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A topological proof of Bott periodicity theorem and a characterization of BU

By

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1. Introduction

The purpose of this paper is to give a simple and topological proof of the Bott periodicity theorem $\mathbf{BU} \simeq \Omega^2 \mathbf{BSU}$ and a characterization of $\mathbf{BU}$.

Let X be a finite type CW complex which is an H-space satisfying the following properties:

(1) as an algebra, $H^*(X; \mathbf{Z}) = \mathbf{Z}[c_1, c_2, \cdots\cdots, c_n, \cdots]$ where $|c_i| = 2i$;

(2) there exist two maps

$$j: \mathbf{CP}^\infty \to X \text{ and}$$

$$\lambda: S^2 \wedge X \to X$$

such that

$$(\lambda \circ (S^2 \wedge j))^*: H^*(X; \mathbf{Z}) \to H^*(S^2 \wedge \mathbf{CP}^\infty; \mathbf{Z}) \text{ is epic and}$$

$$Ad^2\lambda: X \to \Omega^2 X \qquad \text{is an H-map.}$$

Denote the homotopy fiber of $c_1: X \to \mathbf{CP}^\infty = K(\mathbf{Z}, 2)$ by $X\langle 2\rangle$. Then $Ad^2\lambda$ induces a map $Ad^2\lambda: X \to \Omega^2(X\langle 2\rangle)$ since $\Omega^2(X\langle 2\rangle)$ is the connected component of $\Omega^2 X$ containing the constant loop and X is connected. The map $\widetilde{Ad}^2 \lambda$ is a homotopy equivalence (see Theorem 2.1).

Note that $\mathbf{BU}$ is an H-space and satisfies (1). Let λ be the classifying map of

$$\tilde{K}(\cdot) \to \tilde{K}(S^2 \wedge \cdot)$$

defined by

$$x \mapsto (\eta - 1) \hat{\otimes} x$$

and j be the classifying map of $\eta_\infty - 1$ where η (resp. η_∞) is the cannonical line

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bundle over $S^2=\mathbf{CP}^1$ (resp. $\mathbf{CP}^\infty$). In section 3 we show j and λ satisfy (2). Therefore Theorem 2.1 is a topological proof of the Bott periodicity thorem.

Now consider the Segal splitting

$$\epsilon_c: \mathbf{BU}\rightarrow Q(\mathbf{CP}^\infty),$$

where $Q(\cdot)=\lim\limits_{\rightarrow}\Omega^n\Sigma^n(\cdot)$. Since X is an infinite loop space by Theorem 2.1, there is an infinite loop map

$$\xi_X: Q(X)\rightarrow X.$$

In section 4 we show

$$\Phi=\xi_X\circ Q(j)\circ\epsilon_c: \mathbf{BU}\rightarrow X$$

is a homotopy equivalence. Thus the conditions (1) and (2) are a characterization of **BU**.

2. A periodic H-space

In this section, we shall give the topological condition that an H-space have periodicity.

Theorem 2.1. *Let X be a finite type CW-complex which is an H-space. Suppose*

(1) *as an algebra, $H^*(X;\boldsymbol{Z})=\boldsymbol{Z}[c_1, c_2, \cdots, c_n, \cdots]$ where $|c_i|=2i$;*

(2) *there exist two maps*

$$j: \boldsymbol{CP}^\infty\longrightarrow X$$

$$\lambda: S^2\wedge X\longrightarrow X$$

such that

(*a*) $(\lambda\circ(S^2\wedge j))^*: H^*(X;\boldsymbol{Z})\rightarrow H^*(S^2\wedge\boldsymbol{CP}^\infty;\boldsymbol{Z})$ *is an epimorphism;*

(*b*) $Ad^2\lambda: X\longrightarrow\Omega^2X$ *is an H-map.*

Then the map

$$\widetilde{Ad}^2\lambda: X\rightarrow\Omega^2(X\langle 2\rangle)$$

is a homotopy equivalence.

Before we prove this theorem, we will state some remarks. If X satisfies the condition of Theorem 2.1, X is simply connected. Given a simply connected topological space X, we shall denote $X\langle 2\rangle$ to be the 2-connected fiber space of X, and Ad to be a natural equivalence

$$[\Sigma A, B]\overset{\sim}{\longrightarrow}[A, \Omega B].$$

Note that $\Omega^2(X\langle 2\rangle)$ is a connected component of $\Omega^2 X$.

First we show an algebraic lemma. Let R be a PID with unit, and $A=\oplus A_n$, $B=\oplus B_n$ graded R-algebras.

Lemma 2.2. *Let A_n and B_n be free R-modeles with finite rank and* rank $A_n=$rank B_n *for all n. If there exist a homomorphism of R-algebra $f: A\to B$ such that all generators of B is included in the image of f, then $f_n: A_n\to B_n$ is an isomorphism as R-modules for all n.*

Proof. This follows immediately from the fact that f_n is an epimorphism for all n.

Proof of Theorem 2.1. First we write

$$k=\lambda\circ(S^2\wedge j): S^2\wedge \boldsymbol{CP}^\infty\to X,$$

$$k_1=Ad\lambda\circ(S^1\wedge j): S^1\wedge \boldsymbol{CP}^\infty\to \Omega X \text{ and}$$

$$k_2=Ad^2\lambda\circ j: \boldsymbol{CP}^\infty\to \Omega^2 X.$$

Note that $k_1=Adk$, and $k_2=Adk_1$. Let $\alpha_1\in H^*(S^1;\boldsymbol{Z})$, $\alpha_2\in H^*(S^2;\boldsymbol{Z})$ and $\beta\in H^*(\boldsymbol{CP}^\infty;\boldsymbol{Z})$ be generators as cohomology rings. Then we can write

$$H^{2i+1}(S^1\wedge \boldsymbol{CP}^\infty;\boldsymbol{Z})=\langle\alpha_1\times\beta\rangle\hookrightarrow H^{2i+1}(S^1\times \boldsymbol{CP}^\infty;\boldsymbol{Z})\, i=0, 1, 2, \cdots$$

$$H^{2i+2}(S^2\wedge \boldsymbol{CP}^\infty;\boldsymbol{Z})=\langle\alpha_2\times\beta\rangle\hookrightarrow H^{2i+2}(S^2\times \boldsymbol{CP}^\infty;\boldsymbol{Z})\, i=1, 2, \cdots$$

Let σ be the cohomology suspension map: $H^*(X;\boldsymbol{Z})\to H^{*-1}(\Omega X;\boldsymbol{Z})$. Then

$$H^*(\Omega X;\boldsymbol{Z})=\wedge(\sigma(c_1), \sigma(c_2), \cdots, \sigma(c_n), \cdots).$$

Since $k^*: H^*(X;\boldsymbol{Z})\to H^*(S^2\wedge X;\boldsymbol{Z})$ is an epimorphism, $k^*(c_i)=\pm\alpha_2\beta^{i-1}(i=1, 2,\cdots)$, where $\alpha_2\beta^{i-1}$ is $\alpha_2\times\beta^{i-1}$ precisely. From now on we shall often write in this way if there is no risk of misunderstanding. The adjoint map and the suspension map are commutative. Thus $k_1^*(\sigma(c_i))=\pm\alpha_1\beta^{i-1}(i=2, 3, \cdots)$. It follows from this equation that the sequence

$$0\to\ker(k_1^*)\to H^i(\Omega X;\boldsymbol{Z})\xrightarrow{k_1^*} H^i(S^1\wedge \boldsymbol{CP}^\infty;\boldsymbol{Z})\to 0$$

is splitting exact for $i\geq 2$. Therefore its dual sequence

$$0\to H_i(S^1\wedge \boldsymbol{CP}^\infty;\boldsymbol{Z})\xrightarrow{k_{1*}} H_i(\Omega X;\boldsymbol{Z})\to\text{Coker}(k_{1*})\to 0$$

is also splitting exact for $i\geq 2$.

Recall that since ΩX is an H-space $H_*(\Omega X;\boldsymbol{Z})$ is a Hopf algebra over $\boldsymbol{Z}$. Then we can choose generators ξ_i which are primitive elements of degree $2i-1$ such that

$$H_*(\Omega X;\ \boldsymbol{Z})=\wedge(\xi_1,\ \xi_2,\ \cdots,\ \xi_n,\cdots).$$

See [9]. $H_*(S^1\wedge \boldsymbol{CP}^\infty;\ \boldsymbol{Z})$ is a coalgebra over $\boldsymbol{Z}$, and $\theta_i=[\alpha_1\beta^i \mapsto 1]\in H_{2i+1}(S^1 \wedge \boldsymbol{CP}^\infty;\ \boldsymbol{Z})$ is a primitive elememt. Since k_{1*} is a coalgebra homomorphism and the above short exact sequence splits, $k_{1*}(\theta_i)=\pm\xi_{i+1}(i=1,\ 2,\ \cdots)$.

The map k_1 has a lift

$$\tilde{k}_1:\ S^1\wedge \boldsymbol{CP}^\infty\rightarrow\Omega(X\langle 2\rangle)$$

and

$$\tilde{k}_{1*}(\theta_i)=\xi_{i+1}\in H_*(\Omega(X\langle 2\rangle);\ \boldsymbol{Z})=\wedge(\xi_2,\ \xi_3,\ \cdots).$$

Let

$$\tilde{k}_2=\widetilde{Ad}^2\lambda\circ j:\ \mathbf{CP}^\infty\rightarrow\Omega^2(X\langle 2\rangle)$$

be a lift of k_2. Note that $\tilde{k}_2$ is the adjoint map of $\tilde{k}_1$. Let τ be the homology transgression map: $H_*(\Omega(X\langle 2\rangle);\ \mathbf{Z})\rightarrow H_{*-1}(\Omega^2(X\langle 2\rangle);\ \mathbf{Z})$. Since the adjoint map and the transgression map are commutative, the image of the map

$$\tilde{k}_{2*}:\ H_*(\mathbf{CP}^\infty;\ \mathbf{Z})\rightarrow H_*(\Omega^2(X\langle 2\rangle);\ \mathbf{Z})=\mathbf{Z}[\tau(\xi_2),\ \tau(\xi_3),\ \cdots,\ \tau(\xi_n),\ \cdots]$$

includes generators $\tau(\xi_n)$. So does the image of the map $\widetilde{Ad}^2\lambda_*$. By Lemma 2.2, $\widetilde{Ad}^2\lambda_*$ is an isomorphism at each degree. Since X and $\Omega^2(X\langle 2\rangle)$ are simply connected, $\widetilde{Ad}^2\lambda$ is a homotopy equivalence by J.H.C.Whitehead's theorem.

3. A new proof of Bott periodicity

Bott periodicity (the complex case) implys that **BU** is homotopy equivalent to Ω^2 **BSU**. Note that **BSU**$\simeq$**BU** $\langle 2\rangle$. The aim of this section is to give a new proof of Bott periodicity of complex case. By Theorem 2.1, it will be sufficient to construct two maps j: $\mathbf{CP}^\infty\rightarrow\mathbf{BU}$ and λ: $S^2\wedge\mathbf{BU}\rightarrow\mathbf{BU}$ which have the properties given in Theorem 2.1.

Since **BU** is a classifying space of K-thory, it has an H-space structure corresponding to Whitney sum. Let μ: $\mathbf{BU}\times\mathbf{BU}\rightarrow\mathbf{BU}$ be this structure. Denote an algebra generator of $H^*(S^2;\ \mathbf{Z})$(resp. $H^*(\mathbf{CP}^\infty;\ \mathbf{Z})$) by α (resp. β). The Hopf line bundle over $S^2=\mathbf{CP}^1$ is denoted by η whose first Chern class $c_1(\eta)=\alpha$, and the canonical line bundle over $\mathbf{CP}^\infty$ denotes η_∞ whose first Chern class $c_1(\eta_\infty)=\beta$. Now we shall construct maps j and λ as elements of $\tilde{K}$ $(\mathbf{CP}^\infty)$ and $\tilde{K}$ $(S^2\wedge\mathbf{BU})$ respectively. We define $j=\eta_\infty-1$, where 1 is a trivial 1-dim vector bundle.

Define

$$\xi:=\varinjlim(\xi_n-n),$$

where $\xi_n \to \mathbf{BU}_n$ is the universal vector bundle. Note that the restriction of $(\eta - 1)\hat{\otimes}\xi$ to $S^2 \vee \mathbf{BU}$ is trivial. This means that this element is in $\tilde{K}(S^2 \wedge \mathbf{BU})$. Now we define $\lambda = (\eta - 1)\hat{\otimes}\xi$.

Proposition 3.1. *The map*

$$((\lambda \circ (S^2 \wedge j))^* : H^*(\mathbf{BU}\,;\,\mathbf{Z}) \to H^*(S^2 \wedge \mathbf{CP}^\infty\,;\,\mathbf{Z})$$

is an epimorphism.

Proof. We will compute $((\lambda \circ (S^2 \wedge j))^*(c)$, where c is the total Chern class $c = 1 + c_1 + c_2 + \cdots \in H^*(\mathbf{BU}\,;\,\mathbf{Z})$. From now on, we simply write $S^2 \wedge j$ to be j. Then

$$j^* \circ \lambda^*(c) = c(j^*(\lambda)).$$

The right side of the above equation is the total Chern class of the element $j^*(\lambda) \in \tilde{K}(S^2 \wedge \mathbf{CP}^\infty)$. Thus

$$\begin{aligned} c(j^*(\lambda)) &= c((\eta - 1)\hat{\otimes}(\xi_1 - 1)) \\ &= c(\eta\hat{\otimes}\xi_1)c(\eta\hat{\otimes}1)^{-1}c(1\hat{\oplus}\xi_1)^{-1} \\ &= (1 + \alpha + \beta)(1 + \alpha)^{-1}(1 + \beta)^{-1} \\ &= (1 + \alpha)^{-1}(1 + \alpha(1 + \beta)^{-1}) \\ &= (1 - \alpha)(1 + \alpha(\sum_{i=0}^{\infty}(-1)^i\beta^i)) \\ &= 1 + \sum_{i=1}^{\infty}(-1)^i\alpha\beta^i \end{aligned}$$

since $\alpha^2 = 0$ and $j^*(\xi) = \xi_1 - 1$, $\xi_1 = \eta_\infty$. We obtain

$$(\lambda \circ j)^*(c_{i+1}) = (-1)^i\alpha\beta^i.$$

Therefore it is an epimorphism.

Proposition 3.2. *The map*

$$Ad^2\lambda : \mathbf{BU} \to \Omega^2\mathbf{BU}$$

is an H-map.

Proof. We must show that the following diagram

$$\begin{array}{ccc} \mathbf{BU} \times \mathbf{BU} & \xrightarrow{\mu} & \mathbf{BU} \\ {\scriptstyle Ad^2\lambda \times Ad^2\lambda}\downarrow & & \downarrow{\scriptstyle Ad^2\lambda} \\ \Omega^2\mathbf{BU} \times \Omega^2\mathbf{BU} & \xrightarrow{\Omega^2\mu} & \Omega^2\mathbf{BU} \end{array}$$

is homotopy commutative. Its adjoint diagram is the following:

$$\begin{array}{ccc} S^2\wedge(\mathbf{BU}\times\mathbf{BU}) & \xrightarrow{s^2\wedge\mu} & S^2\wedge\mathbf{BU} \\ \downarrow & & \downarrow\lambda \\ \mathbf{BU}\times\mathbf{BU} & \xrightarrow{\mu} & \mathbf{BU}. \end{array}$$

Since these maps are classifying the same element

$$(\eta-1)\hat{\otimes}(\xi\times\xi)\in\tilde{K}(S^2\wedge(\mathbf{BU}\times\mathbf{BU}))=[S^2\wedge(\mathbf{BU}\times\mathbf{BU}),\ \mathbf{BU}],$$

this diagram is obviously homotopy commutative. Then the first diagram is also homotopy commutative.

By above propositions and Theorem 2.1, we obtain Bott periodicity theorem.

Corollary 3.3 (Bott periodicity theorem).

$$\mathbf{BU}\simeq\Omega^2\mathbf{BSU}.$$

4. A topological characterization of *BU*

In fact, X is homotopy equivalent to $\mathbf{BU}$ if X equips the conditons of Theorem 2.1. We shall prove it in this section.

We define

$$Q(X):=\lim_{\to}\Omega^n\Sigma^nX.$$

Note that $Q(X)$ is an infinite loop space. Furthermore, if $f: A\to B$ is a map,

$$Q(f)=\lim_{\to}\Omega^n\Sigma^nf: Q(A)\to Q(B)$$

is an infinite loop map. If X is an infinite loop space, we define an infinite loop map

$$\xi_X=\lim_{\to}\epsilon_n^{-1}\circ\Omega^nAd^{-n}\epsilon_n: Q(X)\to X,$$

where $\epsilon_n: X\to\Omega^nB^n$ is a homotopy equivalence.

By Segal [8] and Becker [3], there exists the Segal map $\epsilon_C: \mathbf{BU}\to Q(\mathbf{CP}^\infty)$ such that $\xi_{\mathbf{BU}}\circ Q(j)\circ\epsilon_C\simeq 1_{\mathbf{BU}}$. The Segal map ϵ_C is a "section" of a principal fiber space. Thus we obtain $Q(\mathbf{CP}^\infty)\simeq\mathbf{BU}\times F$, where F is the homotopy fiber and the homotopy group of F, $\pi_i(F)$ is finite for all i. Thus its reduced homology group $\tilde{H}_i(F;\mathbf{Z})$ is finite for all i.

We note that if X satisfies the conditions of Theorem 2.1, $j=\xi_X\circ Q(j)\circ i$ where i is the inclusion map $\mathbf{CP}^\infty\to Q(\mathbf{CP}^\infty)$. Consider the following commutative diagram:

$$\begin{array}{ccccccc} & & \mathbf{CP}^\infty & & \xrightarrow{\quad\quad j \quad\quad} & & X \\ & & \downarrow{\scriptstyle i} & & & & \| \\ \mathbf{BU} & \xrightarrow{\epsilon_C} & Q(\mathbf{CP}^\infty) & \xrightarrow{Q(j)} & Q(X) & \xrightarrow{\xi_X} & X. \end{array}$$

Since $H_*\ (X\,;\,\mathbf{Z})$ does not contain any torsion elements, there exists the algebra homomorphism φ such that the following diagram is commutative:

$$\begin{array}{ccccc} & & H_*(\mathbf{CP}^\infty\,;\,\mathbf{Z}) & \xrightarrow{\quad j_* \quad} & H_*(X\,;\,\mathbf{Z}) \\ & & \downarrow{\scriptstyle i_*} & & \| \\ H_*(\mathbf{BU}\,;\,\mathbf{Z}) & \xrightarrow{\epsilon_{C*}} & H_*(Q(\mathbf{CP}^\infty)\,;\,\mathbf{Z}) & \xrightarrow{(\xi_X\circ Q(j))_*} & H_*(X\,;\,\mathbf{Z}) \\ \| & & \downarrow{\scriptstyle proj.} & & \| \\ H_*(\mathbf{BU}\,;\,\mathbf{Z}) & \xrightarrow{\cong} & H_*(Q(\mathbf{CP}^\infty)\,;\,\mathbf{Z})/Tor & \xrightarrow{\quad \varphi \quad} & H_*(X\,;\,\mathbf{Z}). \end{array}$$

Then

$$\text{Image } \varphi \supset \text{Image } j_*.$$

Therefore the image of φ includes all algebra generators of $H_*(X\,;\,\mathbf{Z})$. By Lemma 2.2, φ is an isomorphism at each degree. Finally we obtain the following theorem.

Theorem 4.1. *If X satifies the conditions of Theorem* 2.1, *the map*

$$\xi_X\circ Q(j)\circ\epsilon_C:\ \mathbf{BU}\rightarrow X$$

is homotopy equivalent.

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