin{cases} k_1(x,D)k(x,D)=I_d+r(x,D)\\ k(x,D)k_1(x,D)=I_d+r_1(x,D)\end{cases}$$

where $k=e^{\phi}$, $k_1=e^{-\phi}(1+q)$ and $r, r_1\in S^{-\infty}$.

(3) *There exist $C, C'>0$ such that for all $u\in H^{\infty}$*

$$\|u\|_{m_1}\leq C\|k(x,D)u\|+\|r(x,D)u\|_{m_1}\leq C'\|u\|_{m_2}$$

Proof. (1) is verified by simple calculation.

(2) Since $\sigma\ (e^{-\phi}(x,D)e^{\phi}\ (x,D))\sim\sum_{j=0}^{\infty}p_j$ with

$$p_j=\frac{1}{j\,!}\left(\frac{i(D_\xi\cdot D_y-D_\eta\cdot D_x)}{2}\right)^j e^{-\phi}(x,\xi)e^{\phi}(y,\eta)|_{(y,\eta)=(x,\xi)}$$
$$\in S((\langle\xi\rangle^{-1}(\log\langle\xi\rangle)^2)^j),\ j=0,1,\dots,$$

it follows that

$$e^{-\phi}(x,D)e^{\phi}(x,D)=1-p(x,D),\quad p\in S(\langle\xi\rangle^{-2}(\log\langle\xi\rangle)^4).$$

With $q'\sim\sum_{j=0}^{\infty}\sigma\ (p(x,D)^j)$ and $q=\sigma(q'(x,D)e^{-\phi}(x,D))e^{\phi}\in S(\langle\xi\rangle^{-2}(\log\langle\xi\rangle)^4)$ we obtain

$$((1+q)e^{-\phi})\ (x,D)e^{\phi}(x,D)=1+r(x,D),\quad r\in S^{-\infty}.$$

Similarly for some $q_1\in S(\langle\xi\rangle^{-2}(\log\langle\xi\rangle)^4)$

$$e^{\phi}(x,D)\ ((1+q_1)e^{-\phi})\ (x,D)=1+r_1(x,D),\quad r_1\in S^{-\infty}.$$

From these we get $((q-q_1)\,e^{-\phi})\ (x,D)\in\mathrm{Op}\ S^{-\infty}$, that is, $q-q_1\in S^{-\infty}$. This proves (2).

(3) By (2) we obtain

$$\|u\|_{m_1}\leq\|k_1(x,D)k(x,D)u\|_{m_1}+\|r(x,D)u\|_{m_1}$$
$$\leq C\|k(x,D)u\|+\|r(x,D)u\|_{m_1}\leq C'\|u\|_{m_2}.$$

Remark. If $\phi\in C([0,T]: w-S(\log\langle\xi\rangle))$, then $e^{\phi}\in C([0,T]: w-S$

$(\langle\xi\rangle^{m_2}, g))$.
Moreover we can take $q\in C([0, T]: w-S(\langle\xi\rangle^{-2}(\log\langle\xi\rangle)^4))$, $r, r_1\in C([0, T]: S^{-\infty})$.

Lemma 2.2. *Let a satisfy (A0) and let* $\psi\in C^1([0, T]: w-S(\log\langle\xi\rangle))$. *Suppose*

$$(2.1)\quad \partial_t\psi+H_{a_2+\mathrm{Im}a_1}\psi-\mathrm{Re}\ a_1\leq C_0,\quad (t, x, \xi)\in[0, T]\times\mathbf{R}^d\times\mathbf{R}^d$$

$$(2.2)\quad m_1\log\langle\xi\rangle+C_1\leq\psi(t, x, \xi)\leq m_2\log\langle\xi\rangle+C_2,\quad (t, x, \xi)\in[0, T]\times\mathbf{R}^d\times\mathbf{R}^d$$

with $C_0, C_1, C_2, m_1, m_2\in\mathbf{R}$, *then*

$$(2.3)\qquad N(u(t))\leq C\Big(N(u(0)+\int_0^t N(f(\tau))d\tau\Big),\ 0\leq t\leq T$$

$$(2.4)\qquad \int_0^t\widetilde{N}(u(t))^2d\tau\leq C'\Big((N(u(0))^2+\int_0^t N(f(\tau))^2d\tau\Big),\ 0\leq t\leq T$$

for $u\in C^1([0, T]; H^\infty)$. *Hare* $f(t)=(\partial_t+a(x, D))u(t)$, $K=K(t)=e^\psi(t, x, D)$, *and*

$$N(u(t))^2=\|K(t)u(t)\|^2+\|u(t)\|^2_{m_1-1}$$
$$\widetilde{N}(u(t))^2=((-\psi_t-\{a_2+\mathrm{Im}\ a_1, \psi\}+\mathrm{Re}\ a_1)(t, x, D)Ku, Ku)+C''\|Ku\|^2$$

with a constant $C''>0$ *large enough to ensure* $\widetilde{N}(u(t))\geq\|K(t)u(t)\|$.

Proof. Let $u\in C^1([0, T]; H^\infty)$ and set $f=(\partial_t+a(x, D))u$. Put $k(t, x, \xi)=\exp(\psi(t, x, \xi))\in C^1([0, T]: w-S(\langle\xi\rangle^{m_2}, g))$. By Lemma 2.1 there exists $q\in C([0. T]: w-S(\langle\xi\rangle^{-2}(\log\langle\xi\rangle)^4))$ such that

$$\begin{cases}\tilde{k}(t, x, D)k(t, x, D)=\mathrm{Id}+r_1(t, x, D)\\ k(t, x, D)\tilde{k}(t, x, D)=\mathrm{Id}+r_2(t, x, D)\end{cases}$$

where $\tilde{k}=e^{-\psi}(1+q)\in C([0, T]: w-S(\langle\xi\rangle^{-m_1}, g))$ and $r_1, r_2\in C([0, T]: S^{-\infty})$. For simplicity we denote pseudo-differntial operators $p(t, x, D)$ by the corresponding capital letter $P=P(t)$. We have

$$\begin{aligned}\frac{d}{dt}\|K(t)u(t)\|^2&=2\mathrm{Re}(K_t(t)u(t)+K(t)(-Au(t)+f(t)), K(t)u(t))\\&=2\mathrm{Re}((K_t+[A, K]-AK)u, Ku)+2\mathrm{Re}(Kf, Ku)\\&=2\mathrm{Re}(((K_t+[A, K])\widetilde{K}-A)Ku, Ku)\\&\quad+2\mathrm{Re}(R_3u, Ku)+2\mathrm{Re}(Kf, Ku).\end{aligned}$$

Here $r_3\in C([0, T]: S^{-\infty})$. Since

$$\sigma(K_t\widetilde{K})\equiv\psi_t,$$

$$\sigma([A, K])k^{-1}\equiv\frac{1}{i}\{a, \psi\},$$

$$\sigma([A, K]\widetilde{K})\equiv\frac{1}{i}\{a, \psi\}+\frac{1}{2}\{\{a, \psi\}, \psi\},$$

$$\mathrm{Re}\ \sigma([A, K]\widetilde{K})\equiv\{a_2+\mathrm{Im}\ a_1, \psi\}$$

modulo $C([0, T]: w-S(1, g))$, it follows that

$$\frac{d}{dt}\|Ku(t)\|^2\leq 2((\psi_t+\{a_2+\mathrm{Im}\ a_1, \psi\}-\mathrm{Re}\ a_1)(t, x, D)Ku, Ku)$$
$$+2\|R_3u\|\cdot\|Ku\|+2C_1\|Ku\|^2+2\|Kf\|\cdot\|Ku\|$$

with some $C_1>0$. Similarly we have a rougher estimate

$$\frac{d}{dt}\|u(t)\|^2_{m_1-1}\leq 2(C_2\|u\|_{m_1}+\|f\|_{m_1-1})\|u\|_{m_1-1}$$
$$\leq 2(C_3\|Ku\|+C_3\|u\|_{m_1-1}+\|f\|_{m_1-1})\|u\|_{m_1-1}.$$

In the last inequality we use Lemma 2.1 (3). By adding the both estimates we obtain with $\delta>0$

$$(2.5)\qquad \frac{d}{dt}N(u(t))\leq C_4N(u(t))+N(f(t)),$$

$$(2.6)\qquad \frac{d}{dt}N(u(t))^2\leq-\delta\widetilde{N}(u(t))^2+C_5N(u(t))^2+N(f(t))^2.$$

From (2.6)

$$\frac{d}{dt}e^{-C_5t}N(u(t))^2+\delta e^{-C_5t}\widetilde{N}(u(t))^2\leq e^{-C_5t}N(f(t))^2.$$

By Integrating the both sides from 0 to t,

$$e^{-C_5t}N(u(t))^2+\delta\int_0^t e^{-C_5\tau}\widetilde{N}(u(\tau))^2d\tau\leq N(u(0))^2+\int_0^t e^{-C_5\tau}N(f(\tau))^2d\tau$$

wich implies (2.4). Similarly (2.5) leads to (2.3).

Lemma 2.3. *Let* $s\in\mathbf{R}$ *and assume (A0), (A1) and (A3). Then there exist* $\gamma, C_1, C_2>0$ *such that*

$$(2.7)\qquad \|u(t)\|_{s-\gamma}\leq C_1(\|u(0)\|_s+\int_0^t\|(\partial_t+a(x, D))u(\tau)\|_sd\tau),$$

$$(2.8)\qquad \|u(t)\|_{s-\gamma}\leq C_2(\|u(T)\|_s+\int_0^T\|(\partial_t+a(x, D)^*)u(\tau)\|_sd\tau),$$

for all $0\leq t\leq T$ *and* $u\in C^1([0, T]; H^\infty)$. *Here* $\gamma>0$ *is independent of* s.

Proof. Take $m\geq 0$ satisfying

$$H_{a_2+\mathrm{Im}a_1}p+\mathrm{Re}\ a_1+m\ \log\langle\xi\rangle\geq -C$$

with a constant $C>0$. Put

$$\phi(t, x, \xi)=-p(x, \xi)+(s-mt)\log e(x, \xi).$$

Then ϕ satisfies (2.1), (2.2) wich $C_j\in\mathbf{R}$, $m_1=s-M_2-mT$, $m_2=s-M_1$ if

$$M_1\log\langle\xi\rangle+C'\leq p(x, \xi)\leq M_2\log\langle\xi\rangle+C''.$$

By Lemma 2.2 and Lemma 2.1 (3)

$$\|u(t)\|_{m_1}\leq C_5(\|u(0)\|_{m_2}+\int_0^t\|(\partial+a(x, D))u(\tau)\|_{m_2}d\tau)$$

with $C_5>0$, which implies (2.7) with $\gamma=M_2-M_1+mT$. If we replace t by $T-t$, (2.8) is reduced to (2.7) since (A0), (A1) and (A3) are valid for ∂_t+A^* with p replaced by $-p$.

We can prove the next lemma similarly as Lemmas 2.2 and 2.3.

Lemma 2.4. *Let $s\in\mathbf{R}$ and assume (A0), (A1) and (A2). Then the following a priori estimates hold:*

(2.9) $$\|u(t)\|_s\leq C_1(\|u(0)\|_s+\int_0^t\|(\partial_t+a(x, D))u(\tau)\|_s d\tau),$$

(2.10) $$\|u(t)\|_s\leq C_2(\|u(T)\|_s+\int_t^T\|(\partial_t-a(x, D)^*)u(\tau)\|_s d\tau),$$

(2.11) $$\int_0^t((H_{a_2}p+\mathrm{Re}\, a_1)(x, D)\ \langle D\rangle^s u(\tau),\ \langle D\rangle^s u(\tau))d\tau$$
$$\leq C_3(\|u(0)\|_s^2+\int_0^t\|(\partial_t+a(x, D))u(\tau)\|_s^2 d\tau),$$

for all $0\leq t\leq T$ and $u\in C([0, T]; H^{s+2})\cap C^1([0, T]; H^s)$. Here $C_j=C_j(s, T)$.

Lemma 2.5. (1) *If $\lambda(t)$ is a positive non-increasing function in $C([0, \infty))\cap L^1([0, \infty))$, then there exists $\phi(x)=(\phi_1\cdots, \phi_d)$, $\phi_j\in\mathfrak{B}^\infty(\mathbf{R}^d)$ of real value, such that*

$$\phi'_{symm}(x)\equiv\left(\frac{1}{2}(\partial_j\phi_i+\partial_i\phi_j)\right)_{1\leq i, j\leq d}\geq\lambda(|x|)\,\mathrm{Id}>0$$

as possitive definite matrices.

(2) *If $\lambda(t)$ is a positive non-increasing function in $C([0, \infty))$ satisfying $\int_0^t\lambda(\tau)d\tau\leq L\log(t+1)+C$ with $L, C>0$, then there exists $\phi(x)=(\phi_1, \cdots, \phi_d)$ $\phi_j\in C^\infty(\mathbf{R}^d)$ of real value, such that $\partial_i\phi_j\in\mathfrak{B}^\infty(\mathbf{R}^d)$, $|\phi_j(x)|\leq L\log\langle x\rangle+C'$ with $C'>0$ and*

$$\phi'_{symm}(x)\geq\lambda(|x|)\,\mathrm{Id}>0$$

as positive definite matrices.

Proof. Take $\alpha\in C_0^\infty((0, 2))$ such that $\int\alpha(t)dt=1$, $0\leq\alpha\leq1$ and set $\tilde{\lambda}$

$(t)=\int \alpha(\tau)\lambda(t-\tau)d\tau$, where $\lambda(t)=\lambda(0)$ if $t<0$. Then $\lambda(t)\leq\tilde{\lambda}(t)$ and $\int_0^t\lambda(\tau)d\tau+C\geq\int_0^t\tilde{\lambda}(\tau)d\tau$ with $C>0$ if $t\geq 0$. Put $\phi(x)=(f(x_1),\cdots,f(x_d))$ with $f(t)=\int_0^t\tilde{\lambda}(|\tau|)d\tau$. Then

$$\phi'_{symm}(x)=\begin{pmatrix}\tilde{\lambda}(|x_1|) & & \\ & \cdots & \\ & & \tilde{\lambda}(|x_d|)\end{pmatrix}\geq\lambda(|x|)\,\mathrm{Id}.$$

Proof of Corollary 1. 4. Take $\phi(x)$ satisfying Lemma 2.5 (1) for $\lambda(t)$ and set $p(x,\xi)=\phi(x)\cdot\xi\langle\xi\rangle^{-1}$. Then we have

$$\{|\xi|^2, p\}=2\xi\cdot\nabla_x p(x,\xi)=2\phi'_{symm}(x)\xi\cdot\xi\langle\xi\rangle^{-1}\geq 2\lambda(|x|)|\xi|^2\langle\xi\rangle^{-1}$$

which leads to (A2) if $|\mathrm{Re}\, b(x)|\leq\lambda(|x|)$.

Proof of Corollary 1. 5. Let $\alpha\in C^\infty(\mathbf{R}^d)$ such that $\alpha=1\,(|x|<1)$, $\alpha=0\,(|x|>2)$, $0\leq\alpha\leq 1$ and put $\chi(x,\xi)=\alpha\left(\frac{x}{\langle\xi\rangle}\right)\in S(1,\ (1+|x|^2+|\xi|^2)^{-1}(|dx|^2+|d\xi|^2))$. Take $\phi(x)$ satisfying Lemma 2.5 (2) for $\lambda(t)$ and set $p(x,\xi)=\phi(x)\cdot\xi\langle\xi\rangle\chi(x,\xi)$.
Then we obtain

$$\begin{aligned}\{|\xi|^2, p\}&=2(\phi'_{symm}(x)\xi,\xi)\chi(x,\xi)\ \langle\xi\rangle^{-1}+2\phi(x)\cdot\xi\langle\xi\rangle^{-1}(\xi\cdot\nabla_x\chi)(x,\xi)\\&\geq 2(\phi'_{symm}(x)\xi,\xi)\chi(x,\xi)\ \langle\xi\rangle^{-1}-C_1\log\langle\xi\rangle-C_2\\&\geq 2\lambda\ (|x|)\chi(x,\xi)\ |\xi|-C_1\log\langle\xi\rangle-C_3\\&\geq 2\lambda\ (|x|)|\xi|-C_4\log\langle\xi\rangle-C_5.\end{aligned}$$

Here we use the fact that $\lambda(t)=O\left(\frac{\log t}{t}\right)$ as $t\to\infty$. This leads to (A3) if $|\mathrm{Re}\, b(x)|\leq\lambda(|x|)$.

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