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Bass orders in non semisimple algebras

By

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0. Introduction

0. Let R be a Dedekind domain with the quotient field K. Inspired by the works of H. Bass [2], [3] for commutative rings, Drozd-Kirichenko-Roiter [8] introduced the notion of Bass orders in a finite dimensional separable K-algebra A. In this paper, we shall extend most of their local results and also the classification results of ours [9] to an arbitrary finite dimensional K-algebra A.

0.0. In the literature (cf. [8], [7], [15], [5]), hereditary orders, Bass orders and Gorenstein orders are all defined or investigated under the assumption that the ambient algebra A is semisimple or sometimes separable over K. Firstly, the definitions of these three types of orders have senses for an arbitrary A, although we have some options extending the definitions of the former two (cf. 0.2). Secondly, there are ample examples of Gorenstein orders in a non-semisimple A, for example any group algebra RG of any finite group G with the cardinality $\#G$ which is not invertible in K. Thirdly, the method to study Bass orders, especially the one adopted by Drozd-Kirichenko [7] seems in the most part free from the assumption of semisimplicity. All of these observations motivated our investigation.

As for local theory, we can do everything as well as in the semisimple case. Our main results include:
(i) Structure Theorem (4.4.1) which states that any ring indecomposable Bass order is either Morita equivalent to a primary Bass order or else Morita equivalent to one of (explicitly described) very simple Bass orders;
(ii) Classification Theorem (3.7.2) of indecomposable Λ-lattices for any Bass order Λ, withstanding the fact that Λ is no longer of finite representation type;
(iii) Classification Theorem (4.5.2) of primary Bass orders, under the same assumption as semisimple case that the residue algebra $A/\mathrm{rad}\,A$ is central over K and the residue field $R/\mathrm{rad}\,R$ is perfect with coh dim≤ 1.

All in all, the similarity in the results to the semisimple case is rather striking. More remarkably, in the above (ii), we can exlicitly determine the

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projective cover of each indecomposable Λ-lattice. This fact has a pleasing consequence (3.8.2) described in 0.1.3′.

As for global theory, where the separability has been played an indispensable role, we still have a few points left to be cleared up for Bass orders, and we will include in this paper only one basic results (1.7) for hereditary orders.

0.1. In the remaining part of this §0, we give the definitions of various types of orders over a Dedekind domain R, and observe their interrelations. In particular, every characterizing property of such orders is a local property. From §1 and on, excepting only two subsections 1.7 and 3.5, we take our base ring R to be a complete discrete valuation ring and consider an R-order Λ.

0.1.1. In §1, we recollect basic properties of maximal or minimal submodules of Λ-lattices along the line of [7]. One non-standard item we have introduced is the composition series of infinite length (1.4). By dealing with these elementary materials, we can put together almost entire theory of hereditary orders into a single proposition (1.6), which in particular implies that A contains a hereditary order if and only if A is semisimple. Thus our extension to an arbitrary A does not bring any actual gain for the (local) theory of hereditary orders itself. Nevertheless it is necessary for our study of Bass orders in non semisimple algebras, and also it makes the method of Jacobinski [10] for (global) hereditary orders applicable to an arbitrary A (1. 7).

0.1.2. In §2, we study bijective (=projective and injective) Λ-lattices. A bijective indecomposable Λ-lattice P always has the minimum (=unique minimal) Λ-overmodule P' and the maximum Λ-submodule $'P$. Call P *superbijective* if $P' \cong {'P}$ as Λ-lattices and call Λ *superGorenstein* if any projective indecomposable Λ-lattice is superbijective (2.5).

We extend the Rejection Lemma of Drozd-Kirichenko [7] to an arbitrary finite dimensional K-algebra (2.2.1). The Lemma states: If P is indecomposable bijective and not isomorphic to P' as Λ-lattices, then there is a unique overorder Λ' (written as $\Lambda' = \Lambda - (P)$), characterized by the property that an R-lattice L is an indecomposable Λ'-lattice if and only if it is an indecomposable Λ-lattice non-isomorphic to P.

Calling a pair (Λ, P) to be a *superbijective pair* if Λ is ring indecomposable non-hereditary and P is Λ-superbijective, a key fact is proved in 2.6:

(2.6.3) If (Λ, P) is a superbijective pair, then (Λ', P') is again a superbijective pair unless Λ' is hereditary; Λ' is superGorenstein if and only if so is Λ.

Hence, repeating the process, we can make up an increasing sequence of orders (resp. lattices):

(2.7) $\Lambda \subset \Lambda' \subset \Lambda'' \subset \dots$ (resp. $P \subset P' \subset P'' \subset \dots$).

The sequence ends up with some hereditary order if and only if A is semisimple. The semisimplicity comes into our theory only at this point. Examining the sequence, we will draw, directly or indirectly,almost all of the results in this paper. The claims and proofs in 2.6 are directly inspired by and are seemingly quite similar to but subtly different from that of Theorem 3.3 in [7].

0.1.3. In §3, we study Bass orders. As is trivially seen, superGorenstein order is Bass (3.0.2). Less trivially, ring indecomposable non-hereditary Bass order is superGorenstein (3.2.1). Hence, the results of §2 are almost directly applicable to Bass orders, and readily bring:
(3.3.1) A Gorenstein order is a Bass order if and only if any minimal overorder is Gorenstein.
(3.3.2) If Λ is a Bass order in $A=A_{ss}\oplus A_{ns}$, then $\Lambda=\Lambda\cap A_{ss}\oplus\Lambda\cap A_{ns}$ where A_{ss} is semisimple and A_{ns} has no simple ring direct factors.

Thus the theory of Bass orders in an arbitrary A can be separated into the theory in a semisimple A (i.e. $A=A_{ss}$) and that in a totally non-semisimple A (i.e. $A=A_{ns}$). The semisimple case is already covered by [7], [8] and [9](cf. 2.7.3).

0.1.3′. In 3.4-3.7, we study the totally non-semisimple case and prove the following fundamental theorem:
(3.7.1)(i) A contains a Bass order if and only if A is QF-RSZ, i.e., A is quasi-Frobenius with $(\mathrm{rad}A)^2=0$.
(ii) If A is QF-RSZ, an order Λ is Bass if and only if its canonical image $\varphi(\Lambda)$ is a maximal order of $A/\mathrm{rad}A$.

The theorem makes it possible to determine the projective cover of each indecomposable Λ-lattice (3.7.2), and to derive a surprisingly simple explicit formula (14) of $\sup\mu_\Lambda(I)$(3.7.3), where I runs over all left Λ-ideals and $\mu_\Lambda(I)$ is the minimal number of Λ-generators of I. The formula in turn leads to the following natural (local) solution of a problem of Bass (cf.[5] §37) on $\mu_\Lambda(I)$ for quasi-Frobenius K-algebras:
(3.8.2) If A is quasi-Frobenius, the following three properties for Λ are equivalent.
(15) Any overorder $\Gamma(\supseteq\Lambda)$ of Λ is self-dual (i.e. $\Gamma^*\cong\Gamma$).
(16) Λ is a self-dual superGorenstein order.

(17) $$\sup\mu_\Lambda(I)\leq 2.$$

When A is semisimple, the property (17) is in fact a local property by the Swan-Forster Theorem (cf.[6] 41.21), and we already have a global solution: (15)⇔(17). When A is not necessarily semisimple, although the implication (15)⇒(17) is globally no longer true, it still seems to be possible to characterize the orders with the property (17). This last problem will be treated in somewhere else.

0.1.4. In the final section §4, we execute the classification of basic ring indecomposable Bass orbers in a non-semisimple algebra A. According to (3. 7.1), we first classify basic ring indecomposable QF-RSZ algebras A (4.0-4.2), then classify Bass orders Λ in A. Since QF-RSZ algebras make up a special (particularly simple) subclass of generalized uniserial algebras which have a long history of investigation, the structure of such algebras are essentially known. We will only refer to the last paper [14] on this subject among the ones we know. Just like in the semisimple case, if s(=the number of non-isomorphic projective indecomposable lattices of Λ)≥ 2, the result is easy and complete (4.4). In the case of $s=1$(i.e. Λ is primary),we need some assumptions ((0),(c) of 4.5) and some extra effort.

However, our method in [9] is applicable without any substantial change, and leads to the final result (4.5.2), which has a remarkable resemblance to the semisimple case of $A=D\oplus D$.

0.2. Definitions. Let R be a Dedekind domain with the quotient field K. Let A denote a finite dimensional K-algebra and Λ denote an R-order in A. As for terminology, we mostly follow that of [5], in particular R-lattice, R-order, Λ-lattice, Λ-ideal are used in the sense of [5] §23. If M is a left Λ-lattice, $M^*:=\mathrm{Hom}_R(M,R)$ is a right Λ-lattice.

Λ is a left *Gorenstein* order if

(1) Λ^* is a projective right Λ-lattice.

Λ is a left *Bass* order if

(2) any overorder of Λ is left Gorenstein.

Λ is a left *strictly Bass* order if

(3) for any K-algebra epimorphism $\pi: A\rightarrow B$, any overorder of $\pi(\Lambda)$ is a left Gorenstein order of B.

Λ is a left *hereditary* order if

(4) any left Λ-ideal is Λ-projective.

Λ is a left *strictly hereditary* order if

(5) any left Λ-lattice is Λ-projective.

Λ is a *maximal* order if

(6) Λ has no proper overorder.

Interchanging 'left' with 'right' in (1)-(5), we get the properties (1′)-(5′), and get the difinitions of a *right Gorenstein* order *et al*.

0.2.0. As a matter of fact, there are the following implications among these properties, and in general nothing else.

(i) $(i)\Leftrightarrow(i')$ $1\leq i\leq 5$.

(ii) (6)$\Rightarrow$(5)$\Leftrightarrow$(4)$\Rightarrow$(3)$\Rightarrow$(2)$\Rightarrow$(1).

If A is semisimple these implications are well known or easily seen, moreover (2) is equivalent with (3). In general the following are obvious by definitions:

$$(3) \Rightarrow (2) \Rightarrow (1)\ ;\ (5) \Rightarrow (4)\ ;\ (5) \Rightarrow (1).$$

0.2.1. Every property (i) as well as (i') is a local property. Namely Λ has the property (i) if and only if the completion $\Lambda_{\wp}$ at the prime ideal $\wp$ has the property (i) for any $\wp$.

By the Lifting Idempotents Theorem, $\Lambda_{\wp}$ has the property (1) if and only if it has the property (1')(cf. 1.0.2). This implies $(i) \Longleftrightarrow (i')$ for $1 \leq i \leq 3$, namely there are no need to distinguish left and right for Gorensten, Bass or strictly Bass orders. The rest of the implications in 0.2.0 will by proved in 1.6.

0.2.2. Every property (i) as well as (i') is invariant under Morita equivalence. The claim is obvious except perhaps (2) or (3). For these cases, the following observation is sufficient (cf. [5] 37.14): Λ satisfies (3)(resp. (2)) if and only if for any left Λ-lattice L and any left Λ-lattice (resp. faithful left Λ-lattice) M,

$$M > L \text{ implies } M^* > L^*.$$

0.2.3. Let Λ be an R-order in A. If Λ has a certain property, it restricts the K-algebra structure of A. For example the following implications are obvious.

(i) If Λ is Gorenstein, then A is a quasi-Frobenius algebra.

(ii) If Λ is strictly Bass, then A is a Bass algebra (i.e. any homomorphic image of A is quasi-Frobenius).

(iii) If Λ is left (or right) strictly hereditary, then A is a semisimple algebra.
In fact: If Λ is left (or right) hereditary, then A is a semisimple (1.7.1).
However, non-Bass algebra A can contain a Bass order (4.4.3).

1. Maximal submodules and minimal overmodules

From now on, our base ring R is a complete discrete valuation ring with the quotient field K, and the maximal ideal πR. Let A always denote a finite dimensional K-algebra, Λ denote an R-order of A, and $\mathcal{N}$ denote its radical, $\mathcal{N} := \mathrm{rad}\, \Lambda$.

1.0. Notation and Convention. For a finitely generated left (resp. right) A-module V, let ${}_{\Lambda}\mathcal{L}(V)$(resp. $\mathcal{L}_{\Lambda}(V)$) denote the totality of full R-lattices L in V satisfying $\Lambda L = L$ (resp. $L\Lambda = L$).

For $L \in {}_{\Lambda}\mathcal{L}(V)$(resp. $\mathcal{L}_{\Lambda}(V)$), put

$$O_l(L) := \{a \in A\ ;\ aL \subseteq L\} \text{ (resp. } O_r(L) := \{a \in A\ ;\ La \subseteq L\}).$$

Thus in our notation, $O_l(L)$(resp. $O_r(L)$) is an R-subalgebra but not necessarily an R-order of A. $O_l(L)$(resp. $O_r(L)$) is an R-order if and only if

V is a faithful A-module.

1.0.0. Any left Λ-lattice L can naturally be seen as a member of ${}_\Lambda\mathcal{L}(\tilde{L})$ via the canonical injection $L \to \tilde{L} := K \otimes_R L$, $l \mapsto 1 \otimes l$. However, for given two Λ-lattices L_1 and L_2,even if they are rationally equivalent (i.e. $\tilde{L}_1 \cong \tilde{L}_2$ as A-modules), there is no canonical way to indentify $\tilde{L}_1$ with $\tilde{L}_2$.

To remove any ambiguity, let us agree that for a left Λ-lattice L with $\tilde{L} = V$, a Λ-overmodule of L is always referred to the one in ${}_\Lambda\mathcal{L}(V)$.

Let V^*(resp. L^*) denote its dual module, $V^*: =\mathrm{Hom}_K(V, K)$, (resp. $L^*: =\mathrm{Hom}_R(L, R)$). If $L \in {}_\Lambda\mathcal{L}(V)$, we identify as $L^* = \{\chi \in V^* ; \chi(L) \subseteq R\} \subseteq V^*$, by the Frobenius reciprocity. While, by the natural injection $v \mapsto (\chi \mapsto \chi(v))$, we identify as $V = V^{**}$. Then $L \mapsto L^*$ is an inclusion reversing bijection from ${}_\Lambda\mathcal{L}(V)$ onto $\mathcal{L}_\Lambda(V)$, and $L^{**} = L$.

1.0.1 Let $\mathcal{E}$ be a complete set of orthogonal primitive idempotents of Λ, hence $1 = \sum_{e \in \mathcal{E}} e$, and $\Lambda = \oplus_{e \in \mathcal{E}} \Lambda e$(resp. $\oplus_{e \in \mathcal{E}} e\Lambda$) is a decomposition of Λ into indecomposable left (resp. right)Λ-lattices. Let $\{\Lambda e_i ; 1 \le i \le s\}$ be a maximal subset of $\{\Lambda e ; e \in \mathcal{E}\}$ consisting of mutually non-isomorphic lattices. Put

$$P_i := \Lambda e_i,\ \nu_i := \#\{e \in \mathcal{E} ; \Lambda e \cong P_i\}\ 1 \le i \le s.$$

Then $\Lambda \cong \oplus_{i=1}^{s} P_i^{\nu_i}$ as left Λ-lattices.

A left Λ-lattice P is indecomposable projective if and only if there is some $t (1 \le t \le s)$ such that $P \cong P_t$ as Λ-modules. For such a P, put

$$\Lambda(P): = \bigoplus_{\Lambda e \cong P} \Lambda e, \quad \Lambda_P: = \bigoplus_{\Lambda e \not\cong P} \Lambda e$$

$$\Lambda(\tilde{P}): = \bigoplus_{Ae \cong \tilde{P}} \Lambda e, \quad \Lambda_{\tilde{P}}: = \bigoplus_{Ae \not\cong \tilde{P}} \Lambda e$$

$$A(P): = K\Lambda(P),\ A(\tilde{P}): = K\Lambda(\tilde{P}),\ A_P: = K\Lambda_P \text{ and } A_{\tilde{P}}: = K\Lambda_{\tilde{P}}.$$

Note that $\Lambda(P)$ depends not only the isomorphism class of P, but also the choice of $\mathcal{E}$.

Some of the crucial lemmas of [7] which we shall extend to an arbitrary A(cf.1.6.1 and 3.1), are the ones to give a nice criterion for $\Lambda(\tilde{P})$ to be a ring direct factor to Λ.

1.0.2. For a ring $\mathcal{O}$, let $s_l(\mathcal{O})$(resp. $s_r(\mathcal{O})$) denote the number of non-isomorphic indecomposable projective left (resp. right) $\mathcal{O}$-modules. By the Lifting Idempotent Theorem ([5] §6),

$$\Lambda e \cong \Lambda e' \Leftrightarrow \Lambda e/\mathcal{N}e \cong \Lambda e'/\mathcal{N}e' \Leftrightarrow e\Lambda/e\mathcal{N} \cong e'\Lambda/e'\mathcal{N} \Leftrightarrow e\Lambda \cong e'\Lambda,$$

hence $s = s_l(\Lambda) = s_l(\Lambda/\mathcal{N}) = s_r(\Lambda/\mathcal{N}) = s_r(\Lambda)$, and

$\Lambda\cong\bigoplus_{i=1}^{s}(e_i\Lambda)^{\nu_i}$ as right Λ-lattices.

A left Λ-lattice P is called *injective* if its dual lattice P^* is a projective right Λ-lattice. P is called *bijective* if it is projective and injective. Hence P is an indecomposable and bijective left Λ-lattice if and only if there is some t and $\sigma(t)(1\leq t,\ \sigma(t)\leq s)$ such that $P\cong\Lambda e_t$ as left Λ-lattices and $P^*\cong e_{\sigma(t)}\Lambda$ as right Λ-lattices.

Consequently Λ is left Gorenstein if and only if there exists a permutation σ of $\{1,\cdots,s\}$ such that $(\Lambda e_i)^*\cong e_{\sigma(i)}\Lambda$ as right Λ-lattices, or equivalently,

$$\Lambda\cong\bigoplus_{i=1}^{s}(P_i^*)^{\nu_{\sigma(i)}}\quad\text{as right }\Lambda\text{-lattices.}$$

In particular, Λ is left Gorenstein if and only if it is right Gorenstein, and we may drop 'left' or 'right' for 'Gorenstein', 'Bass' or 'strictly Bass'.

1.0.3. The following three conditions for Λ are equivalent.
$\Lambda\cong\Lambda^*$ as left Λ-lattices;
$\Lambda\cong\Lambda^*$ as right Λ-lattices;
Λ is Gorenstein and $\nu_i=\nu_{\sigma(i)}$ for any i $(1\leq i\leq s)$.
If Λ has one of the above properties, we call Λ to be *self-dual*.

1.1. Let L be a left Λ-lattice.
(i) L has the maximum (= unique maximal) Λ-submodule M if and only if $L/\mathcal{N}L$ is Λ-simple. If this is so, L is indecomposable and $M=\mathcal{N}L$.
(ii) L has the minimum (= unique minimal) Λ-overmodule M if and only if $(L^*\mathcal{N})^*/L$ is Λ-simple. If this is so, L is indecomposable and $M=(L^*\mathcal{N})^*$.
(iii) If L is projective (resp. injective), it has the maximum Λ-submodule (resp. minimum Λ-overmodule) if and only if it is indecomposable.

Proof. Since $L/\mathcal{N}L$ is a semisimple Λ-module, (i) is obvious. Taking the dual $(\ )^*$, we get (ii). For (iii), see [5] 6.17.

1.2. Let L be a left Λ-lattice. Suppose that $L/\mathcal{N}L$ is Λ-simple. Let L' be a minimal Λ-overmodule of L.

1.2.0. We have either $\mathcal{N}L'=L$ or $\mathcal{N}L'=\mathcal{N}L$.

a) In case $\mathcal{N}L'=L$: $L'/\mathcal{N}L'$ is Λ-simple, L' is indecomposable and L is the maximum submodule of L'.

b) In case $\mathcal{N}L'=\mathcal{N}L$: $L'/\mathcal{N}L'(=L'/\mathcal{N}L\cong L'/L\oplus L/\mathcal{N}L)$ is a direct sum of two simple Λ-modules, and $\mathcal{N}L$ has no minimum Λ-overmodule.

1.2.1. If L is the minimum overmodule of $\mathcal{N}L$, then L is the maximum submodule of L'.

Proof. The assumption excludes the case (b), hence we meet with the case (a).

1.2.2. Suppose L' is decomposable.
(i) L' is a direct sum of two indecomposable lattices $B_i (i=1, 2)$ such that $B_i/\mathcal{N}B_i \cong L/\mathcal{N}L$.
(ii) Further suppose that L is projective, then B_i is not projective.

Proof. (i) Let $L'=\oplus_{i=1}^r B_i (r\geq 2)$ with indecomposable B_i's. We should have the case b), hence $L'/\mathcal{N}L=\oplus B_i/\mathcal{N}B_i$, and $r=2$. The natural Λ-homomorphism $\pi_i\colon L'\to B_i/\mathcal{N}B_i$ factors through $\bar{\pi}_i\colon L'/\mathcal{N}L\to B_i/\mathcal{N}B_i$. Since L is indecomposable by 1.1(i), Ker $\pi_i=B_j\oplus\mathcal{N}B_i\not\supseteq L, \bar{\pi}_i(L/\mathcal{N}L)\neq 0$ hence $\bar{\pi}_i\colon L/\mathcal{N}L \tilde{\to} B_i/\mathcal{N}B_i$.
(ii) If B_i is projective, by [5] 6.23 $\bar{\pi}_i$ lifts to a projective cover $L\tilde{\to}B_i$, a contradiction.

1.3. Let L be an indecomposable injective left Λ-lattice and $'L$ be a maximal Λ-submodule of L. If $'L$ is decomposable, then $'L=C_1\oplus C_2$ with non-injective indecomposable $C_i (i=1, 2)$ such that $(C_i^*\mathcal{N})^*/C_i\cong(L^*\mathcal{N})^*/L$.

Proof. This is the dual of 1.2.2.

1.4. Composition series. Let $\mathcal{O}$ be a semiperfect ring (cf. [5] 6.23). For a finitely generated left $\mathcal{O}$-module M, a (finite or infinite) strictly decreasing sequence $\{M_i\}$,

$$M=M_0\supset M_1\supset\cdots\supset M_i\supset M_{i+1}\supset\cdots$$

will be called a *composition series* of M, and each M_i/M_{i+1} is called a *composition factor* of M if each M_i/M_{i+1} is $\mathcal{O}$-simple and $\bigcap_i M_i=0$.

If $\mathcal{O}$ itself is artinian, our definitions agree with ordinary ones. The well known argument for artinian case carries over to prove the following (cf. [4] 54.12):

1.4.1. Suppose M has a composition series. Let e be an idempotents of $\mathcal{O}$.
(i) $eM\neq 0$ if and only if $\mathcal{O}e/(\text{rad }\mathcal{O})e$ and M have at least one composition factor in common. In particular, (the set of isomorphism classes of distinct) composition factors are independent of the choice of a composition series.
(ii) If e' is another idempotent of $\mathcal{O}$ such that $\mathcal{O}e'$ has a composition series, then $\mathcal{O}e\mathcal{O}e'\neq 0$ if and only if $\mathcal{O}e/(\text{rad }\mathcal{O})e$ and $\mathcal{O}e'$ have a common composition factor.

1.4.2. If $\mathcal{O}$ is artinian, the length of the composition series is an invariant of M, which we will denote by $l_{\mathcal{O}}(M)$ or $l(M)$. We are mainly interested only in the case where $\mathcal{O}=A$ or Λ. In the former case $\mathcal{O}=A$ is artinian. For a Λ-lattice L, $l_A(\tilde{L})=l(\tilde{L})$ will be called a *rational length* of L. In the latter case, $\mathcal{O}=\Lambda$ is not artinian, but any Λ-lattice L has a composition series. Indeed $\bigcap_{\nu\geq 0}\pi^\nu L=0$, and the composition factor of L is just the compostion factor of the artinian Λ-module $L/\pi L$.

1.4.3. [8] Let L be a left Λ-lattice.
(i) If L has the maximum Λ-submodule and if $l(\tilde{L})\geq l(Ae)$ for any primitive idempotent e of Λ, then L is projective and indecomposable.
(ii) If L has the minimum Λ-overmodule and if $l(\tilde{L})\geq l(eA)$ for any primitive idempotent e of Λ, then L is injective and indecomposable.

Proof. (i) Since $L/\mathcal{N}L$ is Λ-simple, $L/\mathcal{N}L\cong\Lambda e/\mathcal{N}e$ by some primitive idempotent e, and there is a projective cover $\varphi: \Lambda e\rightarrow L$. Then $\tilde{\varphi}: =\mathrm{id}_K\otimes\varphi: Ae\rightarrow\tilde{L}$ is a surjective A-homomorphism. Since $l(\tilde{L})\geq l(Ae)$, $\tilde{\varphi}$ is an isomorphism and so is φ.
(ii) This is the dual of (i).

1.5. Let L be a left Λ-lattice. The following six conditions for (Λ, L) are mutually equivalent.
(1) L has the unique Λ-composition series.
(2) $\mathcal{N}^iL/\mathcal{N}^{i+1}L$ is simple for any $i\geq 0$.
(3) There is some $\nu>0$, such that $\pi L=\mathcal{N}^{\nu}L$ and $\mathcal{N}^iL/\mathcal{N}^{i+1}L$ is simple for $0\leq i<\nu$.
(4) There is some $\nu>0$, such that $\pi L=\mathcal{N}^{\nu}L$ and any $M\in{}_{\Lambda}\mathcal{L}(\tilde{L})$ has the unique expression $M=\pi^a\mathcal{N}^iL$ with $a\in\mathbf{Z}$, $0\leq i<\nu$.
(5) There is some $\nu_0>0$, such that $L\cong\mathcal{N}^{\nu_0}L$, $\mathcal{N}^iL/\mathcal{N}^{i+1}L$ simple, $\mathcal{N}^iL\ncong\mathcal{N}^{i+1}L$ for $0\leq i<\nu_0$.
(6) ${}_{\Lambda}\mathcal{L}(\tilde{L})$ is linearly ordered by inclusion.

The sequence $\{\mathcal{N}^iL;\ 0\leq i\leq\nu_0\}$ of (5) wil be referred to as a *period* of the unique composition series, where the ν_0 is the smallest integer satisfying (5). In that case, we sometimes say that $L\supset\mathcal{N}L\supset\cdots\supset\mathcal{N}^{\nu_0}L\cong L$ is the unique composition series of L.

Proof. (1)$\Leftrightarrow$(2): Obvious by 1.1(i). (1) and (2)$\Leftrightarrow$(3)$\Leftrightarrow$(4)$\Leftrightarrow$(5): Obvious.
(4)$\Rightarrow$(6): $\theta=(M\mapsto a\nu+i)$ is an order reversing (i.e. $M\supseteq M'\Leftrightarrow\theta(M)\leq\theta(M')$) bijection from $\mathcal{L}_{\Lambda}(\tilde{L})$ to $\mathbf{Z}$.
(6)$\Rightarrow$(2): Since $\mathcal{N}^iL/\mathcal{N}^{i+1}L$ is semisimple ($\cong\bigoplus_{j=1}^{r}\bar{L}_j$ with simple $\bar{L}_j$), (6) implies $r=1$.

1.5.1. If L has the unique Λ-composition series, $V=\tilde{L}$ is a simple A-module.

Proof. Suppose V is not simple, $V\supset W\neq 0$, then

$$0\rightarrow W\rightarrow V\overset{\varphi}{\rightarrow}V/W\rightarrow 0 \qquad (A\text{-exact})$$

$$0\rightarrow L\cap W\rightarrow L\rightarrow\varphi(L)\rightarrow 0 \qquad (\Lambda\text{-exact}).$$

Put $L_1: =\pi^{-1}L$, $L_2: =L+\pi^{-b}(L\cap W)\in\mathcal{L}_{\Lambda}(V)$. Then $\varphi(L_1)=\pi^{-1}\varphi(L)\supset\varphi(L)=\varphi(L_2)$, and $L_1\not\subseteq L_2$. If b is big enough, $L_1=\pi^{-1}L\not\supseteq\pi^{-b}(L\cap W)$, and $L_1\not\supseteq L_2$. Hence $\mathcal{L}_{\Lambda}(V)$ is not linearly ordered.

1.6. Theorem. (*Hereditary orders*) *Let A be a finite dimensional K-algebra, Λ be an R-order of A, and $\mathcal{N}: =$* rad Λ. *The following nine*

conditions (H.i) $(0\leq i\leq 8)$ *and the eight conditions* (H.i′) $(1\leq i\leq 8)$ *obtained from* (H.i) *by interchanging 'left' and 'right', are all equivalent to one another.*
(H.0) *If Λ' is an overorder of Λ such that* rad $\Lambda'\supseteq\mathcal{N}$, *then $\Lambda'=\Lambda$ (such a Λ is called an extremal order).*
(H.1) $O_r(\mathcal{N})=\Lambda$.
(H.2) *Any projective indecomposable left Λ-lattice P is the minimum Λ-overmodule of $\mathcal{N}P$.*
(H.3) *Any projective indecomposable left Λ-lattice P has the unique Λ-composition series.*
(H.4) *Any projective indecomposable left Λ-lattice P has the unique Λ-composition series consisting of projective lattices.*
(H.5) *Λ is stirictly left hereditary.*
(H.6) *Λ is left hereditary.*
(H.7) *$\mathcal{N}$ is projective as left Λ-lattice.*
(H.8) *If P is a projective indecomposable left Λ-lattice, then $\mathcal{N}P$ is also projective indecomposable.*

Proof. (H.0)⇒(H.1): Λ': $=O_r(\mathcal{N})=\{x\in A\,;\,\mathcal{N}x\subseteq\mathcal{N}\}$ is an overorder of Λ. Since $\mathcal{N}$ is a topologically nilpotent right ideal of Λ', $\mathcal{N}\subseteq$rad Λ'. Hence (H.0) implies $\Lambda'=\Lambda$.
(H.1)⇒(H.2): Let $\Lambda=\oplus_{e\in\mathcal{E}}\Lambda e$ as in 1.0, and suppose (H.2) does not hold for $P=\Lambda e$. Then there is a minimal Λ-overmodule L of $\mathcal{N}e$ such that $L\neq\Lambda e$ and $\Lambda\not\supseteq L$. Then $\mathcal{L}$: $=L+\mathcal{N}$ is a minimal Λ-overmodule of $\mathcal{N}$, hence $\mathcal{N}\mathcal{L}\subseteq\mathcal{N}$ i. e. $\mathcal{L}\subseteq O_r(\mathcal{N})$, contradicting to (H.1).
(H.2)⇒(H.3): Let P' be a minimal Λ-overmodule of P. By 1.2.1, (H.2) implies that $P=\mathcal{N}P'$ is the maximum Λ-submodule of P'. Suppose $l(\tilde{P})\geq l(Ae)$ for any $e\in\mathcal{E}$. By 1.4.3, P' is projective indecomposable. Repeating the process we get a period $P^{(\nu)}\supset\cdots\supset P'\supset P$, $P^{(\nu)}\cong P$, $\mathcal{N}P^{(i)}=P^{(i-1)}$. By 1.5, $P^{(\nu)}$ (hence P also) has the unique composition series, which in turn implies that $l(\tilde{P})=1$ by 1.5.1. Hence $l(Ae)=1$ for any $e\in\mathcal{E}$.
(H.3)⇒(H.4): By 1.5.1, (H.3) implies that $l(Ae)=1$ for any $e\in\mathcal{E}$, hence again 1.4.3 works.
(H.4)⇒(H.5): Let L be an indecomposable left Λ-lattice. By 1.5.1, (H.4) implies that any simple A-module has the form Ae. Hence by some $e\in\mathcal{E}$, $0\to\mathrm{Ker}\,\pi\to\tilde{L}\xrightarrow{\pi}Ae\to 0$ (A-exact) and $0\to L\cap\mathrm{Ker}\,\pi\to L\to\pi(L)\to 0$ (Λ-exact). By (H.4), $\pi(L)$ is Λ-projective, and $L\cong\pi(L)$.
(H.5)⇒(H.6)⇒(H.7): Obvious by definitions.
(H.7)⇒(H.8): Assuming (H.7), we show (H.8)+(H.4) by the induction on the rational length $l(\tilde{P})$ of P. Suppose $l(\tilde{P})$ is minimal. Then $\mathcal{N}P$, which is projective as a summand of projective $\mathcal{N}$, must be indecomposable. Hence we get a period $P\supset\mathcal{N}P\supset\cdots\supset\mathcal{N}^{\nu}P\cong P$, showing (H.4) for such P.

In general, if $\mathcal{N}P=P_1\oplus Q$, $Q\neq 0$ with a projective indecomposable P_1, since $\tilde{\mathcal{N}}P=\tilde{P}$, we have $0\to\tilde{Q}\to\tilde{P}\xrightarrow{\pi}\tilde{P}_1\to 0$ (A-exact), $0\to\tilde{Q}\cap P\to P\to\pi(P)\to 0$ (Λ-exact). Since $l(\tilde{P}_1)<l(\tilde{P})$, $\pi(P)$ is projective by the induction assumption

resulting a contradiction to the indecomposability of P.
(H.8)$\Rightarrow$(H.1′) : Let $\Lambda\cong\oplus P_i^{\nu_i}$ as in 1.0, then $\mathcal{N}\cong\oplus(\mathcal{N}P_i)^{\nu_i}$. by (H.8), $\mathcal{N}P_i\cong P_{\tau(i)}$ by some $\tau(i)$ $(1\le\tau(i)\le s)$. Since (H.8) obviously implies (H.7), by the proof of (H.7)$\Rightarrow$(H.8), (H.8) also implies (H.4). Hence P_i is the minimum overmodule of $\mathcal{N}P_i$, and τ is a permutation of $\{1, \cdots, s\}$. Thus $O_l(\mathcal{N})=\cap O_l(\mathcal{N}P_i)=\cap O_l(P_i)=O_l(\Lambda)=\Lambda$.
(H.1′)$\Rightarrow$(H.0) : Let Λ' be an overorder of Λ such that $\mathcal{N}'$: $=$rad $\Lambda'\supseteq\mathcal{N}$. Put m: $=\inf\{t\in\mathbf{N}\,;\ \mathcal{N}'^t\subseteq\mathcal{N}\}$. Since $1\notin\mathcal{N}$, $m\ge1$. Assume (H.1′). If $m\ge2$, $\mathcal{N}'^{m-1}\mathcal{N}\subseteq\mathcal{N}'^m\subseteq\mathcal{N}$, i.e. $\mathcal{N}'^{m-1}\subseteq O_l(\mathcal{N})=\Lambda$. Since $\mathcal{N}'^{m-1}$ is a topologically nilpotent twosided ideal of Λ, $\mathcal{N}'^{m-1}$ lies in $\mathcal{N}$, a contradiction. Hence $m=1$, $\mathcal{N}'=\mathcal{N}$ and $\Lambda'\subseteq O_l(\mathcal{N}')=O_l(\mathcal{N})=\Lambda$. Now, by left right symmetry, we completed the proof.

1.6.0. Corollary. *Now we do not need to distinguish 'right hereditary' from 'left hereditary' or 'hereditary' from 'strictly hereditary', and just call 'hereditary'.*
(i) *A maximal order is a hereditary order.*
(ii) *A hereditary order is a strictly Bass order.*

Proof. (i) 'Maximal' implies 'extremal'. (ii) 'Strictly hereditary' obviously implies 'Gorenstein'. An overorder of a strictly hereditary order is also strictly hereditary (cf. 2.0 (iii)), hence is Gorenstein. Thus we have seen that a hereditary order is a Bass order. To see that it is strictly Bass, it suffices to see that the ambient algebra of a hereditary order is always semisimple. The last claim is a part of the next proposition.

1.6.1 Proposition. *Let A be a finite dimensional K-algebra, Λ be an R-order of A, P be a left Λ-lattice. Suppose P has the unique Λ-composition series.*
(i) *The following three conditions for (Λ, P) are mutually equivalent.*
(u1) *Each lattice in the Λ-composition series is Λ-bijective.*
(u2) *At least one lattice in the Λ-composition series is Λ-bijective.*
(u3) *P is a projective indecomposable Λ-lattice, and in the notation of* 1.0. 1 *: $A(\tilde{P})$ is a ring direct factor of A, and is a simple K-algebra ; $\Lambda(\tilde{P})$ is a ring direct factor of Λ, and is a hereditary order of $A(\tilde{P})$.*
(ii) *Suppose (Λ, P) has one of the above properties and put D : $=(\mathrm{End}_A\,\tilde{P})^\circ$. Then D is a finite dimensional division K-algebra, and any $L\in{}_\Lambda\mathcal{L}(\tilde{P})$ is a right O-lattice, where O is the maximal order of D.*

Proof. (i) (u1)$\Rightarrow$(u2) : Obvious.
(u2)$\Rightarrow$(u3) : We may assume P itself in Λ-bijective. By 1.5.1, $\tilde{P}$ is A-simple, hence P is Λ-indecomposable and $P\cong\Lambda e$ by some primitive idempotent e of Λ. To see that $A(\tilde{P})$ (hence $\Lambda(\tilde{P})$ also) is a ring direct factor : By 1.4.1, it suffices to see that $(Ae)V=VAe=0$ for any indecomposable direct factor V of $A_{\tilde{P}}$ as left A-modules. If $(Ae)V\neq0$ or $V(Ae)\neq0$: Again by 1.4.

1, V has an A-composition factor isomorphic to Ae. This implies, since $\tilde{P}\cong Ae$ is bijective simple, $V\cong Ae$, a contradiction to the definition of $A_{\tilde{P}}$.

(u3)⇒(u1): Any $L\in{}_{\Lambda}\mathcal{L}(\tilde{P})$ is an $\Lambda(\tilde{P})$-lattice. $\Lambda(\tilde{P})$ is hereditary, hence strictly hereditary by 1.6, and L is $\Lambda(\tilde{P})$-bijective or equivalently Λ-bijective.
(ii) Since $\tilde{P}$ is simple, D is a division algebra. ${}_{\Lambda}\mathcal{L}(\tilde{P})$ contains at least one right O-lattice. Hence, by 1.5 (4), any $L\in{}_{\Lambda}\mathcal{L}(\tilde{P})$ is an O-lattice.

1.6.2. (i) Let $\Lambda=\oplus_{e\in\mathcal{E}}\Lambda e$ be a ring indecomposable hereditary order in a finite dimensional K-algebra A, and P be a projective indecomposable left Λ-lattice. Then $A=A(\tilde{P})$ is a simple K-algebra. Let $P_0\supset P_1\supset\cdots\supset P_{s-1}\supset P_s\cong P_0$ be a period of ${}_{\Lambda}\mathcal{L}(\tilde{P})$ (cf. 1.5), and put $\nu_i:=\#\{e\in\mathcal{E} : \Lambda e\cong P_i\}$. Then $\Lambda\cong\oplus_{i=1}^{s}P_i^{\nu_i}$ as left Λ-lattices, and $\Lambda=\cap_{i=1}^{s}O_l(P_i)$.

Put $V:=\tilde{P}$, $D:=(\mathrm{End}_A V)^{\circ}$, then the thus obtained triple $(D, V, \{P_i\})$ satisfies the following (1), (2), (3).
(1) D is a division algebra.
(2) V is a finite dimensional right D-module.
(3) Let O (resp. π_D) denote the maximal order of D (resp. a prime of O), then $P_i(0\le i\le s)$ is a full right O-lattice in V such that $P_i\supset P_{i+1}$, $P_s=P_0\pi_D$.

There is a natural K-algebra isomorphism $A\cong\mathrm{End}_D V$, which induces $\Lambda\cong\Lambda(\{P_i\}):=\cap_i \mathrm{End}\,{}_O P_i$.
(ii) Conversely, if a triple $(D, V, \{P_i\})$ has the properties (1), (2), (3), then $\Lambda(\{P_i\})$ is a hereditary order of $A:=\mathrm{End}_D V$. The order $\Lambda(\{P_i\})$ is a maximal order (resp. minimal hereditary order) of A if and only if $s=1$ (resp. $s=\dim_D V$).
(iii) A finite dimensional K-algebra A contains a hereditary (resp. maximal) order if and only if it is semisimple. Maximal orders (resp. minimal hereditary orders) are $A^{\times}$-conjugate to one another.

Proof. (i) Straightforward by 1.6.1.
(ii) To see that $\Lambda=\Lambda(\{P_i\})$ is hereditary, it suffices to see that when $s=\dim_D V$. In this case, one can choose a basis $v_i(0\le i<s)$ of V and $\Pi\in\mathrm{End}_D V$ so that $P_0=\sum_{i=0}^{s-1} v_iO$, $\Pi(v_0)=v_{s-1}\pi_D$, $\Pi(v_i)=v_{i-1}(1\le i<s)$ and $P_i=\Pi^i(P_0)$ $(0\le i<s)$. Then the restriction map $\lambda\mapsto\lambda|_{P_i}(\lambda\in\Lambda)$ induces the following R-algebra exact sequence

$$0\to\Pi\Lambda\to\Lambda\to\bigoplus_{i=0}^{s-1}\mathrm{End}_{\Delta} P_i/P_{i+1}\to 0,$$

where $\Delta:=O/O\pi_D$ is a sfield.

Since $\mathrm{End}_{\Delta} P_i/P_{i+1}\cong\Delta$, this implies that rad $\Lambda=\Pi\Lambda$, hence Λ is hereditary by 1.6 (H′7).
(iii) The first claim is by 1.6.1. The second one is a well-known fact for O-lattices.

1.7. In this subsection, let R be a (global) Dedekind domain with the quotient field K. Let A be a finite dimensional K-algebra, Z its center, and S

the integral closure of R in Z. Let $A_l(1\leq l\leq r)$ be the indecomposable ring direct factors of A, $A=\oplus_{l=1}^{r} A_l$ and Z_l the center of A_l, S_l the integral closure of R in Z_l.

1.7.0. Let O be an R-order of Z. If Λ is a hereditary order of A, then $O\Lambda=\Lambda$, i.e. $O\subseteq\Lambda$.

Proof. Let $K_{\wp}$ (resp. $R_{\wp}$) denote the completion of K (resp. R) by a prime ideal $\wp$ of R. For a K-module V (resp. R-module L), put $V_{\wp}: =V\otimes_K K_{\wp}$ (resp. $L_{\wp}: =L\otimes_R R_{\wp}$). $(O\Lambda)_{\wp}=O_{\wp}\Lambda_{\wp}$ is an overorder of $\Lambda_{\wp}$. $O_{\wp}\,\mathrm{rad}(\Lambda_{\wp})$ is a topologically nilpotent twosided ideal of $O_{\wp}\Lambda_{\wp}$. Hence $\mathrm{rad}((O\Lambda)_{\wp})\supseteq O_{\wp}\,\mathrm{rad}(\Lambda_{\wp})\supseteq\mathrm{rad}(\Lambda_{\wp})$. Since $\Lambda_{\wp}$ is extremal by 1.6, we have $(O\Lambda)_{\wp}=\Lambda_{\wp}$ for any $\wp$, and $O\Lambda=\Lambda$.

1.7.1. Theorem. (i) *The following three conditions for A are equivalent :*
(1) *A contains a maximal order,*
(2) *A contains a hereditary order,*
(3) (3.1) *A is semisimple,*
(3.2) *S is finitely generated as an R-module,*
(ii) *If the above conditions are satisfied, Λ is a maximal (resp. hereditary) R-order of A if and only if it is the direct sum $\Lambda=\oplus_{l=1}^{r}\Lambda_l$ of maximal (resp. hereditary) S_l-orders Λ_l of A_l.*

Proof. (i) (1)⟹(2): By 1.6.0 and 0.2.1.
(2)⟹(3): Suppose A contains a hereditary order Λ. Then $\Lambda_{\wp}$ is a hereditary order of $A_{\wp}$. By 1.6.1 (iv), $A_{\wp}$ is semisimple and so is A. Assume (3.2) does not hold, in particlular $S\not\subseteq\Lambda$. Then there is an R-order O of Z such that $O\not\subseteq\Lambda$, a contradiction to 1.7.0.
(3)⟹(1): By (3.1) $A=\oplus A_l$ is the direct sum of simple A_l's. A_l is central simple over Z_l, and contains a maximal S_l-order Λ_l. By (3.2), Λ_l is an R-order and so is $\Lambda: =\oplus\Lambda_l$. Let Λ' be an R-order of A containing Λ, then $\Lambda'\supseteq S$ and $\Lambda'=\Lambda$. Thus Λ is a maximal R-order of A.
(ii) If Λ is R-hereditary, by 1.7.0, $\Lambda=S\Lambda=(\oplus S_l)\Lambda=\oplus(S_l\Lambda)$.

1.7.2. Remark. (i) By this theorem, the study of hereditary orders in an arbitrary finite dimensional K-algebra A is completely reduced to the situation where A is central simple over K.
(ii) Assume (3.1), hence Z is a direct sum of finite extension fields of K. As is very well-known, if Z is separable over K, then the condition (3.2) is automatically satisfied. This separable case is covered by the existent theory due to Jacobinski [10]. However, there are many important cases covered only by our generalization. For example, if R is a Nagata ring (cf. [13] Chapter 12), (3.2) is certainly satisfied for any Z. In particular, the ring R of integral functions in a field of algebraic functions of one variable over any field is a Nagata ring as well as a Dedekind domain.

2. Bijective lattices and Rejection Lemma

Let A, Λ be as in §1 and let V denote a finitely generated left A-module. As in §1, ${}_\Lambda\mathcal{L}(V)$ denote the totality of full left Λ-lattices in V. Let ${}_\Lambda\mathcal{L}(V)^{ind}$ (resp. ${}_\Lambda\mathcal{L}(V)^{proj}$) denotes the subset of ${}_\Lambda\mathcal{L}(V)$ consisting of indecomposable (resp. projective) Λ-lattices.

2.0. Let Λ and Γ be orders in A.
(i) The following four conditions for (Λ,Γ) are equivalent.
(1) $\Lambda\subseteq\Gamma$
(2) ${}_\Lambda\mathcal{L}(V)^{ind}\cap{}_\Gamma\mathcal{L}(V)={}_\Gamma\mathcal{L}(V)^{ind}$ for any V.
(3) ${}_\Gamma\mathcal{L}(V)^{ind}\subseteq{}_\Lambda\mathcal{L}(V)^{ind}$ for any V.
(4) ${}_\Gamma\mathcal{L}(V)\subseteq{}_\Lambda\mathcal{L}(V)$ for any V.
(ii) $\Lambda=\Gamma$ if and only if ${}_\Lambda\mathcal{L}(V)^{ind}={}_\Gamma\mathcal{L}(V)^{ind}$ for any V.
(iii) If $\Lambda\subseteq\Gamma$, ${}_\Lambda\mathcal{L}(V)^{proj}\cap{}_\Gamma\mathcal{L}(V)\subseteq{}_\Gamma\mathcal{L}(V)^{proj}$.

Proof. (i): (1)⇒(2): The left hand side of (2) is obviously included in the right hand side. If $L\in{}_\Gamma\mathcal{L}(V)$, both of $\mathrm{End}_\Lambda L$ and $\mathrm{End}_\Gamma L$ can be identified with the subset $\{\varphi;\ \varphi(L)\subseteq L\}$ of $\mathrm{End}_A V$. Hence the opposite inclusion holds. (2)⇒(3): Obvious. (3)⇒(4): By Krull-Schmidt-Azumaya Theorem. (4)⇒(1): A special case of (4) with $V=A$ implies $\Lambda\Gamma\subseteq\Gamma$.
(ii): By (i). (iii): By Dual Basis Lemma ([5] 3.46).

2.1. Let P be an indecomposable left Λ-lattice. We ask whether or no there is an overorder Γ of Λ satisfying either one of the following two mutually equivalent conditions for any finitely generated A-module V; (5) as left Λ-lattices and (5*) as right Λ-lattices:

$$(5)\ L\in{}_\Gamma\mathcal{L}(V)^{ind}\Leftrightarrow L\in{}_\Lambda\mathcal{L}(V)^{ind}\text{ and }L\not\cong P,$$

$$(5^*)\ L\in\mathcal{L}_\Gamma(V)^{ind}\Leftrightarrow L\in\mathcal{L}_\Lambda(V)^{ind}\text{ and }L\not\cong P^*.$$

(i) If such a Γ exists, it is unique. Hence we write as $\Gamma:=\Lambda-(P)=\Lambda-(P^*)$, and call it the order obtained by *rejecting P from Λ*.
(ii) If $\Lambda-(P)$ exists, it is a minimal overorder of Λ.
(iii) If $\Lambda-(P)$ exists, P is Λ-bijective.

Proof. (i) By 2.0 (ii). (ii) By (3). (iii) Let $\Lambda=\bigoplus_{e\in\mathcal{E}}\Lambda e$ as in 1.0. Suppose $\Gamma=\Lambda-(P)$ exists. If P is not Λ-projective, then $P\not\cong\Lambda e$. $\Gamma(\Lambda e)=\Lambda e$ for any $e\in\mathcal{E}$, hence $\Gamma\Lambda=\Lambda$ and $\Gamma=\Lambda$. Hence P must be Λ-projective. By (5*), P^* must be right Λ-projective.

2.2. For a bijective indecomposable left Λ-lattice P, put

$$'P:=\mathcal{N}P,\ P':=(P^*\mathcal{N})^*.$$

Then, by 1.1, $'P$ (resp. P') is the maximum Λ-submodule (resp. minimum Λ-overmodule) of P, $P'\supset P\supset{}'P$.

In the notation of 1.0.1, $\Lambda(P)=\oplus_{\Lambda e\cong P}\Lambda e$ etc., put $\Lambda(P)' :=\oplus_{\Lambda e\cong P}(\Lambda e)'$, and

(6) $\Lambda' :=\Lambda(P)'\oplus\Lambda_P$ as left Λ-lattices.

If $\Lambda\cong\oplus_{i=1}^{s}P_i^{\nu_i}$, $P_i=\Lambda e_i(1\leq i\leq s)$ and $P\cong P_t$,

(7) $\Lambda'\cong(P')^{\nu_t}\oplus\bigoplus_{i\neq t}(\Lambda e_i)^{\nu_i}$ as left Λ-lattices.

2.2.0 If $P\cong P_t=\Lambda e_t$ and $(\Lambda e_t)^*\cong e_{\sigma(t)}\Lambda$ by some $\sigma(t)(1\leq\sigma(t)\leq s)$, then $P'/P\cong\bar{\Lambda}\bar{e}_{\sigma(t)} :=\Lambda e_{\sigma(t)}/\mathcal{N}e_{\sigma(t)}$ as left Λ-modules.

Proof. Since $P\supset\pi P'$, $\mathrm{Hom}_{R/\pi R}(P'/P, R/\pi R)=\mathrm{Hom}_R(P'/P, K/R)\cong P^*/P'^*=P^*/P^*\mathcal{N}\cong\bar{e}_{\sigma(t)}\bar{\Lambda}$. Hence $P'/P\cong\mathrm{Hom}_{R/\pi R}(\bar{e}_{\sigma(t)}\bar{\Lambda}, R/\pi R)\cong\bar{\Lambda}\bar{e}_{\sigma(t)}$.

2.2.1. (Rejection Lemma of Drozd-Kirichenko) Let P be a bijective indecomposable left Λ-lattice. Assume P is not isomorphic to P' as left Λ-modules. Then

(i) $\Lambda-(P)$ exists.

(ii) $\Lambda-(P)=\Lambda'$, and

(7*) $\Lambda'\cong(({}'P)^*)^{\nu_{\sigma(t)}}\oplus\bigoplus_{i\neq\sigma(t)}(e_i\Lambda)^{\nu_i}$ as right Λ-lattices

Proof. (i) Put $\Gamma :=\bigcap_M O_l(M)$, where M runs over all indecomposable left Λ-lattices nonisomorphic to P. Let $P'=\oplus B_j$ be the indecomposable decomposition as Λ-lattices, then by (7),

$$\Gamma\subseteq\bigcap_j O_l(B_j)\cap\bigcap_{i\neq t}O_l(P_i)=O_l(\Lambda').$$

Since Λ' is a faithful left Λ-lattice, $O_l(\Lambda')$ is an order of A, hence Γ is also an order of A. We can pick a finite number of indecomposable Λ-lattices $M_u(1\leq u\leq N)$, $M_u\not\cong P$ such that

$$\Gamma=\bigcap_u O_l(M_u)\cap O_l(\Lambda')=O_l(L),\text{ where }L=\bigoplus_u M_u\oplus\Lambda'.$$

We shall prove that $\Gamma\neq\Lambda$, (then since ${}_{\Lambda}\mathcal{L}(V)^{ind}\ni L'\not\cong P\Rightarrow L'\in{}_{\Gamma}\mathcal{L}(V)^{ind}$ for any V, $\Gamma=\Lambda-(P)$). Put $E :=\mathrm{End}_\Lambda L$. E is an order in $\mathrm{End}_A(\tilde{L})$, and L is a noetherian E-module, hence finitely presented, $E^n\to E^m\to L\to 0$ (E-exact). Taking $\mathrm{Hom}_E(\ ,L)$,

$$0\to\mathrm{Hom}_E(L, L)\to\mathrm{Hom}_E(E^m, L)\xrightarrow{\varphi}\mathrm{Hom}_E(E^n, L)\ (\Lambda\text{-exact}).$$

Sisce $\tilde{L}=A\oplus(\oplus_t\tilde{M}_t)$, $\mathrm{Hom}_E(L, L)\cong O_l(L)=\Gamma$. If $\Gamma=\Lambda\cong P\oplus X$, since $\mathrm{Hom}_E(E^m, L)\cong L^m$, φ induces Λ-isomorphisms:

$$\mathrm{Im}\ \varphi\cong\frac{L^m}{P\oplus X}\cong\frac{L^m/X}{P\oplus X/X}\cong\frac{L^m/X}{P}.$$

Sitting in $\mathrm{Hom}_E(E^n, L)\cong L^n$, $\mathrm{Im}\,\varphi$ is R-torsionfree, hence L^m/X is also an Λ-lattice, and injectivity of P implies $L^m/X\cong P\oplus\mathrm{Im}\,\varphi$. Since L has no P-factor, we have got a contradiction by the Krull-Schmidt-Azumaya Theorem.
(ii) If $P\cong\Lambda e$, $(\Lambda e)'=\Gamma(\Lambda e)'\supseteq\Gamma(\Lambda e)\supset\Lambda e$, and $\Gamma(\Lambda e)=(\Lambda e)'$. If $P\ncong\Lambda e$, $\Gamma(\Lambda e)=\Lambda e$. Hence $\Gamma=\Gamma\Lambda=\Lambda'$. Since $('P)^*$ is the minimum overmodule of P^*, (7*) is the right module version of (7).

2.2.2. Remark. In Lemma 2.9 [7], the condition $P'\ncong P$ is not mentioned (perhaps by their definition of 'overring'). Suppose $P'\cong P$. Then, by 1.6.1, Λ is the ring direct sum of $\Lambda(\tilde{P})$ and $\Lambda_{\tilde{P}}$, with the maximal order $\Lambda(\tilde{P})$. Hence any overorder Γ of Λ is a ring direct sum of the form $\Gamma=\Lambda(\tilde{P})\oplus\Gamma_1$, and $P\in {}_{\Gamma}\mathcal{L}(\tilde{P})$. In particular $\Lambda-(P)$ does not exist in our definition. However there is no need to consider $\Lambda-(P)$ for such a lattice P.

2.2.3. Let $\Lambda=\oplus_{j=1}^m\Lambda_j$ be a ring direct sum, and P be a bijective indecomposable direct factor of $\Lambda_l(1\leq l\leq m)$, such that $P'\ncong P$. Then

$$\Lambda-(P)=(\Lambda_l-(P))\oplus\bigoplus_{j\neq l}\Lambda_j.$$

Proof. Obvious by 2.2.1 and (6).

2.2.4. Suppose $P'\ncong P$. Recall that $\Lambda(P)$ or Λ_P depends on the choice of the c.s.o.p.i. $\mathcal{E}$ of Λ. To describe $\Lambda'=\Lambda-(P)$ in terms of $\mathcal{E}$, we naturally choose the c.s.o.p.i. $\mathcal{E}'$ of Λ' in the following way:

If P' is Λ-indecomposable, put $\mathcal{E}':=\mathcal{E}$.

If P' is Λ-decomposable, hence $P'=B_1\oplus B_2$ by 1.2.2, put $\mathcal{E}':=\{e\in\mathcal{E}\,;\,\Lambda e\ncong P\}\cup\{e', e''\,;\,e'+e''=e\in\mathcal{E},\,\Lambda e\cong P\}$.

(i) $$\Lambda(P)'=\begin{cases}\Lambda'(P') & \text{if } P' \text{ is } \Lambda\text{-indecomposable}\\ \Lambda'(B_1)\oplus\Lambda'(B_2) & \text{if } P'=B_1\oplus B_2.\end{cases}$$

(ii) $$s(\Lambda')=s(\Lambda)+\begin{cases}1 & \text{if } P'\cong B_1\oplus B_2,\ B_1\ncong B_2\\ -1 & \text{if } P' \text{ is } \Lambda\text{-projective}\\ 0 & \text{if otherwise.}\end{cases}$$

(iii) If Λ is ring indecomposable and P' is Λ-indecomposable, then Λ' is ring indecomposable.

Proof. (i) Obvious from the choice of $\mathcal{E}'$.
(ii) By 1.2.2, $B_j\ncong P_i$ for all $i\neq t$; P' is Λ-projective if and only if $P'\cong P_i$ for some $i\neq t$.
(iii) Any indecomposable ring direct factor of Λ' is a block, hence a sum of $\Lambda'(P')$ and $\Lambda(P_i)(i\neq t)$. Hence if Λ' is ring decomposable, then so is Λ by (i).

2.3. Let P be a bijective indecomposable left Λ-lattice and P' (resp. $'P$) be its minimum Λ-over (resp. maximum Λ-sub) module of P.

2.3.1. P' is projective (as left Λ-lattices) if and only if $'P$ is injective. If that is so, both of P' and $'P$ are indecomposable.

Proof. If $P\cong P'$, then $'P\cong '(P')=P$. Suppose $P\not\cong P'$, then $\Lambda' := \Lambda-(P)$ exists and:

$$P' : \text{projective} \overset{1.2.2}{\Longleftrightarrow} P' : \text{indecomposable projective} \overset{(7)}{\Longleftrightarrow} s_l(\Lambda')=s-1$$
$$\overset{1.0.2}{\Longleftrightarrow} s_r(\Lambda')=s-1 \overset{(7^*)}{\Longleftrightarrow} ('P)^* \text{ is isomorphic to a direct summand of } \bigoplus_{i\neq\sigma(t)}(e_i\Lambda)^{\mu_i}$$
$$\Longleftrightarrow {}'P : \text{injective} \overset{1.3}{\Longleftrightarrow} {}'P : \text{indecomposable injective.}$$

2.3.2. (i) P' is Λ-decomposable if and only if $'P$ is Λ-decomposable. (ii) Suppose P' is Λ-decomposable. Then $P'=B_1\oplus B_2$ with indecomposable nonprojective Λ-lattices $B_j(1\leq j\leq 2)$. In particular $P\not\cong P'$, and $\Lambda-(P)$ exists. Put $\Lambda' := \Lambda-(P)$, $\mathcal{N}' := \operatorname{rad}\Lambda'$. Then $'P=\mathcal{N}B_1\oplus\mathcal{N}B_2$, $\mathcal{N}B_j$ is an indecomposable noninjective Λ-lattice, and $\mathcal{N}B_j=\mathcal{N}'B_j$ for $j=1, 2$.

Proof. By 1.2.2, we already know the first claim of (ii), $P'=B_1\oplus B_2$. Hence $\mathcal{N}B_1\oplus\mathcal{N}B_2=\mathcal{N}P'$ which coincides with $\mathcal{N}P='P$ by 1.2.0, proving the only if part of (i). If $'P$ is decomposable, then so is $('P)^*$ which is the minimum Λ-overmodule of P^*, hence $P^*\mathcal{N}$ is decomposable and so is $P'=(P^*\mathcal{N})^*$. Since $\mathcal{N}B_j$ is noninjective by 1.3, it is not isomorphic to P, and so an Λ'-submodule of B_j. Since B_j is Λ'-projective indecomposable, $\mathcal{N}'B_j$ is the maximum Λ'-submodule of B_j, and $\mathcal{N}'B_j\supseteq\mathcal{N}B_j$. Since $B_j/\mathcal{N}B_j$ is Λ-simple by 1.2.2, we have $\mathcal{N}'B_j=\mathcal{N}B_j$.

2.4. Let P, Q be mutually nonisomorphic bijective indecomposable left Λ-lattices. Assume $P\not\cong P'$, and put $\Lambda' := \Lambda-(P)$, $\mathcal{N}' := \operatorname{rad}\Lambda'$. By 2.2, Q is also a bijective indecomposable Λ'-lattice.

2.4.1. The following three conditions (8), (9), (10) (resp. (8*), (9*), (10*)) are mutually equivalent.

(8)	$\mathcal{N}Q\not\cong P$	(8*)	$Q^*\mathcal{N}\not\cong P^*$
(9)	$\mathcal{N}Q$ is an Λ'-lattice	(9*)	$Q^*\mathcal{N}$ is an Λ'-lattice
(10)	$\mathcal{N}Q=(\mathcal{N}')Q$	(10*)	$Q^*\mathcal{N}=Q^*\mathcal{N}'$

Proof. (8)⇒(9): If $\mathcal{N}Q$ is Λ-indecomposable, it is an Λ'-lattice by the definition of $\Lambda-(P)$. If $\mathcal{N}Q$ is decomposable, then by 1.3, $\mathcal{N}Q=C_1\oplus C_2$ with noninjective indecomposable Λ-lattices C_j, hence $C_j\not\cong P$, and C_j is an Λ'-lattice. (9)⇒(10): The maximum Λ-submodule $\mathcal{N}Q$ always contains $(\mathcal{N}')Q$. If $\mathcal{N}Q$ is an Λ'-lattice, the maximum Λ'-submodule $(\mathcal{N}')Q$ contains $\mathcal{N}Q$, hence $\mathcal{N}Q=(\mathcal{N}')Q$. (10)⇒(8): P is not an Λ'-lattice, and cannot be isomorphic to an Λ'-lattice $\mathcal{N}Q=(\mathcal{N}')Q$.

2.4.2. (i) If $Q' \not\cong P$ as Λ-lattices, the minimum Λ-overmodule Q' is also the minimum Λ'-overmodule of Q. If $'Q \not\cong P$ as Λ-lattices, the maximum Λ-submodule $'Q$ is also the maximum Λ'-submodule of Q.
(ii) Suppose $Q' \not\cong Q$ as Λ'-lattices, or equivalently $Q' \not\cong Q$ as Λ-lattices. Then $\Lambda-(Q)$ and $(\Lambda-(P))-(Q)$ exist. If $Q' \not\cong P$ as Λ-lattices, in the notation of 2. 2, we have

$$(\Lambda-(P))-(Q)=\Lambda(P)' \oplus \Lambda(Q)' \oplus X,$$

where X is the sum over Λe such that $\Lambda e \not\cong P, Q$.

Poof. (i) $Q' \not\cong P \Leftrightarrow (8^*) \Leftrightarrow (10^*) \Leftrightarrow Q'=(Q^* \mathcal{N}')^*$. $'Q \not\cong P \Leftrightarrow (8) \Leftrightarrow (10) \Leftrightarrow 'Q=(\mathcal{N}')Q$. (ii) Obvious by (i) and 2.2.

2.4.3. (Commutativity of Rejections) Suppose $P \not\cong Q$, $P \not\cong P'$, $Q \not\cong Q'$, $P \not\cong Q'$, and $Q \not\cong P'$ as left Λ-lattices. Then $\Lambda-(P)$, $\Lambda-(Q)$, $(\Lambda-(P))-(Q)$, $(\Lambda-(Q))-(P)$ exist, and

$$(\Lambda-(P))-(Q)=(\Lambda-(Q))-(P)$$

Poof. Changing the role of P and Q in 2.4.2, $(\Lambda-(Q))-(P)=\Lambda(Q)' \oplus \Lambda(P)' \oplus X$.

2.5. Definitions. A left Λ-lattice P will be called *superbijective* if it is indecomposable bijective and moreover $P' := (P\mathcal{N})^*$ is Λ-isomorphic to $'P := \mathcal{N}P$. An order Λ will be called *superGorenstein* if any indecomposable projective Λ-lattice is superbijective.

2.5.1. If P is a superbijective left Λ-lattice and $P' \not\cong P$, then $\Lambda' := \Lambda-(P)$ exists and P' is Λ'-bijective.

Proof. Comparing (7) with (7*), this is obvious.

2.5.2. Let Λ be a ring indecompsable order and P be a superbijective left Λ-lattice. The following five conditions for (Λ, P) are equivalent:

(11) P' is Λ-bijective.
(12) P' is Λ-projective.
(13) $\mathcal{N}P'=P$.
(14) Λ is hereditary with $s(\Lambda) \leq 2$.
(15) Λ is hereditary.

Proof. (11)$\Rightarrow$(12)$\Rightarrow$(13); (14)$\Rightarrow$(15): Obvious.

(13)$\Rightarrow$(14): $P \supset 'P \cong P$ or $P \supset 'P \cong P' \supset P$ is the unique Λ-composition series of P. By 1.6.1 and 1.6.2, Λ is hereditary with $s(\Lambda) \leq 2$. (15)$\Rightarrow$(11): By 1.6.1, Λ is strictly hereditary, and any Λ-lattice is bijective.

2.5.3. Let $\Lambda \cong \oplus_{i=1}^{s} (\Lambda e_i)^{\nu_i}$ be a ring indecomposable hereditary order.
(i) Λ is superGorenstein if and only if $s(\Lambda) \leq 2$.
(ii) Λ is self-dual (cf. 1.0.3) if and only if Λ is of equimultiplicity (i.e. $\nu_i = \nu_j$ for any i, j).

(iii) For a finitely generated left Λ-module M, let $\mu_\Lambda(M)$ denote the minimal number of Λ-generators of M. For a real number x, let $\{x\}$ denote the least integer$\geq x$. If I runs over all left Λ-ideals, we have the following formulas.

(16) $$\sup_I \mu_\Lambda(I)=\sup_i\{\nu_i^{-1}\sum_i \nu_i\},$$

(17) $$\sup_I \mu_\Lambda(I)=1 \text{ if } s(\Lambda)=1,$$

(18) $$\sup_I \mu_\Lambda(I)=\sup_i\{1+\nu_i^{-1}\nu_{\sigma(i)}\} \text{ if } s(\Lambda)=2.$$

Proof. (i) Only if part is in 2.5.2. If $s(\Lambda)=1$, then $P'\cong P\cong{}'P$; if $s(\Lambda)=2$, $P'\ncong P$, $P'\cong{}'P$ for any indecomposable P.
(ii) By 1.6.0, Λ is Gorenstein and let σ be a permutation such that $(\Lambda e_i)^*\cong e_{\sigma(i)}\Lambda$. Let $P\cong\Lambda e_t$ and m be the minimal number such that $\sigma^m(t)=t$. By 2.2.0, $P'/P\cong\bar{\Lambda}\bar{e}_{\sigma(t)}$ and $P'\cong\Lambda e_{\sigma(t)}$. Thus $\Lambda e_{\sigma^{m-1}(t)}\supset\cdots\supset\Lambda e_{\sigma(t)}\supset\Lambda e_t\cong P$ is the unique Λ-composition series of P. Ring indecomposability of Λ implies that σ is transitive. Our claim follows from 1.0.3.
(iii) For a left Λ-lattice $I\cong\oplus P_i^{a_i}$, $\mu_\Lambda(I)$ is the least integer μ such that I is a direct factor of Λ^μ, hence $\mu_\Lambda(I)=\sup_i\{\nu_i^{-1}a_i\}$. I is a Λ-ideal if and only if $\sum a_i\leq\sum\nu_i$, hence $\sup_I\mu_\Lambda(I)\leq\sup_i\{\nu_i^{-1}\sum\nu_i\}$. The value $\{\nu_i^{-1}\sum\nu_i\}$ can be actuallly attained, for example, by $I\cong P_i^{\Sigma\nu_i}$.

2.6. A pair (Λ,P) will be called a *superbijective pair* if Λ is a nonhereditary ring indecomposable order and P is a superbijective left Λ-lattice.

Suppose (Λ,P) is a superbijective pair. By 2.5.2, P' is not Λ-bijective, hence $P'\ncong P$ and Λ': $=\Lambda-(P)$ exists. Put $\mathcal{N}'$: $=\mathrm{rad}\Lambda'$.

2.6.0. (i) We have the following relations:

(19) $\mathcal{N}P'=\mathcal{N}P$ (19*) $('P)^*\mathcal{N}=P^*\mathcal{N}$

(20) $\mathcal{N}\Lambda'=\mathcal{N}$ (20*) $\Lambda'\mathcal{N}=\mathcal{N}$

(21) $\mathcal{N}=\mathcal{N}'\cap\Lambda$

(ii) Let Q be a bijective indecomposable Λ-lattice non-isomorphic to P. Then Q is Λ-superbijective if and only if it is Λ'-superbijective. If that is so, $(\mathcal{N}')Q=\mathcal{N}Q$.

Proof. (i) By 1.2, $\mathcal{N}P'=P$ or $\mathcal{N}P$. Since Λ is not hereditary, 2.5.2 implies (19), which in turn implies (20). The similar argument for right Λ-lattices implies (19*) and (20*). By (20) and (20*), $\mathcal{N}$ is a topologically nilpotent twosided ideal of Λ', hence (21).
(ii) If $Q'\cong P\cong{}'Q$, Q is Λ-superbijective, and by 2.5.2, Λ is hereditary , a contradiction. Hence we always have either (8) $'Q\ncong P$ or (8*) $Q'\ncong P$ of 2.4.1. Since $Q'\cong{}'Q$, both of (8) and (8*) should hold. By 2.4.2 (i), Q' is the minimum Λ'-overmodule of Q and $'Q$ is the maximum Λ'-submdule of Q. By 2.4.1, we

also have (10) $\mathcal{N}Q=(\mathcal{N}')Q$.

2.6.1. Suppose P' is Λ-decomposable. By 2.3.2, $P'=B_1\oplus B_2$, $'P=\mathcal{N}B_1\oplus\mathcal{N}B_2$ and $\mathcal{N}B_j=\mathcal{N}'B_j(j=1,2)$. By 2.5.1, B_j and $\mathcal{N}B_j$ are bijective indecomposable Λ'-lattices.

Our assumption $P'\cong'P$ is equivalent to one and only one of the following three sets of relations as Λ-lattices:

(I) $B_1\cong B_2\cong\mathcal{N}B_1\cong\mathcal{N}B_2$,

(II) $B_1\not\cong B_2$, $B_1\cong\mathcal{N}B_2$, $B_2\cong\mathcal{N}B_1$,

(III) $B_1\not\cong B_2$, $B_1\cong\mathcal{N}B_1$, $B_2\cong\mathcal{N}B_2$.

Accordingly, one can make up a period of the unique Λ'-composition series of B_j:

$B_1\supset\mathcal{N}'B_1\cong B_2$ case (I),

$B_1\supset\mathcal{N}'B_1(\cong B_2)\supset(\mathcal{N}')^2B_1\cong B_1$ case (II),

$B_j\supset\mathcal{N}'B_j\cong B_j(j=1,2)$ case (III).

By 1.6.1, $\Lambda'(B_j)$ is a hereditary ring direct factor of Λ'.

Recalling our convention 2.2.4, we have $\Lambda(P)'=\Lambda'(B_1)\oplus\Lambda'(B_2)=\Lambda'(\tilde{B}_1)$(resp. $\Lambda'(\tilde{B}_1)\oplus\Lambda'(\tilde{B}_2)$) in case (I) or (II)(resp. in case (III)). Hence $\Lambda(P)'$ is a hereditary ring direct factor of Λ', consequently $\Lambda(P)$ is a ring direct factor of Λ. The ring indecomposability of Λ implies $\Lambda=\Lambda(P)$ and $\Lambda'=\Lambda(P)'$. Thus we have seen:

(i) A is semisimple, Λ' is hereditary, $\mathcal{N}'$ coincides with $\mathcal{N}$.

(ii) $$s(A)=\begin{cases}1\\1\\2\end{cases}\qquad s(\Lambda')=\begin{cases}1\\2\\2\end{cases}\qquad s(\Lambda)=\begin{cases}1 & \text{case (I)}\\1 & \text{case (II)}\\1 & \text{case (III)}\end{cases}$$

2.6.2. Suppose P' is Λ-indecomposable. By 2.2.4 and 2.5.1, we have:

(22) Λ' is ring indecomposable and P' is a bijective indecomposable Λ'-lattice.

There are associated an Λ-composition series $P'\supset P\supset'P=\mathcal{N}P\cong P'$, and an Λ'-series $P'\supset\mathcal{N}'P'\supseteq'P$. By (19), $P'/'P$ is an $\Lambda/\mathcal{N}$-module and we have the following Λ-isomorphisms:

(23) $$P'/'P\cong P'/P\oplus P/'P\cong P'/\mathcal{N}'P'\oplus\mathcal{N}'P'/'P.$$

We devide the case into the following two disjoint subcases, where, to be compatible with the notation of our classification paper [9], we call the first case an (IVa):

(IVa) $P\supseteq\mathcal{N}'P'$,

(b) $P \not\supseteq \mathcal{N}'P'$.

Since P is not a Λ'-lattice, (IVa) is equivalent with each of the following two:

(a1) $P \supset \mathcal{N}'P'$, (a2) $\mathcal{N}'P' = {}'P$.

(i) The condition (b) is equivalent with each of the following four conditions, where isomorphisms are that of Λ-modules:

(b1) $P' = P + \mathcal{N}'P'$ (b2) $P'/\mathcal{N}'P' \cong P/\mathcal{N}P$

(b3) $P'/P \cong \mathcal{N}'P'/\mathcal{N}P$ (b4) $\mathcal{N}'P'$ is the minimum Λ'-overmodule of $'P$.

(ii) Let P'' be the minimum Λ'-overmodule of P' (which exists by (22)), and let $\Lambda'' := \Lambda' - (P'')$. Suppose the case (b) occurs. Then:

(B) P' is a superbijective Λ'-lattice.

(B′) $P''/P' \cong P'/P$ as Λ-lattices.

(B″) $\mathcal{N}\Lambda'' = \mathcal{N}'$.

(iii) In case (IVa), Λ' is hereditary with $s(\Lambda') = s(\Lambda) = 1$, $\mathcal{N}'^2 = \mathcal{N}$.

Proof. (i) (b)$\Leftrightarrow P + \mathcal{N}'P' \supset P \Leftrightarrow$(b1). (b1)$\Rightarrow$(b2)$\overset{(23)}{\Longleftrightarrow}$(b3)$\Rightarrow \mathcal{N}'P' \supset {}'P \Rightarrow$ not (a2)=(b). (b3)$\overset{(23)}{\Longleftrightarrow} l_\Lambda(\mathcal{N}'P'/{}'P) = 1 \Leftrightarrow l_{\Lambda'}(\mathcal{N}'P'/{}'P) = 1 \overset{(22)}{\Longleftrightarrow}$(b4).
(ii) $P' \cong {}'P$ as Λ-modules$\Rightarrow P' \cong {}'P$ as Λ'-modules. By (b4), we have (B) and (B′). To see (B″): $\mathcal{N}'P' =_{(19)} \mathcal{N}'P'' \supset_{(21)} \mathcal{N}P'' \supset \mathcal{N}P' =_{(19)} \mathcal{N}P$. By (b3), $\mathcal{N}P'' = \mathcal{N}'P'$ or $\mathcal{N}P$.

If $\mathcal{N}P'' = \mathcal{N}P$, P''/P is an $\Lambda/\mathcal{N}$-module and $P''/P = P''/P' \oplus P'/P \cong P'/P \oplus P'/P$ as Λ-modules, hence P has no minimum Λ-overmodules, a contradiction. Hence $\mathcal{N}P'' = N'P'$ and $\mathcal{N}\Lambda'' = \mathcal{N}'$.
(iii) By (a2), $P' \supset \mathcal{N}'P' = {}'P \cong P'$ is the unique Λ'-composition series of P' and Λ' is hereditary with $s(\Lambda') = s(\Lambda) = 1$. Since $(\mathcal{N}')^2P' = {}'P = \mathcal{N}P$, we have $\mathcal{N}'^2 = \mathcal{N}$.

2.6.3. Theorem. *Let (Λ, P) be a superbijective pair.*
(i) *If Λ' is not hereditary, then (Λ', P') is a superbijective pair.*
(ii) *Λ is superGorenstein if and only if Λ' is so.*

Proof. (i) By 2.6.1 (i), and (ii) (B), (iii) of 2.6.2.
(ii) In case (I)-(IVa), Λ' is hereditary with $s(\Lambda') \leq 2$ and $s(\Lambda) = 1$, hence both Λ and Λ' are always superGorenstein. In case (b), by (B) and 2.6.0 (ii).

2.6.4. The preceding theorem is just adequate to study Bass orders in non-simisimple algebras. However, there are some readily available extra informations (due to [7]), which are necessary for the detailed study of Bass orders in semisimple algebras.

Let (Λ, P) be as in 2.6.2 and suppose the case (b) occurs. By 2.5.2 and 2.6.2, Λ' is hereditary if and only if $\mathcal{N}'P'$ is Λ'-bijective. We devide the case into the following two subcases, where isomorphisms are that of Λ'-(or equivalently Λ-) modules:

(IVb) $\mathcal{N}'P' \cong P'$,

(V) $\mathcal{N}'P' \not\cong P'$.

(i) In case (IVb), Λ' is hereditary with $s(\Lambda')=s(\Lambda)=1$ and $\mathcal{N}'^2=\mathcal{N}$.
(ii) In case (V), Λ' is hereditary with $s(\Lambda')=s(\Lambda)=2$; $\sigma(t)\neq t$; $\Lambda e_{\sigma(t)}$ is a superbijective Λ-lattice and $\mathcal{N}'P' \cong \Lambda e_{\sigma(t)}$.
(iii) Summing up, if (Λ, P) is a superbijective pair, then $\Lambda' :=\Lambda-(P)$ is hereditary if and only if one of the cases (I)-(V) occurs. If that is so, both of Λ and Λ' are superGorenstein.

Proof. (i) $P' \supset \mathcal{N}'P' \cong P'$ is the unique Λ'-composition series of P'. Since $(\mathcal{N}')^2P'='P=\mathcal{N}P$, $(\mathcal{N}')^2=\mathcal{N}$.
(ii) Put $Q:=\mathcal{N}'P'$. $P' \supset Q \supset 'P \cong P'$ is the unique Λ'-composition series of P', hence $s(\Lambda')=s(\Lambda)=2$. By 2.5.3, Λ' is superGorenstein, hence Q' is Λ'-superbijective. Since Q is an Λ'-lattice, $Q \not\cong P$. By 2.6.0 (ii), Q is Λ-superbijective and $\mathcal{N}Q=\mathcal{N}'Q$. Hence $Q/\mathcal{N}Q=Q/\mathcal{N}'Q=\mathcal{N}'P'/'P \cong P'/P$ by (b3) 2.6.2. By 2.2.0, $Q/\mathcal{N}Q \cong \bar{\Lambda}\bar{e}_{\sigma(t)}$ and $Q \cong \Lambda e_{\sigma(t)}$.
(iii) Since $s(\Lambda')\leq 2$, Λ' is superGorenstein. In case (I)-(IV), $s(\Lambda)=1$, and Λ is obviously superGorenstein. In case (V), Λ is superGorenstein by (ii).

2.7. Let (Λ, P) be a superbijective pair. Define the pairs $(\Lambda^{(m)}, P^{(m)})$ inductively as follows:

$(\Lambda^{(0)}, P^{(0)}) :=(\Lambda, P)$. For $m\geq 1$, if $(\Lambda^{(m-1)}, P^{(m-1)})$ is a superbijective pair, putting $\mathcal{N}^{(m-1)} :=\mathrm{rad}\,\Lambda^{(m-1)}$, define as

$$(\Lambda^{(m)}, P^{(m)}) :=(\Lambda^{(m-1)}-(P^{(m-1)}), (P^{(m-1)*}\mathcal{N}^{(m-1)})^*).$$

If $(\Lambda^{(m)}, P^{(m)})$ is not a superbijective pair, hence $\Lambda^{(m)}$ is hereditary by 2.6.3, put $m(\Lambda, P) :=m$. If such an m does not exists, put $m(\Lambda, P) :=\infty$. By definition, $1\leq m(\Lambda, P)\leq\infty$, and we have an increasing sequence of superbijective pairs $\{(\Lambda^{(m)}, P^{(m)}); 0\leq m<m(\Lambda, P)\}$.

2.7.0. Lemma. *Let m be a natural number such that $0\leq m\leq m(\Lambda, P)$ and let $P\cong\Lambda e_t$, $P^*\cong e_{\sigma(t)}\Lambda$, $\bar{\Lambda}\bar{e} :=\Lambda e_t/\mathcal{N}e_t$.*
(i) We have the following equalities or Λ-isomorphisms:

(24) $\mathcal{N}^{(m-1)}P^{(m)}=\mathcal{N}^{(m-1)}P^{(m-1)}$ $(m\geq 1)$

(25) $P^{(m)}/\mathcal{N}^{(m)}P^{(m)}\cong\bar{\Lambda}\bar{e}_t$ $(m\geq 0)$

(26) $P^{(m)}/P^{(m-1)}\cong\bar{\Lambda}\bar{e}_{\sigma(t)}$ $(m\geq 1)$

(27) $\mathcal{N}\Lambda^{(m)}=\mathcal{N}^{(m-1)}$ $(m\geq 1)$.

(ii) *$\Lambda e_t\oplus\Lambda e_{\sigma(t)}$ is the projective cover of $P^{(m)}$ $(m\geq 1)$ as Λ-modules.*

Proof. (i) (24): By (19) 2.6.0. (25): Induction by (b2) 2.6.2. (26): If $m=1$, by 2.2.0. If $m\geq 2$, induction using (B′). (27): If $m=1$, by (20). If $m\geq 2$, induction by (B″).
(ii) By applying (27), (24), (25)+(26), in this order;

$$P^{(m)}/\mathcal{N}P^{(m)}=P^{(m)}/\mathcal{N}^{(m-1)}P^{(m)}=P^{(m)}/\mathcal{N}^{(m-1)}P^{(m-1)}$$

$$\cong P^{(m)}/P^{(m-1)}\oplus P^{(m-1)}/\mathcal{N}^{(m-1)}P^{(m-1)}\cong\bar{\Lambda}\bar{e}_{\sigma(t)}\oplus\bar{\Lambda}\bar{e}_t.$$

2.7.1. (i) $m(\Lambda, P)<\infty$ if and only if A is semisimple.
(ii) If A is semisimple, Λ is superGorenstein.
Proof. If $n=m(\Lambda, P)$, $\Lambda^{(n)}$ is hereditary, hence A is semisimple by 1.6.1. If A is semisimple, the increasing sequence $\Lambda\subset\Lambda^{(1)}\subset\ldots$ must terminates. Then, by 2.6.3 (ii) and 2.6.4 (iii) Λ is superGorenstein.

2.7.2. Suppose $n:=m(\Lambda, P)<\infty$. According to which case of (I)-(V) occurs at the stage from $\Lambda^{(n-1)}$ to $\Lambda^{(n)}$, we have:

(i) Any indecomposable left Λ-lattice L is Λ-isomorphic to one and only one of $P^{(m)}(0\leq m<n)$ or
B_1(case (I)), B_1 or B_2 (case (II), (III)),$P^{(n)}$(case (IV)), Q(case (V)).

(ii) Let M be a left Λ-lattice and $m(L)$ denote the multiplicity of an indecomposable lattice L in M, $M\cong\oplus L^{m(L)}$. Then the Λ-projective cover $\wp(M)$ is given by

$$\wp(M)=P^{p(M)}\text{ case (I)-(IV)},\quad P^{p_1(M)}\oplus Q^{p_2(M)}\text{ case (V)}$$

with

$$p(M)=m(P)+2\sum_{0<m<n}m(P^{(m)})+\begin{cases}m(B_1) & \text{case (I)}\\ m(B_1)+m(B_2) & \text{case (II)(III)}\\ 2m(P^{(n)}) & \text{case (IV)}\end{cases}$$

$$p_1(M)=\sum_{0\leq m\leq n}m(P^{(m)})\qquad p_2(M)=\sum_{0<m\leq n}m(P^{(m)})+m(Q).$$

(iii) If M is a left Λ-ideal, then $p(M)$, $p_1(M)$, $p_2(M)\leq\nu_t+\nu_{\sigma(t)}$ and $\sup_I\mu_\Lambda(I)=\sup_i\{1+\nu_i^{-1}\nu_{\sigma(i)}\}$, where I runs over all left ideals and $\mu_\Lambda(I)$ denotes the minimal number of Λ-generators of I.

Proof. Using 3.0.1, 2.6 and 2.7.0 (ii), the proof is straightforward (cf. Proof of 3.7.3) except parhaps the case (V). In case (V), although there is some other intrinsic way, the fastest way is (as in [7]), assuming Λ to be basic identify Λ as the well-known Hecke type order $\begin{pmatrix}O & O\\ \wp^{n+1} & O\end{pmatrix}$ in $M_2(D)$, where D:

$=(\mathrm{End}_A \tilde{P})^{\circ}$, O is the maximal order with the radical $\wp$. To do this, since $\Lambda = P \oplus Q$, $\Lambda^{(n)} = P^{(n)} \oplus Q$, $\tilde{P} \cong \tilde{Q}$ and $\Lambda = O_l(P) \cap O_l(Q)$, it suffices to see that P and Q are right O-lattices.

Since $\Lambda^{(n)}$ is hereditary, Q is an O-lattice by 1.6.2. By (ii) 2.7.1, Q is also Λ-superbijective. Put $n' := m(\Lambda, Q)$. At the n'-th stage of $\Lambda-(Q)$, $\Lambda-(Q')$, ..., since $s(\Lambda)=2$, the case (V) is only possible. Hence, by the same reason as Q, P is an O-lattice.

2.7.3. Remark. (i) Assuming Λ to be basic, according to the case (I)-(V), the pair $(\Lambda^{(n)}, \mathcal{N}^{(n-1)})$ can be identified as the $(\Omega, \mathcal{N})$ in [9] 4.0.4. Since 'superGorenstein' is synonymous with 'Bass' for ring indecomposable non-hereditary orders (cf. 3.2.1) this was a basic structure theory of Bass orders in semisimple algebras obtained by [8] and [7], and was the starting point of our classification paper [9].

(ii) The method of [7](to look at only one projective indecomposable Λ-lattice P) which we have been pursueing was able to govern the whole theory of Bass orders in semisimple case, because of the validity of (ii) 2.7.1. It is not the case in general (cf. 3.8.3).

In §3, we shall consider all indecomposable projectives simultenously.

2.8. Suppose $m(\Lambda, P) = \infty$

(i) A full Λ-lattice L in $\tilde{P}$ is an $\Lambda^{(n)}$-lattice if and only if $L \not\cong P^{(m)}$ for any $m (0 \leq m < n)$, as Λ-lattices.

(ii) For a full R-lattice L in $\tilde{P}$, $L \in {}_{\Lambda}\mathcal{L}(\tilde{P})$ if and only if $L \cong P^{(m)}$ for some $m \geq 0$, as Λ-lattices.

(iii) $\tilde{P}$ is an indecomposable A-module.

Proof. (i) If L is Λ-decomposable, its direct factor cannot Λ-isomorphic to P, hence the claim follows from the definition of $\Lambda-(P)$ and 2.1 (5).

(ii) Suppose $L \not\cong P^{(m)}$ for any m. By (i) L is an $\Lambda^{(m)}$-lattice for any m. We may assume $L \supseteq P$. If $L \supseteq P^{(m)}$, then $L \neq P^{(m)}$, and L contains the minimum $\Lambda^{(m)}$-overmodule $P^{(m+1)}$ of $P^{(m)}$. This is absurd since $l_{\Lambda}(L/P) < \infty$.

(iii) If $\tilde{P} = V_1 \oplus V_2$, $L := (P \cap V_1) \oplus (P \cap V_2) \in \mathcal{L}_{\Lambda}(\tilde{P})$, hence by (ii), $L \cong P^{(m)}$ for some m. This contradicts to the indecomposability of $P^{(m)}$.

3. Gorenstein orders and Bass orders

In this section, unless otherwise stated, Λ always denote a Gorenstein order in a finite dimensional K-algebra A, and $\mathcal{N} := \mathrm{rad}\,\Lambda$. P always denote an indecomposable projective (hence bijective) left Λ-lattice, and $P' := (P^* \mathcal{N})^*$(resp. $'P := \mathcal{N}P$) denote the minimum Λ-overmodule (resp. the maximum Λ-submodule) of P.

As in 1.0.2, we write

$$\Lambda = \bigoplus_{e \in \mathcal{E}} \Lambda e \cong \bigoplus_{i=1}^{s} P_i^{\nu_i},\ P_i = \Lambda e_i \text{ as left } \Lambda\text{-lattices,}$$

$$\Lambda=\bigoplus_{e\in\mathscr{E}} e\Lambda\cong\bigoplus_{i=1}^{s}(P_i^*)^{\nu_{\sigma(i)}} \text{ as right } \Lambda\text{-lattices.}$$

by some permutation σ of $\{1, \cdots, s\}$.

3.0. Thus $P\cong P_t=\Lambda e_t$, $P^*\cong P_t^*\cong e_{\sigma(t)}\Lambda$ by some t $(1\leq t\leq s)$.

3.0.1. Γ is a minimal overorder of Λ if and only if $\Gamma=\Lambda-(P)$ by some indecomposable projective left Λ-lattice $P\cong\Lambda e_t$ such that $P'\ncong P$. If that is so, we have:

(0) Λ': $=\Lambda-(P)\cong P'^{\nu_t}\oplus\bigoplus_{i\neq t} P_i^{\nu_i}$ as left Λ-lattices,

(0*) $\Lambda'\cong('P)^{*\nu_{\sigma(t)}}\oplus\bigoplus_{i\neq t}(P_i^*)^{\nu_{\sigma(i)}}$ as right Λ-lattices,

Proof. If $\Gamma\subseteq O_l(P_i)$ for any P_i $(1\leq i\leq s)$, $\Gamma\subseteq\cap O_l(P_i)=\Lambda$. Hence there is some $P\cong P_t$, such that $\Gamma\nsubseteq O_l(P)$ i.e., $P\notin_{\Gamma}\mathscr{L}(\tilde{P})$. Then $P\ncong P'$ by 2.2.2, $\Gamma\supseteq\Lambda-(P)$, and the minimality of Γ implies $\Gamma=\Lambda-(P)$. The formula (0) (resp. (0*)) is identical with (7) (resp. (7*)) of 2.2 up to notation.

3.0.2. Suppose Λ is superGorenstein.
(i) $P_i=\Lambda e_i$ has the composition factor $\{\bar{\Lambda}\bar{e}_i, \bar{\Lambda}\bar{e}_{\sigma(i)}\}$.
(ii) Decompose $\{1, \cdots, s\}$ into σ-orbits, $\{1, \cdots, s\}=\cup_{j=1}^{t}X_j$, and put $\mathscr{E}_j$: $=\{e\in\mathscr{E}$; $\Lambda e\cong\Lambda e_j$ for some $i\in X_j\}$. Then the indecomposable ring direct factors (=blocks) of Λ are given by Λ_j: $=\bigoplus_{e\in\mathscr{E}_j}\Lambda e$ $(1\leq j\leq t)$. In particular Λ is ring indecomposable if and only if σ is a transitive cycle.
(iii) Any overorder Γ of Λ is superGorenstein. In particular, Λ is Bass.
(iv) If Λ is ring indecomposable, then Λ is self-dual if and only if Λ is of equimultiplicity.

Proof. (i) Since $P_i'\supset P_i\supset 'P_i=\mathscr{N}P_i\cong P_i'$ is a Λ-composition series of P_i, the claim follows from 3.0.1. (ii) By 1.4.1.
(iii) By 3.0.1, it suffices to see that $\Lambda-(P)$ is superGorenstein if $P'\ncong P$. By 2.2.3, we may assume Λ is ring indecomposable. If Λ is hereditary, then $s(\Lambda)\leq 2$ by 2.5.3, hence $s(\Lambda')\leq 2$, and Λ' is superGorenstein again by 2.5.3. If Λ is non-hereditary, then (Λ, P) is a superbijective pair and the claim is in (ii) 2.6.3. (iv) By (ii) and 1.0.3.

3.1. Proposition. *The following three conditions for* (Λ, P) *are equivalent.*

(1) $\mathscr{N}P'=P$.

(2) P' *is projective as a left* Λ*-lattice.*

(3) *In the notation of* 1.0.1, $A=A(\tilde{P})\oplus A_{\tilde{P}}$, $\Lambda=\Lambda(\tilde{P})\oplus\Lambda_{\tilde{P}}$ *as rings,* $A(\tilde{P})$ *is a simple K-algebra,* $\Lambda(\tilde{P})$ *is a hereditary order of* $A(\tilde{P})$.

Proof. (1)⇒(2): Suppose $\mathscr{N}P'=P$. Then $P'/\mathscr{N}P'=P'/P$ is Λ-simple and there is a projective cover φ: $P_i\rightarrow P'$ by some i $(1\leq i\leq s)$, consequently a surjective Λ-homomorphism $\mathcal{N}P_i \to \mathcal{N}P' = P$, hence $\mathcal{N}P_i \cong P \oplus X$ by some Λ-lattice X. Since Λ is Gorenstein by our assumption, P_i is injective and its maximum submodule $'P_i = \mathcal{N}P_i$ cannot have an injective direct factor P unless $X=0$ by 1.3. Thus $\mathcal{N}P_i \cong P$, $l(\tilde{P}_i) = l(\mathcal{N}\tilde{P}_i) = l(\tilde{P}) = l(\tilde{P}')$, and φ is an isomorphism.

(2)⇒(1): If P' is projective, since P'/P is Λ-simple, $\mathcal{N}P' = P$.

(2)⇒(3): Suppose P' is projective, hence bijective indecomposable. Since P'/P is Λ-simple, $'(P')$ coincides with P, which is injective. Hence, by 2.3.1, $(P')'$ is bijective indecomposable. Repeating the process we get an increasing sequence of bijective Λ-lattices $\{P^{(\nu)}; 0 \leq \nu \leq m\}$, $P^{(\nu+1)} = (P^{(\nu)})'$, $P^{(0)} = P \cong P^{(m)}$. By the equivalence of (1) and (2), $P^{(\nu)} = \mathcal{N}P^{(\nu+1)}$. Hence P has the unique composition series (cf. 1.5), we have (3) by 1.6.1.

(3)⇒(1): (3) implies that P has the unique composition series, and $\mathcal{N}P' = P$.

3.2. Suppose P' is not Λ-projective. Then $\Lambda - (P)$ is Gorenstein if and only if $P' \cong {'P}$ as Λ-lattices.

Proof. Recalling the fact that P' (resp. $'P$) has no Λ-projective (resp. Λ-injective) direct factor by 1.2.2 (resp. 1.3):

Λ' is left Gorenstein $\overset{0.2}{\Longleftrightarrow} (P')^*$ is right Λ'-projective

$\overset{(0^*)}{\Longleftrightarrow} P' | ('P)^{\nu\sigma(t)}$, similarly Λ' is right Gorenstein $\Leftrightarrow {'P} | (P')^{\nu t}$. Thus, if P' (hence $'P$) is Λ-indecomposable, the claim is obvious. If P' is decomposable, then $P' = B_1 \oplus B_2$, $'P = \mathcal{N}B_1 \oplus \mathcal{N}B_2$ by 2.3.2, hence the claim is still obvious.

3.2.1. Lemma. *Let Λ be a ring indecomposable non-hereditary Gorenstein order. The following four conditions for Λ are equivalent:*
(MG) *Any minimal overorder of Λ is Gorenstein.*
(ΛP) *If $P' \not\cong P$, $\Lambda - (P)$ is Gorenstein.*
(SG) *Λ is superGorenstein.*
(B) *Λ is Bass.*

Proof. (MG)⇔(ΛP): By 3.0.1. (ΛP)⇔(SG): By 3.1 and 3.2. (SG)⇒(B): By (iii) 3.0.2. (B)⇒(MG): By definitions.

3.2.2. Lemma. *Let Λ be a ring indecomposable Bass order in a non-semisimple K-algebra A. Then A is ring indecomposable.*

Proof. Since A is not semisimple, Λ is not hereditary by 1.6.1. By 3.2.1, (Λ, P) is a superbijective pair for any P. $\Lambda' = \Lambda - (P)$ cannot be hereditary, hence by 2.6.3, Λ' is again ring indecomposable superGorenstein. Hence, by 3.0.1, any overorder of Λ is ring indecomposable, which implies that A is ring indecomposable.

3.3. Let A be a finite dimensional K-algebra and Λ be a Gorenstein order.

3.3.1. Theorem. *The following conditions are equivalent.*
(MG) *Any minimal overorder of Λ is Gorenstein.*
(B) *Λ is Bass.*

Proof. To prove the theorem, we may assume Λ to be ring indecomposable. If Λ is hereditary, then both of (MG) and (B) are always valid. If Λ is not hereditary, the claim is in 3.2.1.

3.3.2. Theorem. *Let $A=A_{ss}\oplus A_{ns}$ be the (unique) direct sum decomposition as rings such that A_{ss} is semisimple and A_{ns} has no simple ring direct factors. If Λ is a Bass order of A, then $\Lambda=\Lambda\cap A_{ss}\oplus\Lambda\cap A_{ns}$.*

Proof. Let $\{\Lambda_l;\ l\in I\}$ be the indecomposable ring direct factors of Λ. Put I_{ss} (resp. I_{ns}) be the subset of I consisting of l such that $K\Lambda_l$ is semisimple (resp. non-semisimple). Put A_0 (resp. A_1) be the sum $\oplus K\Lambda_l$ over I_{ss} (resp. I_{ns}). Then A_0 is semisimple, and, by 3.2.2,A_1 has no simple ring factors, hence $A_0=A_{ss}$ and $A_1=A_{ns}$.

3.3.3. Remark. Thus we can separate the study of Bass orders in A into that in A_{ss} and that in A_{ns}.

As for the former semisimple case, we have already fairly complete results 2.7.2 as well as the ones in [7], [8] and [9].

As for the latter non-semisimple case, 'Bass' is synonymous with 'superGorenstein', and from the next section and on, we shall concentrate to study such orders.

3.4. In the rest of this section, unless otherwise stated, let A always denote a finite dimensional K-algebra without simple ring direct factors, and $\mathfrak{R}$ its radical. Thus $\mathfrak{R}^{r-1}\neq 0$, $\mathfrak{R}^r=0$ by some $r\geq 2$. Let $\varphi: A\to A/\mathfrak{R}$ denote the canonical projection. Let Λ denote an order in A, and $\Lambda=\oplus_{e\in\mathscr{E}}\Lambda e\cong\oplus_{i=1}^{s} P_i^{\nu_i}$, $\Lambda e_i=P_i$ as in 1.0.1.

Suppose Λ is Bass.

3.4.0. Since each ring direct factor Λ_l of Λ is superGorenstein, in view of 2.2.3, according to 2.7, one can define $P_i^{(m)}$ for any $i(1\leq i\leq s)$ and any $m\geq 0$. Since $P_i'\not\cong P_j$ if $i\neq j$, by 2.4.3,we have

(1) $(\Lambda-(P_i))-(P_j)=(\Lambda-(P_j))-(P_i)$ if $i\neq j$,

and inductively

(2) $\qquad P_i^{(m)}\cong P_j^{(n)}$ as Λ-lattices if and only if $(i, m)=(j, n)$.

Now one can define the overorder

(3) $\qquad \Lambda(n_1, \cdots, n_s):=\Lambda-(P_i^{(m_i)};\ 1\leq i\leq s,\ 0\leq m_i<n_i)$

obtained from Λ by rejecting $P_i^{(m_i)}$'s, where the ordering of rejections has no effect by (1), as far as it has a sense, namely one must reject $P_i^{(m+1)}$ only after $P_i^{(m)}$ has been already rejected. By definition, it has the properties

(4) An indecomposable left Λ-lattice L is an $\Lambda(n_1, \cdots, n_s)$-lattice if and only if $L \not\cong P_i^{(m_i)} (1 \leq i \leq s,\ 0 \leq m_i < n_i)$ as Λ-lattices.

(5) $\Lambda(n_1, \cdots, n_s) \cong \bigoplus_{i=1}^{s} (P_i^{(n_i)})^{\nu_i}$ as left Λ-(as well as $\Lambda(n_1, \cdots, n_s)$-) lattices.

(6) Any overorder of Λ coincides with some $\Lambda(n_1, \cdots, n_s)$.

Put $\mathcal{M}_l := \pi^{-l}(\Lambda \cap \mathcal{R})$, where π is a prime of our base ring R. Then $\mathcal{R} = \bigcup_{l \geq 0} \mathcal{M}_l$, $M_l := \sum_{j=1}^{r-1} \mathcal{M}_l^j$ is a full two sided Λ-lattice of $\mathcal{R}$, $M_l M_l \subseteq M_l$ and $\bigcup_{l \geq 0} M_l = \mathcal{R}$. Hence

(7) $\Lambda_l := \Lambda + M_l = \bigoplus_{e \in \varepsilon} (\Lambda e + M_l e)$ is an overorder of Λ such that $\bigcup_{l \geq 0} \Lambda_l = \Lambda + \mathcal{R}$.

(8) $\Lambda + \mathcal{R}$ is a union of (infinitely many) $\Lambda(m_1, \cdots, m_s)$'s.

3.4.1. Suppose V is a finitely generated left A-module, nonisomorphic to any $Ae_i (1 \leq i \leq s)$. If L is a full indecomposable A-lattice in V (i.e. $L \in {}_{\Lambda}\mathcal{L}(V)^{ind}$) then $\mathcal{R}L = \mathcal{R}V = 0$. In other words, if ${}_{\Lambda}\mathcal{L}(V)^{ind} \neq \phi$, ${}_{\Lambda}\mathcal{L}(V)^{ind} = {}_{\varphi(\Lambda)}\mathcal{L}(V)^{ind}$.

Proof. Since $V \not\cong Ae_i$, L is not Λ-isomorphic to $P_i^{(n)}$. By (4), L is an $\Lambda(n_1, \cdots, n_s)$-lattice for any $(n_1, \cdots, n_s)$. By (8), $\mathcal{R}L \subseteq L$. If $0 \neq \mathcal{R}L \ni xy \neq 0$, $x \in \mathcal{R}$, $y \in L$, then $Kxy \subseteq L$. This is absurd since L is an R-lattice.

3.4.2. $\mathcal{R}^2 = 0$.

Proof. Since $\mathcal{R} = \bigoplus \mathcal{R}e$, it suffices to show $\mathcal{R}^2 e = 0$ for any $e \in \mathcal{E}$. Let L be an Λ-direct summand of an Λ-lattice in $\Lambda \cap \mathcal{R}e$. Since $Ae \supset \mathcal{R}e$, $Ae \supset V := \tilde{L}$. If $V \cong Ae_i$, V is A-bijective. This is impossible since Ae is indecomposable by 2.8 (iii). Hence $V \not\cong Ae_i$, and $\mathcal{R}V = 0$ by 3.4.1.

3.5. Definition. We will call a K-algebra A to be QF-RSZ (Quasi-Frobenius with the radical square zero) if A is quasi-Frobenius and $\mathcal{R}^2 = 0$.

Under the assumption that A has no simple ring direct factor we have just seen: suppose A contains a Bass order Λ, then

(i) A is QF-RSZ (by 0.2.3 (i) and 3.4.2),

(ii) If $\mathcal{E}$ is a c.s.o.p.i. (=complete system of orthogonal primitive idempotents) of Λ, then $\mathcal{E}$ is also a c.s.o.p.i. of A (by 2.8 (iii)).

To proceed further, in particular to give a sort of converse of the above statements, we shall insert a few (more or less well-known) elementary lemmas. The first two are valid without any assumption on A. While the latter two are valid for any finite dimensional K-algebra A without simple ring direct factors, over an arbitrary field K.

3.5.1. Let L be a full R-lattice in A, and Ω' be an order in $A/\mathfrak{R}$, containing $\varphi(L)$. Then there is an order Ω of A, containing L and $\varphi(\Omega)=\Omega'$.

Proof. Pick a finite number of u_i $(1\leq i\leq m)$ in A, so that $S:=\sum_{i=1}^m Ru_i \supseteq L$, $\varphi(S)=\Omega'$. Then there are a_{ij}^l $(1\leq i, j, l\leq m)$ in R such that $x_{ij}:=u_iu_j-\sum_l a_{ij}^l u_l\in\mathfrak{R}$. Put $W:=\sum_{i,j} Rx_{ij}$ and observe: $SS\subseteq S+W$, $WS\subseteq S+W+SW$. Hence $\Omega:=S+SW+\cdots+(SW)^{r-1}$ (where $\mathfrak{R}^r=0$) has the required properties.

3.5.2. Let Λ be an order of A, and suppose that $\varphi(\Lambda)$ is a maximal order of $\bar{A}:=A/\mathfrak{R}$.
(i) If $\mathcal{E}$ is a c.s.o.p.i. of Λ, then $\mathcal{E}$ is also a c.s.o.p.i. of A.
(ii) Two projective indecomposable Λ-lattices L_i $(i=1, 2)$ are Λ-isomorphic if and only if $\tilde{L}_i$ are A-isomorphic.

Proof. (i) $\varphi(\mathcal{E})$ is a c.s.o.p.i. of $\varphi(\Lambda)$. Since $\varphi(\Lambda)$ is maximal, $\varphi(\mathcal{E})$ is a c.s.o.p.i. of $\bar{A}$. Being a lift of $\varphi(\mathcal{E})$, $\mathcal{E}$ is a c.s.o.p.i. of A.
(ii) By (i), we may assume $\tilde{L}_i=Ae_i$ with $e\in\mathcal{E}$. Then

$$Ae_1=\tilde{L}_1\cong\tilde{L}_2=Ae_2\Leftrightarrow\bar{A}\varphi(e_1)\cong\bar{A}\varphi(e_2)\Leftrightarrow\varphi(\Lambda)\varphi(e_1)\cong\varphi(\Lambda)\varphi(e_2)\Leftrightarrow\Lambda e_1\cong\Lambda e_2$$

by the Lifting Idempotents Theorem.

3.5.3. Let $\mathcal{E}$ be a c.s.o.p.i. of A, $\{Ae_i;\ 1\leq i\leq s\}$ be a maximal subset of $\{Ae;\ e\in\mathcal{E}\}$ consisting of mutually nonisomorphic A-modules. The following four conditions for A are equivalent.

(l) $l(Ae)=l(eA)=2$ for any $e\in\mathcal{E}$.

($l1$) A is QF-RSZ.

($l2$) There is a permutation σ of $\{1, \cdots, s\}$ such that $\mathfrak{R}e_i\cong\bar{A}\bar{e}_{\sigma(i)}:=\varphi(A)\varphi(e_{\sigma(i)})$, $e_{\sigma(i)}\mathfrak{R}\cong\bar{e}_i\bar{A}$.

($l3$) There are permutations σ, τ such that $\mathfrak{R}e_i\cong\bar{A}\bar{e}_{\sigma(i)}$, $e_i\mathfrak{R}\cong\bar{e}_{\tau(i)}\bar{A}$.

If the above conditions are satisfied, A is ring indecomposable if and only if σ is transitive.

Proof. (l)$\Rightarrow$($l1$): The condition (l) obviously implies that $\mathfrak{R}^2=0$. We shall show that $(Ae_i)^*$ is right A-projective. Since Ae_i contains the minimum left A-submodule $\mathfrak{R}e_i$, $(Ae_i)^*$ contains the annihilator $(\mathfrak{R}e_i)^\perp$ of $\mathfrak{R}e_i$ as the maximum right A-submodule, hence $(\mathfrak{R}e_i)^\perp=(Ae_i)^*\mathcal{N}$ and $(Ae_i)^*/(Ae_i)^*\mathcal{N}$ is right A-simple. Consequently there is a projective cover $f: e_jA\to(Ae_i)^*$, by some j $(1\leq j\leq s)$. Since $l(e_jA)=2=l(Ae_i)=l((Ae_i)^*)$, f is an isomorphism.

($l1$)$\Rightarrow$($l2$): Assume ($l1$). $\mathfrak{R}e_i$ is the maximum left A-submodule of Ae_i and is completely reducible since $\mathfrak{R}^2=0$. Since A is quasi-Frobenius, $(Ae_i)^*\cong e_{\sigma(i)}A$ by some permutation σ. Ae_i contains $(e_{\sigma(i)}\mathfrak{R})^\perp$ as the minimum left A-submodule. Therefore we have either a) $\mathfrak{R}e_i=(e_{\sigma(i)}\mathfrak{R})^\perp\cong\bar{A}\bar{e}_{\sigma(i)}$, or b) $Ae_i=(e_{\sigma(i)}\mathfrak{R})^\perp\cong\bar{A}\bar{e}_{\sigma(i)}$, $\mathfrak{R}e_i=0$. In any case, Ae_i has the composition factors $\{\bar{A}\bar{e}_i, \bar{A}\bar{e}_{\sigma(i)}\}$. If the case b) occurs, $\bigoplus_{Ae\cong Ae_i}Ae$ is a simple ring direct factor of A by 1.4.1. Thus we should always have the case a), this (and the similar arguments for $e_{\sigma(i)}A$) proves ($l2$).

$(l2)\Rightarrow(l3)\Rightarrow(l)$: Obvious.

Since $Ae_i\supset\mathfrak{R}e_i\cong\bar{A}\bar{e}_{\sigma(i)}\neq0$ is a composition series of Ae_i, the proof of the last statement is identical with that of 3.0.2 (ii).

3.5.4. Further assume that A is basic.
(i) The condition (l) is equivalent to each of the following (L) and (L3).
(L3) $\mathfrak{R}\cong\bar{A}$ as left A-modules and also as right A-modules.
(L) There are $\xi\in A$, and $\psi\in\mathrm{Aut}_{K\text{-alg}}\bar{A}$ such that

(L_1) $\mathfrak{R}=A\xi$

(L_2) $\xi a=a'\xi\Leftrightarrow\psi(\bar{a})=\bar{a}'$ for $a, a'\in A$.

(ii) If the codition (L) is satisfied, then we have

(L_2') $\psi(\bar{e}_i)=\bar{e}_{\sigma(i)}$, $\xi e_i=e_{\sigma(i)}\xi$

(L_3) $e_j\mathfrak{R}e_i=\begin{cases}0 & \text{if } j\neq\sigma(i)\\ e_j\mathfrak{R}=\mathfrak{R}e_i & \text{if } j=\sigma(i)\end{cases}$

In particular, if A is ring indecomposable and $s\geq2$,

(L_3') $e_i\mathfrak{R}e_i=0$ for any $i(1\leq i\leq s)$.

Proof. (i) $(l3)\Leftrightarrow$(L3): Obvious. (L3)$\Rightarrow$(L): Lift the A-isomorphism $\bar{A}\xrightarrow{\sim}\mathfrak{R}$ to a projective cover $\theta: A\to\mathfrak{R}$, and put $\xi:=\theta(1)$. (L)$\Rightarrow$(L3): $a\mapsto a\xi$(resp. ξa) induces $\bar{A}\xrightarrow{\sim}\mathfrak{R}$ as left (resp. right) A-modules.
(ii) $\psi(\bar{e}_i)=\bar{e}_{\bar{\psi}(i)}$ by some permutation $\bar{\psi}$. By (L_2), $\xi e_i=e_{\bar{\psi}(i)}\xi$ and $\mathfrak{R}e_i=A\xi e_i=Ae_{\bar{\psi}(i)}\xi$. Since $\mathfrak{R}e_i$ is A-simple, $\mathfrak{R}e_i\cong\bar{A}_{\bar{\psi}(i)}$. By $(l2)$, $\bar{\psi}=\sigma$ hence (L_2'). (L_2') together with $(l2)$ implies (L_3). If A is ring indecomposable, σ is transitive and $\sigma(i)\neq i$ if $s\geq2$. Hence (L_3) implies (L_3').

3.6. Returning to the setting of 3.4, let Λ be a Bass order of A.

3.6.1. Any overorder of Λ is contained in $\Lambda+\mathfrak{R}$.
Proof. By 3.5, A has the property (l), in particular $\mathfrak{R}e\neq0$ for any $e\in\mathcal{E}$. In the notation of 3.4.0, $\bigcup_{l\geq0}(\Lambda e+M_le)=\Lambda e+\mathfrak{R}e$ is not an R-lattice since $\mathfrak{R}e\neq0$. Hence there is some l such that $\Lambda e+M_le\supset\Lambda e=P$, and $\Lambda e+M_le$ contains the minimum Λ-overmodule P' of P, thus $\Lambda+\mathfrak{R}\supseteq\Lambda-(P)$. Inductively, $\Lambda+\mathfrak{R}$ contains any $\Lambda(n_1, \cdots, n_s)$. The claim follows from (6) 3.4.0.

3.6.2. $\varphi(\Lambda)$ is a maximal oredr of $A/\mathfrak{R}$.
Proof. Let Ω' be a maximal order of $A/\mathfrak{R}$ containing $\varphi(\Lambda)$. By 3.5.1, there is an overorder Ω of Λ such that $\varphi(\Omega)=\Omega'$. Since $\Omega\subseteq\Lambda+\mathfrak{R}$ by 3.6.1, $\varphi(\Lambda)=\varphi(\Omega)=\Omega'$.

3.7. Let A be a finite dimensional K-algebra having no simple ring direct factors, with the radical $\mathfrak{R}$ and the canonical projection $\varphi: A\to A/\mathfrak{R}$.

3.7.1. Theorem. (i) *A contains a Bass order if and only if A is QF-RSZ, i.e. A is quasi-Frobenius and $\mathcal{R}^2=0$.*
(ii) *Suppose A is QF-RSZ. An order Λ of A is Bass if and only if $\varphi(\Lambda)$ is a maximal order of $A/\mathcal{R}$.*

Proof. 'Only if part' of (i) is given in 3.5. 'Only if part' of (ii) is given in 3.6.2. We will show:
$(*)$ If A is QF-RSZ and $\varphi(\Lambda)$ maximal, then Λ is Gorenstein.

This proves the 'if part' of (ii)(and also of (i)). Indeed, if Γ is an overorder of Λ, then $\varphi(\Gamma)=\varphi(\Lambda)$ is maximal, and by $(*)$, Γ is Gorenstein.

Let P be a projective indecomposable left Λ-lattice. We shall show that P is Λ-injective. Let P' be a minimal Λ-overmodule of P. We have either a) $\mathcal{N}P'=P$ or b) $\mathcal{N}P'=\mathcal{N}P$.

a) Suppose $\mathcal{N}P'=P$. Then P' has the maximum Λ-submodule P. This implies, by 1.4.3(i) together with the property (l)3.5.3, that P' is a projective indecomposable Λ-lattice. Since $\tilde{P}=\tilde{P'}$, we have $P'\cong P$ by 3.5.2(ii). Thus $P\cong P'\supset\mathcal{N}P'=P$ gives a period of the unique Λ-composition series of P in the sense of 1.5. Hence P has the minimum Λ-overmodule P'. Thanks again to our assumption (l) on A, P is injective by 1.4.3(ii). Note however that this case in fact does not occur because of 1.6.2.

b) Suppose $\mathcal{N}P'=\mathcal{N}P$. Putting $P=\Lambda e$, $\mathcal{N}P'=\mathcal{N}P\Rightarrow\mathcal{N}P'\subseteq\mathcal{N}\Rightarrow P'\subseteq O_r(\mathcal{N})$ $\overset{3.6.1}{\Rightarrow} P'\subseteq\Lambda+\mathcal{R}\Rightarrow P'\subseteq Ae\cap(\Lambda+\mathcal{R})=\Lambda e+\mathcal{R}e\Rightarrow P'=P+P'\cap\mathcal{R}e$. Since $\varphi(\Lambda)$ is maximal, ${}_{\Lambda}\mathcal{L}(\mathcal{R}e)={}_{\varphi(\Lambda)}\mathcal{L}(\mathcal{R}e)$ is linearly ordered by inclusion. Hence P' is the minimum Λ-overmodule of P. By (l) together with 1.4.3(ii), P is Λ-injective.

3.7.2. Theorem. *Let $\Lambda\cong\oplus P_i^{\nu_i}$ be a Bass order, and $P_i\cong\Lambda e_i$, $P_i^*\cong e_{\sigma(i)}\Lambda(1\leq i\leq s)$.*

Any indecomposable left Λ-lattice L is Λ-isomorphic to one and only one of the following lattices:

$$P_i^{(m)}(1\leq i\leq s,\ 0\leq m),\ Q_i:=\Lambda\cap\mathcal{R}e_i(1\leq i\leq s).$$

Furthermore, its ambient K-module $\tilde{L}:=KL$ or its Λ-projective cover $\mathcal{P}(L)$ is given as follows:

$$KP_i^{(m)}=Ae_i,\ KQ_i=\mathcal{R}e_i(1\leq i\leq s,\ 0\leq m), \tag{9}$$

$$\mathcal{P}(P_i^{(m)})=\begin{cases}\Lambda e_i\oplus\Lambda e_{\sigma(i)} & (m>0)\\ \Lambda e_i & (m=0)\end{cases} \tag{10}$$

$$\mathcal{P}(Q_i)=\Lambda e_{\sigma(i)}(1\leq i\leq s) \tag{11}$$

Proof. If $\tilde{L}\cong Ae_i$ by some i, then $L\cong P_i^{(m)}$ by 2.8 (ii), and (10) by 2.7.0(ii).

Suppose $\tilde{L}\ncong Ae_i$ for any i. By 3.4.1, $L\in{}_{\varphi(\Lambda)}\mathcal{L}(\tilde{L})^{ind}$. Since $\varphi(\Lambda)$ is a maximal order of $\varphi(A)$, $\tilde{L}$ is $\varphi(A)$-simple, and L is unique up to $\varphi(\Lambda)$ (as well as Λ)-isomorphism.

We may suppose $\tilde{L}=\varphi(A)\varphi(e_j)=\varphi(Ae_j)$, $L=\varphi(\Lambda)\varphi(e_j)=\varphi(\Lambda e_j)$ by some j $(1\leq j\leq s)$. Putting $i:=\sigma^{-1}(j)$, $\tilde{L}\cong\mathfrak{R}e_i$ and $L\cong\Lambda\cap\mathfrak{R}e_i$ by (l2) 3.5.3.

Since Ker $\varphi\cap\Lambda e_j=\mathfrak{R}\cap\Lambda e_j\supset\Lambda\cap\mathfrak{R}\cap\Lambda e_j\supset\mathcal{N}\cap\Lambda e_j=\mathcal{N}e_j$, we have $L/\mathcal{N}L=\varphi(\Lambda e_j)/\mathcal{N}\varphi(\Lambda e_j)=\varphi(\Lambda e_j)/\varphi(\mathcal{N}e_j)\cong\Lambda e_j/\mathcal{N}e_j$, which proves (11).

3.7.3. Corollary. *Let Λ be as in 3.7.2, and M be a left Λ-lattice. Let $a_M(i, m)=a(i, m)$ (resp. $b_M(i)=b(i)$) denote the multiplicity of $P_i^{(m)}$ (resp. Q_i) in M. Let $\mu_\Lambda(M)=\mu(M)$ denote the minimal number of Λ-generators of M. Put :*

$$a_M(i)=a(i):=\sum_{m\geq 0}a(i, m),\ p_M(i)=p(i):=a(i)+\sum_{m>0}a(\sigma(i), m)+b(\sigma(i)).$$

(i) *We have*

$$\tilde{M}\cong\bigoplus_i(Ae_i)^{a(i)}\oplus\bigoplus_i(\mathfrak{R}e_i)^{b(i)} \tag{12}$$

$$M \text{ is a } \Lambda\text{-ideal}\Leftrightarrow a_M(i)+b_M(i)\leq\nu_i\ (1\leq i\leq s) \tag{12'}$$

$$\mathcal{P}(M)\cong\bigoplus_i P_i^{p(i)} \tag{13}$$

$$\mu(M)=\sup_i\{\nu_i^{-1}p_M(i)\}, \tag{13'}$$

where $\{x\}$ denotes the least integer such that $\geq x$.
(ii) *Let I runs over all left Λ-ideals, then*

$$\sup_I\mu(I)=\sup_i\{1+\nu_i^{-1}\nu_{\sigma(i)}\}. \tag{14}$$

Proof. (i) (12): by (9). (12′): For an A-module V, put $V_0:=\{v\in V;\ \mathfrak{R}v=0\}$. Since $l(Ae_i)=2$, $(Ae_i)_0=\mathfrak{R}e_i$ and $(\mathfrak{R}e_i)_0=\mathfrak{R}e_i$. If $A\supset\tilde{M}$, then $A_0\supset(\tilde{M})_0$, which implies (12′). (13): By (10)+(11). If there is a Λ-epimorphism f: $\Lambda^n\to M$, then it induces a Λ-epimorphism f': $\Lambda^n\to\mathcal{P}(M)$ such that $f=g\circ f'$, where g: $\mathcal{P}(M)\to M$ is a projective cover. Thus, $\mu(M)=\mu(\mathcal{P}(M))$ and $\mu(\mathcal{P}(M))$ is the least number μ such that $\mathcal{P}(M)$ is a direct factor of Λ^μ, i.e. $\mu\nu_i\geq p_M(i)$, hence (13′).
(ii) Since $p(i)\leq a(i)+b(i)+a(\sigma(i))+b(\sigma(i))\leq\nu_i+\nu_{\sigma(i)}$ we have $\mu(I)=\sup_i\{\nu_i^{-1}p(i)\}\leq\sup_i\{1+\nu_i^{-1}\nu_{\sigma(i)}\}$. The right hand side is actually attained, for example by $I=(P_i^{(1)})^{\nu_i}\oplus(P_{\sigma(i)}^{(1)})^{\nu_{\sigma(i)}}$.

3.7.4. Remark. Let A be QF-RSZ and Λ be a Bass order of A. By 3.7.1 together with 3.5.2, if A is the ring direct sum of A_ι, then Λ is the ring direct sum of $\Lambda\cap A_\iota$, and A is basic if and only if Λ is basic. Therefore, to know all Bass orders up to Morita equivalence we shall firstly classify all non-semisimple ring indecomposable basic QF-RSZ K-algebra A, then secondly, for each A, classify all Bass orders of A up to $A^\times$-conjugacy. This is our next task and will be carried out in the next section.

3.8. Let $\Lambda \cong \oplus_{i=1}^{s}(\Lambda e_i)^{\nu_i}$ be an R-order in a finite dimensional K-algebra A.

3.8.1. Theorem. *If Λ is a non-maximal superGorenstein order and $(\Lambda e_i)^* \cong e_{\sigma(i)}\Lambda$ $(1 \leq i \leq s)$, then*

$$\sup_I \mu_\Lambda(I) = \sup_i \{1 + \nu_i^{-1}\nu_{\sigma(i)}\}.$$

In particular, $\sup_I \mu_\Lambda(I) \geq 2$ *and* $\sup_I \mu_\Lambda(I) = 2$ *if and only if Λ is self-dual.*

Proof. If A has no simple ring direct factor, the formula is already established by (14) 3.7.3. In general, it suffices to prove the formula for ring indecomposable Λ. If Λ is hereditary, it is done in (18) 2.5.3. If Λ is non-hereditary with A semisimple, it is done in (iii) 2.7.2. The last claim is by (iv) 3.0.2.

3.8.2. Theorem. *If A is quasi-Frobenius, the following three properties for Λ are mutually equivalent :*
(15) *Any overorder $\Gamma(\supseteq \Lambda)$ of Λ is self-dual.*
(16) *Λ is a self-dual superGorenstein order.*
(17) $\sup_I \mu_\Lambda(I) \leq 2$.

Proof. Since Λ has each one of the above three properties if and only if so does each ring direct factor of Λ, we may assume Λ to be ring indecomposable in proving the implications of those properties.

(15)⇒(16): Assume (15). Then Λ is obviously Bass, and self-dual. If Λ is nonhereditary, then Λ is superGorenstein by 3.2.1. Assume Λ is hereditary. By 2.5.3 (ii), Λ is of equimultiplicity, $\Lambda \cong \oplus_{i=1}^{s} P_i^{\nu}$. If $s = s(\Lambda) \geq 3$, and $P_1' \cong P_2$, then $\Lambda - (P_1) \cong P_2^{2\nu} \oplus (\oplus_{i \neq 1,2} P_i^{\nu})$ is not self-dual. Hence $s(\Lambda) \leq 2$, and Λ is superGorenstein by (i) 2.5.3.

(16)⇒(17): Assume (16). If Λ is maximal, $\sup_I \mu(I) = 1$. If Λ is non-maximal $\sup_I \mu(I) = 2$ by 3.8.1.

(17)⇒(15): Let $A \cong \oplus (Ae_i)^{\nu_i}$ and $(Ae_i)^* \cong e_{\sigma(i)}A$. Ae_i has the minimum submodule $V_i \cong \bar{A}\bar{e}_{\sigma(i)}$, the annihilator of $e_{\sigma(i)}\mathcal{R}$. If A is not Forbenius, there is some i such that $\nu_i > \nu_{\sigma(i)}$. Then $V := (Ae_{\sigma(i)})^{\nu_{\sigma(i)}} + V_i^{\nu_i}$ is an A-ideal with the projective cover $\mathcal{P}(V) = Ae_{\sigma(i)}^{\nu_{\sigma(i)} + \nu_i}$, hence $\mu_A(V) > 2$ and (17) can not hold. If A is Frobenius, the proof of Roiter (cf. [5], 37.17) works (without the separability assumption on A presupposed in [5]). The Frobenius assumption on A is necessary to conclude that "$W_i = \Lambda y$" from "$W_i = y\Lambda$" at the bottom of p. 787 in [5].

3.8.3. Remark. Let $A \cong \oplus_{i=1}^{s}(Ae_i)^{\nu_i}$ be a ring indecomposable QF-RSZ algebra with nonzero radical and $s \geq 2$. Let $\mathcal{A}_j$ $(1 \leq j \leq s)$ be the annihilator of

$\bigoplus_{i\neq j} Ae_i+\mathfrak{R}e_j$. Then the algebra $A_j:=A/\mathscr{A}_j$ is not quasi-Frobenius. Let $\Lambda\cong\bigoplus(\Lambda e_i)^{\nu_i}$ be a Bass order of A and put $\Lambda_j:=\Lambda/\Lambda\cap\mathscr{A}_j$. It is not difficult to see that Λ_j is a ring indecomposable non-Gorenstein order of A_j and:
(i) If $i\neq j$, $(\Lambda_j, \Lambda e_i)$ is a superbijective pair.
(ii) If $\Lambda\cong\Lambda^*$, Λ_j has the property (17).

Consequently: 2.7.1 (ii) does not hold without the semisimplicity assumption on A; the implication (17)$\Rightarrow$(15) in 3.8.2 does not hold without the quasi-Frobenius assumption on A.

4. Classification

In this section let A always denote a finite dimensional basic K-algebra without simple ring direct factors. Being basic the residue algebra $A/\mathfrak{R}$ is a ring direct sum of division K-algebras.

Fix a direct sum $B=\bigoplus_{i=1}^{s} B_i$ of division algebras B_i, and let $\mathscr{A}(B)$ denote the set of isomorphism classes of QF-RSZ algebras with the residue algebra isomorphic to B. By abuse of notation, let $A\in\mathscr{A}(B)$ denote that A is such an algebra.

Let $\mathscr{A}_0(B)$ (resp. $\mathscr{A}(B)^{ind}$) denote the subset of $\mathscr{A}(B)$ consisting of the classes of cleft (resp. ring indecomposable) algebras. Our first aim is to explicitly describe $\mathscr{A}(B)^{ind}$ for any B. However, in general, we can only describe $\mathscr{A}_0(B)^{ind}:=\mathscr{A}_0(B)\cap\mathscr{A}(B)^{ind}$ (cf. 4.0.3). If $s\geq 2$, $\mathscr{A}(B)^{ind}$ coincides with $\mathscr{A}_0(B)^{ind}$, and we are through. If $s=1$, our result remains partial (cf. 4.2).

4.0. By 3.5.4, $A\in\mathscr{A}(B)$ means that there are $\varphi\in\mathrm{Hom}_{K\text{-alg}}(A, B)$, $\xi\in A$, $\psi\in\mathrm{Aut}_{K\text{-alg}}B$ such that

(0) $$0\longrightarrow\mathfrak{R}\longrightarrow A\xrightarrow{\varphi}B\longrightarrow 0 \text{ (exact)}$$

(1) $$A\xi=\mathfrak{R}$$

(2) $$\xi a=a'\xi\Longleftrightarrow\psi\circ\varphi(a)=\varphi(a') \text{ for } a, a'\in A.$$

4.0.1. Suppose $A_1\in\mathscr{A}(B)$, and let φ_1, ξ_1, ψ_1 be the corresponding data of A_1, and $f: A\to A_1$ be a K-algebra isomorphism.

Then there is an automorphism $\bar{f}$ of B, and $c_1\in A_1^{\times}$ such that

(3) $$\varphi_1\circ f=\bar{f}\circ\varphi,$$

(4) $$f(\xi)=c_1\xi_1.$$

Applying f to the first equality of (2), we have $c_1\xi_1 f(a)=f(\xi a)=f(a'\xi)=f(a')c_1\xi_1$. By the relation (2) for A', this equality amounts to

(5) $$\psi_1\circ\bar{f}=I(c)\circ\bar{f}\circ\psi \text{ where } c:=\varphi(c_1)^{-1}\in B^{\times} \text{ and } I(c):=(b\mapsto cbc^{-1})\in$$

Aut B.

Define an equivalence relation $\sim$ in Aut B, by

(6) $\qquad \psi_1 \sim \psi \Longleftrightarrow \psi_1 = I(c) \circ \rho \circ \psi \circ \rho^{-1}$ for some $c \in B^\times$, $\rho \in \mathrm{Aut}\, B$.

In other words $\psi_1 \sim \psi$ if and only if (the images) of ψ_1, ψ are conjugate in the outer automorphism group Out B: $=\mathrm{Aut}\, B/I(B^\times)$.

We have seen that A $\mapsto \psi$ defines a well-defined map

(7) $\qquad \Psi : \mathscr{A}(B) \to \mathrm{Aut}\, B/\sim$

4.0.2. For a given $\psi \in \mathrm{Aut}\, B$, define the K-subalgebra A_ψ of $M_2(B)$, by

(8) $$A_\psi : = \left\{ \begin{pmatrix} b & x \\ 0 & \psi(b) \end{pmatrix} ;\ b,\ x \in B \right\}$$

Then $A_\psi \in \mathscr{A}_0(B)$, for example by taking $\xi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\varphi = \left(\begin{pmatrix} b & x \\ 0 & \psi(b) \end{pmatrix} \mapsto b \right)$.

Conversely, if $A \in \mathscr{A}_0(B)$, i.e. if $\varphi : A \to B$ admits a cross section homomorphism $\eta : B \to A$, identifying Imη with B, $A = B + B\xi$, $\xi^2 = 0$, $\xi b = \psi(b)\xi$, and $b + x\xi \mapsto \begin{pmatrix} b & x \\ 0 & \psi(b) \end{pmatrix}$ is an isomorphism of A onto A_ψ. Thus we have seen that $A \mapsto \psi$ (or $\psi \mapsto A_\psi$) induces the bijection:

(7′) $\qquad \Psi : \mathscr{A}_0(B) \xrightarrow{\sim} \mathrm{Aut}\, B/\sim.$

4.0.3. By 3.5.3, A is ring indecomposable if and only if ψ is transitive on B_i's, hence $\mathscr{A}(B)^{ind}$ is not empty only if $B \cong D^s$ by some division algebra D. Suppose $B = D^s$. Then by obvious identification, Aut $B \cong \mathscr{S}_s \times (\mathrm{Aut}\, D)^s$ where we denote the symmetric group by $\mathscr{S}_s$, and it is not difficult to determine $\sim$class of transitive ψ's in Aut B. For $\alpha \in \mathrm{Aut}\, D$, $(x_1, \cdots, x_s) \in D^s$, define $\psi_\alpha \in$ Aut D^s, by

(9) $\qquad \psi_\alpha(x_1, \cdots, x_s) : = (\alpha(x_s), x_1, \cdots, x_{s-1}).$

If $\psi(\in \mathrm{Aut}\, D^s)$ permutes the direct factors transitively, then $\psi \sim \psi_\alpha$ for some $\alpha \in \mathrm{Aut}\, D$. While $\psi_\alpha \sim \psi_\beta$ if and omly if $\alpha^{-1}\beta \in I(D^\times)$. Thus the map Ψ of (7′), or its inverse map $\psi \mapsto A_\psi$, $\alpha \mapsto A_{\psi_\alpha}$ induces the bijection

(7″) $\qquad \Psi : \mathscr{A}_0(D^s)^{ind} \xrightarrow{\sim} \mathrm{Aut}\, D/I(D^\times).$

4.1. Suppose $s \geq 2$, and $A \in \mathscr{A}(D^s)^{ind}$. Let $\mathscr{E} = \{e_1, \cdots, e_s\}$ be a c.s.o.p.i. of A. Define the K-linear map $\zeta = \zeta_{\mathscr{E}} : A \to A$, by

$$\zeta(a) : = \sum_{i=1}^{s} e_i a e_i. \quad \text{Put } \tilde{\mathscr{E}} : = \mathrm{Im}\, \zeta.$$

4.1.1. There is a unique K-algebra homomorphism $\bar{\zeta}: B \to A$ such that $\zeta = \bar{\zeta} \circ \varphi$. $\bar{\zeta}$ is a cross section of $\varphi: A \to B$.

Proof. The relation (L_3') of 3.5.4 implies $\zeta(\mathcal{R})=0$, and $e_iae_ibe_i - e_iabe_i \in e_iAe_i \cap \mathcal{R} = e_i\mathcal{R}e_i = 0$, i.e. $\zeta(a)\zeta(b)=\zeta(ab)$. Hence $\zeta = \bar{\zeta}\circ\varphi$ by some K-algebra homomorphism $\bar{\zeta}: B \to A$. Since $\varphi\circ\zeta=\varphi$, we have $\varphi\circ\bar{\zeta}\circ\varphi = \varphi\circ\zeta = \varphi$, which means that $\bar{\zeta}$ is a cross section of φ.

4.1.2. (i) $\tilde{\mathcal{E}} = \mathrm{Im}\,\zeta = \mathrm{Im}\,\bar{\zeta}$ is a K-subalgebra of A isomorphic to B. We have

$$(10) \qquad A = \tilde{\mathcal{E}} \oplus \mathcal{R}, \ \tilde{\mathcal{E}} = \bigoplus_{i=1}^{s} e_iAe_i, \ \mathcal{R} = \bigoplus_{i=1}^{s} e_{\sigma(i)}Ae_i \text{ as } \tilde{\mathcal{E}}\text{-bimodules.}$$

(ii) If M is a subring of A containing $\mathcal{E}$, then $M = M\cap\tilde{\mathcal{E}} + M\cap\mathcal{R}$.

Proof. (i) Since $\bar{\zeta}$ is a cross section homomorphism of φ, $\tilde{\mathcal{E}} \cong B$ and $A = \tilde{\mathcal{E}}\oplus\mathcal{R}$. $Ae_i = (\tilde{\mathcal{E}}\oplus\mathcal{R})e_i = e_iAe_i + \mathcal{R}e_i$, $e_{\sigma(i)}Ae_i = e_{\sigma(i)}(\tilde{\mathcal{E}}\oplus\mathcal{R})e_i = e_{\sigma(i)}\mathcal{R}e_i = e_{\sigma(i)}\mathcal{R} = \mathcal{R}e_i$ by (L_3) of 3.5.4. (ii) is obvious from (i).

4.1.3. If $\mathcal{E}'$ (resp. η) is a c.s.o.p.i. of A (resp. a cross section homomorphism of φ), then there exists $a \in A^\times$ such that $\mathcal{E}' = a\mathcal{E}a^{-1}$ (resp. $\mathrm{Im}\,\eta = a\tilde{\mathcal{E}}a^{-1}$).

Proof. By Krull-Schmidt Theorem, we may assume $\mathcal{E}' = \{e_i'; 1\le i\le s\}$ and $Ae_i'a = Ae_i$ by some $a\in A^\times$. Hence we may further assume $Ae_i' = Ae_i$ and write $e_i' = x_i + y_i$ with $x_i \in e_iAe_i$, $y_i \in e_{\sigma(i)}Ae_i$. Now $e_i'^2 = e_i'$ implies $e_i' = e_i$.

$\mathrm{Im}\,\eta$ contains some c.s.o.p.i. $\mathcal{E}'$ of A, hence we may assume $\mathrm{Im}\,\eta \supseteq \mathcal{E}$. By 4.1.2 (ii), $\mathrm{Im}\,\eta = \mathrm{Im}\,\eta\cap\tilde{\mathcal{E}} + \mathrm{Im}\,\eta\cap\mathcal{R} = \mathrm{Im}\,\eta\cap\tilde{\mathcal{E}}$, hence $\mathrm{Im}\,\eta = \tilde{\mathcal{E}}$.

4.2. (i) Suppose either $s\ge 2$ or $s=1$ but D is separable over K. In the former case by 4.1.1, in the latter case by the Wedderburn's Theorem, we have $\mathcal{A}(D^s)^{ind} = \mathcal{A}_0(D^s)^{ind}$. Thus under the above assumption, we have completely classified $\mathcal{A}(D^s)^{ind}$, i.e. all the K-algebra which contain basic ring indecomposable Bass orders (by 3.7 and 4.0.3).
(ii) If D is central over K, $\mathcal{A}_0(D^s)^{ind} = \mathcal{A}(D^s)^{ind}$ is a singleton set (by 4.0.3). Further if $s=1$, $A := D\otimes_K(K[X]/(X^2))$ is the only member of $\mathcal{A}_0(D^s)^{ind}$.

4.3. Now we turn to the classification of indecomposable Bass orders. Suppose

$$A \in \mathcal{A}_0(B)^{ind}, \ B = \bigoplus_{i=1}^{s} B_i \text{ and } B_i \cong D.$$

Up to K-isomorphism, we may assume $A = A_\psi$ of (8), or may even assume $\psi = \psi_a$ of (9). However, to make the computations smoother, we fix one cross section η of $\varphi: A \to B$, and identify $b\in B$ with $\eta(b)\in A$. Thus (0), (1), (2) of 4.0 turn into

$$A = B\oplus\mathcal{R}, \ \mathcal{R} = B\xi, \ \xi b = \psi(b)\xi, \text{ and } \xi^2 = 0.$$

Let O_i (resp. $\wp_i=\pi_i O_i$) denote the maximal order of B_i (resp. the maximal ideal of O_i), and put

$$O:=\bigoplus_{i=1}^{s} O_i,\ \wp(n_1,\cdots,n_s):=\bigoplus_{i=1}^{s}\wp_i^{n_i}\text{ for } n_i\in\mathbf{Z}.$$

Then O is the unique maximal order of B, and any full left (or right) O-lattice of $\mathfrak{R}=B\xi=\xi B$ has the form $\wp(n_1,\cdots,n_s)\xi$. Consequently any order of A containg O has the following form

$$\Lambda(n_1,\cdots,n_s):=O+\wp(n_1,\cdots,n_s)\xi.$$

4.3.1. (i) $\Lambda(n_1,\cdots,n_s)$ is a Bass order of A for any $n_i\in\mathbf{Z}$.
(ii) $\Lambda(n_1,\cdots,n_s)$ is $A^\times$-conjugate to $\Lambda(n'_1,\cdots,n'_s)$ if and only if $n_1+\cdots+n_s=n'_1+\cdots+n'_s$.
(iii) If $s\geq 2$, any Bass order of A is $A^\times$-conjugate to one and only one of $\Lambda(n,0,\cdots,0)$ for some $n\in\mathbf{Z}$.

Proof. (i) By 3.7.1 (ii).
(ii) If Λ is $A^\times$-conjugate to Λ', then $\Lambda\cap\mathfrak{R}$ is $A^\times$-conjugate to $\Lambda'\cap\mathfrak{R}$. Our claim follows from the following observation: $A^\times=B^\times(1+B\xi)$; the inner action $I(x)$ of $x\in O^\times(1+B\xi)$ stabilizes $\wp(n_1,\cdots,n_s)\xi$; $I(\sum_{i=1}^{s}\pi^{m_i})$ transforms $\wp(n_1,\cdots,n_s)\xi$ (resp. $\Lambda(n_1,\cdots,n_s)$) into $\wp(n'_1,\cdots,n'_s)\xi$ (resp. $\Lambda(n'_1,\cdots,n'_s)$) with $n'_i=n_i+m_i-m_{\sigma^{-1}(i)}$.
(iii) Let $\mathscr{E}$ be a c.s.o.p.i. of A. By 4.1.3, up to $A^\times$-conjugacy, we may assume $B=\mathrm{Im}\,\eta=\tilde{\mathscr{E}}$. Let Λ be a Bass order of A. Since $\varphi(\Lambda)$ is maximal by 3.6.2, Λ contains a c.s.o.p.i. of A. Hence, again by 4.1.3, we may assume $\Lambda\supseteq\mathscr{E}$. Then, by 4.1.2(ii), $\Lambda=\Lambda\cap B+\Lambda\cap\mathfrak{R}$. The maximality of $\varphi(\Lambda)=\varphi(\Lambda\cap B)$ implies $\Lambda\cap B=O$. Thus $\Lambda\supseteq O$, and $\Lambda=\Lambda(n_1,\cdots,n_s)$ by some n_i. Now our claim is an obvious consequence of (ii).

4.4. Theorem. *Summing up the results* 4.0-4.3, *we have the following main results of this paper which will be described in the matrix representation to avoid any inaccuracy.*

4.4.0. Let A be a finite dimensional K-algebra with the nonzero radical $\mathfrak{R}$. Suppose A contains at least one ring indecomposable basic Bass order.
(i) The residue algebra $A/\mathfrak{R}$ is isomorphic to the direct sum D^s of s copies of some division K-algebra D(cf. 4.0.3).
(ii) If either $s\geq 2$ or $s=1$ but D is separable over K, then A is isomorphic to one and only one of

$$A(s,D,\alpha):=\{(x_{ij})\in M_2(D^s);\ x_{21}=0,\ x_{22}=\psi_\alpha(x_{11})\},$$

where α runs over the representatives of $\mathrm{Aut}_{K\text{-alg}}D/I(D^\times)$, and $\psi_\alpha(x_1,\cdots,x_s):=(\alpha(x_s),x_1,\cdots,x_{s-1})$ for $(x_1,\cdots,x_s)\in D^s$. (cf. 4.0.1-4.0.2).
(iii) Let O(resp. $\wp$) denote the maximal order of D(resp. the maximal ideal of

O). Put

$$\Lambda(s, D, \alpha; n) := \{(x_{ij}) \in A(s, D, \alpha);\ x_{11} \in O^s,\ x_{12} \in \wp^n \oplus \overbrace{O \oplus \cdots \oplus O}^{s-1}\}.$$

Then, $\Lambda(s, D, \alpha; n)$ is a ring indecomposable basic Bass order of $A = A(s, D, \alpha)$, it is $A^\times$-conjugate to $\Lambda(s, D, \alpha; n')$ if and only if $n' = n$.

If $s \geq 2$, any Bass order of $A = A(s, D, \alpha)$ is $A^\times$-conjugate to one and only one of $\Lambda(s, D, \alpha; n)$(cf. 4.3.1).

4.4.1. (Structure Theorem) *Any ring indecomposable Bass order in a non-semisimple K-algebra is either Morita equivalent to one of $\Lambda(s, D, \alpha; n)$, or else Morita equivalent to a primary Bass order.*

Proof. Obvious from 4.4.0.

4.4.2. Remark. Note that the above result is in good analogy with the semisimple case of [8]. However there is a difference to the semisimple case that $\Lambda(n) := \Lambda(s, D, \alpha; n)$ may isomorphic to $\Lambda(n')$ as R-algebras.

Let $e = e_{D/Z}$ be the ramification index of D over its center Z. It is easy to see that:

$$n \equiv n' \bmod e \Rightarrow \Lambda(n) \cong \Lambda(n') \text{ as } R\text{-algebras.}$$

If D is central over K, it is not difficult to see that the converse $\Leftarrow$ is also true.

4.4.3. (Strictly Bass orders) *A ring indecomposable R-order Λ of a non-semisimple algebra A is strictly Bass if and only if it is isomorphic to a total matrix algebra $M_m(\Lambda_0)$ over some primary Bass order Λ_0.*

Proof. If Λ is a Bass order in A, then A is a Bass algebra. If Λ has at least two non-isomorphic projective indecomposable modules, then so is A by 3.7 and 3.5.2, hence A is not a Bass algebra by [7] Theorem 7.7.

Assume $\Lambda = M_m(\Lambda_0)$, and put $A_0 := K\Lambda_0$, $\varphi_0: A_0 \to A_0/\mathrm{rad}A_0$. Then the only non-trivial quotient of $A := K\Lambda = M_m(A_0)$ is $\varphi: A \to A/\mathrm{rad}A \cong M_m(A_0/\mathrm{rad}A_0)$. Since Λ_0 is primary Bass, by 3.7, $\varphi(\Lambda) = M_m(\varphi_0(\Lambda_0))$ is maximal, hence is Bass.

4.4.4. Thus everything was settled for $s \geq 2$, and just like in the semisimple case, there remains to investigate primary Bass orders. Here our method of [9] for the semisimple case works without any substantial change. In the rest of this paper, we will state the results only with brief indications of proofs.

Call a sequence $\{\Lambda_i;\ i \in \mathrm{N}\}$ a (downward) infinite primary Bass chain, if each Λ_i is a primary Bass order, $\Lambda_i \supset \Lambda_{i+1}$, and Λ_i is the minimum overorder of Λ_{i+1}.

4.4.5. (Infinite primary Bass chain) *A non-semisimple K-algebra A contains an infinite primary Bass chain if and only if A is indecomposable as a left A-module, $l(A)=2$ and the residual map $\varphi: A \to A/\mathfrak{R} \cong D$ admits a cross section homomorphism.*

If that is so, and moreover if D is central over K, then any such chain is $A^{\times}$-conjugate to $\{\Lambda(1, D, \mathrm{id}_D; n); n \geq n_0\}$ by some n_0, in the notation of 4.4. 0(iii).

Proof. By §2 of [9], $\Lambda_\infty := \bigcap \Lambda_i$ is the maximal order of a division algebra $B := K\Lambda_\infty$, and $\dim_B A=2$, hence φ splits. The coverse is in 4.4.0.

If D is central, then any derivation of D is inner [11], chapter 5, Theorem 18. This implies that the splitting homomorphism η(hence its image B also) is unique up to $A^{\times}$-conjugacy, hence our last claim.

4.5. We shall classify primary Bass orders in a non-semisimple K-algebra A, as in the semisimple case, under the following two assumptions.

(0) Each direct factor of $A/\mathfrak{R}$ is central over K.

(c) $R/\pi R$ is perfect and of coh. dim. ≤ 1.

The first assumtion implies (4.3.2 (ii)) that $A \cong D \otimes_K K[X]/(X^2)$, by some central division K-algebra D. Identifying $D \otimes 1$ with D, and taking $1 \otimes (X \bmod (X^2))$ as ξ, we have

$$A=D+D\xi, \ \xi^2=0, \ x\xi=\xi x \text{ for } x \in D.$$

Let d, O, $\mathcal{P}$ denote the index, the maximal order, the maximal ideal of D. If Λ is a Bass order of A, it is necessarily primary, and $\Lambda \cap \mathfrak{R}$ has the form $\mathcal{P}^{\mu}\xi$. The integer μ is an invariant of $A^{\times}$-conjugacy class of Λ, and will be called the *rank* of Λ. Let $\mathcal{B}_\mu$ (resp. $\mathcal{M}_\mu$) denote the set of $A^{\times}$-conjugacy classes of primary (resp. minimal primary) Bass orders of A, of rank μ. We shall explicitly describe the set $\mathcal{B}_\mu$ and $\mathcal{M}_\mu$.

4.5.1. The second assumtion (c) implies (cf. [9] §4) that D contains a cyclic unramified subfield L of degree d, and a prime π_D such that

$$D=\bigoplus_{i=0}^{d-1} L\pi_D^i, \ \pi_D x \pi_D^{-1}=\sigma(x) \text{ for } x \in L, \ \mathrm{Gal}(L/K)=\langle\sigma\rangle.$$

Fix a generator λ of the maximal order O_L of L, $O_L=R[\lambda]$, and let $f(X) \in R[X]$ be the monic minimal polynomial of λ over K. One successively proves the following facts (1)-(7).

(1) For $x \in A$, $f(x)=0$ if and only if x is $A^{\times}$-conjugate to λ.

Let Λ be a primary Bass order of A. Since $\varphi(\Lambda)$ is maximal, up to $A^{\times}$-conjugacy, one may suppose $\lambda \in \Lambda$. Let Λ' be the minimum overorder of

Λ and $\mathcal{N}'$ be its radical.

(2) There exists $n \in \mathcal{N}'$ such that $n\Lambda' = \mathcal{N}$, $n\lambda n^{-1} = \sigma(\lambda)$.

Again up to $A^\times$-conjugacy (still supposing $\lambda \in \Lambda$) one may suppose

(3) n has the form $n = n_a := \pi_D(1 + a\xi)$ with $a \in L$.

For any $a \in L$ and $\mu \in \mathbf{Z}$, put

$$(*)\ \Lambda_\mu(a) := O_L[n_a] + \wp^\mu \xi, \text{ where } O_L[n_a] := \sum_{i=0}^{\infty} R n_a^i.$$

$\Lambda_\mu(a)$ is an R-subalgebra of A, but not necessarily an order. Let $T_{L/K}$: $L \to K$ be the trace map, $v_K(\ )$ be the normalized valuation of K, and put $\nu = \nu(a) := v_K(T_{L/K}(a))$. One observes:

(4) If $\mu \leq d\nu + 1$, then $\Lambda_\mu(a)$ is $A^\times$-conjugate to $\Lambda_\mu(0)$, which is obviously a primary Bass order of A.

Then one applies 6.0 of [9], taking the pair ($\Lambda_{d\nu}(0)$, rad $\Lambda_{d\nu+1}(0)$) in here as the pair (Ω, $\mathcal{N}$) in there, and gets

(5) $\Lambda_{d(\nu+1)}(a)$ is a minimal Bass order.

The above (4), (5), together with the surjectivity of $T_{L/K}: \wp_L^\nu \to \wp_K^\nu$ imply that the map $a \mapsto \Lambda_\mu(a)$ induces a surjection $\theta = \theta_{\nu,\mu}: \wp_L^\nu \to \mathcal{B}_\mu$ if $d_\nu + 1 < \mu \leq d(\nu+1)$. One easily sees that θ factors through the composite map of $T_{L/K}$: $\wp_L^\nu \to \wp_K^\nu$ with the quotient map $\rho: \wp_K^\nu \to \wp_K^\nu/\wp_K^{\nu+1}$, and results $\bar{\theta}$, $\theta = \bar{\theta} \circ \rho \circ T_{L/K}$. Finally, in a similar way as described in the last part of [9] 6.2.2, with the aid of the determination of the normalizer Nor $\Lambda_\mu(0)$ of $\Lambda_\mu(0)$ in $A^\times$,

$$(6)\qquad \text{Nor } \Lambda_\mu(0) = \begin{cases} D^\times(1 + (K + \wp^\mu)\xi) & \text{if } \mu \not\equiv 1 \bmod d \\ D^\times(1 + (K + \wp^{\mu-1})\xi & \text{if } \mu \equiv 1 \bmod d, \end{cases}$$

One proves

(7) $\bar{\theta} = \bar{\theta}_{\nu,\mu}: \wp_L^\nu/\wp_L^{\nu+1} \to \mathcal{B}_\mu$ is a bijection for $d_\nu + 1 < \mu \leq d(\nu+1)$.

4.5.2. Theorem (Classification of primary Bass Orders)
(*i*) *If $\mu \equiv 1 \bmod d$, there is one and only one primary Bass order $\Lambda_\mu(0) = O + \wp^\mu \xi$, up to $A^\times$-conjugacy. If $\mu \not\equiv 1 \bmod d$, the map $a \mapsto \Lambda_\mu(a)$, defined by $(*)$, induces a bijection from $\wp_K^\nu/\wp_K^{\nu+1}$ onto the set $\mathcal{B}_\mu$ of the $A^\times$-conjugacy classes of primary Bass orders of rank μ, where μ is determined by $d\nu + 1 < \mu \leq d(\nu+1)$.*
(*ii*) *If $\mu \not\equiv 0 \bmod d$, there is no minimal Bass orders of rank μ in A. If $\mu = d(\nu+1)$, the map $a \mapsto \Lambda_\mu(a)$ induces a bijection between $\wp_K^\nu/\wp_K^{\nu-1} - \{0\}$ and the set $\mathcal{M}_\mu$ of $A^\times$-conjugacy classes of minimal primary Bass orders of rank μ in A.*

4.5.3. Remark. (i) The above result for $A=D\otimes_K K[X]/(X^2)$ is entirely similar to the case of $A=D\oplus D$.
(ii) In the non-semisimple case, our classification is complete, at least under the assumption (0) and (c) of 4.5. There remains no problem like the ones remarked in [9] 0.3.3, for the semisimple case. In other words, non-semisimple case is substantially simpler than the semisimple case.

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