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On the Picard number of Fano 3-folds with terminal singularities

To memory of Boris Moishezon

By

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Introduction

Here we continue investigations started in [N6], [N7].

Algebraic varieties we consider are defined over field $\mathbf{C}$ of complex numbers.

In this paper, we get a final result on estimating the Picard number $\rho=\dim N_1(X)$ of a Fano 3-fold X with terminal $\mathbf{Q}$-factorial singularities if X does not have small extremal rays and its Mori polyhedron does not have faces with Kodaira dimension 1 or 2. One can consider this class as a generalization of the class of Fano 3-folds with Picard number 1. There are many non-singular Fano 3-folds satisfying this condition and with Picard number 2 (see [Mo-Mu] and also [Ma]). We also think that studying the Picard number of this calss may be important for studying Fano 3-folds with Picard number 1, too (see Corollary 2 below).

Let X be a Fano 3-fold with $\mathbf{Q}$-factorial terminal singularities. Let R be an extremal ray of the Mori polyhedron $\overline{NE}(X)$ of X. We say that R has the *type* (I) (respectively (II)) if curves of R fill an irreducible divisor $D(R)$ of X and the contraction of the ray R contracts the divisor $D(R)$ to a point (respectively to a curve). An extremal ray R is called *small* if curves of this ray fill a curve on X.

A pair $\{R_1, R_2\}$ of extremal rays has the type $\mathfrak{B}_2$ if extremal rays R_1, R_2 are different, both have the type (II), and have the same divisor $D(R_1)=D(R_2)$.

We recall that a face γ of Mori polyhedron $\overline{NE}(X)$ defines a contraction f_γ: $X\to X'$ (see [Ka1] and [Sh]) such that $f(C)$ is a point for an irreducible curve C if and only if C belongs to γ. The $\dim X'$ is called the Kodaira dimension of the γ. A set $\mathcal{E}$ of extremal rays is called extremal if it is contained in a face of Mori polyhedron.

Basic Theorem. *Let X be a Fano 3-fold with terminal $\mathbf{Q}$-factorial sing-*

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ularities. Assume that X does not have a small extremal ray, and Mori polyhedron $\overline{NE}(X)$ does not have a face of Kodaira dimension 1 or 2.

Then the following statements for the X hold:

(1) The X does not have a pair of extremal rays of the type $\mathfrak{B}_2$ and Mori polyhedron $\overline{NE}(X)$ is simplicial;

(2) The X does not have more than one extremal ray of the type (I).

(3) If $\mathscr{E}$ is an extremal set of k extremal rays of X, then the $\mathscr{E}$ has one of the types: $\mathfrak{A}_1\amalg(k-1)\mathfrak{C}_1$, $\mathfrak{D}_2\amalg(k-2)\mathfrak{C}_1$, $\mathfrak{C}_2\amalg(k-2)\mathfrak{C}_1$, $k\mathfrak{C}_1$ (we use notation of Theorem 2.3.3).

(4) We have the inequality for the Picard number of the X:

$$\rho(X)=\dim N_1(X)\leq 7 .$$

Proof. See Theorem 2.5.8.

It follows from (4):

Corollary 1. *Let X be a Fano 3-fold with terminal $\mathbf{Q}$-factorial singularities and $\rho(X)>7$. Then X has either a small extremal ray or a face of Kodaira dimension 1 or 2 for Mori polyhedron.*

We mention that non-singular Fano 3-folds do not have a small extremal ray (by Mori [Mo1]), and their maximal Picard number is equal to 10 according to their classification by Mori and Mukai [Mo-Mu]. Thus, all these statements already work for non-singular Fano 3-folds.

From the statement (2) of the Theorem, we also get the following application of Basic Theorem to geometry of Fano 3-folds.

Let us consider a Fano 3-fold X and its blow-up X_p at different non-singular points $\{x_1, ..., x_p\}$ of X. We say that this is a Fano blow-up if X_p is Fano. We have the following very simple

Proposition. *Let X be a Fano 3-fold with terminal $\mathbf{Q}$-factorial singularities and without small extremal rays. Let X_p be a Fano blow-up of X. Then for any small extremal ray S on X_p, the S has a non-empty intersection with one of exceptional divisors $E_1, ..., E_p$ of this blow up and does not belong to any of them. The divisors $E_1, ..., E_p$ define p extremal rays of the type (I) on X_p.*

Proof. See Proposition 2.2.14.

It is known that a contraction of a face of Kodaira dimension 1 or 2 of $\overline{NE}(X)$ of a Fano 3-fold X has a general fiber which is a rational surface or curve respectively, because this contraction has relatively negative canonical class. See [Ka1], [Sh]. It is also known that a small extremal ray is rational [Mo2].

Then, using Basic Theorem and Proposition, we can divide Fano 3-folds of Basic Theorem on the following 3 classes:

Corollary 2. *Let X be a Fano 3-fold with terminal $\mathbf{Q}$-factorial singular-*

ities and without small extremal rays, and without faces of Kodaira dimension 1 or 2 for the Mori polyhedron. Let ε *be the number of extremal rays of the type* (*I*) *on* X (*by Basic Theorem, the* $\varepsilon \leq 1$).

Then there exists p, $1 \leq p \leq 2-\varepsilon$, *such that* X *belongs to one of calsses* (*A*), (*B*) or (*C*) *below*:

(*A*) *There exists a Fano blow-up* X_p *of* X *with a face of Kodaira dimension* 1 *or* 2. *Thus, birationally,* X *is a fibration on rational surfaces over a curve or rational curves over a surface.*

(*B*) *There exist Fano blow-ups* X_p *of* X *for general* p *points on* X *such that for all these blow-ups the* X_p *has a small extremal ray* S. *Then images of curves of* S *on* X *give a system of rational curves on* X *which cover a Zariski open subset of* X.

(*C*) *There do not exist Fano blow-ups* X_p *of* X *for general* p *points.*

We remark that for Fano 3-folds with Picard number 1, the $\varepsilon=0$. *Thus,* $1 \leq p \leq 2$.

Using statements (2), (3) and (4) of Basic Theorem, one can formulate similar results for Fano blow-ups along curves.

To prove Basic Theorem, we classify appropriate so called extremal sets and E-sets of extremal rays of the type (I) or (II). We use so called diagram method to deduce from this classification the statement (4) of the Basic Theorem.

A set $\mathscr{E}$ of extremal rays is called *extremal* if it is contained in a face of Mori polyhedron. The $\mathscr{E}$ has *Kodaira dimension 3* if a contraction of this face gives a morphism on a 3-fold. For Fano 3-folds with $\mathbf{Q}$-factorial terminal singularities, we give a description of extremal sets $\mathscr{E}$ of Kodaira dimension 3 which contain extremal rays of the types (I) or (II) only.

A set $\mathscr{L}$ of extremal rays is called *E-set* if $\mathscr{L}$ is not extremal, but any proper subset of $\mathscr{L}$ is extremal. Thus, the $\mathscr{L}$ is minimal non-extremal. For Fano 3-folds with $\mathbf{Q}$-factorial terminal singularities, we give a description of *E*-set $\mathscr{L}$ such that $\mathscr{L}$ contains extremal rays of the types (I) or (II) only, and any proper subset of $\mathscr{L}$ is extremal of Kodaira dimension 3.

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Preliminary variant of this paper was published as a preprint [N8]. Generalizations of results here one can find in a preprint [N9].

CHAPTER I. Diagram Method

Here we give the simplest variant of the diagram method for multi-dimensional algebraic varieties. We shall use this method in the next chapter. This part also contains some corrections and generalizations of the corresponding parts of our papers [N6] and [N7].

Let X be a projective algebraic variety with $\mathbf{Q}$-factorial singularities over an algebraically closed field. Let $\dim X \geq 2$. Let $N_1(X)$ be the $\mathbf{R}$-linear space generated by the numerical equivalence classes of all algebraic curves on X, and let $N^1(X)$ be the $\mathbf{R}$-linear space generated by the numerical equivalence classes of all Cartier (or Weil) divisors on X. Linear spaces $N_1(X)$ and $N^1(X)$ are dual to one another by the intersection pairing. Let $NE(X)$ be a convex cone in $N_1(X)$ generated by all effective curves on X. Let $\overline{NE}(X)$ be the closure of the cone $NE(X)$ in $N_1(X)$. It is called *Mori cone* (or *polyhedron*) of X. A non-zero element $x \in N^1(X)$ is called *nef* if $x \cdot \overline{NE}(X) \geq 0$. Let $NEF(X)$ be the set of all nef elements of X and the zero. It is the convex cone in $N^1(X)$ dual to Mori cone $\overline{NE}(X)$. A ray $R \subset \overline{NE}(X)$ with origin 0 is called *extremal* if from $C_1 \in \overline{NE}(X)$, $C_2 \in \overline{NE}(X)$ and $C_1 + C_2 \in R$ it follows that $C_1 \in R$ and $C_2 \in R$.

We consider a condition (i) for a set $\mathfrak{R}$ of extremal rays on X.

(i) *If $R \in \mathfrak{R}$, then all curves $C \in R$ fill out an irreducible divisor $D(R)$ on X.*

In this case, an oriented graph $G(\mathfrak{R})$ corresponds to $\mathfrak{R}$ in the following way: Two different rays R_1 and R_2 are joined by an arrow R_1R_2 from R_1 to R_2 if $R_1 \cdot D(R_2) > 0$. Here and in what follows, for an extremal ray R and a divisor D we write $R \cdot D > 0$ if $r \cdot D > 0$ for $r \in R$ and $r \neq 0$. (The same convention is applied for the symbols $\leq$, $\geq$ and $<$.)

A set $\mathscr{E}$ of extremal rays is called *extermal* if it is contained in a face of $\overline{NE}(X)$. Equivalently, there exists a nef element $H \in N^1(X)$ such that $\mathscr{E} \cdot H = 0$. Evidently, a subset of an extremal set is extremal, too.

We consider the following condition (ii) for extremal sets $\mathscr{E}$ of extremal rays.

(ii) *An extremal set $\mathscr{E} = \{R_1, \ldots, R_n\}$ satisfies the condition* (i), *and for any real numbers $m_1 \geq 0, \ldots, m_n \geq 0$ which are not all equal to 0, there exists a ray $R_j \in \mathscr{E}$ such that $R_j \cdot (m_1 D(R_1) + m_2 D(R_2) + \cdots + m_n D(R_n)) < 0$. In particular, the effective divisor $m_1 D(R_1) + m_2 D(R_2) + \cdots + m_n D(R_n)$ is not nef.*

A set $\mathscr{L}$ of extremal rays is called *E-set* (extremal in a different sense) if the $\mathscr{L}$ is not extremal but every proper subset of $\mathscr{L}$ is extremal. Thus, $\mathscr{L}$ is a minimal non-extremal set of extremal rays. Evidently, an E-set $\mathscr{L}$ contains at least two elements.

We consider the following condition (iii) for E-sets $\mathscr{L}$.

(iii) *Any proper subset of an E-set* $\mathcal{L}=\{Q_1, \ldots, Q_m\}$ *satisfies the condition* (ii), *and there exists a non-zero effective nef divisor* $D(\mathcal{L})=a_1D(Q_1)+a_2D(Q_2)+\cdots+a_mD(Q_m)$.

The following statement is very important.

Lemma 1.1. *An E-set* $\mathcal{L}$ *satisfying the condition* (iii) *is connected in the following sense: For any decomposition* $\mathcal{L}=\mathcal{L}_1\amalg\mathcal{L}_2$, *where* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *are non-empty, there exists an arrow* Q_1Q_2 *such that* $Q_1\in\mathcal{L}_1$ and $Q_2\in\mathcal{L}_2$.

If $\mathcal{L}$ *and* $\mathcal{M}$ *are two different E-sets satisfying the condition* (iii), *then* there exists an arrow *LM where* $L\in\mathcal{L}$ and $M\in\mathcal{M}$.

Proof. Let $\mathcal{L}=\{Q_1, \ldots, Q_m\}$. By (iii), there exists a nef divisor $D(\mathcal{L})=a_1D(Q_1)+a_2D(Q_2)+\cdots+a_mD(Q_m)$. If one of the coefficients $a_1, \ldots, a_m$ is equal to zero, we get a contradiction to the conditions (ii) and (iii). It follows that all the coefficients $a_1, \ldots, a_m$ are positive. Let $\mathcal{L}=\mathcal{L}_1\amalg\mathcal{L}_2$ where $\mathcal{L}_1=\{Q_1, \ldots, Q_k\}$ and $\mathcal{L}_2=\{Q_{k+1}, \ldots, Q_m\}$. The divisors $D_1=a_1D(Q_1)+\cdots+a_kD(Q_k)$ and $D_2=a_{k+1}D(Q_{k+1})+\cdots a_mD(Q_m)$ are non zero. By (ii), there exists a ray Q_i, $1\leq i\leq k$, such that $Q_i\cdot D_1<0$. On the other hand, $Q_i\cdot D(\mathcal{L})=Q_1\cdot(D_1+D_2)\geq 0$. It follows that there exists j, $k+1\leq j\leq m$, such that $Q_i\cdot D(Q_j)>0$. It means that Q_iQ_j is an arrow.

Let us prove the second statement. By the condition (iii), for every ray $R\in\mathcal{L}$, we have the inequality $R\cdot D(\mathcal{M})\geq 0$. If $R\cdot D(\mathcal{M})=0$ for any $R\in\mathcal{L}$, then the set $\mathcal{L}$ is extremal, and we get the contradiction. It follows that there exists a ray $R\in\mathcal{L}$ such that $R\cdot D(\mathcal{M})>0$. It follows our assertion.

Let $NEF(X)=\overline{NE}(X)^*\subset N^1(X)$ be the cone of nef elements of X and $\mathcal{M}(X)=NEF(X)/\mathbf{R}^+$ its projectivization. We use usual relations of orthogonality between subsets of $\mathcal{M}(X)$ and $\overline{NE}(X)$. So, for $U\subset\mathcal{M}(X)$ and $V\subset\overline{NE}(X)$ we write $U\perp V$ if $x\cdot y=0$ for any $\mathbf{R}^+x\in U$ and any $y\in V$. Thus, for $U\subset\mathcal{M}(X)$, $V\subset\overline{NE}(X)$ we denote

$$U^\perp=\{y\in\overline{NE}(X)\,|\,U\perp y\}\quad,\quad V^\perp=\{x\in\mathcal{M}(X)\,|\,x\perp V\}\quad.$$

A subset $\gamma\subset\mathcal{M}(X)$ is called a *face* of $\mathcal{M}(X)$ if there exists a non-zero element $r\in\overline{NE}(X)$ such that $\gamma=r^\perp$.

A convex set is called a *closed polyhedron* if it is a convex hull of a finite set of points. A convex closed polyhedron is called *simplicial if* all its proper faces are simplexes. A convex closed polyhedron is called *simple* (equivalently, it has simplicial angles) if it is dual to a simplicial one. In other words, any its face of codimension k is contained exactly in k faces of γ of the highest dimension. Similar names we use for convex cones and cones over polyhedra. For example, a convex cone is called *simplex*, *simplicial and simple* if it is a cone over a simplex, simplicial or simple polyhedron respectively.

We need some relative notions of the notions above.

We say that $\mathcal{M}(X)$ is a *closed plyhedron in its face* $\gamma\subset\mathcal{M}(X)$ if γ is a closed polyhedron and $\mathcal{M}(X)$ is a closed polyhedron in a neighbourhood T of γ. Thus, there should exist a closed polyhedron $\mathcal{M}'$ such that $\mathcal{M}' \cap T = \mathcal{M}(X) \cap T$.

We will use the following notation. Let $\mathfrak{R}(X)$ be the set of all extremal rays of X. For a face $\gamma \subset \mathcal{M}(X)$,

$$\mathfrak{R}(\gamma) = \{R \in \mathfrak{R}(X) \mid \exists\, \mathbf{R}^+ H \in \gamma: R \cdot H = 0\}$$

and

$$\mathfrak{R}(\gamma^\perp) = \{R \in \mathfrak{R}(X) \mid \gamma \perp R\} \quad .$$

Let us assume that $\mathcal{M}(X)$ is a closed polyhedron in its face γ. Then sets $\mathfrak{R}(\gamma_1)$ and $\mathfrak{R}(\gamma_1^\perp)$ are finite for any face $\gamma_1 \subset \gamma$. Evidently, the face γ is simple if

$$(1) \qquad \#\mathfrak{R}(\gamma_1^\perp) - \#\mathfrak{R}(\gamma^\perp) = \mathrm{codim}_\gamma \gamma_1$$

for any face γ_1 of γ. Then we say that the polyhedron $\mathcal{M}(X)$ is *simple in its face* γ. Evidently, this condition is equivalent to the condition:

$$(2) \qquad \dim[\mathcal{E}] - \dim[\mathfrak{R}(\gamma^\perp)] = \#\mathcal{E} - \#\mathfrak{R}(\gamma^\perp)$$

for any extremal set $\mathcal{E}$ such that $R(\gamma^\perp) \subset \mathcal{E}$. Here $[\cdot]$ denotes a linear hull. (In [N6], we required a more strong condition for a polyhedron $\mathcal{M}(X)$ to be simple in its face γ: $\#\mathfrak{R}(\gamma_1^\perp) = \dim \mathcal{M}(X) - \dim \gamma_1$ for any face γ_1 of γ.)

Let A, B be two vertices of an oriented graph G. The *distance* $\rho(A, B)$ in G is a length (the number of links) of a shortest oriented path of the graph G from A to B. The distance is $+\infty$ if this path does not exist. The *diameter* diam G of an oriented graph G is the maximum distance between ordered pairs of its vertices. By the Lemma 1.1, the diameter of an E-set is a finite number if this set satisfies the condition (iii).

Theorem 1.2 below is an analog for algebraic varieties of arbitrary dimension of the Lemma 3.4 of [N2] and the Lemma 1.4.1 of [N5], which were devoted to surfaces.

Theorem 1.2. *Let X be a projective algebraic variety with $\mathbf{Q}$-factorial singularities and* $\dim X \geq 2$. *Let us suppose that $\mathcal{M}(X)$ is closed and simple in its face γ. Assume that the set $\mathfrak{R}(\gamma)$ satisfies the condition* (i) *above. Assume that there are some constants d, C_1, C_2 such that the conditions* (a) *and* (b) *below hold:*

(a) *For any E-set $\mathcal{L} \subset \mathfrak{R}(\gamma)$ such that $\mathcal{L}$ contains at least two elements which don't belong to $\mathfrak{R}(\gamma^\perp)$ and for any proper subset $\mathcal{L}' \subset \mathcal{L}$ the set $R(\gamma^\perp) \cup \mathcal{L}'$ is extremal, the condition* (iii) *is valid and*

$$\mathrm{diam}\, G(\mathcal{L}) \leq d \quad .$$

(b) *For any extremal subset $\mathcal{E}$ such that $\mathfrak{R}(\gamma^\perp) \subset \mathcal{E} \subset \mathfrak{R}(\gamma)$, we have: the $\mathcal{E}$ satisfies the condition* (ii) *and for the distance in the oriented graph $G(\mathcal{E})$*

$$\#\,\{(R_1, R_2) \in (\mathcal{E} - \mathfrak{R}(\gamma^\perp)) \times (\mathcal{E} - \mathfrak{R}(\gamma^\perp)) \mid 1 \leq \rho(R_1, R_2) \leq d\} \leq C_1 \#\,(\mathcal{E} - \mathfrak{R}(\gamma^\perp));$$

and

$$\#\,\{(R_1, R_2) \in (\mathcal{E} - \mathfrak{R}(\gamma^\perp)) \times (\mathcal{E} - \mathfrak{R}(\gamma^\perp)) \mid d+1 \leq \rho(R_1, R_2) \leq 2d+1\} \leq C_2 \#\,(\mathcal{E} - \mathfrak{R}(\gamma^\perp)) \quad .$$

Then $\dim\gamma < (16/3)C_1 + 4C_2 + 6$.

Proof. We use the following Lemma 1.3 which was proved in [N1]. The lemma was used in [N1] to get a bound (≤ 9) of the dimension of a hyperbolic (Lobachevsky) space admitting an action of an arithmetic reflection group with a field of definition of the degree $> N$. Here N is some absolute constant.

Lemma 1.3. *Let* $\mathcal{M}$ *be a convex closed simple polyhedron of a dimension* n, *and* $A_n^{i,k}$ *the average number of* i*-dimensional faces of* k*-dimensional faces of* $\mathcal{M}$. *Then for* $n \geq 2k-1$

$$A_n^{i,k} < \frac{\binom{n-i}{n-k} \cdot \left(\binom{[n/2]}{i} + \binom{n-[n/2]}{i}\right)}{\binom{[n/2]}{k} + \binom{n-[n/2]}{k}}.$$

In particular, if $n \geq 3$

$$A_n^{0,2} < \begin{cases} \dfrac{4(n-1)}{n-2} & \text{if } n \text{ is even}, \\ \dfrac{4n}{n-1} & \text{if } n \text{ is odd}. \end{cases}$$

Proof. See [N1]. We mention that the right side of the inequality of the Lemma 1.3 decreases and tends to the number $2^{k-i}\binom{k}{i}$ of i-dimensional faces of k-dimensional cube if n increases.

From the estimate of $A_n^{0,2}$ of the Lemma, it follows the following analog of Vinberg's Lemma from [V]. Vinberg's Lemma was used by him to obtain an estimate ($\dim < 30$) for the dimension of a hyperbolic space admitting an action of a discrete reflection group with a bounded fundamental polyhedron.

By definition, an *angle* of a polyhedron T is an angle of a 2-dimensional face of T. Thus, the angle is defined by a vertex A of T, a plane containing A and a 2-dimensional face γ_2 of T, and two rays with the beginning at A which contain two corresponding sides of the γ_2. To define an *oriented angle* of T, one should in addition put in order two rays of the angle.

Lemma 1.4. *Let* $\mathcal{M}$ *be a convex simple polyhedron of a dimension* n. *Let* C *and* D *are some numbers. Suppose that oriented angles (2-dimensional, plane) of* $\mathcal{M}$ *are supplied with weights and the following conditions* (1) *and* (2) *hold*:

(1) *The sum of weights of all oriented angles at any vertex of* M *is not greater than* $Cn+D$.

(2) *The sum of weights of all oriented angles of any 2-dimensional face of* $\mathcal{M}$ *is at least* $5-k$ *where* k *is the number of vertices of the 2-dimensional face.*

Then

$$n < 8C + 5 + \begin{cases} 1 + 8D/n & \text{if } n \text{ is even}, \\ (8C+8D)/(n-1) & \text{if } n \text{ is odd} \end{cases}.$$

In particular, for $C \geq 0$ *and* $D=0$, *we have*

$$n < 8C+6 .$$

Proof. We correspond to a non-oriented plane angle of $\mathcal{M}$ a weight which is equal to the sum of weights of two corresponding oriented angles. Evidently, the conditions of the Lemma hold for the weights of non-oriented angles too if we forget about the word "oriented". Then we obtain Vinberg's lemma from [V] which we formulate a little bit more precisely here. Since the proof is simple, we give the proof here.

Let Σ be the sum of weights of all (non-oriented) angles of the polyhedron $\mathcal{M}$. Let α_0 be the number of vertices of $\mathcal{M}$ and α_2 the number of 2-dimensional faces of $\mathcal{M}$. Since $\mathcal{M}$ is simple,

$$\alpha_0 \frac{n(n-1)}{2} = \alpha_2 A_n^{0,2} .$$

From this equality and conditions of the Lemma, we get inequalities

$$(Cn+D)\alpha_0 \geq \Sigma \geq \sum \alpha_{2,k}(5-k) = 5\alpha_2 - \alpha_2 A_n^{0,2} =$$
$$= \alpha_2(5-A_n^{0,2}) = \alpha_0 (n(n-1)/2)(5/A_n^{0,2}-1) .$$

Here $\alpha_{2,k}$ is the number of 2-dimensional faces with k vertices of $\mathcal{M}$. Thus, from this inequality and Lemma 1.3, we get

$$Cn+D \geq (n(n-1)/2)(5/A_n^{0,2}-1) > \begin{cases} n(n-6)/8 & \text{if } n \text{ is even,} \\ (n-1)(n-5)/8 & \text{if } n \text{ is odd.} \end{cases}$$

From this calculations, Lemma 1.4 follows.

The proof of Theorem 1.2. (Compare with [V].) Let $\angle$ be an oriented angle of γ. Let $\mathfrak{R}(\angle) \subset \mathfrak{R}(\gamma)$ be the set of all extremal rays of $\mathcal{M}(X)$ which are orthogonal to the vertex of $\angle$. Since $\mathcal{M}(X)$ is simple in γ, the set $\mathfrak{R}(\angle)$ is a disjoint union

$$\mathfrak{R}(\angle) = \mathfrak{R}(\angle^{\perp}) \cup \{R_1(\angle)\} \cup \{R_2(\angle)\}$$

where $\mathfrak{R}(\angle^{\perp})$ contains all rays orthogonal to the plane of the angle $\angle$, the rays $R_1(\angle)$ and $R_2(\angle)$ are orthogonal to the first and second side of the oriented angle $\angle$, respectively. Evidently, the set $\mathfrak{R}(\angle)$ and the ordered pair of rays $(R_1(\angle), R_2(\angle))$ define the oriented angle $\angle$ uniquely. We define the weight $\sigma(\angle)$ by the formula:

$$\sigma(\angle) = \begin{cases} 2/3, & \text{if } 1 \leq \rho(R_1(\angle), R_2(\angle)) \leq d, \\ 1/2, & \text{if } d+1 \leq \rho(R_1(\angle), R_2(\angle)) \leq 2d+1, \\ 0, & \text{if } 2d+2 \leq \rho(R_1(\angle), R_2(\angle)). \end{cases}$$

Here we take the distance in the graph $G(\mathfrak{R}(\angle))$. Let us prove conditions of the Lemma 1.4 with the constants $C=(2/3)C_1+C_2/2$ and $D=0$.

The condition (1) follows from the condition (b) of the theorem. We remark that rays $R_1(\angle)$, $R_2(\angle)$ do not belong to the set $R(\gamma^{\perp})$.

Let us prove the condition (2).

Let γ_3 be a 2-dimensional triangle face (triangle) of γ. The set $\mathfrak{R}(\gamma_3)$ of all extremal rays orthogonal to points of γ_3 is the union of the set $\mathfrak{R}(\gamma_3^{\perp})$ of extremal rays, which are orthogonal to the plane of the triangle γ_3, and rays R_1, R_2, R_3, which are orthogonal to the sides of the triangle γ_3. Union of the set $\mathfrak{R}(\gamma_3^{\perp})$ with any two rays of R_1, R_2, R_3 is extremal, since it is orthogonal to a vertex of γ_3. On the other hand, the set $\mathfrak{R}(\gamma_3)=\mathfrak{R}(\gamma_3^{\perp})\cup\{R_1, R_2, R_3\}$ is not extremal, since it is not orthogonal to a point of $\mathcal{M}(X)$. Indeed, the set of all points of $\mathcal{M}(X)$, which are orthogonal to the set $\mathfrak{R}(\gamma_3^{\perp})\cup\{R_2, R_3\}$, $\mathfrak{R}(\gamma_3^{\perp})\cup\{R_1, R_3\}$, or $\mathfrak{R}(\gamma_3^{\perp})\cup\{R_1, R_2\}$ is the vertex A_1, A_2, or A_3 respectively of the triangle γ_3, and the intersection of these sets of vertices is empty. Thus, there exists an E-set $\mathcal{L}\subset\mathfrak{R}(\gamma_3)$, which contains the set of rays $\{R_1, R_2, R_3\}$. By the condition (a), the graph $G(\mathcal{L})$ contains a shortest oriented path s of the length $\leq d$ which connects the rays R_1, R_3. If this path does not contain the ray R_2, then the oriented angle of γ_3 defined by the set $\mathfrak{R}(\gamma_3^{\perp})\cup\{R_1, R_3\}$ and the pair (R_1, R_3) has the weight 2/3. If this path contains the ray R_2, then the oriented angle of γ_3 defined by the set $\mathfrak{R}(\gamma_3^{\perp})\cup\{R_1, R_2\}$ and the pair (R_1, R_2) has the weight 2/3. Thus, we proved that the side A_2A_3 of the triangle γ_3 defines an oriented angle of the triangle with the weight 2/3 and the first side A_2A_3 of the oriented angle. The triangle has three sides. It follows the condition (2) of the Lemma 1.4 for the triangle.

Let γ_4 be a 2-dimensional quadrangle face (quadrangle) of γ. In this case,

$$\mathfrak{R}(\gamma_4)=\mathfrak{R}(\gamma_4^{\perp})\cup\{R_1, R_2, R_3, R_4\}$$

where $\mathfrak{R}(\gamma_4^{\perp})$ is the set of all extremal rays which are orthogonal to the plane of the quadrangle and the rays R_1, R_2, R_3, R_4 are orthogonal to the consecutive sides of the quadrangle. As above, one can see that the sets $\mathfrak{R}(\gamma_4^{\perp})\cup\{R_1, R_3\}$, $\mathfrak{R}(\gamma_4^{\perp})\cup\{R_2, R_4\}$ are not extremal, but the sets $\mathfrak{R}(\gamma_4^{\perp})\cup\{R_1, R_2\}$, $\mathfrak{R}(\gamma_4^{\perp})\cup\{R_2, R_3\}$, $\mathfrak{R}(\gamma_4^{\perp})\cup\{R_3, R_4\}$ and $\mathfrak{R}(\gamma_4^{\perp})\cup\{R_4, R_1\}$ are extremal. It follows that there are E-sets $\mathcal{L}$, $\mathcal{N}$ such that $\{R_1, R_3\}\subset\mathcal{L}\subset\mathfrak{R}(\gamma_4^{\perp})\cup\{R_1, R_3\}$ and $\{R_2, R_4\}\subset\mathcal{N}\subset\mathfrak{R}(\gamma_4^{\perp})\cup\{R_2, R_4\}$. By Lemma 1.1, there exist rays $R\in\mathcal{L}$ and $Q\in\mathcal{N}$ such that RQ is an arrow. By the condition (a) of the theorem, one of the rays R_1, R_3 is joined by an oriented path s_1 of the length $\leq d$ with the ray R and this path does not contain another ray from R_1, R_3 (here R is the terminal of the path s_1). We can suppose that this ray is R_1 (otherwise, one should replace the ray R_1 by the ray R_3). As above, we can suppose that the ray Q is connected by the oriented path s_2 of the length $\leq d$ with the ray R_2 and this path does not contain the ray R_4. The path $s_1\, RQ\, s_2$ is an oriented path of the length $\leq 2d+1$ in the oriented graph $G(\mathfrak{R}(\gamma_4^{\perp})\cup\{R_1, R_2\})$. It follows that the oriented angle of the quadrangle γ_4, such that consecutive sides of this angle are orthogonal to the rays R_1 and R_2 respectively, has the weight $\geq 1/2$. Thus, we proved that for a pair of opposite sides of γ_4 there exists an oriented angle with weight $\geq 1/2$ such that the first side of this oriented angle is one of this opposite sides of the quadrangle. A quadrangle has two pairs of opposite sides. It follows that the sum of weights of oriented angles of γ_4 is ≥ 1. It proves the condition (2) of the Lemma 1.4 and the theorem.

In the sequel, we apply Theorem 1.2 to 3-folds.

CHAPTER II. Threefolds

1. Contractible extremal rays

We consider normal projective 3-folds X with $\mathbf{Q}$-factorial singularities. Let R be an extremal ray of Mori polyhedron $\overline{NE}(X)$ of X. A morphism f: $X \to Y$ onto a normal projective variety Y is called the *contraction* of the ray R if for an irreducible curve C of X the image $f(C)$ is a point if and only if $C \in R$. The contraction f is defined by a linear system H on X (H give rise to a nef element of $N^1(X)$, which we also denote by H). It follows that an irreducible curve C is contracted if and only if $C \cdot H = 0$. We assume that the contraction f has properties: $f\mathcal{O}_X = \mathcal{O}_Y$ and the sequence

$$(1.1) \qquad 0 \to \mathbf{R}R \to N_1(X) \to N_1(Y) \to 0$$

is exact where the arrow $N_1(X) \to N_1(Y)$ is f_*. An extremal ray R is called *contractible* if there exists its contraction f with these properties.

The number $\kappa(R) = \dim Y$ is called *Kodaira dimension* of the contractible extremal ray R.

A face γ of $\overline{NE}(X)$ is called *contractible* if there exists a morphism f: $X \to Y$ onto a normal projective variety Y such that $f_*\gamma = 0$, $f_*\mathcal{O}_X = \mathcal{O}_Y$ and f contracts curves lying in γ only. The $\kappa(\gamma) = \dim Y$ is called *Kodaira dimension of* γ.

Let H be a general nef element orthogonal to a face γ of Mori polyhedron. *Numerical Kodaira dimension of* γ is defined by the formula

$$\kappa_{num}(\gamma) = \begin{cases} 3, & \text{if } H^3 > 0; \\ 2, & \text{if } H^3 = 0 \text{ and } H^2 \not\equiv 0; \\ 1, & \text{if } H^2 \equiv 0\ . \end{cases}$$

It is obvious that for a contractible face γ we have $\kappa_{num}(\gamma) \geq \kappa(\gamma)$. In particular, $\kappa_{num}(\gamma) = \kappa(\gamma)$ for a contractible face γ of Kodaira dimension $\kappa(\gamma) = 3$.

2. Paris of extremal rays of Kodaira dimension three lying in contractible faces of $\overline{NE}(X)$ of Kodaira dimension three

We assume further that X is a projective normal threefold with $\mathbf{Q}$-factorial singularities.

Lemma 2.2.1. *Let R be a contractible extremal ray of Kodaira dimension*

3 and $f: X \to Y$ its contraction.

Then there are three possibilities:

(I) *All curves $C \in R$ fill an irreducible Weil divisor $D(R)$, the contraction f contracts $D(R)$ to a point and $R \cdot D(R) < 0$.*

(II) *All curves $C \in R$ fill an irreducible Weil divisor $D(R)$, the contraction f contracts $D(R)$ to an irreducible curve and $R \cdot D(R) < 0$.*

(III) *(small extremal ray) All curves $C \in R$ give a finite set of irreducible curves and the contraction f contracts these curves to points.*

Proof. Assume that some curves of R fill an irreducible divisor D. Then $R \cdot D < 0$ (this inequality follows from the Proposition 2.2.6 below). Suppose that $C \in R$ and D does not contain C. It follows that $R \cdot D \geq 0$. We get a contradiction. It follows the lemma.

According to Lemma 2.2.1, we say that an extremal ray R has the *type (I), (II) or (III) (small)* if it is contractible of Kodaira dimension 3 and the statement (I), (II) or (III) respectively holds.

Lemma 2.2.2. *Let R_1 and R_2 are two different extremal rays of the type (I). Then divisors $D(R_1)$ and $D(R_2)$ do not intersect one another.*

Proof. Otherwise, $D(R_1)$ and $D(R_2)$ have a common curve and the rays R_1 and R_2 are not different.

For an irreducible Weil divisor D on X let

$$\overline{NE}(X, D) = (\text{image} \overline{NE}(D)) \subset \overline{NE}(X) \ .$$

Lemma 2.2.3. *Let R be an extremal ray of the type (II), and f its contraction. Then $\overline{NE}(X, D(R)) = R + \mathbf{R}^+ S$, where $\mathbf{R}^+ f_* S = \mathbf{R}^+ (f(D))$.*

Proof. This follows at once from the exact sequence (1.1).

Lemma 2.2.4. *Let R_1 and R_2 are two different extremal rays of the type (II) such that the divisors $D(R_1)$, $D(R_2)$ coincide. Then for $D = D(R_1) = D(R_2)$ we have: $\overline{NE}(X, D) = R_1 + R_2$. In particular, do not exist three different extremal rays of the type (II) such that their divisors coincide one another.*

Proof. This follows from the Lemma 2.2.3.

Lemma 2.2.5. *Let R be an extremal ray of the type (II) and f its cotraction. Then there does not exist more than one extremal ray Q of the type (I) such that $D(R) \cap D(Q)$ is not empty. If Q is this ray, then $D(R) \cap D(Q)$ is a curve and any irreducible component of this curve is not contained in fibers of f.*

Proof. The last assertion is obvious. Let us prove the first one. Suppose that Q_1 and Q_2 are two different extremal rays of the type (I) such that $D(Q_1) \cap D(R)$ and $D(Q_2) \cap D(R)$ are not empty. Then the plane angle $\overline{NE}(X, D(R))$ (see the Lemma 2.2.3) contains three different extremal rays: Q_1, Q_2

and R. It is impossible.

The following key proposition is very important.

Proposition 2.2.6. *Let X be a projective 3-fold with $\mathbf{Q}$-factorial singularities, $D_1, \ldots, D_m$ irreducible divisors on X and $f\colon X \to Y$ a surjective morphism such that $\dim X = \dim Y$ and $\dim f(D_i) < \dim D_i$. Let $y \in f(D_1) \cap \ldots \cap f(D_m)$. Then there are $a_1 > 0, \ldots, a_m > 0$ and an open U, $y \in U \subset f(D_1) \cup \ldots \cup f(D_m)$, such that*

$$C \cdot (a_1D_1 + \ldots + a_mD_m) < 0$$

if a curve $C \subset D_1 \cup \ldots \cup D_m$ belongs to a non-trivial algebraic family of curves on $D_1 \cup \ldots \cup D_m$ and $f(C) = \text{point} \in U$.

Proof. The proof is the same as the well-known case of surfaces (but, for surfaces, it is not necessary to suppose that C belong to a nontrivial algebraic family). Let H be an irreducible ample divisor on X and $H' = f_*H$. Since $\dim f(D_i) < \dim D_i$, it follows that $f(D_1) \cup \ldots \cup f(D_m) \subset H'$. Let ϕ be a non-zero rational function on Y which is regular in a neighbourhood U of y on Y and is equal to zero on the divisor H'. In the open set $f^{-1}(U)$ the divisor $(f^*\phi)$ can be written in a form

$$(f^*\phi) = \sum_{i=1}^{m} a_iD_i + \sum_{j=1}^{n} b_jZ_j \ .$$

where all $a_i > 0$ and all $b_j > 0$. Here every divisor Z_j is different from any divisor D_i. We have

$$0 = C \cdot \sum_{i=1}^{m} a_iD_i + C \cdot \sum_{j=1}^{n} b_jZ_j \ .$$

Here $C \cdot (\sum_{j=1}^{n} b_jZ_j) > 0$ since C belongs to a nontrivial algebraic family of curves on a surface $D_1 \cup \ldots \cup D_m$ and one of the Z_j is the hyperplane section H.

Lemma 2.2.7. *Let R_1, R_2 are two extremal rays of the type (II), divisors $D(R_1)$, $D(R_2)$ are different and $D(R_1) \cap D(R_2) \neq 0$. Assume that R_1, R_2 belong to a contractible face of $\overline{NE}(X)$ of Kodaira dimension 3. Let $0 \neq F_1 \in R_1$ and $0 \neq F_2 \in R_2$. Then*

$$(F_1 \cdot D(R_2))\,(F_2 \cdot D(R_1)) < (F_1 \cdot D(R_1))\,(F_2 \cdot D(R_2)) \ .$$

Proof. Let f be the contraction of a face of Kodaira dimension 3, which contains both rays R_1, R_2. By Proposition 2.2.6, there are $a_1 > 0$, $a_2 > 0$ such that

$$a_1(F_1 \cdot D(R_1)) + a_2(F_1 \cdot D(R_2)) < 0 \quad \text{and} \quad a_1(F_2 \cdot D(R_1)) + a_2(F_2 \cdot D(R_2)) < 0$$

or

$$-a_1(F_1 \cdot D(R_1)) > a_2(F_1 \cdot D(R_2)) \quad \text{and} \quad -a_2(F_2 \cdot D(R_2)) > a_1(F_2 \cdot D(R_1))$$

where $F_1 \cdot D(R_1) < 0$. $F_2 \cdot D(R_2) < 0$ and $F_1 \cdot D(R_2) > 0$, $F_2 \cdot D(R_1) > 0$. Multiplying inequalities above, we obtain the lemma.

3. A classification of extremal sets of extremal rays which contain extremal rays of the type (I) and simple extremal rays of the type (II)

As above, we assume that X is a projective normal 3-fold with $\mathbf{Q}$-factorial singularities.

Definition 2.3.1. An extremal ray R of the type (II) is called *simple* if

$$R\cdot(D(R)+D)\geq 0$$

for any irreducible divisor D such that $R\cdot D>0$.

The following proposition gives a simple sufficient condition for an extremal ray to be simple.

Proposition 2.3.2. *Let R be an extremal ray of the type (II) and $f\colon X\to Y$ the contraction of R. Suppose that the curve $f(D(R))$ is not contained in the set of singularities of Y. Then*

(1) *the ray R is simple;*

(2) *if X has only isolated singularities, then a general element C of the ray R (a general fiber of the morphism $f|D(R)$) is isomorphic to $\mathbf{P}^1$ and the divisor $D(R)$ is non-singular along C. If additionally $R\cdot K_X<0$, then $C\cdot D(R)=C\cdot K_X=-1$.*

(3) *In particular, both statements* (1) *and* (2) *are true if X has terminal singularities and $R\cdot K_X<0$.*

Proof. Let D be an irreducible divisor on X such that $R\cdot D>0$. Since $R\cdot D(R)<0$, the divisor D is different from $D(R)$ and the intersection $D\cap D(R)$ is a curve which does not belong to R. Then $D'=f_*(D)$ is an irreducible divisor on Y and $\Gamma=f(D(R))$ is a curve on D'. Let $y\in\Gamma$ be a non-singular point of Y. Then the divisor D' is defined by some local equation ϕ in a neighbourhood U of y. Evidently, in the open set $f^{-1}(U)$ we can write

$$(f^*\phi)=D+m\,(D(R))$$

where the integer $m\geq 1$. Let a curve $C\in R$ and $f(C)=y\in U\cap f(D(R))$. Then $0=C\cdot(D+m(D(R)))=C\cdot(D+D(R))+C\cdot(m-1)(D(R))$. Since $m\geq 1$ and $C\cdot D(R)<0$, it follows that $C\cdot(D+D(R))\geq 0$.

Let us prove (2). Let us consider a linear system $|H|$ of hyperplane sections on Y and the corresponding linear systems on resolutions of singularities of Y and X. Let us apply Bertini's theorem (see, for example, [Ha, ch. III, Corollary 10.9 and the Exercise 11.3]) to these linear systems. Singularities of X and Y are isolated. Then by Bertini theorem, for a general element H of $|H|$ we obtain that (a) H and $H'=f^{-1}(H)$ are irreducible and non-singular; (b) H intersects Γ transversely in non-singular points of Γ. Let us consider the corresponding birational morphism $f'=f|H'\colon H'\to H$ of the non-singular irreducible surfaces. It is a composition of blowing ups at non-singular points. Thus, fibers of f' over $H\cap\Gamma$ are trees of non-singular rational curves. The exceptional curve of the first of these blowing ups is identified with the fiber of the projectivization of the normal bundle $\mathbf{P}(\mathcal{N}_{\Gamma/Y})$. Thus, we obtain a rational map over the curve Γ

$$\phi: \mathbf{P}(\mathcal{N}_{\Gamma/Y}) \to D(R)$$

of the irreducible surfaces. Evidently, it is an injection at general points of $\mathbf{P}(\mathcal{N}_{\Gamma/Y})$. It follows that ϕ is a birational isomorphism of the surfaces. Since ϕ is a birational map over the curve Γ, it follows that the general fibers of this maps are birationally isomorphic. It follows that a general fiber of f' is $C \simeq \mathbf{P}^1$. Since C is non-singular and is an intersection of the non-singular surface H' with the surface $D(R)$, and since X has only isolated singularities, it follows that $D(R)$ is non-singular along the general curve C.

The X and $D(R)$ are non-singular along $C \simeq \mathbf{P}^1$ and the curve C is non-singular. Then the canonical class $K_C = (K_X + D(R))|C$ where both divisors K_X and $D(R)$ are Cartier divisors on X along C. It follows that $-2 = \deg K_C = K_X \cdot C + D(R) \cdot C$, where the both numbers $K_X \cdot C$ and $D(R) \cdot C$ are negative integers. Then $D(R) \cdot C = K_X \cdot C = -1$.

If X has terminal singularities and $R \cdot K_X < 0$, then Y has terminal singularities too (see, for example, [Ka1]). Moreover, 3-dimensional terminal singularities are isolated. From (1), (2), the last statement of the Proposition follows.

In connection with Proposition 2.3.2, see also [Mo2, 1.3 and 2.3.2] and [I, Lemma 1].

Let R_1, R_2 are two extremal rays of the type (I) or (II). They are joined if $D(R_1) \cap D(R_2) \neq 0$. It defines *connected components* of a set of extremal rays of the type (I) or (II).

We recall (see Chapter I) that a set $\mathscr{E}$ of extremal rays is called *extremal* if it is contained in a face of $\overline{NE}(X)$. We say that $\mathscr{E}$ is *extremal of Kodaira dimension 3* if it is contained in a face of numerical Kodaira dimension 3 of $\overline{NE}(X)$.

We prove the following classification result.

Theorem 2.3.3. *Let $\mathscr{E} = \{R_1, R_2, \ldots, R_n\}$ be an extremal set of extremal rays of the type (I) or (II). Suppose that every extremal ray of $\mathscr{E}$ of the type (II) is simple. Assume that $\mathscr{E}$ is contained in a contractible face with Kodaira dimension 3 of $\overline{NE}(X)$. (Thus, $\mathscr{E}$ is extremal of Kodaira dimension 3.) Then every connected component of $\mathscr{E}$ has a type $\mathfrak{A}_1$, $\mathfrak{B}_2$, $\mathfrak{C}_m$ or $\mathfrak{D}_2$ below (see figure 1).*

($\mathfrak{A}_1$) *One extremal ray of the type (I).*

($\mathfrak{B}_2$) *Two different extremal rays S_1, S_2 of the type (II) such that their divisors $D(S_1) = D(S_2)$ coincided.*

($\mathfrak{C}_m$) *$m \geq 1$ extremal rays $S_1, S_2, \ldots, S_m$ of the type (II) such that their divisors $D(S_2), D(S_3), \ldots, D(S_m)$ do not intersect one another, and $S_1 \cdot D(S_i) = 0$ and $S_i \cdot D(S_1) > 0$ for $i = 2, \ldots, m$.*

($\mathfrak{D}_2$) *Two extremal rays S_1, S_2, where S_1 is of the type (II) and S_2 of the* *type (I), $S_1 \cdot D(S_2) > 0$ and $S_2 \cdot D(S_1) > 0$. Either $S_1 \cdot (b_1 D(S_1) + b_2 D(S_2)) < 0$ or $S_2 \cdot (b_1 D(S_1) + b_2 D(S_2)) < 0$ for any b_1, b_2 such that $b_1 \geq 0$, $b_2 \geq 0$ and one of b_1, b_2 is not zero.*

The following inverse statement is true: If $\mathscr{E} = \{R_1, R_2, \ldots, R_n\}$ is a connected set of extremal rays of the type (I) or (II) and $\mathscr{E}$ has a type $\mathfrak{A}_1$, $\mathfrak{B}_2$, $\mathfrak{C}_m$ or $\mathfrak{D}_2$ above, then $\mathscr{E}$ generates a simplex face $R_1 + \cdots + R_n$ of the dimension n and numerical Kodaira dimension 3 of $\overline{NE}(X)$. In particular, extremal rays of the set $\mathscr{E}$ are linearly independent.

Proof. Let us prove the first statement. We can suppose that $\mathscr{E}$ is connected. We have to prove that $\mathscr{E}$ has the type $\mathfrak{A}_1$, $\mathfrak{B}_2$, $\mathfrak{C}_m$ or $\mathfrak{D}_2$. If $n=1$, this is obvious.

Let $n=2$. From Lemma 2.2.2, it follows that one of the rays R_1, R_2 has the type (II). Let R_1 have the type (II) and R_2 the type (I). Since $D(R_1) \cap D(R_2) \neq \emptyset$, evidently $R_2 \cdot D(R_1) > 0$. If $R_1 \cdot D(R_2) = 0$, then the curve $D(R_1) \cap D(R_2)$ belongs to the ray R_1. It follows that the rays R_1 and R_2 contain the same curve. We get a contradiction. Thus, $R_1 \cdot D(R_2) > 0$. The rays R_1, R_2 belong to a contractible face of Kodaira dimension 3 of Mori polyhedron. Let f be a contraction of this face. By the Lemma 2.2.3, f contracts the divisors $D(R_1)$, $D(R_2)$ to the same point. By Proposition 2.2.6, there exist positive a_1, a_2 such that $R_1 \cdot (a_1 D(R_1) + a_2 D(R_2)) < 0$ and $R_2 \cdot (a_1 D(R_1) + a_2 D(R_2)) < 0$. Now suppose that for some $b_1 > 0$ and $b_2 > 0$ the inequalities $R_1 \cdot (b_1 D(R_1) + b_2 D(R_2)) \geq 0$ and $R_2 \cdot (b_1 D(R_1) + b_2 D(R_2)) \geq 0$ hold. There exists $\lambda > 0$ such that $\lambda b_1 \leq a_1$, $\lambda b_2 \leq a_2$ and one of these inequalities is an equality. For example, let $\lambda b_1 = a_1$. Then

$$R_1 \cdot (a_1 D(R_1) + a_2 D(R_2)) = R_1 \cdot \lambda (b_1 D(R_1) + b_2 D(R_2)) + R_1 \cdot (a_2 - \lambda b_2) D(R_2) \geq 0 .$$

We get a contradiction. It proves that in this case $\mathscr{E}$ has the type $\mathfrak{D}_2$.

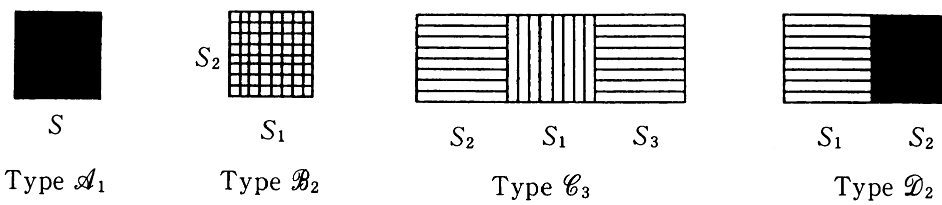

Figure 1.

Now assume that both rays R_1, R_2 have the type (II). Since the rays R_1, R_2 are simple, from Lemma 2.2.7, it follows that either $R_1 \cdot D(R_2) = 0$ or $R_2 \cdot D(R_1) = 0$. If both these equalities hold, the rays R_1, R_2 have a common curve. We get a contradiction. Thus, in this case, $\mathscr{E}$ has the type $\mathfrak{C}_2$.

Let $n=3$. Every proper subset of $\mathscr{E}$ has connected components of types

$\mathfrak{A}_1$, $\mathfrak{B}_2$, $\mathfrak{C}_m$ or $\mathfrak{D}_2$. Using Lemmas 2.2.2—2.2.5, one can see very easily that either $\mathscr{E}$ has the type $\mathfrak{C}_3$ or we have the following case:

The rays R_1, R_2, R_3 have the type (II), every two elements subset of $\mathscr{E}$ has the type $\mathfrak{C}_2$ and we can find a numeration such that $R_1 \cdot D(R_2) > 0$, $R_2 \cdot D(R_3) > 0$, $R_3 \cdot D(R_1) > 0$. Let f be a contraction of the face γ. By Lemma 2.2.3, f contracts the divisoras $D(R_1)$, $D(R_2)$, $D(R_3)$ to a one point. By Proposition 2.2.6, there are positive a_1, a_2, a_3 such that

$$R_i \cdot (a_1 D(R_1) + a_2 D(R_2) + a_3 D(R_3)) < 0$$

for $i=1, 2, 3$. On the other hand, from simplicity of the rays R_1, R_2, R_3, it follows that

$$R_i \cdot (D(R_1) + D(R_2) + D(R_3)) \geq 0 \ .$$

Let $a_1 = \min \{a_1, a_2, a_3\}$. From the last inequality,

$$R_1 \cdot (a_1 D(R_1) + a_2 D(R_2) + a_3 D(R_3)) =$$
$$= R_1 \cdot a_1 (D(R_1) + D(R_2) + D(R_3)) + R_1((a_2 - a_1) D(R_2) + (a_3 - a_1) D(R_3)) \geq 0 \ .$$

We get a contradiction with the inequality above.

Let $n > 3$. We have proven that every two or three elements subset of $\mathscr{E}$ has connected components of types $\mathfrak{A}_1$, $\mathfrak{B}_2$, $\mathfrak{C}_m$ or $\mathfrak{D}_2$. It follows very easily that $\mathscr{E}$ has the type $\mathfrak{C}_n$ (we suppose that $\mathscr{E}$ is connected).

Let us prove the inverse statement. For the type $\mathfrak{A}_1$ this is obvious.

Let $\mathscr{E}$ have the type $\mathfrak{B}_2$. Since the rays S_1, S_2 are extremal of Kodaira dimension 3, there are *nef* elements H_1, H_2 such that $H_1 \cdot S_1 = H_2 \cdot S_2 = 0$, $H_1^3 > 0$, $H_2^3 > 0$. Let $0 \neq C_1 \in S_1$ and $0 \neq C_2 \in S_2$. Let D be a divisor of the rays S_1 and S_2. Let us consider a map

$$(H_1, H_2) \to H = (-D \cdot C_2)(H_2 \cdot C_1) H_1 + \\ + (-D \cdot C_1)(H_1 \cdot C_2) H_2 + (H_2 \cdot C_1)(H_1 \cdot C_2) D \ . \tag{3.1}$$

For a fixed H_1, we get a linear map $H_2 \to H$ of the set of *nef* elements H_2 orthogonal to S_2 into the set of *nef* elements H orthogonal to S_1 and S_2. This map has a one dimensional kernel generated by $(-D \cdot C_2) H_1 + (H_1 \cdot C_2) D$. It follows that $S_1 + S_2$ is a 2-dimensional face of $\overline{NE}(X)$.

For a general *nef* element $H = a_1 H_1 + a_2 H_2 + bD$ orthogonal to this face, where $a_1, a_2, b > 0$, we have $H^3 = (a_1 H_1 + a_2 H_2 + bD)^3 \geq (a_1 H_1 + a_2 H_2 + bD)^2 \cdot (a_1 H_1 + a_2 H_2) = (a_1 H_1 + a_2 H_2 + bD) \cdot (a_1 H_1 + a_2 H_2 + bD) \cdot (a_1 H_1 + a_2 H_2) \geq (a_1 H_1 + a_2 H_2)^2 \cdot (a_1 H_1 + a_2 H_2 + bD) \geq (a_1 H_1 + a_2 H_2)^3 > 0$, since $a_1 H_1 + a_2 H_2 + bD$ and $a_1 H_1 + a_2 H_2$ are *nef*. It follows that the face $S_1 + S_2$ is of the numerical Kodaira dimension 3.

Let $\mathscr{E}$ have the type $\mathfrak{C}_m$. Let H be a *nef* element orthogonal to the ray S_1. Let $0 \neq C_i \in S_i$. Let us consider a map

$$H \to H' = H + \sum_{i=2}^{m} (-(H \cdot C_i)/(C_i \cdot D(S_i))) D(S_i) \ . \tag{3.2}$$

It is a linear map of the set of *nef* elements H orthogonal to S_1 into the set of *nef* elements H' orthogonal to the rays $S_1, S_2, \ldots, S_m$. The kernel of the map has the dimension $m-1$. It follows that the rays $S_1, S_2, \ldots, S_m$ belong to face of $\overline{NE}(X)$ of a dimension $\leq m$. On the other hand, multiplying the divisors $D(S_1), \ldots, D(S_m)$ by rays $S_1, \ldots, S_m$, one can see very easily that the rays $S_1, \ldots, S_m$ are linearly independent. Thus, they generate an m-dimensional face of $\overline{NE}(X)$. Let us show that this face is $S_1+S_2+\cdots+S_m$. To prove this, we show that every $m-1$ subset of $\mathscr{E}$ is contained in a face of $\overline{NE}(X)$ of a dimension $\leq m-1$.

If this subset contains the ray S_1, this subset has the type $\mathfrak{C}_{m-1}$. By induction, we can suppose that this subset belongs to a face of $\overline{NE}(X)$ of dimension $m-1$. Let us consider the subset $\{S_2, S_3, \ldots, S_m\}$. Let H be an ample element of X. For the element H, the map (3.2) gives an element H' which is orthogonal to the rays $S_2, \ldots, S_m$, but is not orthogonal to the ray S_1. It follows that the set $\{S_2, \ldots, S_m\}$ belongs to a face of the Mori polyhedron of the dimension $<m$. Like the above, one can see that for a general H orthogonal to S_1, the element H' has $(H')^3 \geq H^3>0$.

Let $\mathscr{E}$ have the type $\mathfrak{D}_2$. Let H be a *nef* element orthogonal to the ray S_2. Let $0\neq C_i \in S_i$. Let us consider a map

$$H \rightarrow H'=H+\frac{(H\cdot C_1)\left((-D(S_2)\cdot C_2)D(S_1)+(D(S_1)\cdot C_2)D(S_2)\right)}{(D(S_2)\cdot C_2)(D(S_1)\cdot C_1)-(D(S_1)\cdot C_2)(D(S_2)\cdot C_1)} . \tag{3.3}$$

Evidently, $C_2\cdot\left((-D(S_2)\cdot C_2)D(S_1)+(D(S_1)\cdot C_2)D(S_2)\right)=0$. From this equality and the inequality of the definition of the system $\mathfrak{D}_2$, it follows that $C_1\cdot\left((-D(S_2)\cdot C_2)D(S_1)+(D(S_1)\cdot C_2)D(S_2)\right)<0$. Thus, the denominator of the formula (3.3) is positive. Then (3.3) is a linear map of the set of *nef* elements H orthogonal to the ray S_2 into the set of *nef* elements H' orthogonal to the rays S_1, S_2. Evidently, the map has a one dimensional kernel. Thus, the rays S_1 and S_2 generate a two dimensional face S_1+S_2 of Mori polyhedron. As above, for a general element H orthogonal to S_2 we have $(H')^3\geq H^3>0$.

Corollary 2.3.4. *Let $\mathscr{E}=\{R_1, R_2, \ldots, R_n\}$ be an extremal set of extremal rays of the type (I) or (II) and every extremal ray of $\mathscr{E}$ of the type (II) is simple. Assume that $\mathscr{E}$ is contained in a contractible face with Kodaira dimension* 3 *of the $\overline{NE}(X)$. Let $m_1\geq 0, m_2\geq 0, \ldots, m_n\geq 0$ and at least one of $m_1, \ldots, m_n$ is positive.*

Then there exists i, $1\leq i\leq n$, such that

$$R_i\cdot(m_1D(R_1)+\cdots+m_nD(R_n))<0 .$$

Thus, the condition (ii) of Chapter I is valid.

Proof. It is sufficient to prove this statement for the connected $\mathscr{E}$. For every type $\mathfrak{A}_1$, $\mathfrak{B}_2$, $\mathfrak{C}_m$ and $\mathfrak{D}_2$ of the Theorem 2.3.3, one can prove it very easily.

Unfortunately, in general, the inverse statement of the Theorem 2.3.3 holds only for connected extremal sets $\mathscr{E}$. We will give two cases where it is true for a non-connected $\mathscr{E}$.

Definition 2.3.5. A threefold X is called *strongly projective* (respectively *very strongly projective*) if the following statement holds: a set $\{Q_1, \dots, Q_n\}$ of extremal rays of the type (II) is extremal of Kodaira dimension 3 (respectively generates the simplex face $Q_1+\cdots+Q_n$ of $\overline{NE}(X)$ of dimension n and Kodaira dimension 3) if its divisors $D(Q_1), \dots, D(Q_n)$ do not intersect one another.

Theorem 2.3.6. *Let $\mathcal{E}=\{R_1, R_2, \dots, R_n\}$ be a set of extremal rays of the type (I) or (II) such that every connected component of $\mathcal{E}$ has the type $\mathfrak{A}_1$, $\mathfrak{B}_2$, $\mathfrak{C}_m$ or $\mathfrak{D}_2$. Then:*

(1) *$\mathcal{E}$ is extremal of numerical Kodaira dimension 3 if and only if the same is true for any subset of $\mathcal{E}$ containing only extremal rays of the type (II) whose divisors do not intersect one another. In particular, it holds if X is strongly projective.*

(2) *$\mathcal{E}$ generates a simplex face $R_1+\cdots+R_n$ with numerical Kodaira dimension 3 of the Mori polyhedron if and only if the same is true for any subset of $\mathcal{E}$ containing only extremal rays of the type (II) whose divisors do not intersect one another. In particular, it is true if X is very strongly projective.*

Proof. Let us prove (1). Only the inverse statement is non-trivial. We prove it by induction on n. For $n=1$, the statement is obviously true.

Assume that some connected component of $\mathcal{E}$ has the type $\mathfrak{A}_1$. Suppose that this component contains the ray R_1. By our induction hypothesis, there exists a nef element H such that $H^3>0$ and $H\cdot R_i=0$ if $i>1$. Then there exists $k\geq 0$, such that $H'=H+kD(R_1)$ is nef and $H'\cdot\mathcal{E}=0$. As above, one can prove that $(H')^3\geq H^3>0$.

Assume that some connected component of $\mathcal{E}$ has the type $\mathfrak{B}_2$. Suppose that this component contains the rays R_1, R_2 and $D(R_1)=D(R_2)=D$. Then, by induction, there are nef elements H_1 and H_2 such that $H_1^3>0$, $H_2^3>0$ and $H_1\cdot\{R_1, R_3, \dots, R_n\}=0$, $H_2\cdot\{R_2, R_3, \dots, R_n\}=0$. As for the proof of the inverse statement of the Theorem 2.3.3 in the case $\mathfrak{B}_2$, there are $k_1\geq 0$, $k_2\geq 0$, $k_3\geq 0$ such that the element $H=k_1H_1+k_2H_2+k_3D$ is *nef*, $H\cdot\mathcal{E}=0$ and $H^3>0$.

Assume that some connected component of $\mathcal{E}$ has the type $\mathfrak{C}_m$, $m>1$. We use the notation of Theorem 2.3.3 for this connected component. Let this be $\{S_1, S_2, \dots, S_m\}$. By induction, there exists a nef element H such that H is orthogonal to $\mathcal{E}-\{S_2, \dots, S_m\}$ and $H^3>0$. As for the proof of the inverse statement of the Theorem 2.3.3 in the case $\mathfrak{C}_m$, there are $k_2\geq 0, \dots, k_m\geq 0$ such that $H'=H+k_2D(S_2)+\cdots+k_mD(S_m)$ is nef, $H'\cdot\mathcal{E}=0$ and $(H')^3\geq H^3>0$.

Assume that some connected component of $\mathcal{E}$ has the type $\mathfrak{D}_2$. We use the notation of Theorem 2.3.3 for this connected component. Let this be $\{S_1, S_2\}$. By induction, there exists a nef element H such that $H^3>0$ and H is orthogonal to $\mathcal{E}-\{S_1\}$. As for Theorem 2.3.3, there are $k_1\geq 0$, $k_2\geq 0$ such that $H'=H+k_1D(S_1)+k_2D(S_2)$ is *nef*, $H'\cdot\mathcal{E}=0$ and $(H')^3\geq H^3>0$.

If every connected component of $\mathcal{E}$ has the type $\mathfrak{C}_1$, then the statement holds by the condition of the theorem.

Let us prove (2). Only the inverse statement is non-trivial. We prove it by induction on n. For $n=1$ the statement is true. It is sufficient to prove that $\mathscr{E}$ is contained in a face of a dimension $\leq n$ of Mori polyhedron because, by our induction hypothesis, any its $n-1$ elements subset generates a simplex face of the dimension $n-1$ of Mori polyhedron.

Assume that some connected component of $\mathscr{E}$ has the type $\mathfrak{A}_1$. Suppose that the ray R_1 belongs to this component and $0\neq C_1\in R_1$. Let us consider the map

$$H\rightarrow H'=H'+((H\cdot C_1)/(-D(R_1)\cdot C_1))D(R_1)\ .$$

of the set of nef elements H orthogonal to the set $\{R_2, \ldots, R_n\}$ into the set of nef elements H' orthogonal to the $\mathscr{E}$. It is the linear map with one dimensional kernel. Since, by the induction, the set $\{R_2, \ldots, R_n\}$ is contained in a face of Mori polyhedron of the dimension $n-1$, it follows that $\mathscr{E}$ is contained in a face of the dimension n.

If $\mathscr{E}$ has a connected component of the type $\mathfrak{B}_2$, $\mathfrak{C}_m$, $m>1$, or $\mathfrak{D}_2$, the proof is the same if one uses the maps (3.1), (3.2) and (3.3) above.

If all connected components of $\mathscr{E}$ have the type $\mathfrak{C}_1$, the statement holds by the condition.

Remark 2.3.7. Like the statement (1) of Theorem 2.3.6, one can prove that a set $\mathscr{E}$ of extremal rays with connected components of the type $\mathfrak{A}_1$, $\mathfrak{B}_2$, $\mathfrak{C}_m$ or $\mathfrak{D}_2$ is extremal if and only if the same is true for any subset of $\mathscr{E}$ containing only extremal rays of the type (II) whose divisors do not intersect one another.

The next proposition is simple but important. To simplify the notation, we say that for a fixed $a_1, \ldots, a_n$, we have a *linear dependence condition*

$$a_1R_1+\cdots+a_nR_n=0$$

between extremal rays $R_1, \ldots, R_n$ if there exist non-zero $C_i\in R_i$ such that

$$a_1C_1+\cdots+a_nC_n=0\ .$$

Proposition 2.3.8. *Assume that a set $\mathscr{E}=\{R_1, R_2, \ldots, R_m\}$ of extremal rays has connected components of the type $\mathfrak{A}_1$, $\mathfrak{B}_2$, $\mathfrak{C}_m$ or $\mathfrak{D}_2$ and there exists a linear dependence condition $a_1R_1+a_2R_2+\cdots+a_mR_m=0$ with all $a_i\neq 0$. Then all connected components of $\mathscr{E}$ have the type $\mathfrak{B}_2$. Let these components be $\mathfrak{B}^1, \ldots, \mathfrak{B}^t$. Then $t\geq 2$, and we can choose a numeration such that $\mathfrak{B}^i=\{R_{i1}, R_{i2}\}$ and the linear dependence has a form*

$$a_{11}R_{11}+a_{21}R_{21}+\cdots+a_{t1}R_{t1}=a_{12}R_{12}+a_{22}R_{22}+\cdots+a_{t2}R_{t2}\ .$$

where all $a_{ij}>0$.

Proof. Let us multiply the divisors $D(R_1), \ldots, D(R_m)$ by the equality $a_1R_1+a_2R_2+\cdots+a_mR_m=0$. Then we get that $a_k=0$ if the ray R_k belongs to a connected component of the type $\mathfrak{A}_1$, $\mathfrak{C}_m$ or $\mathfrak{D}_2$. Thus, all connected components of $\mathscr{E}$ have the type $\mathfrak{B}_2$. Let these components be

$$\mathfrak{B}^1=\{R_{11}, R_{12}\}\ , \mathfrak{B}^2=\{R_{21}, R_{22}\}\ , \ldots, \mathfrak{B}^t=\{R_{t1}, R_{t2}\}\ .$$

Obviously, $t \geq 2$, and we can rewrite the linear dependence as

$$a_{11}R_{11}+a_{12}R_{12}+a_{21}R_{21}+a_{22}R_{22}+\cdots+a_{t1}R_{t1}+a_{t2}R_{t2}=0\ ,$$

where all $a_{ij} \neq 0$. Multiplying all divisors $D(R_{ij})$ by this equation and using inequalities $R_{ij} \cdot D(R_{ij}) < 0$, we get the last statement of the proposition.

4. A classification of E-sets of extremal rays of type (I) or (II)

As in the above, we suppose that X is a projective normal 3-fold with $\mathbf{Q}$-factorial singularities.

We recall that a set $\mathcal{L}$ of extremal rays is called an *E-set* if it is not extremal but any proper subset of $\mathcal{L}$ is extremal (it is contained in a face of $\overline{NE}(X)$). Thus, an E-set is a minimal non-extremal set of extremal rays.

Theorem 2.4.1. *Let $\mathcal{L}$ be an E-set of extremal rays of the type (I) or (II). Suppose that every ray of the type (II) of $\mathcal{L}$ is simple and every proper subset of $\mathcal{L}$ is contained in a contractible face of Kodaira dimension* 3 *of Mori polyhedron. Then we have one of the following cases*:

(a) *$\mathcal{L}$ is connected and $\mathcal{L}=\{R_1, R_2, R_3\}$, where any R_i has the type (II) and each of 2-element subsets $\{R_1, R_2\}$, $\{R_2, R_3\}$, $\{R_3, R_1\}$ of $\mathcal{L}$ has the type $\mathfrak{C}_2$. Here $R_1 \cdot D(R_2) > 0$, $R_2 \cdot D(R_3) > 0$, $R_3 \cdot D(R_1) > 0$ but $R_2 \cdot D(R_1) = R_3 \cdot D(R_2) = R_1 \cdot D(R_3) = 0$. The divisor $D(\mathcal{L}) = D(R_1) + D(R_2) + D(R_3)$ is nef.*

(b) *$\mathcal{L}$ is connected and $\mathcal{L}=\{R_1, R_2\}$, where at least one of the rays R_1, R_2 has the type (II). There are positive m_1, m_2 such that $R \cdot (m_1D(R_1) + m_2D(R_2)) \geq 0$ for any extremal ray R of the type (I) or simple extremal ray of type (II) on X. If the divisor $m_1D(R_1) + m_2D(R_2)$ is not nef, both the extremal rays R_1, R_2 have the type (II).*

(c) *$\mathcal{L}$ is connected and $\mathcal{L}=\{R_1, R_2\}$ where both R_1, and R_2 have the type (II) and there exists a simple extremal ray S_1 of the type (II) such that the rays R_1, S_1 define the extremal set of the type $\mathfrak{B}_2$ (it means that $S_1 \neq R_1$ but the divisors $D(S_1) = D(R_1)$) and the rays S_1, R_2 define the extremal set of the type $\mathfrak{C}_2$, where $S_1 \cdot D(R_2) = 0$ but $R_2 \cdot D(S_1) > 0$. Here there do not exist positive m_1, m_2 such that the divisor $m_1D(R_1) + m_2D(R_2)$ is nef, since evidently $S_1 \cdot (m_1D(R_1) + m_2D(R_2)) < 0$. See figure* 2 *below.*

(d) *$\mathcal{L}=\{R_1, \ldots, R_k\}$ where $k \geq 2$, all rays $R_1, \ldots, R_k$ have the type (II) and the divisors $D(R_1), \ldots, D(R_k)$ do not intersect one another. Any proper subset of $\mathcal{L}$ is contained in a contractible face of Kodaira dimension* 3 *of Mori polyhedron but $\mathcal{L}$ is not contained in a face of Mori polyhedron.*

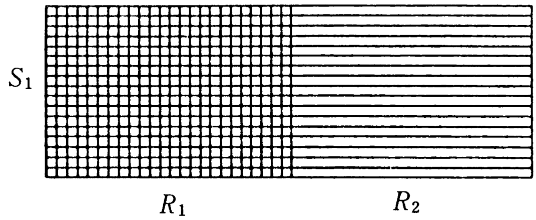

Figure 2.

Proof. Let $\mathscr{L}=\{R_1, \ldots, R_n\}$ be an E-set of extremal rays satisfying the conditions of the theorem. Let us consider two cases.

The case 1. Let $\mathscr{L}$ is not connected. Then every connected component of $\mathscr{L}$ is extremal and, by Theorem 2.3.3, it has the type $\mathfrak{A}_1$, $\mathfrak{B}_2$, $\mathfrak{C}_m$ or $\mathfrak{D}_2$. If some of these components does not have the type $\mathfrak{C}_1$, then, by the statement (1) of Theorem 2.3.6, $\mathscr{L}$ is extremal and we get a contradiction. Thus, we get the case (d) of the theorem.

The case 2. Let $\mathscr{L}=\{R_1, \ldots, R_n\}$ is connected. Let $n \geq 4$. By Theorem 2.3.3, any proper subset of $\mathscr{L}$ has connected components of the type $\mathfrak{A}_1$, $\mathfrak{B}_2$, $\mathfrak{C}_m$ or $\mathfrak{D}_2$. Like for the proof of Theorem 2.3.3, it follows that $\mathscr{L}$ has the type $\mathfrak{C}_n$. By Theorem 2.3.3, then $\mathscr{L}$ is extremal. We get a contradiciton.

Let $n=3$. Then, like for the proof of Theorem 2.3.3, we get that $\mathscr{L}$ has the type (a).

Let $n=2$ and $\mathscr{L}=\{R_1, R_2\}$. If both rays R_1, R_2 have the type (I), then, by Lemma 2.2.2, $\mathscr{L}$ is not connected and we get a contradiction.

Let R_1 has the type (I) and R_2 has the type (II). Since the set $\mathscr{L}$ is not extremal, by Theorem 2.3.3, there are positive m_1, m_2 such that $R_1 \cdot (m_1 D(R_1)+m_2 D(R_2)) \geq 0$ and $R_2 \cdot (m_1 D(R_1)+m_2 D(R_2)) \geq 0$. By Lemma 2.2.3, it follows that $C \cdot (m_1 D(R_1)+m_2 D(R_2)) \geq 0$ if the curve C is contained in the $D(R_1) \cup D(R_2)$. If C is not contained in $D(R_1) \cup D(R_2)$, then obviously $C \cdot (m_1 D(R_1)+m_2 D(R_2)) \geq 0$. It follows, that the divisor $m_1 D(R_1)+m_2 D(R_2)$ is nef. Thus, we get the case (b).

Let both rays R_1, R_2 have the type (II). If $D(R_1)=D(R_2)$, then we get an extremal set $\{R_1, R_2\}$ by Theorem 2.3.3. Thus, the divisors $D(R_1)$ and $D(R_2)$ are different. By Lemma 2.2.1, the curve $D(R_1) \cap D(R_2)$ does not have an irreducible component which belongs to both rays R_1 and R_2. Since rays R_1, R_2 are simple, it follows that $R_1 \cdot (D(R_1)+D(R_2)) \geq 0$ and $R_2 \cdot (D(R_1)+D(R_2)) \geq 0$. Let R be an extremal ray of type (I) or simple extremal ray of the type (II). If the divisor $D(R)$ does not coincide with the divisor $D(R_1)$ or $D(R_2)$, then obviously $R \cdot (D(R_1)+D(R_2)) \geq 0$. Thus, if there does not exist an extremal ray R which has the same divisor as the ray R_1 or R_2, we get the case (b).

Assume that $D(R)=D(R_1)$. Then, by Lemma 2.2.5, the ray R has the type (II), too. If $R \cdot D(R_2)=0$, we get the case (c) of the theorem where $S_1=R$. If $R \cdot D(R_2)>0$, then $R \cdot (D(R_1)+D(R_2)) \geq 0$ since the ray R is simple. Then we get the case (b) of the theorem.

5. An application of the diagram method to Fano 3-folds with terminal singularities

We restrict ourselves to considering Fano 3-folds with $\mathbf{Q}$-factorial terminal singularities, but it is possible to formulate and prove corresponding results for a negative part of Mori cone of 3-dimensional variety with $\mathbf{Q}$-factorial terminal singularities like in [N7].

We recall that an algebraic 3-fold X over $\mathbf{C}$ with $\mathbf{Q}$-factorial singularities is called Fano if the anticanonical class $-K_X$ is ample. By results of Kawamata [Ka1] and Shokurov [Sh], any face of $\overline{NE}(X)$ is contractible and $\overline{NE}(X)$ is generated by a finite set of extremal rays if X is a Fano 3-fold with terminal $\mathbf{Q}$-factorial singularities.

5.1. Preliminary results. We need the following

Lemma 2.5.1 *Let X be a Fano 3-fold with $\mathbf{Q}$-factorial terminal singularities. Let $\mathscr{E}=\{R_1, ..., R_n\}$ be a set of n extremal rays of the type (II) and with disjoint divisors $D(R_1), ..., D(R_n)$ on X. (Thus, $\mathscr{E}$ has the type $n\mathfrak{C}_1$).*

If we suppose that the set $\mathscr{E}$ is not extremal, then there exists a small extremal ray S and i, $1\le i\le n$, such that $S\cdot(-K_X+D(R_i))<0$ and $S\cdot D(R_j)=0$ if $j\ne i$.

It follows that any curve of the ray S belongs to the divisor $D(R_i)$.

Proof. By Proposition 2.3.2, the divisor $H=-K_X+D(R_1)+\cdots+D(R_n)$ is orthogonal to $\mathscr{E}$. Besides, H is *nef* and $H^3>0$ if there does not exist a small extremal ray S with the property above. Then, $\mathscr{E}$ is extremal of Kodaira dimension 3.

Definition 2.5.2. A set $\{R, S\}$ of extremal rays has the type $\mathfrak{C}_2$ if the ray R has type (II), the extremal ray S is small and $S\cdot D(R)<0$. (See Figure 3.)

Thus, by Lemma 2.5.1, the set $R_1, ..., R_n, S$ of extremal rays contains a subset of the type $\mathfrak{C}_2$.

By Proposition 2.3.2, any extremal ray of X of the type (II) is simple, and by results of Sections 3 and 4 we get a classification of extremal sets and E-sets of extremal rays of the type (I) and (II) on X.

We have the following general theorem.

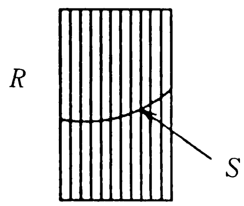

Figure 3.

Theorem 2.5.3. *Let X be a Fano 3-fold with $\mathbf{Q}$-factorial terminal singularities. Let α be a face of $\overline{NE}(X)$. Then we have the following possibilities:*

(1) *There exists a small extremal ray S such that $\alpha+S$ is contained in a face of $\overline{NE}(X)$ of Kodaira dimension 3.*

(2) *There are extremal rays R_1, R_2 of the type (II) and small extremal ray S such that $\alpha+R_1$ and $\alpha+R_2$ are contained in faces of $\overline{NE}(X)$ of Kodaira dimension*

3, *the ray* R_2 *does not belong to* α, *and one of the sets* $\{R_1, S\}$ *or* $\{R_2, S\}$ *has the type* $\mathfrak{C}_2$.

(3) *The face* α *is contained in a face of* $\overline{NE}(X)$ *of Kodaira dimension* 1 *or* 2.

(4) *There exists an E-set* $\mathscr{L}=\{R_1, R_2\}$ *such that* $R_1 \not\subset \alpha$, $R_2 \not\subset \alpha$, *but* $\alpha+R_1$ *and* $\alpha+R_1$ *are contained in faces of* $\overline{NE}(X)$ *of Kodaira dimension* 3. *The* $\mathscr{L}$ *satisfies the condition* (*c*) *of Theorem* 2.4.1: *Thus, both extremal rays* R_1, R_2 *have the type* (*II*) $R_1 \cdot D(R_2)>0$ and $R_2 \cdot D(R_1)>0$ *and there exists an extremal ray* R'_1 *of the type* (*II*) *such that* $D(R_1)=D(R'_1)$ *and* $R'_1 \cdot D.(R_2)=0$.

(5) *There are extremal rays* $R_1, \ldots, R_n$ *of the type* (*II*) *such that any of them does not belong to* α, $\alpha+R_1+\cdots+R_n$ *is contained in face of* $\overline{NE}(X)$ *of Kodaira dimension* 3 *and*

$$\dim \alpha+R_1+\cdots+R_n<\dim \alpha+n \ .$$

(6) $\dim N_1(X)-\dim \alpha \leq 12$.

Proof. Let us consider the face $\gamma=\alpha^{\perp}$ of $\mathscr{M}(X)$ and apply Theorem 1.2 to this face γ. We have $\dim \gamma=\dim N_1(X)-1-\dim \alpha$.

Assume that α does not satisfy the conditions (1), (3) and (5). Then $\mathfrak{R}(\gamma)$ contains extremal rays of the type (I) or (II) only and $\mathscr{M}(X)$ is closed and simple in the face γ. By Proposition 2.3.2 and Theorem 2.3.3, any extremal subset $\mathscr{E}$ of $\mathfrak{R}(\gamma)$ has connected components of the types $\mathfrak{A}_1$, $\mathfrak{B}_2$, $\mathfrak{C}_n$ or $\mathfrak{D}_2$. By Corollary 2.3.4, the condition (ii) is valid for extremal subsets of $R(\gamma)$. Let $\mathscr{L} \subset \mathfrak{R}(\gamma)$ be a E-set. Assume that at least two elements $R_1, R_2 \in \mathscr{L}$ don't belong to $\mathfrak{R}(\gamma^{\perp})$ and for any proper subset $\mathscr{L}' \subset \mathscr{L}$ we have that $\mathscr{L}' \cup \mathfrak{R}(\gamma^{\perp})$ is extremal. Let us apply Theorem 2.4.1 to $\mathscr{L}$.

Assume that $\mathscr{L}$ has the type (d). By Lemma 2.5.1, one of extremal rays R_1 of $\mathscr{L}$ together with some small extremal ray S define a set of the type $\mathfrak{C}_2$. Since $\{R_1\} \subset \mathscr{L}$ is a proper subset of $\mathscr{L}$, the $\mathfrak{R}(\gamma^{\perp}) \cup \{R_1\}$ is extremal. Or $\alpha+R_1$ is contained in a face of $\overline{NE}(X)$. Since $\mathscr{L}$ has at least 2 elements which do not belong to $\mathfrak{R}(\gamma^{\perp})$, there exists another extremal ray R_2 of $\mathscr{L}$ which does not belong to $\mathfrak{R}(\gamma^{\perp})$. Like the above, $\alpha+R_2$ is contained in a face of $\overline{NE}(X)$ of Kodaira dimension 3. By definition of the case (d), both extremal rays R_1, R_2 have the type (II). Thus, we get the case (2) of the theorem.

Assume that $\mathscr{L}$ has the type (c). Then we get the case (4) of the theorem.

Assume that $\mathscr{L}=\{R_1, R_2\}$ has the type (b). Suppose that the divisor $m_1D(R_1)+m_2D(R_2)$ is not *nef* (see the case (b) of Theorem 2.4.1). Then there exists a small extremal ray S such that $S \cdot (m_1D(R_1)+m_2D(R_2))<0$. It follows that one of the sets $\{R_1, S\}$ or $\{R_2, S\}$ has the type $\mathfrak{C}_2$. Thus, we get the case (2).

Assume that $\mathscr{L}=\{R_1, R_2, R_3\}$ has the type (a). Then the divisor $D(R_1)+D(R_2)+D(R_3)$ is *nef*.

Thus, if we additionally exclude the cases (2) and (4), then all conditions of Theorem 1.2 are satisfied. By Theorems 2.4.1 and 2.3.3, we can take $d = 2$, $C_1 = 1$ and $C_2 = 0$. (See Figure 4 for graphs $G(\mathscr{E})$ corresponding to extremal sets $\mathscr{E}$ of the types $\mathfrak{A}_1$, $\mathfrak{B}_2$, $\mathfrak{C}_m$ and $\mathfrak{D}_2$.) Thus, by Theorem 1.2, $\dim\gamma < 34/3$. It follows that $\dim N_1(X) - \dim\alpha \leq 12$.

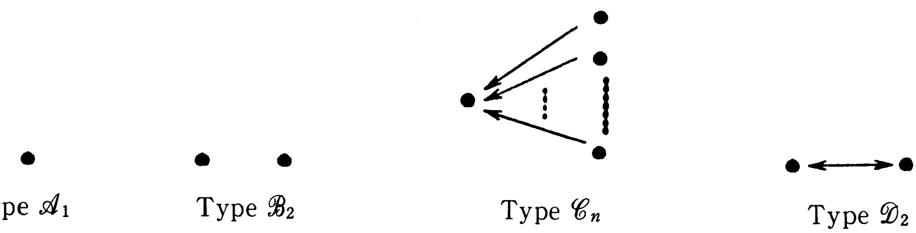

Figure 4.

5.2. General properties of configurations of extremal rays of the type $\mathfrak{B}_2$. Let $\{R_{11}, R_{12}\}$ be a set of extremal rays of the type $\mathfrak{B}_2$. By Theorem 2.3.3, they define a 2-dimensional face $R_{11} + R_{12}$ of $\overline{NE}(X)$. Let $\{R_{21}, R_{22}\}$ be another set of extremal rays of the type $\mathfrak{B}_2$. Sinse two different 2-dimensional faces of $\overline{NE}(X)$ may have only a common extremal ray, the divisors $D(R_{11}) = D(R_{12})$ and $D(R_{21}) = D(R_{22})$ don't have a common point. There exists the maximal set $\{R_{11}, R_{12}\}$, $\{R_{21}, R_{22}\}$, ..., $\{R_{n1}, R_{n2}\}$ of pairs of extremal rays of the type $\mathfrak{B}_2$.

Lemma 2.5.4. *Any t pairs $\{R_{11}, R_{12}\}$, $\{R_{21}, R_{22}\}$, ..., $\{R_{t1}, R_{t2}\}$ of extremal rays of the type $\mathfrak{B}_2$ generate a face*

$$\sum_{i=1}^{t}\sum_{j=1}^{2} R_{ij} \subset \overline{NE}(X) \subset N_1(X)$$

of the Kodaira dimension 3 *of* $\overline{NE}(X)$.

Proof. This face is orthogonal to the nef divisor $H = -K_X + D(R_{11}) + \cdots + D(R_{t1})$ with $H^3 \geq (-K_X)^3 > 0$.

Lemma 2.5.5. *Under the above notation, there exists a changing of order of pairs of extremal rays R_{i1}, R_{i2} such that $R_{11} + \cdots + R_{t1}$ is a simplex face of* $\overline{NE}(X)$.

Proof. For $t = 1$, it is obvious. Let us suppose that $\theta = R_{11} + \cdots + R_{(t-1)1}$ is a simplex face of the face

$$\alpha_{t-1} = \sum_{i=1}^{t-1}\sum_{j=1}^{2} R_{ij} .$$

The face

$$\alpha_t = \sum_{i=1}^{t}\sum_{j=1}^{2} R_{ij}$$

has α_{t-1} as its face and does not coincide with the face α_{t-1}. It follows that there exists a face β of α_t of the dimension t such that $\beta \not\subset \alpha_{t-1}$ but $\theta \subset \beta$ is a face of β. It follows that all extremal rays of β are the extremal rays $R_{11}, \ldots, R_{(t-1)1}$ and some of extremal rays R_{t1}, R_{t2}. Assume that both extremal rays R_{t1}, R_{t2} belong to β. Then the extremal rays $R_{11}, \ldots, R_{(t-1)1}, R_{t1}, R_{t2}$ are linearly dependent, since $\dim\beta = t$. By Proposition 2.3.8, it is impossible. Thus, only one of extremal rays R_{t1}, R_{t2} belongs to the face β. Suppose that this is R_{t1}. Then $\beta = R_{11} + \cdots + R_{(t-1)1} + R_{t1}$ will be the face we were looking for.

We divide the maximal set $\{R_{11}, R_{12}\}$, $\{R_{21}, R_{22}\}$, ..., $\{R_{n1}, R_{n2}\}$ of pairs of extremal rays of the type $\mathfrak{B}_2$ into two parts:

$$\{R_{11}, R_{12}\}, \quad \{R_{21}, R_{22}\}, \quad \ldots, \quad \{R_{m1}, R_{m2}\}$$

and

$$\{R_{(m+1)1}, R_{(m+1)2}\}, \quad \{R_{(m+2)1}, R_{(m+2)2}\}, \quad \ldots, \quad \{R_{(m+k)1}, R_{(m+k)2}\}$$

where $n = m + k$. By definition, here the extremal rays R_{i1}, R_{i2} belong to the first part if and only if they are linearly independent of other extremal rays from the set $\{R_{11}, R_{12}\}$, $\{R_{21}, R_{22}\}$, ..., $\{R_{n1}, R_{n2}\}$. Thus, extremal rays R_{j1}, R_{j2} belong to the second part if they are linearly dependent of other extremal rays from the set $\{R_{11}, R_{12}\}$, $\{R_{21}, R_{22}\}$, ..., $\{R_{n1}, R_{n2}\}$.

Lemma 2.5.6. *Let S be an extremal ray of the type (II) such that $\{R_{i1}, S\}$ define a configulation (c) of the Theorem 2.4.1. Thus: $R_{i1} \cdot D(S) > 0$, $S \cdot D(R_{i1}) > 0$ and $R_{i2} \cdot D(S) = 0$. Then the extremal ray R_{i1}, R_{i2} are linearly independent from all other extremal rays in $\{R_{11}, R_{12}\}$, $\{R_{21}, R_{22}\}$, ..., $\{R_{n1}, R_{n2}\}$. Thus, $1 \leq i \leq m$. There does not exist a configulation of this type with the ray R_{i2}. Thus, there does not exist an extremal ray S' of the type (II) such that $R_{i2} \cdot D(S') > 0$, $S' \cdot D(R_{i2}) > 0$ and $R_{i1} \cdot D(S') = 0$.*

Proof. The $R_{i1} + R_{i2}$ and $R_{i2} + S$ are 2-dimensional faces of $\overline{NE}(X)$ with intersection by the extremal ray R_{i2}. It follows that any curve of $D(S)$ belongs to the face $R_{i2} + S$ (by Lemma 2.2.3). It follows that the divisor $D(S)$ has no common point with the divisor $D(R_{j1})$ for any other pair R_{j1}, R_{j2} for $j \neq i$. Multiplying $D(S)$ by a linear relation of extremal rays R_{i2}, R_{i2} with other extremal rays $\{R_{11}, R_{12}\}$, $\{R_{21}, R_{22}\}$, ..., $\{R_{n1}, R_{n2}\}$ and using Proposition 2.3.8, we get that this linear relation does not exist.

Let us suppose that there exists an extremal ray S' (see formulation of the lemma). Then $R_{i1} + S'$ is another 2-dimensional face of $\overline{NE}(X)$. Evidently, divisors $D(S)$ and $D(S')$ have a non-empty intersection. Thus, faces $R_{i2} + S$ and $R_{i1} + S'$ have a common ray. But it is possible only if $S = S'$. Thus, we get a contradiction, because $R_{i1} \cdot D(S) > 0$ but $R_{i1} \cdot D(S') = 0$.

Using this Lemma 2.5.6, we can subdivide the first set

$$\{R_{11}, R_{12}\}, \{R_{21}, R_{22}\}, \ldots, \{R_{m1}, R_{m2}\} \quad .$$

into sets

$$\{R_{11}, R_{12}\}, \{R_{21}, R_{22}\}, \ldots, \{R_{m_11}, R_{m_12}\}$$

and

$$\{R_{(m_1+1)1}, R_{(m_1+1)2}\}, \ldots, \{R_{(m_1+m_2)1}, R_{(m_1+m_2)2}\}$$

where $m_1+m_2=m$. Here R_{i1}, R_{i2} belong to the first part if and only if there exists an extremal ray S such that R_{i1}, S satisfy the condition of Lemma 2.5.6. By Lemma 2.5.6, the order between extremal rays R_{i1}, and R_{i2} is then canonical.

Let us consider the second set

$$\{R_{(m+1)1}, R_{(m+1)2}\}, \{R_{(m+2)1}, R_{(m+2)2}\}, \ldots, \{R_{(m+k)1}, R_{(m+k)2}\} .$$

We introduce an invariant

$$\delta = \dim \sum_{i=m+1}^{m+k} \sum_{j=1}^{2} R_{ij} - k$$

of X. Evidently,

$$\text{either} \quad k=\delta=0 \quad \text{or} \quad k \geq 2 \quad \text{and} \quad 1 \leq \delta < k .$$

Thus,

$$\dim \sum_{i=m+1}^{m+k} \sum_{j=1}^{2} R_{ij} = k+\delta$$

Let

$$\rho_0(X) = \dim N_1(X) - \dim \sum_{i=1}^{n=m+k} \sum_{j=1}^{2} R_{ij} .$$

Then

$$\rho(X) = \dim N_1(X) = \rho_0(X) + 2m + k + \delta$$

The invariants: $\rho_0(X)$, n, m, k, δ, m_1, m_2 are important invariants of a Fano 3-fold X.

The following lemma will be very useful:

Lemma 2.5.7. *Let X be a Fano 3-fold with $\mathbf{Q}$-factorial terminal singularities. Let $\mathcal{E}$ be the set of all extremal rays of a proper face $[\mathcal{E}]$ of $\overline{NE}(X)$. Let*

$$\{R_{11}, R_{12}\} \cup \ldots \cup \{R_{t1}, R_{t2}\}$$

be a set of different pairs of extremal rays of the type $\mathfrak{B}_2$. Assume that $R \cdot D(R_{i1}) = 0$ for any $R \in \mathcal{E}$ and any i, $1 \leq i \leq t$. Then there are extremal rays $Q_1, \ldots, Q_r$ such that the following statements hold:

(a) $r \leq t$;

(b) *For any i, $1 \leq i \leq r$, there exists j, $1 \leq j \leq t$, such that $Q_i \cdot D(R_{j1}) > 0$ (in particular, Q_i is different from extremal rays of pairs of extremal rays $\{R_{u1}, R_{u2}\}$ of the type $\mathfrak{B}_2$)*;

(c) *For any j, $1 \leq j \leq r$, there exists an extremal ray Q_i, $1 \leq i \leq r$, such that $Q_i \cdot D(R_{j1}) > 0$* ;

(d) *The set $\mathcal{E} \cup \{Q_1, \ldots, Q_r\}$ is extremal, and extremal rays $\{Q_1, \ldots, Q_r\}$ are linearly independent.*

Proof. If $t=0$, we can take $r=0$. Thus, we assume that $t\geq 1$.

Since $R_{ij}\cdot D(R_{ij})<0$, $1\leq i\leq t$, $1\leq j\leq 2$, the set $\mathscr{E}$ does not contain the rays R_{ij}. Let H be a general *nef* element orthogonal to $[\mathscr{E}]$. Since $t\geq 1$, there exists $a>0$ such that $H'=H+aD(R_{11})$ is *nef* and H' is orthogonal to $\mathscr{E}$ and one of the rays R_{11}, R_{12}. Let this ray be R_{11}. Then the set $\mathscr{E}\cup\{R_{11}\}$ is extremal and is contained in a (proper) face of $\overline{NE}(X)$. It follows, $\dim[\mathscr{E}]<\dim[\mathscr{E}\subset\{R_{11}\}]<\dim\overline{NE}(X)$, and $\dim[\mathscr{E}]<\dim\overline{NE}(X)-1$. Let us consider a linear subspace $V(\mathscr{E})\subset N_1(X)$ generated by all extremal rays $\mathscr{E}$. By our condition, $V(\mathscr{E})$ is a linear envelope of the face $[\mathscr{E}]$ of $\overline{NE}(X)$.

Let us consider the factorization map π: $N_1(X)\rightarrow N_1(X)/V(\mathscr{E})$. Since the cone $\overline{NE}(X)$ is polyhedral, the cone $\pi(\overline{NE}(X))$ is generated by images of extremal rays T such that the set $\mathscr{E}\cup\{T\}$ is contained in a face $[\mathscr{E}\subset\{T\}]$ of $\overline{NE}(X)$ of the dimension $\dim[\mathscr{E}]+1$. In particular, since $\dim[\mathscr{E}]<\dim N_1(X)-1$, the face $[\mathscr{E}\cup\{T\}]$ is proper, and the set $\mathscr{E}\cup\{T\}$ is extremal.

There exists a curve C on X such that $C\cdot D(R_{11})>0$. This curve C (as any element $x\in\overline{NE}(X)$) is a linear combination of extremal rays T with non-negative coefficients and extremal rays from $\mathscr{E}$ with real coefficients. We have $R\cdot D(R_{11})=0$ for any extremal ray $R\in\mathscr{E}$. Thus, there exists an extremal ray T above such that $T\cdot D(R_{11})>0$. It follows that T is different from extremal rays of pairs of the type $\mathfrak{B}_2$. We take $Q_1=T$. By our construction, the set $\mathscr{E}\cup\{Q_1\}$ is extremal. If $Q_1\cdot D(R_{j1})>0$ for any j such that $1\leq j\leq t$, then $r=1$, and the set $\{Q_1\}$ gives the set we were looking for. Otherwise, there exists a minimal j such that $2\leq j\leq t$ and $Q_1\cdot D(R_{j1})=0$. Then we replace $\mathscr{E}$ by the set $\mathscr{E}_1$ of all extremal rays in the face $[\mathscr{E}\cup\{Q_1\}]$ of the dimension $\dim[\mathscr{E}_1]=\dim[\mathscr{E}]+1$, and the set

$$\{R_{11}, R_{12}\}\cup\ldots\cup\{R_{t1}, R_{t2}\}$$

by

$$\{R_{j1}, R_{j2}|1\leq j\leq t,\ Q_1\cdot D(R_{j1})=0\}\ ,$$

and repeat this procedure.

5.3. Basic Theorems We want to prove the following basic theorem.

Basic Theorem 2.5.8. *Let X be a Fano 3-fold with terminal* $\mathbf{Q}$*-factorial singularities. Assume that X does not have a small extremal ray, and Mori polyhedron $\overline{NE}(X)$ does not have a face of Kodaira dimension* 1 *or* 2.

Then we have the following for the X:

(1) *The X does not have a pair of extremal rays of the type $\mathfrak{B}_2$ (thus, in notation above, the invariant $n=0$) and Mori polyhedron $\overline{NE}(X)$ is simplicial.*

(2) *The X does not have more than one extremal ray of the type (I).*

(3) *If $\mathscr{E}$ is an extremal set of k extremal rays of X, then the $\mathscr{E}$ has one of the types*: $\mathfrak{A}_1\mathrm{II}\ (k-1)\ \mathfrak{C}_1$, $\mathfrak{D}_2\mathrm{II}\ (k-2)\ \mathfrak{C}_1$, $\mathfrak{C}_2\mathrm{II}\ (k-2)\ \mathfrak{C}_1$, $k\mathfrak{C}_1$ *(we use notation of*

Theorem 2.3.3).

(4) *We have the inequality for the Picard number of* X:

$\rho(X)=\dim N_1(X)\leq 7$.

Proof. We use notations introduced in the Section 5.2. We divide the proof into several steps.

Let us consider extremal rays

$$\mathscr{E}_0=\{R_{11}, R_{12}\}\cup\{R_{21}, R_{22}\}\cup\ldots\cup\{R_{n1}, R_{n2}\} .$$

Let

$$\mathscr{E}_0^{ind}=\{R_{11}, R_{12}\}\cup\{R_{21}, R_{22}\}\cup\ldots\cup\{R_{m1}, R_{m2}\} ,$$

and

$$\mathscr{E}_0^{dep}=\{R_{(m+1)1}, R_{(m+1)2}\}\cup\{R_{(m+2)1}, R_{(m+2)2}\}\cup\ldots\cup\{R_{n1}, R_{n2}\} .$$

By Lemma 2.5.4, the set $\mathscr{E}_0$ is extremal. Let $\mathscr{E}$ be a maximal extremal set of extremal rays which contains $\mathscr{E}_0$. Let $\mathscr{E}_1=\mathscr{E}-\mathscr{E}_0$. By Proposition 2.3.8, $\#\mathscr{E}_1=\rho(X)-1-\dim[\mathscr{E}_0]$. By Theorem 2.3.3, for $S\in\mathscr{E}_1$, the divisor $D(S)$ has no a common point with divisors $D(R_{i1})$, $1\leq i\leq n$.

Lemma 2.5.9. *Assume that* X *satisfies the conditions of Theorem* 2.5.8. *Let* Q *be an extremal ray such that* Q *is different from extremal rays* R_{ij}, $1\leq i\leq n$, $1\leq j\leq 2$, *and the set* $\mathscr{E}_1\cup\{Q\}$ *is extremal. Then the* Q *has the type* (*II*) *and there exists exactly one* i *such that* $1\leq i\leq n$ *and* $Q\cdot D(R_{i1})>0$ *and* $D(Q)\cap D(R_{j1})=\emptyset$ *if* $j\neq i$.

Proof. Assume that Q has the type (I). Then the divisor $D(Q)$ has no common point with the divisors $D(R_{i1})$, $1\leq i\leq n$. By Theorems 2.3.3, 2.3.6 and Lemma 2.5.1, the set $\{Q\}\cup\mathscr{E}_1\cup\mathscr{E}_0$ is extremal. We then get a contradiction with the condition that $\mathscr{E}_1\cup\mathscr{E}_0$ is a maximal extremal set. Thus, the extremal ray Q has the type (II).

If $D(Q)$ has no common point with the divisors $D(R_{i1})$, $1\leq i\leq n$, we get a contradiction by the same way. Thus, there exists i such that $1\leq i\leq n$ and $D(Q)\cap D(R_{i1})\neq\emptyset$. Let us consider a projectivization $\overline{PNE}(X)$. By Lemma 2.2.2, $\overline{PNE}(X, D(Q))$ is an interval with two ends. Its first end is the vertex PQ and its second end is a point of the edge $P(R_{i1}+R_{i2})$ of the convex polyhedron $\overline{PNE}(X)$. Thus, the i is defined by the extremal ray Q. Evidently, $Q\cdot D(R_{i1})>0$.

Lemma 2.5.10. *With the conditions of Lemma* 2.5.9 *above, assume that* $m+1\leq i\leq n$. *Then there exists exactly one extremal ray* $Q=Q_i$ *with the conditions of Lemma* 2.5.9: thus, the set $\mathscr{E}_1\cup\{Q_i\}$ is extremal and $Q_i\cdot D(R_{i1})>0$, and $D(Q_i)\cap D(R_{j1})=\emptyset$ *if* $j\neq i$.

Proof. The

$$\beta=\sum_{S\in\mathscr{E}_1}S+\sum_{R\in\mathscr{E}_1}R$$

is a face of $\overline{NE}(X)$ of highest dimension $\rho(X)-1$, and

$$\beta_i=\sum_{S\in\mathscr{E}_1} S+\sum_{R\in\mathscr{E}_0-\{R_{i1},R_{i2}\}} R$$

is a face $\beta_i\subset\beta\subset\overline{NE}(X)$ of dimension $\rho(X)-2$ and of the codimension one in β (Here we use that $m+1\leq i\leq m+k$). It follows that there exists exactly one face β'_i of $\overline{NE}(X)$ such that β'_i contains β_i, $\dim\beta'_i=\rho(X)-1$, and $\beta'_i\neq\beta$. By Theorems 2.3.3 and 2.3.6, and Lemma 2.5.9, $\beta'_i=\beta_i+Q_i$ where Q_i is an extremal ray such that the set $\mathscr{E}_1\cup\{Q_i\}\cup(\mathscr{E}_0-\{R_{i1},R_{i2}\})$ is extremal, and the ray Q_i has the properties of Lemma 2.5.10. It follows that the Q_i is unique and does exist.

Lemma 2.5.11. *Under the above notation, the set $\mathscr{E}_1\cup\mathscr{E}_0^{ind}\cup\{Q_{m+1},\dots,Q_n\}$ is extremal.*

Proof. By Theorems 2.3.3, 2.3.6, Proposition 2.3.8 and Lemma 2.5.1, the set $\mathscr{E}=\mathscr{E}_1\cup\mathscr{E}_0^{ind}$ is extremal and generates a face of $\overline{NE}(X)$. We apply Lemma 2.5.7 to this $\mathscr{E}$ and $\mathscr{E}_0^{dep}$. By Lemma 2.5.7, there are extremal rays $Q'_{m+1},\dots,Q'_{m+r}$ such that the set $\mathscr{E}_1\cup\mathscr{E}_0^{ind}\cup\{Q'_{m+1},\dots,Q'_{m+r}\}$ is extremal and for any i, $m+1\leq i\leq m+r$, there exists j, $m+1\leq j\leq n$, such that $Q'_i\cdot D(R_{j1})>0$. Moreover, for any j, $m+1\leq j\leq n$, there exists an extremal ray Q_i, $m+1\leq i\leq m+r$, such that

$$Q'_i\cdot D(R_{j1})>0\ .$$

By Lemmas 2.5.9 and 2.5.10, $r=k$ and $\mathscr{E}_1\cup\mathscr{E}_0^{ind}\cup\{Q'_{m+1},\dots,Q'_{m+r}\}=\mathscr{E}_1\cup\mathscr{E}_0^{ind}\cup\{Q_{m+1},\dots,Q_n\}$.

Lemma 2.5.12. *The set $\mathscr{E}_0^{dep}$ is empty.*

Proof. By Lemmas 2.5.9, 2.5.10 and 2.5.11, the set of extremal rays $U=\mathscr{E}_1\cup\mathscr{E}_0^{ind}\cup\{Q_{m+1},\dots,Q_n\}$ is a maximal extremal set which contains $\mathscr{E}_1\cup\mathscr{E}_0^{ind}$ and does not contain extremal rays from $\mathscr{E}_0^{dep}$. Assume that $k=n-m\neq 0$. Then $k\geq 2$ and $\dim U=\rho_0(X)-1+2m+k$. But the dimension of a face of $\overline{NE}(X)$ of highest dimension is equal to $\rho(X)-1=\rho_0(X)-1+2m+k+\delta$ where $\delta\geq 1$. Thus, the extremal set U is not maximal, and there exists another extremal ray S such that $U\cup\{S\}$ is extremal. By definition of U, the $S\in\mathscr{E}_0^{dep}$. Let $S=R_{i1}$ where $m+1\leq i\leq n$. Since $Q_i\cdot D(R_{i1})>0$, by Theorem 2.3.3, the extremal set $\{Q_i,R_{i1}\}$ has the type $\mathfrak{C}_2$. Thus, $R_{i1}\cdot D(Q_i)=0$. By definition of the set $\mathscr{E}_0^{dep}$, there exists a linear dependence $\sum_{l=m+1}^{l=n}a_{l1}R_{l1}+a_{l2}R_{l2}=0$ where $a_{i1}\neq 0$ and $a_{i2}\neq 0$. Multiplying $D(Q_i)$ by the equality above, we get $a_{i2}=0$. Thus, we get a contradiction. (Compare with Lemma 2.5.6.)

Lemma 2.5.13 *The set $\mathscr{E}_0^{ind}$ is empty.*

Proof. Since $\mathscr{E}_0^{dep}=\emptyset$, the set $U=\mathscr{E}_1\cup\mathscr{E}_0^{ind}=\mathscr{E}_1\cup\{R_{11},R_{12}\}\cup\ldots\cup\{R_{m1},R_{m2}\}$ is a maximal extremal set. It follows that U generates a simplex face of $\overline{NE}(X)$ of codimension 1. Thus, $U_1=\mathscr{E}_1\cup\mathscr{E}_0^{ind}-\{R_{m2}\}=\mathscr{E}_1\cup\{R_{11},R_{12}\}\cup\ldots\cup\{R_{(m-1)1},R_{(m-1)2}\}\cup\{R_{m1}\}$ generates a simplex face of $\overline{NE}(X)$ of codimension 2. It follows that there exists an extremal ray Q_{m2} such that $U'_1=\mathscr{E}_1\cup\{R_{11},R_{12}\}\cup\ldots\cup\{R_{(m-1)1},R_{(m-1)2}\}\cup\{R_{m1}\}\cup\{Q_{m2}\}$ generates a simplex face of $\overline{NE}(X)$ of codimension 1, and Q_{m2} is different from R_{m2}. By Lemma 2.5.9, $Q_{m2}\cdot D(R_{m1})>0$. Thus, by Theorem 2.3.3, $\{Q_{m2},R_{m1}\}$ is an extremal set of the type $\mathfrak{C}_2$ where $R_{m1}\cdot D(Q_{m2})=0$.

Similarly, we can find an extremal ray Q_{m1} such that the set $\{Q_{m1},R_{m2}\}$ is extremal of the type $\mathfrak{C}_2$ where $R_{m2}\cdot D(Q_{m1})=0$. Then we get a contradiction to Lemma 2.5.6. Thus, $m=0$, and the set $\mathscr{E}_0^{ind}=\emptyset$.

Thus, we proved that X does not have a pair of extremal rays of the type $\mathfrak{B}_2$. By Theorem 2.3.3 and Proposition 2.3.8, the Mori polyhedron $\overline{NE}(X)$ is then simplicial. Thus, we have proven the statement (1).

Now let us prove (2): X does not have more than one extremal ray of the type (I).

By Lemma 2.2.2, divisors of different extremal rays of the type (I) do not have a common point. By Theorem 2.3.6, any set of extremal rays of the type (I) generates a simplex face of $\overline{NE}(X)$ of Kodaira dimension 3. It follows that the set of extremal rays of the type (I) is finite. Let

$$\{R_1,\ldots,R_s\}$$

be the whole set of extremal rays of the type (I) on X. We should prove that $s\leq 1$.

Let $\mathscr{E}$ be a maximal extremal set of extremal rays on X containing the set $\{R_1,\ldots,R_s\}$ and such that each connected component of $\mathscr{E}$ contains one of extremal rays $R_1,\ldots,R_s$ (see the definition of connected components before Theorem 2.3.3). By Theorem 2.3.3, then $\mathscr{E}$ has exactly s connected components $T_1,\ldots,T_s$ such that T_i contains the extremal ray R_i. The T_i has either the type $\mathfrak{A}_1$ (thus, $T_i=\{R_i\}$) or $\mathfrak{D}_2$ (thus, T_i contains two extremal rays: the R_i and another extremal ray which has the type (II)). Evidently, the maximal $\mathscr{E}$ does exist.

By [Ka1] and [Sh], any face of $\overline{NE}(X)$ is contractible, and by our conditions, it has Kodaira dimension 3. By Proposition 2.2.6, for any $1\leq i\leq s$, there exists an effective divisor $D(T_i)$ which is a linear combination of divisors of rays from T_i with positive coefficients and $R\cdot D(T_i)<0$ for any $R\in T_i$. Since T_i has the type $\mathfrak{A}_1$ or $\mathfrak{D}_2$, one can see easily by Lemma 2.2.3, that the same it true for each curve of divisors of rays of T_i because this curve belongs to the sum of extremal rays of T_i with positive coefficients.

Using the divisors $D(T_i)$, similarly to Lemma 2.5.7, we can find extremal rays

$\{Q_1, \ldots, Q_r\}$

with properties:

(a) $r \leq s$;

(b) For any i, $1 \leq i \leq r$, there exists j, $1 \leq j \leq t$, such that $Q_i \cdot D(T_j) > 0$ (in particular, Q_i is different from extremal rays of $\mathscr{E}$ and does not have the type (I));

(c) For any j, $1 \leq j \leq s$, there exists an extremal ray Q_i, $1 \leq i \leq r$, such that

$Q_i \cdot D(T_j) > 0$;

(d) The set $\{Q_1, \ldots, Q_r\}$ of extremal rays is extremal.

By our conditions, all extremal rays on X are divisorial. Thus, by (b), the extremal rays $Q_1, \ldots, Q_r$ have the type (II).

Let us take the ray Q_i, and let $Q_i \cdot D(T_j) > 0$. By Theorem 2.3.3, the set T_j generates a simplex face γ_j of $\overline{NE}(X)$. We have mentioned above that each curve of divisors of rays from T_j belongs to this face. It follows that $\overline{NE}(X, D(Q_i))$ is a 2-dimensional angle bounded by the ray Q_i and a ray from the face γ_j since the divisor $D(Q_i)$ evidently has a common curve with one of divisors $D(R)$, $R \in T_j$. Since any two sets of $T_1, \ldots, T_s$ do not have a common extremal ray, the faces $\gamma_1, \ldots, \gamma_s$ do not have a common ray (not necessarily extremal). It follows that the angle $\overline{NE}(X, D(Q_i))$ does not have a common ray with the face γ_k for $k \neq j$. Thus, the divisor $D(Q_i)$ does not have a common point with divisors of rays T_k. It follows that $r=s$ and we can choose an order $Q_1, \ldots, Q_s$ such that $Q_i \cdot D(T_i) > 0$ but $D(Q_i)$ do not have a common point with divisors of extremal rays T_j if $j \neq i$.

Let us fix i, $1 \leq i \leq s$. By our construction, the set $\mathscr{E} \cup \{Q_i\}$ has connected components

$T_1, \ldots, T_{i-1}, T_i \cup \{Q_i\}, T_{i+1}, T_s$.

By definition of $\mathscr{E}$, then the $\mathscr{E} \cup \{Q_i\}$ is not extremal. Thus, it contains an E-set (minimal non-extremal) $\mathscr{L}_i$ which contains Q_i. By Theorem 2.4.1 and Lemma 1.1, the $\mathscr{L}_i$ is connected. Thus, $\{Q_i\} \subset \mathscr{L}_i \subset T_i \cup \{Q_i\}$. Let us consider the sets $\mathscr{L}_1, \ldots, \mathscr{L}_s$. By Lemma 1.1, the $\mathscr{L}_i$, $\mathscr{L}_j$ are joint by arrows. By our construction, it follows that Q_i, Q_j are joint by arrows Q_iQ_j and Q_jQ_i for any $1 \leq i < j \leq s$. By Theorem 2.3.3, for the extremal set $\{Q_1, \ldots, Q_s\}$ of extremal rays of the type (II), this is possible only if $s \leq 1$. This proves the statement (2).

To prove (3) we use the following.

Statement. *The contraction of a ray R of the type (II) on X gives a Fano 3-fold X' with terminal $\mathbf{Q}$-factorial singularities and without small extremal rays and without faces of Kodaira dimension 1 or 2 for $\overline{NE}(X')$. Extremal sets $\mathscr{E}'$ on X' are in one to one correspodence with extremal sets $\mathscr{E}$ on X which contain the ray R.*

Proof. Let σ: $X \to X'$ be a contraction of R. The X' has terminal $\mathbf{Q}$-factorial singularities by [Ka1] and [Sh]. We have, $K_X = \sigma^*(K_{X'}) +$

$dD(R)$. Multiplying this equality by R and using Proposition 2.3.2, we get that $d=1$. By the statement (1), it follows that $\sigma^*(-K_{X'})=-K_X+D(R)$ is nef and only contracts the extremal ray R. Then $-K_{X'}$ is ample on X' and X' is a Fano 3-fold with terminal $\mathbf{Q}$-factorial singularities. Faces of $\overline{NE}(X')$ are in one to one correspondence with faces of $\overline{NE}(X)$ which contain the R. Contractions of faces of $\overline{NE}(X')$ are dominated by these of the corresponding faces of $\overline{NE}(X)$. This proves the last statement.

Let $\mathscr{E}=\{R_1, \ldots, R_k\}$ be an extremal set on X. By Theorem 2.3.3, it has connected components of the type $\mathfrak{A}_1$, $\mathfrak{B}_2$, $\mathfrak{C}_m$ or $\mathfrak{D}_2$. Moreover, by (1) and (2), it does not have a connected component of the type $\mathfrak{B}_2$ and does not have more than one connected component of the type $\mathfrak{A}_1$. By Statement above, the same should be true for the extremal set $\mathscr{E}'$ which one gets by the contraction of any extremal ray R_i of the type (II) of $\mathscr{E}$. This shows the statement (3).

Now we prove (4): $\rho(X)\le 7$.

First, we show how to prove $\rho(X)\le 8$ applying Theorem 1.2 to the face $\gamma=\mathcal{M}(X)$ of $\dim\mathcal{M}(X)=m=\rho(X)-1$. By the statement (1) of Theorem 2.5.8 and Theorems 2.3.3 and 2.4.1, the $\mathcal{M}(X)$ is simple and all conditions of Theorem 1.2 are valid for some constants d, C_1, C_2. By Theorem 2.4.1, we can take $d=2$. By the proof of Theorem 1.2, we should find the constants C_1, and C_2 for maximal extremal sets $\mathscr{E}$ only (only this sets we really use). Thus, $\#\mathscr{E}=m$. By the statement (3), then the constants $C_1\le 2/m$ and $C_2=0$. Thus, we get $m<(16/3)2/m+6$. Then, $m=\rho(X)-1\le 7$, and $\rho(X)\le 8$.

To prove the better inequality $\rho(X)\le 7$, we should analyze the proof of Theorem 1.2 for our case more carefully. We will show that the conditions of Lemma 1.4 hold for the $\mathcal{M}(X)$ with the constants $C=0$ and $D=2/3$. By Lemma 1.4, we then get the inequality $\rho(X)\le 7$ we want to prove.

Like for the proof of Theorem 1.2, we introduce a weight of an oriented angle, but using a new formula: $\sigma(\angle)=2/3$ if $\rho(R_1(\angle), R_2(\angle))=1$, and $\sigma(\angle)=0$ otherwise.

By (3) of Theorem 2.5.8, the condition (1) of Lemma 1.4 holds with constants $C=0$ and $D=2/3$.

Let us prove the condition (2) of Lemma 1.4. For $k=3$ (triangle) it is true since an E-set which has at least 3 elements has the type (a) of Theorem 2.4.1 (see the proof of Theorem 1.2). Thus, the triangle has at least three oriented angles with the weight 2/3. For $k=4$ (quadrangle), we proved (when we were proving Theorem 1.2) that one can find at least two oriented angles of the quadrangle such that any of them has finite $\rho(R_1(\angle), R_2(\angle))$. By (3) of Theorem 2.5.8, then $\rho(R_1(\angle), R_2(\angle))=1$. Thus, the quadrangle has at least two oriented angles of the weight 2/3. This finishes the proof of Theorem 2.5.8.

Now, we give an application of (2) of Theorem 2.5.8 to the geometry of Fano 3-folds.

Let us consider a Fano 3-fold X and blow-ups X_p at different non-singular points $\{x_1, \ldots, x_p\}$ of X. We say that this is a Fano blow-up if X_p is Fano. We have the following very simple

Proposition 2.5.14. *Let X be a Fano 3-fold with terminal $\mathbf{Q}$-factorial singularities and without small extremal rays. Let X_p be a Fano blow up of X. Then for any small extremal ray S on X_p, the S has a non-empty intersection with one of exceptional divisors $E_1, \ldots, E_p$ of this blow up and does not belong to any of them. Moreover, the exceptional divisors $E_1, \ldots, E_p$ define p extremal rays $Q_1, \ldots, Q_p$ of the type (I) on X_p such that $E_i = D(Q_i)$.*

Proof. The last statement is clear. Let S be a small extremal ray on X_p which does not intersect divisors $E_1, \ldots, E_p$. Let H be a general *nef* element orthogonal to S. Let $l_1, \ldots, l_n$ be lines which generate extremal rays $Q_1, \ldots, Q_p$. Then the divisor $H' = H + (l_1 \cdot H)/(-l_1 \cdot E_1)E_1 + \cdots + (l_p \cdot H)/(-l_p \cdot E_p)E_p$ is a nef divisor on X_p orthogonal to all extremal rays $Q_1, \ldots, Q_p, S$, and $(H')^3 > H^3 > 0$. This proves that the extremal rays $Q_1, \ldots, Q_p, S$ generate a face of $\overline{NE}(X_p)$ of Kodaira dimension 3. Then, by the contraction of the extremal rays $Q_1, \ldots, Q_p$, the image of S gives a small extremal ray on X. This gives a contradiction.

It is known that a contraction of a face of Kodaira dimension 1 or 2 of $\overline{NE}(Y)$ of a Fano 3-fold Y has a general fiber which is rational surface or curve respectively, because this contraction has relatively negative canonical class. See [Ka1], [Sh]. It is known that a small extremal ray is rational [Mo2].

Then, using the statement (2) of Theorem 2.5.8 and Proposition 2.5.14, we can divide Fano 3-folds of Theorem 2.5.8 into the following 3 classes:

Corollary 2.5.15. *Let X be a Fano 3-fold with terminal $\mathbf{Q}$-factorial singularities and without small extremal rays, and without faces of Kodaira dimension 1 or 2 for Mori polyhedron. Let ε be the number of extremal rays of the type (I) on X (by Theorem 2.5.8, the $\varepsilon \leq 1$).*

Then there exists p, $1 \leq p \leq 2 - \varepsilon$, such that X belongs to one of classes (A), (B) or (C) below:

(A) There exists a Fano blow-up X_p of X with a face of Kodaira dimension 1 or 2. Thus, birationally, X is a fibration of rational surfaces over a curve or of rational curves over a surface.

(B) There exist Fano blow-ups X_p of X for general p points on X such that for all these blow-ups the X_p has a small extremal ray S. Then images of curves of S on X give a system of rational curves on X which cover a Zariski open subset of X.

(C) There do not exist Fano blow-ups X_p of X for general p points.

We remark that for Fano 3-folds with Picard number 1 the $\varepsilon = 0$. Thus, $1 \leq p \leq 2$.

We mention that that statements (3) and (4) of Theorem 2.5.8 give similar information for blow ups of X along curves. Of course, it is more difficult to formulate these statements.

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