J. Math. Kyoto Univ. (JMKYAZ)
34-2 (1994) 263-271

Uniform upper bounds for hypoelliptic kernels with drift

By

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Let G be a nilpotentLie group and let X_i be left invariant vector fields satisfying the Hörmander hypothesis. We consider the Haar measure over the Lie group. In this case, [V] have shown that the heat kernel of $\Delta=\frac{1}{2}\sum X_i^2$ satisfies for all t, x, y:

$$p_t(x, y) \leq \frac{C}{(\sqrt{t})^N} \tag{0.1}$$

where N is the grad of the Lie Algebra. Moreover that the uniform upper bound is relevant because in all t, $p_t(x, x) \geq \frac{C}{(\sqrt{t})^N}$. In other term, the long time and the small time estimates of the heat kernel are of the same type. Our purpose is concerned with a more complicated phenomenon which comes from the influence of the drift.

Let us suppose that $\Delta=\frac{1}{2}\sum X_i^2+X_0$ satisfies uniformly the weak Hörmander hypothesis over $\boldsymbol{R}^d$. There exists an r such that the space of the Lie Bracket of X_0, X_i, $i\neq 0$ of length $\leq r$, X_0 alone excluded, spanns uniformly in $x \in \boldsymbol{R}^d$. Moreover the vector fields X_i have bounded derivatives. In this situation, [Le1] introduced a "metric" depending on time t and solved the problem of knowing when for $t\leq t(x)$.

$$\frac{C}{\operatorname{Vol} B_t(x, \sqrt{t})} \leq p_t(x, y) \leq \frac{C}{\operatorname{Vol} B_t(x, \sqrt{t})} \tag{0.2}$$

for the balls $B_t(x, r)$ associated with this "metric" d_t. In particular, we can find a family of points $y_t \to x$ such the lower bound in (0.2) is true for $p_t(x, y_t)$. We show that these balls are relevant in order to get uniform estimates in small time $t\leq 1$ of $p_t(x, y)$. We get:

Theorem 1. *For all $x, y, t\leq 1$:*

$$p_t(x, y) \leq \frac{C}{\operatorname{Vol} B_t(x, \sqrt{t})}. \tag{0.3}$$

Communicated by Prof. S. Watanabe, July 13, 1992, Revised September 27, 1993

Let us compare these balls with the ball corresponding to $\frac{1}{2}\sum X_i^2$. Their volume is much bigger than these last (under weak Hörmander hypothesis, their volume can be zero). So in small time, we need to change the structure of the balls in order to get a relevant upper-bound of $p_t(x, y)$.

In order to show the theorem 1, we need to use the uniform estimates established in [HL] of the volume of the balls $B_t(x, \sqrt{t})$ and we use a refinement to techniques of [KS1] by using an upper-bound of the density of some components of hypoelliptic kernels with drift over nilpotent groups. This upper-bound ignores the problems which arise from the Bismut condition and which were deeply involved in the lower bound of (0.2). This leads to a change of a metric: the new metric $\tilde{d}_t(x, y)$ is not comparable to the old one $d_t(x, y)$. $\tilde{d}_t(x, y)$ is finite when $d_t(x, y)$ can be infinite (see [H]). Although this, the balls for $d_t(x, y)$ satisfy still the estimates of [HL].
Our second theorem is involved with the perturbation theory of a symmetric second order operator $\Delta=\frac{1}{2}\sum X_i^* X_i$ over a Riemannian manifold. Let us suppose that $p_t(x, y)<\frac{C}{\sqrt{t^N}}$ for $t \leq 1$ and that $p_t(x, y)<\frac{C}{\sqrt{t^M}}$ for $t \geq 1$ for $M \leq N$. $p_t(x, y)$ is the density of $\exp[-t\Delta]$ for the Riemannian measure. Let us suppose that $\exp[-t\Delta]1=1$. We introduce a divergence free vector field X_0 and the perturbated Laplacian $\Delta_p=\Delta+X_0$.

Theorem 2. *Let us suppose that* $\exp[-t\Delta_p]1=1$ and $\exp[-t\Delta_p^*]1=1$. $\exp[-t\Delta_p]$ *has a heat kernel* $p_{t,p}(x, y)$ *which satisfies the same estimates as* $p_t(x, y)$.

The proof of this theorem is based upon the Nash inequality ([CKS], [KS2]). The theorem 1 shows that the perturbation theory does not give the right estimates of the perturbated semi-group, at least in small time.

We thank S. Albeverio and L. Saloff-Coste for helpful discussions as well as the von Humboldt foundation for financial support.

1. Proof of the first theorem

Let h be and element of the Cameron-Martin space and let $X_0, X_i, i=1, \ldots, m$ vector fields satisfying uniformly the weak Hörmander hypothesis over $\boldsymbol{R}^d$. Let $x_{s,t}(h)$ be the horizontal curve associated with the operator $\frac{1}{2}\sum X_i^2+X_0$ in time t:

$$(1.1)\qquad \begin{aligned} dx_{s,t}(h) &= \sum X_i(x_{s,t}(h))\,dh_s^i + tX_0(x_{s,t}(h))\,ds \\ x_{0,t}(h) &= x\ . \end{aligned}$$

We suppose that the vector fields have bounded derivatives. These horizontal curves were introduced in [BAL1], [BAL2], [Le1]. Let $d_t(x, y)$ be the "distance" associated with $x_{1,t}(h)$, $(d_t^2(x, y) = \inf_{x_{1,t}(h)=y}\|h\|^2)$ and let $B_t(x, r)$ and let $B_t(x, r)$ be the associated balls. In order to get estimates of the volume of these balls ([Le1] in the non-uniform case, [HL] in the uniform case), let us give some definitions:

If $\alpha = (i_1, \dots, i_m)$ is a multi-index, let us write $X_{[\alpha]} = [X_{i_1}[X_{i_2}\dots[X_{i_{m-1}}, X_{i_m}]\dots]$. Its weight $\|\alpha\|$ is $m +$ *number of* X_0 in α. If $I = (\alpha_1, \dots, \alpha_d)$ is a d-uple of multi-indices, $\alpha_i = 0$ excluded, we put $\lambda_I(x) = \det(X_{[\alpha_i]}(x))$. The weight of I is by definition $\|I\| = \sum\|\alpha_i\|$. x_t is $x_{1,t}(0)$ the point in (1.1) corresponding to $h = 0$. Following [HL], we have when $t \leq 1$:

$$C\sum_{\|I\|\leq 2r}|\lambda_I(x_t)|\sqrt{t}^{\|I\|} \leq \mathrm{Vol}B_t(x, \sqrt{t}) \leq C\sum_{\|I\|\leq 2r}|\lambda_I(x_t)|\sqrt{t}^{\|I\|}\,. \tag{1.2}$$

If we use the lemma (2.5) of [NSW], (1.2) is equivalent to:

$$C\sum_{\|I\|\leq 2r}|\lambda_I(x)|\sqrt{t}^{\|I\|} \leq \mathrm{Vol}B_t(x, \sqrt{t}) \leq C\sum_{\|I\|\leq 2r}|\lambda_I(x)|\sqrt{t}^{\|I\|}\,. \tag{1.3}$$

Let us now introduce the solution of the Stratonovitch differential equation:

$$\begin{aligned} &dx_{s,t}(\sqrt{t}dw) = \sqrt{t}\sum X_i(x_{s,t}(\sqrt{t}dw))dw_s^i + tX_0(x_{s,t}(\sqrt{t}dw))ds \\ &x_{0,t}(\sqrt{t}dw) = x \end{aligned} \tag{1.4}$$

which has a smooth density $p_t(x, y)$. It is also the heat kernel in the time t associated with the operator $\frac{1}{2}\sum X_i^2 + X_0$. We begin now to prove (0.3). We write:

$$x_{1,t}(\sqrt{tdw}) = \exp\Big[tX_0 + \sum_{\|\alpha\|\leq r'}\sqrt{t}^{\|\alpha\|}X_{[\alpha]}F_{(\alpha)}(dw)\Big](x) + \mathrm{Rest}. \tag{1.5}$$

where r' is big enough, independent of x, and where we have taken the summation over an Hull basis, such that the $F_{(\alpha)}(dw)$ have a nice density which comes from the Malliavin calculus ([Le1], [Ta]). Following [Le1], it is enough to show that the density $q_{\sqrt{t}}(x, y)$ of the measure $\mu_{\delta,\sqrt{t}}(x)$

$$f \to E\Big[f\Big(\exp\Big[tX_0 + \Sigma\sqrt{t}^{\|\alpha\|}X_{[\alpha]}F_{(\alpha)}(dw)\Big](x)\Big)\,;\,|F_{(\alpha)}(dw)| \leq \sqrt{t}^{\delta\|\alpha\|}\Big] \tag{1.6}$$

is uniformly bounded by $\frac{C}{\mathrm{Vol}\,B_t(x, t)}$ for some $\delta < 1$. Moreover, each $F_{(\alpha)}(dw)$ is a sum of iterated integrals of the length of α and containing exactly the same indices as α. From [BAL1], appendix, we deduce that there exists a constant $C > 0$ such that:

$$E[\exp[C\sup_{\alpha}|F_{(\alpha)}(dw)|^{\frac{2}{\|\alpha\|}}]] < \infty \tag{1.7}$$

If we regularize the function $z_\alpha \to \sup_\alpha |z_\alpha|^{\frac{2}{\|\alpha\|}}$, and if we proceed as in [Le2], final remark, we deduce that the density $\tilde{q}(z)$ of $F_{(\alpha)}(dw)$ satisfies:

$$q(z) \leq C\exp\left[-C\sup_\alpha |z_\alpha|^{\frac{2}{\|\alpha\|}}\right] . \tag{1.8}$$

So we have only to give an upper-bound of the density $\tilde{q}_{\sqrt{t}}(x, y)$ of the measure $\tilde{\mu}$:

$$f \to \frac{1}{\sqrt{t}^{\Sigma\|\alpha\|}} \int_{|z_\alpha| \leq \sqrt{t}^{(1-\delta)\|\alpha\|}} f\Big(\exp\Big[tX_0 + \sum X_{[\alpha]} z_\alpha\Big](x)\Big) \exp\left[-C\sup_\alpha \frac{|z_\alpha|^{\frac{2}{\|\alpha\|}}}{t}\right] \prod dz_\alpha . \tag{1.9}$$

But in (1.9), we come back to a finite dimensional problem. We follow the method of [KS1], section 3, with some modifications which come from the algebraic structure of (1.9).

The first modification is that we need to modify the distance. We define $\tilde{d}_t(x, y)$ by:

$$\tilde{d}_t^2(x, y) = \inf_{\exp[tX_0 + \Sigma X_{[\alpha]} z_{[\alpha]}](x) = y} \sup |z_{(\alpha)}|^{\frac{2}{\|\alpha\|}} \tag{1.10}$$

Let us define by $m_{t,x,y}$ the set of $|z_{(\alpha)}| \leq \sqrt{t}^{(1-\delta)\|\alpha\|}$ for δ close to zero such that $\exp[tX_0 + \Sigma X_{[\alpha]} z_{(\alpha)}](x) = y$ and by $m_{0,x,x}$ the set of z_α, $|z_\alpha| \leq \sqrt{t}^{(1-\delta)\|\alpha\|}$ such that $\exp[\Sigma X_{[\alpha]} z_{(\alpha)}](x) = x$. We have a diffeomorphism between these two sets if δ is close to zero. Namely, let us choose u such that $\exp[tX_0 + \Sigma u_\alpha X_{[\alpha]}](x) = y$ and such that $\sup |u_\alpha|^{\frac{2}{\|\alpha\|}} = \tilde{d}_t^2(x, y)$. Let us write if $z_\alpha \in m_{0,x,x}$:

$$\exp\Big[tX_0 + \sum u_\alpha X_{[\alpha]}\Big] \exp\Big[\sum z_\alpha X_{[\alpha]}\Big](x) = y . \tag{1.11}$$

Let us apply the Campbell-Hausdorff formula in (1.11). We get

$$\exp\Big[tX_0 + \sum_{\|\alpha\| \leq r'} (u_\alpha + z_\alpha) X_{[\alpha]} + \sum_{\|\Gamma\| \leq r'} X_{[\Gamma]} \prod_{0 \in \Gamma, \alpha \in \Gamma, \alpha' \in \Gamma} (t u_\alpha z_{\alpha'})\Big](x) + \text{Rest} = y. \tag{1.12}$$

where Γ in (1.12) is a bracket of brackets, therefore a sum of Lie Brackets whose weight is the sum of weights from the component of Γ. If $|u_\alpha| \leq \sqrt{t}^{(1-\delta_1)\|\alpha\|}$ for δ_1 close enough to zero, the remaining part is smaller than $\sqrt{t}^N$ and N is a higher power than all the powers which appear in the bound of z_α. More precisely, $\sqrt{t}^N < C\sqrt{t}^{\|\alpha\|}$ for all α appearing in the Campbell-Hausdorff formula. So in (1.12), we can put the rest inside by using the implicit function theorem. We get:

$$\text{(1.13)}\quad \begin{aligned}&\exp\Big[tX_0+\sum_{\|\alpha\|\le r'}(u_\alpha+z_\alpha(t))X^{[\alpha]}+\\ &\sum_{\|\Gamma\|\le r'}X_{[\Gamma]}\prod_{0\in\Gamma,\alpha\in\Gamma,\alpha'\in\Gamma}(tu_\alpha z_\alpha(t))\Big](x)=y\ .\end{aligned}$$

We write now for a^α_Γ constant:

$$\text{(1.14)}\qquad X_{[\Gamma]}=\sum_{\|\alpha\|=\|\Gamma\|}a^\alpha_\Gamma X_{[\alpha]}$$

and since Γ is composed from at least two multi-indices, we can put:

$$\text{(1.15)}\qquad z'_\alpha(t)=u_\alpha+z_\alpha(t)+\sum_{\|\alpha\|=\|\Gamma\|}a^\alpha_\Gamma\prod_{0\in\Gamma,\beta\in\Gamma,\beta'\in\Gamma}tu_\beta z_{\beta'}(t)$$

Since only z'_β with $\|\beta'\|<\|\alpha\|$ appear in the product, $z_\alpha\to z'_\alpha(t)$ is a local diffeomorphism from $m_{0,x,x}$ into $m_{t,x,y}$. Moreover:

$$\text{(1.16)}\qquad |z_\alpha-z_\alpha(t)|^{\frac{2}{\|\alpha\|}}<Ct$$

if δ and δ_1 are close enough to zero: Let us call this diffeomorphism $S_{x,y,t}$, (1.16) playing an important role later. Namely, we have:

$$\text{(1.17)}\quad \begin{aligned}&|z'_\alpha(t)|^{\frac{2}{\|\alpha\|}}\ge C_1|z_\alpha|^{\frac{2}{\|\alpha\|}}-C_2|u_\alpha|^{\frac{2}{\|\alpha\|}}\\ &-C_3\sum_{\|\Gamma\|=\|\alpha\|}\prod_{0\in\Gamma,\beta\in\Gamma,\beta'\in\Gamma}|u_\beta|^{\frac{2}{\|\beta\|}\frac{\|\beta\|}{\|\alpha\|}}|z_{\beta'}|^{\frac{2}{\|\beta'\|}\frac{\|\beta'\|}{\|\alpha\|}}t^{\frac{2}{\|\alpha\|}}-C_4t\ .\end{aligned}$$

The product $\prod$ contains at least 2 elements and does not contain only the same type of variables. If we make the distinction between $\sup|u_\alpha|^{\frac{2}{\|\alpha\|}}\ge C\sup|z_\alpha|^{\frac{2}{\|\alpha\|}}$ or not for a nice choice of C, we deduce that either:

$$\text{(1.18)}\quad \begin{aligned}&\sup_\alpha|z'_\alpha(t)|^{\frac{2}{\|\alpha\|}}\ge C_1\sup_\alpha|z_\alpha|^{\frac{2}{\|\alpha\|}}\\ &-C_2\sup_\alpha|u_\alpha|^{\frac{2}{\|\alpha\|}}-C_3\sup_{\alpha,p,q}|u_\alpha|^{\frac{2}{\|\alpha\|}p}t^{2q}-C_4t\ .\end{aligned}$$

for $p+q=1$ describing a finite set or

$$\text{(1.19)}\quad \begin{aligned}&\sup_\alpha|z'_\alpha(t)|^{\frac{2}{\|\alpha\|}}\ge C_1\sup_\alpha|z_\alpha|^{\frac{2}{\|\alpha\|}}\\ &-C_2\sup_a|u_\alpha|^{\frac{2}{\|\alpha\|}}-C'_3\sup_{\alpha,p,q}|u_\alpha|^{\frac{2}{\|\alpha\|}p}t^{2q}-C_4t\ .\end{aligned}$$

for $p+q=1$ describing a finite set and for C'_3 small. Now, if we distinguish between $\sup|u_\alpha|^{\frac{2}{\|\alpha\|}}<Ct$ or not and if we make the same distinction in the second case, we deduce that in all the cases:

$$\sup_\alpha |z'_\alpha(t)|^{\frac{2}{\|\alpha\|}} \geq C_1 \sup |z_\alpha|^{\frac{2}{\|\alpha\|}} - C\tilde{d}_t^2(x, y) - Ct \tag{1.20}$$

under our assumption over z_α and u_α (This is true if $\tilde{d}_t(x, y) \leq \sqrt{t}^{(1-\delta_1)}$ for δ_1 close to zero).
Moreover, we have by definition since u_α reachs the distance $\tilde{d}$:

$$\sup_\alpha |z'_\alpha(t)|^{\frac{2}{\|\alpha\|}} \geq \sup_\alpha |u_\alpha|^{\frac{2}{\|\alpha\|}} = \tilde{d}_t^2(x, y) \ . \tag{1.21}$$

We deduce from this an analoguous of the lemma (3.23) in [KS1]:

$$\begin{aligned} C_1\Big[\sup_\alpha |z_\alpha|^{\frac{2}{\|\alpha\|}} + \tilde{d}_t^2(x, y)\Big] - Ct &\leq \sup_\alpha |z'_\alpha(t)|^{\frac{2}{\|\alpha\|}} \\ &\leq C_2\Big[\sup_\alpha |z_\alpha|^{\frac{2}{\|\alpha\|}} + \tilde{d}_t^2(x, y)\Big] + Ct \ . \end{aligned} \tag{1.22}$$

From this we deduce as in [KS1] that provided $\tilde{d}_t^2(x, y) \leq t^{(1-\delta_1)}$

$$\tilde{q}_t(x, y) \leq \frac{1}{\sqrt{t}^{\Sigma\|\alpha\|}} \int_{m_0(x,x)} \exp\Big[-C\frac{\sup_\alpha |z_\alpha|^{\frac{2}{\|\alpha\|}}}{t}\Big] \prod dz_\alpha \ . \tag{1.23}$$

Let us remark that if $\tilde{d}_t^2\,(x, y) > t^{(1-\delta_1)}$, we have clearly a rapid decay of $\tilde{q}_t(x, y)$. Moreover if $\tilde{d}_t^2(x, y) < t$, we deduce as in [KS1] that:

$$\tilde{q}_t(x, y) \geq \frac{1}{\sqrt{t}^{\Sigma\|\alpha\|}} \int_{m_0(x,x)} \exp\Big[-C\frac{\sup_\alpha |z_\alpha|^{\frac{2}{\|\alpha\|}}}{t}\Big] \prod dz_\alpha \ . \tag{1.24}$$

But

$$\int_{\tilde{d}_t(x, y) < \sqrt{t}} \tilde{q}_t(x, y)\, dy < 1 \tag{1.25}$$

If we denote by $\tilde{B}_t(x, r)$ the ball associated to $\tilde{d}_t(x, r)$, we deduce as in [KS1] 3.34:

$$\begin{aligned} &\sqrt{t}^{\Sigma\|\alpha\|} \int_{m_0(x, x)} 1_{|z_\alpha|^{\frac{2}{\|\alpha\|}} < \sqrt{t}} \prod dz_\alpha \mathrm{Vol}\, \tilde{B}_t(x, \sqrt{t}) \\ &= \sqrt{t}^{\Sigma\|\alpha\|} M_0(x, \sqrt{t})\, \mathrm{Vol}\, \tilde{B}_t(x, \sqrt{t}) \leq C \ . \end{aligned} \tag{1.26}$$

We follow [KS1]. For $\tilde{d}_t^2(x, y) < t^{(1-\delta_1)}$, we get:

$$\begin{aligned} \tilde{q}_t(x, y) &\leq \frac{1}{\sqrt{t}^{\Sigma\|\alpha\|}} \int_{m_0(x, x)} \exp\Big[-C\frac{\sup_\alpha |z_\alpha|^{\frac{2}{\|\alpha\|}}}{t}\Big] \prod dz_\alpha \\ &\leq \frac{C}{\sqrt{t}^{\|\alpha\|+2}} \int_0^1 s \exp\Big[-\frac{Cs^2}{t}\Big] M_0(x, s)\, ds \\ &\leq C \int_0^{\frac{1}{\sqrt{t}}} s \exp[-Cs^2]\, s^{\Sigma\|\alpha\|} \frac{1}{\mathrm{Vol}\, \tilde{B}_{s^2 t}(x, \sqrt{t}s)} ds \ . \end{aligned} \tag{1.27}$$

Moreover ([HL]), the volume of $\tilde{B}_t(x, \sqrt{t})$ satisfies the same extimates as

(1.2), (1.3) for $B_t(x, \sqrt{t})$. In particular, it satisfies the doubling property:

$$\mathrm{Vol}\,\tilde{B}_{4t}(x, 2\sqrt{t}) \leq C \mathrm{Vol}\,\tilde{B}_t(x, \sqrt{t}) \ . \tag{1.28}$$

We conclude from this as in [KS1] that if $d_t^2(x, y) < t^{(1-\delta_1)}$

$$\tilde{q}_t(x, y) \leq \frac{C}{\mathrm{Vol}\,\tilde{B}_t(x, \sqrt{t})} \leq \frac{C}{\mathrm{Vol}\,B_t(x, \sqrt{t})} \ . \tag{1.29}$$

Remark: Let us consider a function $g(x)$ from $\boldsymbol{R}$ into $[0, 1]$ such that $g(x) = x^2$ over a neighborhood of 0, equals at 0 only in 0 with bounded derivatives. Let us consider the generator $\frac{\partial}{\partial x^2} + g(x)\frac{\partial}{\partial y}$ over $\boldsymbol{R}^2$. $d_t((0, 0), (0, y)) = \infty$ if $y < 0$ and $d_t((0, 0), (0, y)) < \infty$ if $y \geq 0$ over a small neighborhood of 0. Moreover, if we consider the Bismut distance $d_{R,t}$ as in [HL] (or [BAL2]) over a little neighborhood of the departure, $d_t((0, 0), (0, 0))$ differs from $d_{R,t}((0, 0), (0, 0)) = \infty$. This shows us that the following estimate for $t \leq 1$ is wrong:

$$\begin{aligned} &\frac{C}{\mathrm{Vol}\,B_t((0, 0), \sqrt{t})} \exp\left[-\frac{d_{R,t}^2((0, 0), (0, y))}{C_1 t}\right] \\ &\leq pt((0, 0), (0, y)) \\ &\frac{C'}{\mathrm{Vol}\,B_t((0, 0), \sqrt{t})} \exp\left[-\frac{d_{R,t}^2((0, 0), (0, y))}{C_1' t}\right] . \end{aligned} \tag{1.30}$$

Namely the right side implies $p_t((0, 0), (0, 0)) = 0$ and since $d_{R,t}^2((0, 0), (0, y)) \to 0$ when $y > 0 \to 0$, the left side will imply $p_t((0, 0), (0, 0)) > 0$ by the continuity of the heat kernel (see [Le4] for logarithmic estimates of such kernels when $g(x) = x^n$).

2. Proof of the second theorem

Since Δ is symmetric, we have ([CKS]) the following Nash inequality:

$$\|f\|_2^2 \leq C\left(\mathscr{E}(f, f)^{\frac{M}{M+2}} + \mathscr{E}(f, f)^{\frac{N}{N+2}}\right) \tag{2.1}$$

where $\| \ \|_2^2$ is the L^2-norm, $E(f, f)$ the Dirichlet form of Δ for f of L^1-norm 1. Let $P_{t,p}$ be the perturbated semi-group. Since the divergence of X_0 is null, $\Delta_p + \Delta_p^* = 2\Delta$. Therefore

$$\frac{\partial}{\partial t}\left\|P_{t,p}f\right\|_2^2 = 2\mathscr{E}(P_{t,p}f, P_{t,p}f) \ . \tag{2.2}$$

Moreover there is no potential in Δ_p. We deduce as in [CKS] that the $L_{1\to 2}$ norm of $P_{t,p}$ is smaller than $\frac{C}{t^{\frac{N}{4}}}$ for $t < 1$ and than $\frac{C}{t^{\frac{M}{4}}}$ for $t > 1$. It is the same for $P^*_{t,p}$ which is associated with Δ^*_p. The proof follows as in [CKS].

Remark. The hypothesis that X_0 has no divergence can be interpreted in the following way, under suitable hypothesis; let Φ_t the flow of diffeomorphism associated to the equation

(2.3) $$dy_s = X_0(y_s)ds$$

and let us write

(2.4) $$x_t = \Phi_t(\tilde{y}_t)$$

where

(2.5) $$dx_s = \sum_{i>0} X_i(x_s)d\omega^i_s + X_0(x_s)ds + Y(x_s)ds\ .$$

We have ([Bi1])

(2.6) $$d\tilde{y}_s = \sum_{i>0} \Phi^{*-1}_s X_i(\tilde{y}_s)d\omega^i_s + \Phi^{*-1}_s Y_i(\tilde{y}_s)ds$$

since we consider a Stratonovich equation. Let $\tilde{p}_t\ (x, y)$ the heat kernel associated to $\tilde{y}_t$ and $p_t(x, y)$ the heat kernel associated with x_t. We have:

(2.7) $$E[f(x_t)] = \int p_t(x, y)f(y)dy = \int f(\Phi_t(y))\tilde{p}_t(x, y)dy\ .$$

This shows us since the flow Φ_t keeps the volume:

(2.8) $$p_t(x, y) = \tilde{p}_t(x, \Phi^{-1}_t(y))$$

for all $x, y, t>0$.

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