J. Math. Kyoto Univ. (JMKYAZ)
34-3 (1994) 531-538

Stationary measures for automaton rules 90 and 150

By

Munemi MIYAMOTO

This is a continuation of [3]. Let $\Omega = \{0,+1\}^Z$. A transformation Λ: $\Omega \rightarrow \Omega$ is defined as follows;

$$\Lambda x\ (i) = x\ (i-1) + x\ (i+1) \bmod 2\ ,$$

where $x \in \Omega$ and $i \in Z$. In [3] Λ was called *one-dimensional life game*. According to the classification of one-dimensional automata by Wolfram [5], this is rule 90. We are interested in the Λ-invariant measures on Ω. For $0 \leqq p \leqq 1$, let β_p be the distribution of the Bernoulli trials with density p. It is shown in [2,3,4] that $\beta_{1/2}$, the distribution of coin tossing, is Λ-invariant.

Furthermore, let M be the set of translation-invariant mixing measures on Ω and let Conv (M) be the convex hull of M, i.e., the set of convex combinations of measures in M. If we replace the adjective "mixing" with "ergodic", we have the set Conv (E) of all translation-invariant measures (the ergodic decomposition theorem). The behaviour of $\Lambda^n P$ as $n \rightarrow \infty$ for $P \in$ Conv (M) is quite different from that for $P \in$ Conv $(E) \setminus$Conv (M). First we see the behaviour for $P \in$ Conv (M). The following theorem is an improvement of Theorem 3 in [3].

Theorem 1. *Assume $P \in$ Conv (M). Then, $\Lambda^n P$ converges as $n \rightarrow \infty$ if and only if P is a convex combination of β_0, $\beta_{1/2}$ and β_1.*

Collorary (Theorem 1 in [3]). *Assume $P \in$ Conv (M). P is Λ-invariant if and only if P is a convex combination of β_0 and $\beta_{1/2}$.*

Remark that $\Lambda^n\ \beta_p$ does not converge as $n \rightarrow \infty$ unless $p = 0, 1/2, 1$. But Theorem 4 in [3] says that if $0 < p < 1$

$$\lim 1/N \sum_{n=0}^{N-1} \Lambda^n \beta_p = \beta_{1/2}.$$

It is natural to ask if there are any other Λ-invariant measures outside Conv (M) [1]. The answer is "Yes, there are infinitely many" [4]. Let us show this in more general setting.

Let $n \geqq 3$ be an odd integer. A configuration x_n in Ω is defined as follows;

$$x_n\ (i) = \begin{cases} 0 & \text{if } i = 0 \bmod n, \\ 1 & \text{otherwise.} \end{cases}$$

Received July 7, 1993

This x_n is periodic in space. Furthermore, x_n is periodic in time, i.e., we have the following lemma (see the proof of Theorem 1 in [4]).

Lemma 2. *For each odd $n \geqq 3$, there exists $m \geqq 1$ such that $\Lambda^m x_n = x_n$.*

Set $\nu_n = \sum_{j=0}^{n-1} \theta^j \delta_{x_n}/n$, where θ is the translation operator, and set $\mu_n = \sum_{j=0}^{m-1} \Lambda^j \nu_n/m$. It is clear that μ_n is Λ- and translation-invariant. We see that $\Lambda x_3 = x_3$, i.e., x_3 is a fixed point of Λ. The measure $\mu_3 = (\delta_{x_3} + \delta_{\theta x_3} + \delta_{\theta^2 x_3})/3$ is, therefore, ergodic. But, if $n \geqq 5$,

$$E = \{x_n, \theta x_n, \theta^2 x_n, \cdots\cdots, \theta^{n-1} x_n\}$$

is a translation-invariant set with $0 < \mu_n(E) < 1$. The inequality $\mu_n(E) < 1$ follows from $\Lambda x_n \notin E$ and $\mu_n(\{\Lambda x_n\}) > 0$. Thus we have

Theorem 2. *For each odd $n \geqq 3$, μ_n is Λ- and translation-invariant. The measure μ_3 is ergodic, but μ_n $(n \geqq 5)$ are not ergodic.*

If $n \geqq 5$, μ_n is a convex combination of the *ergodic* measures $\Lambda^j \nu_n$ $(0 \leqq j \leqq m - 1)$. Thus, the Λ-invariance of $\mu_n \in \mathrm{Conv}(E)$ does not imply the Λ-invariance of its ergodic components. On the contrary, Collorary to Theorem 1 says that the Λ-invariance of a convex combination of *mixing* measures implies the Λ-invariance of its components. In fact, its components must be β_0 and $\beta_{1/2}$.

We have Λ-invariant ergodic measures β_0, $\beta_{1/2}$ and μ_3. It is natural to ask if there are any other Λ-invariant ergodic measures. The answer is again "Yes, there are infinitely many". Let $p \geqq 2$ be an integer. For $1 \leqq i \leqq 2^p$, set $y_p(i) = 1$. For $i \geqq 2^p + 1$, define $y_p(i)$ successively as follows:

$$y_p(i) = \Lambda y_p(i - 2^p + 1) .$$

Lemma 2. 1) *y_p can be extended to $\{i \leqq 0\}$ so that y_p is periodic in space, i.e., $y_p = \theta^u y_p$ for some $u \geqq 1$.*
2) *$\Lambda y_p = \theta^v y_p$, where $v = 2^p - 1$.*

Set $\varepsilon_p = \sum_{j=0}^{u-1} \theta^j \delta_{y_p}/u$. Since y_p is periodic in space, it is clear that

$$\theta \varepsilon_p = \varepsilon_p,$$

$$\Lambda \varepsilon_p = \sum_{j=0}^{u-1} \theta^{j+v} \delta_{y_p}/u = \varepsilon_p .$$

If $E \subset \Omega$ is translation-invariant and $\varepsilon_p(E) > 0$, then $\varepsilon_p(E) = 1$. Thus we have

Theorem 3. *For each $p \geqq 2$, ε_p is Λ-invariant and ergodic.*

Let us prove Theorem 1 and Lemmata 1, 2. The following lemma plays the key role in the computation of Λ^n.

Lemma 3. *For any k it holds that*

$$\Lambda^{2^k} x(i) = x(i-2^k) + x(i+2^k) \mod 2.$$

Proof is easy.

To prove Theorem 1 let us introduce the Fourier transform of a probability measure μ on Ω. Let $\xi = (\xi(i);\ -\infty < i < +\infty)$ be a sequence of 0 and 1 with only finitely many 1's. For $\omega = (\omega(i);\ -\infty < i < +\infty) \in \Omega$, set $\langle \xi, \omega \rangle = \sum_{i=-\infty}^{+\infty} \xi(i)\omega(i)$. Denote the Fourier transform of μ by F (μ) or $\hat{\mu}$, i.e.,

$$\mathrm{F}\ (\mu)\ (\xi) = \hat{\mu}(\xi) = \int_\Omega (-1)^{\langle \xi, \omega \rangle} \mu(d\omega)\ .$$

We have, by Lemma 3,

$$\begin{aligned}\mathrm{F}(\Lambda^{2^n}\mu)\ (\xi) &= \int_\Omega (-1)^{\langle \xi, \Lambda^{2^n}\omega \rangle} \mu(d\omega) \\ &= \int_\Omega (-1)^{\langle \xi, \theta^{-2^n}\omega \rangle + \langle \xi, \theta^{2^n}\omega \rangle} \mu(d\omega)\ .\end{aligned}$$

If μ is in M, i.e., if μ is mixing and translation-invariant, then,

$$\lim \mathrm{F}\ (\Lambda^{2^n}\mu)\ (\xi) = \hat{\mu}(\xi)^2\ .$$

By the same argument we have

$$\lim \mathrm{F}(\Lambda^{2^{2n}+2^n}\mu)\ (\xi) = \hat{\mu}(\xi)^4\ .$$

Proof of Theorem 1. Take a probability measure π on M. Set

$$P\ (\cdot) = \int_M \mu(\cdot)\, d\pi(\mu) \in \mathrm{Conv}(M)\ .$$

By the above argument we see

$$\lim \mathrm{F}(\Lambda^{2^n}P)\ (\xi) = \int_M \lim \mathrm{F}\ (\Lambda^{2^n}\mu)\ (\xi)\, d\pi(\mu) = \int_M \hat{\mu}(\xi)^2 d\pi(\mu),$$

$$\lim \mathrm{F}\ (\Lambda^{2^{2n}+2^n}P)\ (\xi) = \int_M \lim \mathrm{F}\ (\Lambda^{2^{2n}+2^n}\mu)\ (\xi)\, d\pi(\mu) =$$

$$= \int_M \hat{\mu}(\xi)^4 d\pi(\mu)\ .$$

Assume $\Lambda^n P$ converges as $n \to \infty$. Since

$$\lim \mathrm{F}\ (\Lambda^{2^n}P)\ (\xi) = \lim \mathrm{F}(\Lambda^{2^{2n}+2^n}P)\ (\xi)\ ,$$

we have

$$\int_M \{\hat{\mu}(\xi)^2 - \hat{\mu}(\xi)^4\}\, d\pi(\mu) = 0\ ,$$

which implies $\hat{\mu}(\xi) = 0, \pm 1$ for a. a. $(\pi)\,\mu$.

Since $\lim \Lambda^n \beta_0 = \lim \Lambda^n \beta_1 = \beta_0$, we can assume $\pi(\{\beta_0, \beta_1\}) = 0$ .We have $\hat{\mu}(\xi) = 0$ for any $\xi \neq \cdots 000 \cdots$and for a. a. $(\pi)\,\mu$, which means $\mu = \beta_{1/2}$ for a.a. $(\pi)\,\mu$, i.e., $P = \beta_{1/2}$. The "only if" part of Theorem 1 is thus proved. The "if" part is clear, because $\beta_{1/2}$ is Λ-invariant.

Proof of Lemma 1. Let us prove Lemma 1 for odd $n \geqq 3$. We can write

$$\Lambda x(i) = \sum_{j \in \{\pm 1\}+i} x(j) \bmod 2 ,$$

$$\Lambda^2 x(i) = \sum_{j \in \{\pm 2\}+i} x(j) \bmod 2 .$$

Therefore,

$$\begin{aligned} \Lambda^3 x(i) &= \Lambda^2 \Lambda x(i) \\ &= \sum_{j \in \{\pm 2\}+i} \Lambda x(j) \bmod 2 \\ &= \sum_{j \in \{\pm 2\}+i} \{x(j-1) + x(j+1)\} \bmod 2 \\ &= \sum_{j \in \{\pm 2 \pm 1\}+i} x(j) \bmod 2 . \end{aligned}$$

Let $m = 2^k - 1 = 2^{k-1} + 2^{k-2} \cdots + 2 + 1$, where k will be specified later. Let

$$\begin{aligned} S &= \{\pm 2^{k-1} \pm 2^{k-2} \cdots \pm 2 \pm 1\} \\ &= \{-2^k+1, -2^k+3, \cdots, -1, +1, \cdots, 2^k-3, 2^k-1\} . \end{aligned}$$

We can easily see by Lemma 3

$$\Lambda^m(i) = \sum_{j \in S+i} x(j) \bmod 2 .$$

Since S and x_n are symmetric with respect to 0, it holds that

$$\begin{aligned} \Lambda^m x_n(0) &= \sum_{j \in S} x_n(j) \bmod 2 \\ &= 0 . \end{aligned}$$

Next we must show that

$$\Lambda^m x_n(i) = 1 \bmod 2 \quad (1 \leqq i \leqq n-1) .$$

We consider the pairs $\{-j+2i, j\}$. Remark that if j is in $S+i$ then $-j+2i$ is in $S+i$ and vice versa. We say that a pair $\{-j+2i, j\}$ in $S+i$ is *positive* if

$$x_n(-j+2i) + x_n(j) = 1 \bmod 2 .$$

If neither $-j+2i$ nor j is divisible by n, then the pair $\{-j+2i, j\}$ is not positive. It is impossible that both $-j+2i$ and j are divisible by n. So that it is sufficient to consider only pairs $\{-tn+2i, tn\}$ and $\{-tn, tn+2i\}$ with $t \geqq 0$. Let $\#_+$ ($\#_-$) be the number of pairs $\{-tn+2i, tn\}$ ($\{-tn, tn+2i\}$) in $S+i$ with $t > 0$, i.e., the number of t such that $0 < tn \leqq m+i$ ($0 < tn+2i \leqq m+i$). We separate the case $t=0$.

In case that i is odd, $S+i \subset 2Z$. Therefore, $\{-0n+2i, 0n\} = \{-0n, 0n+2i\}$ is in $S+i$. Since the pair $\{2i, 0\}$ is positive,

$$\Lambda^m x_n(i) = 1 + \#_+ + \#_- \bmod 2 .$$

We see that

$\#_+ - \#_-$

$=$the number of even t which satisfies $m-i<tn\leqq m+i$.

On the other hand we have

Lemma 4. *We can choose k so that $m=2^k-1$ is divisible by n.*

Set $q=m/n$, i.e., $m=nq$. Remark that q is odd. The inequality $m-i<tn\leqq m+i$ is equivalent to $-i<n\ (t-q)\leqq i$. Since q is odd but t must be even, it holds $|t-q|\geqq 1$, which implies $|n\,(t-q)|\geqq n>i$. Thus the inequality $m-i<tn\leqq m+i$ has no solution, i.e., $\#_+ - \#_- =0$. We have

$$\begin{aligned}\Lambda^m x_n(i) &= 1+\#_+ + \#_- \bmod 2\\ &= 1+\#_+ - \#_- \bmod 2\\ &= 1\ .\end{aligned}$$

In case that i is even, $S+i\subset 2Z+1$. The pair $\{-0n+2i, 0n\}=\{-0n, 0n+2i\}$ is not in $S+i$. Therefore,

$$\Lambda^m x_n(i)=\#_+ + \#_- \bmod 2\ .$$

We have

$\#_+ - \#_-$

$=$the number of odd t which satisfies $m-i<tn\leqq m+i$.

The inequality $m-i<tn\leqq m+i$, which is equivalent to $-i<n\,(t-q)\leqq i$, has the unique odd solution $t=q$. Thus $\#_+ - \#_- =1$. Therefore,

$$\begin{aligned}&\Lambda^m x_n(i)\\ &= \#_+ + \#_- \bmod 2\\ &= \#_+ - \#_- \bmod 2\\ &= 1\ .\end{aligned}$$

Lemma 1 is thus proved.

Proof of Lemma 4. Let p be a prime and let e be a natural number. Let us regard $Z/p^e Z$ as a group with multiplication. The multiples of p should be taken away, because they are nilpotent. The number of them is p^{e-1}. Therefore, the order of this group is equal to $p^e-p^{e-1}=(p-1)\ p^{e-1}$. 2 is an element of this group. Therefore, $2^{(p-1)p^{e-1}}=1$ in $Z/p^e Z$, hence $2^{s(p-1)p^{e-1}}=1$ in $Z/p^e Z$ for any $s\geqq 0$. Thus

$$2^{s(p-1)p^{e-1}}-1$$

is divisible by p^e for any $s\geqq 0$.

Let $n=p_1^{e_1}\,p_2^{e_2}\cdots p_r^{e_r}$ be the factorization of n into prime factors. Set $k=(p_1-1)p_1^{e_1-1}\ (p_2-1)p_2^{e_2-1}\cdots(p_r-1)p_r^{e_r-1}$. By the above argument 2^k-1 is divisible by $p_j^{e_j}$ for $1\leqq j\leqq r$, hence it is divisible by n.

Poof of Lemma 2. First remark that by definition of y_p

$$\begin{aligned}&y_p\ (i)=1\ \ (1\leqq i\leqq 2^p),\\ &y_p\ (i)=0\ \ (2^p+1\leqq i\leqq 2^{p+1}-2),\\ &y_p\ (2^{p+1}-1)=y_p(2^{p+1})=1\ .\end{aligned}$$

It is easy to see that for $k\geqq 1$ and $i>k\ 2^p$

$$y_p\ (i)=\Lambda^k y_p\ (i-k\ (2^p-1))\ .$$

For $k=2^{p-1}$ and $i>2^{2p-1}$, we have by Lemma 3

$$\begin{aligned} y_p\ (i) &=\Lambda^{2^{p-1}}\ y_p\ (i-2^{p-1}(2^p-1)) \\ &=y_p(i-2^{p-1}(2^p-1)-2^{p-1})+y_p\ (i-2^{p-1}(2^p-1)+2^{p-1}) \bmod 2 \\ &=y_p(i-2^{2p-1})+y_p\ (i-2^{2p-1}+2^p) \bmod 2\ . \end{aligned}$$

Using this, we have

$$\begin{aligned} y_p\ (i+2^{2p-1}) &=y_p\ (i)+y_p\ (i+2^p) \bmod 2 \\ &=1+0 \bmod 2\ (1\leqq i\leqq 2^p-2) \\ &=1\ . \end{aligned}$$

$$\begin{aligned} y_p(2^p-1+2^{2p-1}) &=y_p(2^p-1)+y_p\ (2^{p+1}-1) \bmod 2 \\ &=1+1 \bmod 2 \\ &=0, \end{aligned}$$

$$\begin{aligned} y_p(2^p+2^{2p-1}) &=y_p(2^p)+y_p\ (2^{p+1}) \bmod 2 \\ &=1+1 \bmod 2 \\ &=0\ . \end{aligned}$$

Therefore, we can see that for $1\leqq i\leqq 2^p$

$$y_p(i+2^{2p-1})=y_p\ (i+2)\ ,$$

which implies that $\{y_p\ (i) : i\geqq 1\}$ has the period $u=2^{2p-1}-2$. It is easy to extend y_p to $\{i\leqq 0\}$.

The second assertion in Lemma 2 is obvious by definition of y_p.

Analogous arguments are possible also for rule 150:

$$\tilde{\Lambda}x(i)=x(i-1)+x(i)+x(i+1) \bmod 2\ .$$

As to $\tilde{\Lambda}$ we have

Lemma 3'. *For any k it holds that*

$$\Lambda^{2^k}x\ (i)=x\ (i-2^k)+x\ (i)+x\ (i+2^k) \bmod 2\ .$$

Theorem 1'. *Assume $P\in \mathrm{Conv}(M)$. The following three conditions are equivalent to each other.*

1) *$\tilde{\Lambda}^n P$ converges as $n\to\infty$.*
2) *P is $\tilde{\Lambda}$-invariant.*
3) *P is a convex combination of β_0, $\beta_{1/2}$ and β_1.*

Outline of Proof. Take a probability measure π on M. Set

$$P(\cdot)=\int_M \mu(\cdot)\,\mathrm{d}\pi(\mu)\in \mathrm{Conv}\ (M)\ .$$

Assume $\tilde{\Lambda}^n P$ converges as $n\to\infty$. By the same argument as in the proof of Theorem 1, we see

$$\int_M \{\hat{\mu}(\xi)^3-\hat{\mu}(\xi)^9\}\ \mathrm{d}\pi(\mu)=0\ .$$

Let ξ_0 be a finite sequence of 0 and 1 and let

$$\xi=\cdots 000\xi_0 0^n\ \xi_0 000\cdots\ .$$

The above equality holds for this ξ. Since μ is mixing, letting $n\to\infty$, we have

$$\int_M \{\hat{\mu}(\xi_0)^6-\hat{\mu}(\xi_0)^{18}\}\ d\pi(\mu)=0\ .$$

This implies that P is a convex combination of β_0, $\beta_{1/2}$ and β_1.

The convergence of the Cesaro means for $\tilde{\Lambda}^n P$ can be proved by the Fourier transformation method [2].

We have infinitely many $\tilde{\Lambda}$-invariant measures outside Conv (M). Let $n\geqq 5$ be an odd integer. A configuration $\tilde{x}_n$ in Ω is defined as follows;

$$\tilde{x}_n\ (i)=\begin{cases} 0 \text{ if } i=0,\pm 1 \bmod n, \\ 1 \text{ otherwise.}\end{cases}$$

Lemma 1'. *For each odd $n\geqq 5$, there exists $m\geqq 1$ such that $\Lambda^m\tilde{x}_n=\tilde{x}_n$.*

Proof. By Lemma 4 we can choose k so that $m=2^{2k}-1$ is divisible by n. By Lemma 3' we see

$$\tilde{\Lambda}^m\tilde{x}(i)=\tilde{x}(i)+\sum_{0<3h\leqq 2^{2k}-1}\{\tilde{x}(3h-1+i)+\tilde{x}(3h+i)\}$$
$$+\sum_{0<3h\leqq 2^{2k}-1}\{\tilde{x}(-3h+i)+\tilde{x}(-3h+1+i)\} \bmod 2.$$

Setting $\tilde{z}\ (j)=\tilde{x}_n\ (j)+\tilde{x}_n\ (j+1)$ mod 2, we have

$$\tilde{\Lambda}^m\tilde{x}_n\ (i)=\tilde{x}_n\ (i)+\sum_{0<3h\leqq 2^{2k}-1}\{\tilde{z}(3h-1+i)+\tilde{z}(-3h+i)\} \bmod 2\ .$$

Remark that $\tilde{z}(3h-1+i)=1$ if and only if $3h-1+i=-2,+1$ mod n and that $3h-1+i=-2$ mod n means $3\ (h+1)-1+i=+1$ mod n. Let

$$h_0=\min\ \{h;\ 3h-1+i=-2 \bmod n,\ h\geqq 0\}\ ,$$
$$h_1=\max\ \{h;\ 3h-1+i=-2 \bmod n,\ 3h\leqq 2^{2k-1}\}\ .$$

We have

$$\sum_{0<3h\leqq 2^{2k}-1}\tilde{z}(3h-1+i)=\sum_{0<3h\leqq 2^{2k}-1,\ 3h-1+i=-2,+1 \bmod n}\tilde{z}(3h-1+i)$$
$$=\tilde{z}(3h_0-1+i)+\tilde{z}(3(h_0+1)-1+i)$$
$$+\sum_{h_0<h<h_1,\ 3h-1+i=-2 \bmod n}\{\tilde{z}(3h-1+i)+\tilde{z}(3(h+1)-1+i)\}$$
$$+\tilde{z}(3h_1-1+i)+\tilde{z}(3(h_1+1)-1+i) \bmod 2\ .$$

Note that $m=2^{2k}-1=(1+3)^k-1$ is a multiple of 3 and that three equalities $h_0=0$, $h_1=2^{2k}-1$ and $i=-1$ mod n are mutually equivalent. In case $h_0=0$, the first and the last terms must be omitted. In any case

$$\sum_{0<3h\leqq 2^{2k}-1} \tilde{z}(3h-1+i)=0 \bmod 2 .$$

In the same way we can see

$$\sum_{0<3h\leqq 2^{2k}-1} \tilde{z}(-3h+i)=0 \bmod 2 .$$

Thus we have

$$\tilde{\Lambda}^m \tilde{x}_n (i)=\tilde{x}_n (i) .$$

DIVISION OF MATHEMATICS,
FACULTY OF INTEGRATED HUMAN STUDIES,
KYOTO UNIVERSITY

References

[1] R. Durrett, Lecture notes on particle systems and percolation, Wadsworth, 1988.

[2] D. A. Lind, Applications of ergodic theory and sofic systems to cellular automata, Physica D, **10** (1984), 36-44.

[3] M. Miyamoto, An equilibrium state for a one-dimensional life game, J. Math. Kyoto Univ., **19** (1979), 525-540.

[4] S. S. Vallander, One system of automata with local interactions, Multicomponent random systems, Adv. Prob. relat. Top., **6** (1980), 577-587.

[5] S. Wolfram, Statistical mechanics of cellular automata, Rev. of Modern Phys., **55** (1983), 601-644.