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On equivalence of product measures by random translation

By

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1. Introduction

Let $\mathbf{X}=\{X_k\}$ be an i.i.d. real random sequence and $\mathbf{Y}=\{Y_k\}$ an independent random sequence also independent of $\mathbf{X}$. Throughout this paper, the notations $\mathbf{X}$ and $\mathbf{Y}$ are used to denote these notions. $\mathbf{X}$ and $\mathbf{X}+\mathbf{Y}=\{X_k+Y_k\}$ induce probability measures $\mu_{\mathbf{X}}$ and $\mu_{\mathbf{X}+\mathbf{Y}}$ on $\boldsymbol{R}^{\mathrm{N}}$, respectively. For each k, denote the distributions of X_k, Y_k and X_k+Y_k by μ_{X_k}, μ_{Y_k} and $\mu_{X_k+Y_k}$, respectively. If $\mu_{X_k+Y_k}\sim\mu_{X_k}$ for every k, then the Kakutani dichotomy [1] implies that we have either $\mu_{\mathbf{X}+\mathbf{Y}}\sim\mu_{\mathbf{X}}$ or $\mu_{\mathbf{X}+\mathbf{Y}}\perp\mu_{\mathbf{X}}$. Our main problem is to describe the conditions on $\mathbf{Y}$ so that $\mu_{\mathbf{X}+\mathbf{Y}}\sim\mu_{\mathbf{X}}$ holds.

We say that $\mathbf{X}$ satisfies the condition (A) if μ_{X_1} is equivalent to the Lebesgue measure and the density function f satisfies

$$\int_{-\infty}^{+\infty}\frac{f'(x)^2}{f(x)}dx<\infty.$$

Similarly, the condition (C) for $\mathbf{X}$ is defined by replacing the condition of the density by

$$\int_{-\infty}^{+\infty}\frac{f''(x)^2}{f(x)}dx<\infty.$$

It is known that (C) implies (A) (Sato-Watari [8]).

When $\mathbf{Y}$ is a deterministic sequence or a symmetric random sequence, there are systematic studies which may be summarized as follows:

Theorem 1.1 (*Shepp* [9]). *Suppose* $\mathbf{y}\in\boldsymbol{R}^{\mathrm{N}}$. *Then*
(i) $\mu_{\mathbf{X}+\mathbf{y}}\sim\mu_{\mathbf{X}}$ *implies* $\mathbf{y}\in\ell_2$.
(ii) *If* $\mathbf{X}$ *satisfies the condition (A), then* $\mathbf{y}\in\ell_2$ *implies* $\mu_{\mathbf{X}+\mathbf{y}}\sim\mu_{\mathbf{X}}$.
(iii) *If* $\mu_{\mathbf{X}+\mathbf{y}}\sim\mu_{\mathbf{X}}$ *for all* $\mathbf{y}\in\ell_2$, *then* $\mathbf{X}$ *satisfies (A).*

Theorem 1.2 (*Okazaki-Sato* [4], *Sato-Watari* [8], *Okazaki* [3]).
(i) *Assume that* $\mathbf{Y}=\mathbf{a}\boldsymbol{\epsilon}=\{a_k\epsilon_k\}$, *where* $\mathbf{a}=\{a_k\}$ *is a real sequence and* $\boldsymbol{\epsilon}=\{\epsilon_k\}$

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is a Rademacher sequence, that is, an independent random sequence with distributions $\mathbf{P}[\epsilon_k=1]=\mathbf{P}[\epsilon_k=-1]=\frac{1}{2}$ *for each k. Then* $\mu_{\mathbf{X}+\mathbf{Y}}\sim\mu_{\mathbf{X}}$ *implies* $\mathbf{a}\in\ell_4$, *in other words,* $\mathbf{Y}\in\ell_4$ *a.s.*

(ii) *If* $\mathbf{X}$ *satisfies* (*C*) *and* $\mathbf{Y}$ *is symmetric, then* $\mathbf{Y}\in\ell_4$ *a.s. implies* $\mu_{\mathbf{X}+\mathbf{Y}}\sim\mu_{\mathbf{X}}$.

(iii) *If* $\mu_{\mathbf{X}+\mathbf{Y}}\sim\mu_{\mathbf{X}}$ *for all* $\mathbf{Y}=\mathbf{a}\boldsymbol{\epsilon}\in\ell_4$ *a.s., then* $\mathbf{X}$ *satisfies* (*C*).

In this paper, we treat the general case and generalize the theorems above. The following conditions for a random sequence $\mathbf{Y}$ and $\epsilon>0$ play important roles and are often referred to:

$(a)_\epsilon\ \sum_k \mathbf{E}[Y_k^4: |Y_k|\le\epsilon]<\infty,\quad (b)_\epsilon\ \sum_k \mathbf{E}[Y_k^2: |Y_k|\le\epsilon]^2<\infty,$
$(c)_\epsilon\ \sum_k \mathbf{E}[Y_k: |Y_k|\le\epsilon]^2<\infty,$
$(d)_\epsilon\ \sum_k \mathbf{P}[|Y_k|>\epsilon]<\infty,\quad (e)_\epsilon\ \sum_k \mathbf{P}[|Y_k|>\epsilon]^2<\infty.$

We give some remarks. It is easy to see that $(a)_\epsilon$ implies $(b)_\epsilon$ and $(d)_\epsilon$ implies $(e)_\epsilon$. And by Kolmogorov's three series theorem, the following three statements are equivalent to each other:

- $\mathbf{Y}\in\ell_4$ *a.s.*;
- $(a)_\epsilon$ and $(d)_\epsilon$ hold for some $\epsilon>0$;
- $(a)_\epsilon$ and $(d)_\epsilon$ hold for every $\epsilon>0$.

In particular, when $\mathbf{Y}=\mathbf{a}\boldsymbol{\epsilon}$, $\mathbf{Y}\in\ell_4$ *a.s.* if and only if $(b)_\epsilon(e)_\epsilon$ hold for some (or every) $\epsilon>0$. Note also that when $\mathbf{Y}$ is deterministic, $\mathbf{Y}\in\ell_2$ if and only if $(c)_\epsilon(e)_\epsilon$ hold for some (or every) $\epsilon>0$.

Particularly when $\mathbf{X}$ is a standard Gaussian sequence, i.e. μ_{X_1} is Gaussian with mean zero and variance 1, $\mathbf{X}$ satisfies (C) and we have detailed results such as

Theorem 1.3 (*Kitada-Sato* [2, *Theorems* 7 *and* 9]).
(i) *If* $\mathbf{Y}$ *is symmetric and* $\mu_{\mathbf{X}+\mathbf{Y}}\sim\mu_{\mathbf{X}}$, *then* $(b)_\epsilon(e)_\epsilon$ *hold for every* $\epsilon>0$.
(ii) *If* $(a)_\epsilon(c)_\epsilon(d)_\epsilon$ *hold for some* $\epsilon>0$, *then* $\mu_{\mathbf{X}+\mathbf{Y}}\sim\mu_{\mathbf{X}}$.
(iii) *If* $\mathbf{Y}$ *is symmetric and* $(b)_\epsilon(d)_\epsilon$ *hold for some* $\epsilon>0$, *then* $\mu_{\mathbf{X}+\mathbf{Y}}\sim\mu_{\mathbf{X}}$.

How can the assumptions of $\mathbf{X}$ be weaken in Theorem 1.3? Concerning this, there are some related results as follows:

Theorem 1.4 ([2, *Theorem* 5]). *Suppose* μ_{X_1} *is equivalent to the Lebesgue measure and the density function f is in* C^2, $\mathbf{Y}$ *is symmetric, and* $\lim_{k\to\infty} Y_k=0$ *a.s. Then* $\mu_{\mathbf{X}+\mathbf{Y}}\sim\mu_{\mathbf{X}}$ *implies* $(b)_\epsilon$ *for some* $\epsilon>0$.

Theorem 1.5 (*Sato-Tamashiro* [6, *Theorem* 2.1(*A*)]). *Suppose* $\mathbf{X}$ *and* $\mathbf{Y}$ *take values in* $\boldsymbol{Z}_+^N$ *and* $\mathbf{P}[X_1=0]>0$. *Then* $\mu_{\mathbf{X}+\mathbf{Y}}\sim\mu_{\mathbf{X}}$ *implies* $(e)_\epsilon$ *for every* $\epsilon>0$.

Theorem 1.6 (*Sato-Tamashiro* [7, *Theorem* 1]). *Suppose* $\mathbf{X}$ *is standard Gaussian and* $\mathbf{Y}$ *takes values in* $\boldsymbol{R}_+^N$. *Then*

(i) $\mu_{X+Y} \sim \mu_X$ *implies* $(c)_\epsilon (e)_\epsilon$ *for every* $\epsilon > 0$.
(ii) $(c)_\epsilon (d)_\epsilon$ *for some* $\epsilon > 0$ *imply* $\mu_{X+Y} \sim \mu_X$.

Theorem 1.7 ([2, *Theorem* 4]). *Suppose* $\mathbf{X}$ *satisfies* (C), $\mathbf{Y}$ *is symmetric and moreover*

$$\int_{-\infty}^{+\infty} \sup_{|z|<\epsilon} \frac{f''(x+z)^2}{f(x)} dx < \infty$$

for some $\epsilon > 0$. (*When* $\mathbf{X}$ *is standard Gaussian, it holds for any* $\epsilon > 0$.) *Then* $(b)_\epsilon (d)_\epsilon$ *for this* ϵ *imply* $\mu_{X+Y} \sim \mu_X$.

We shall show in fact the statement of Theorem 1.3(i) is true under no extra assumptions of $\mathbf{X}$ and $\mathbf{Y}$, and moreover an additional conclusion holds. We shall also show (ii) and (iii) (with slight modification) under weaker conditions than ever proved. Our main theorem is the following

Theorem 1.8. (i) $\mu_{X+Y} \sim \mu_X$ *implies* $(b)_\epsilon$ $(c)_\epsilon$ $(e)_\epsilon$ *for every* $\epsilon > 0$.
(ii) *If* $\mathbf{X}$ *satisfies* (C), *then* $(a)_\epsilon$ $(c)_\epsilon$ $(d)_\epsilon$ *for some* $\epsilon > 0$ *imply* $\mu_{X+Y} \sim \mu_X$.
(iii) *Suppose* $\mathbf{X}$ *satisfies* (C) *and moreover*

$$\sup_{|z|<\epsilon} (\epsilon - |z|)^2 \int_{-\infty}^{+\infty} \frac{f''(x+z)^2}{f(x)} dx < \infty$$

for some $\epsilon > 0$. *Then* $(b)_\epsilon$ $(c)_\epsilon$ $(d)_\epsilon$ *for this* ϵ *imply* $\mu_{X+Y} \sim \mu_X$.

The condition (C) is necessary in (ii) (iii) because of Theorem 1.2 (iii). Theorem 1.8 involves Theorem 1.1 (i) (ii) (under the condition (C)) and Theorem 1.2 (i)(ii), as well as all results from Theorem 1.3 to Theorem 1.7.

This paper is organized as follows. In section 2 we prove Thorem 1.8. In section 3 we discuss the possibility of improvement of the theorem and give some negative examples.

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2. Proof of Theorem 1.8

First we state a theorem which will be needed in the proof.

Theorem 2.1 ([2, *Theorem* 2]). *Suppose* $\mu_{X_k+Y_k} \sim \mu_{X_k}$ *for every* k. *Define* $p_k(x) = \frac{d\mu_{X_k+Y_k}}{d\mu_{X_k}}(x)$, $A_k = \{x \in \boldsymbol{R}\ ;\ p_k(x) - 1 < 1\}$ *and* $\lambda = \mu_{X_1}$. *Then the following three statements are equivalent.*

- $\mu_{X+Y} \sim \mu_X$.
- $\sum_k (p_k(X_k) - 1)$ *converges a.s.*
- *the following two series are convergent* :

(2.1) $$\sum_k \int_{A_k^c} (p_k(x)-1)d\lambda(x)<\infty,$$

(2.2) $$\sum_k \int_{A_k} (p_k(x)-1)^2 d\lambda(x)<\infty.$$

Proof of Theorem 1.8. We use the notation of Theorem 2.1.

(i) Since $\mu_{X+Y}\sim\mu_X$, we have $\mu_{X_k+Y_k}\sim\mu_{X_k}$ for every k. Clearly it is enough to show $(b)_R$ $(c)_R$ $(e)_R$ for some $R>0$. Define $g(x)=\lambda([x-1, x+1])$. We see $0\le g(x)\le 1$ and $\lim_{x\to\pm\infty} g(x)=0$. Put $\alpha=\sup_{x\in\boldsymbol{R}} g(x)$. Then $0<\alpha\le 1$ and we can take a real sequence $\{w_m\}$ and $w\in\boldsymbol{R}$ such that $|w-w_m|\le 1$ for every m and $g(w_m)\uparrow \alpha$ as m goes to infinity. We can also take $R>0$ such that

$$\lambda([w-R+2, w+R-2]^c)<\frac{\alpha}{3}.$$

We prove the claim for this R. First we shall show $(e)_R$. Put $l_k=\mathbf{P}[|Y_k|>R]$ for each k. Without loss of generality, we may assume $l_k>0$ for every k.

There is a subsequence $\{w_{m(k)}\}$ such that

(2.3) $$g(w_{m(k)})>\alpha\left(1+\frac{l_k}{3}\right)^{-1}.$$

For convenience we replace the notation $w_{m(k)}$ by w_k. Then for each k,

$$\int_{[w_k-1,w_k+1]} p_k(x)d\lambda(x)=\mu_{X_k+Y_k}([w_k-1, w_k+1])$$
$$=\int_{\boldsymbol{R}}\mu_{X_k}([w_k-1-y, w_k+1-y])d\mu_{Y_k}(y)$$
$$=\left(\int_{[-R,R]}+\int_{[-R,R]^c}\right)\lambda([w_k-1-y, w_k+1-y])d\mu_{Y_k}(y).$$

If $y\in[-R, R]^c$ then $[w_k-1-y, w_k+1-y]\subset[w-R+2, w+R-2]^c$, so the integrals are not greater than

$$\int_{[-R,R]}\alpha d\mu_{Y_k}+\int_{[-R,R]^c}\frac{\alpha}{3}d\mu_{Y_k}=\alpha(1-l_k)+\frac{\alpha}{3}l_k=\alpha(1-\frac{2}{3}l_k).$$

By (2.3), this is dominated by

$$g(w_k)\left(1+\frac{l_k}{3}\right)\left(1-\frac{2}{3}l_k\right)\le g(w_k)\left(1-\frac{l_k}{3}\right).$$

Now we define $\Psi(t)=(1-t)^2\cdot 1_{(t\le 1)}$. Since Ψ is convex and monotone decreasing, we get

$$\int_{A_k}(p_k(x)-1)^2 d\lambda(x)\ge\int_{\boldsymbol{R}}\Psi(p_k(x))d\lambda(x)\ge\int_{[w_k-1,w_k+1]}\Psi(p_k(x))d\lambda(x)$$

$$\geq g(w_k)\Psi\left(\frac{1}{g(w_k)}\int_{[w_k-1,w_k+1]} p_k(x)d\lambda(x)\right)$$

$$\geq g(w_k)\Psi\left(\frac{1}{g(w_k)}\cdot g(w_k)\left(1-\frac{l_k}{3}\right)\right)$$

$$\geq\frac{3}{4}\alpha\cdot\frac{1}{9}l_k^2.$$

Hence we have $\sum_k l_k^2<\infty$ from (2.2).

Next, we shall show $(b)_R$ and $(c)_R$. Fix $\delta\in(0, R^{-1})$ such that $\mathbf{E}[e^{i\delta X_1}]\neq 0$. From (2.1) and (2.2), we have

$$\infty>\sum_k\left\{\left(\int_{A_k{}^c}(p_k(x)-1)d\lambda(x)\right)^2+\int_{A_k}(p_k(x)-1)^2d\lambda(x)\right\}$$

$$\geq\sum_k\left\{\left(\int_{A_k{}^c}(p_k(x)-1)d\lambda(x)\right)^2+\left(\int_{A_k}|p_k(x)-1|d\lambda(x)\right)^2\right\}$$

$$\geq\frac{1}{2}\sum_k\left(\int_R|p_k(x)-1|d\lambda(x)\right)^2$$

$$\geq\frac{1}{2}\sum_k\left|\int_R e^{i\delta x}(1-p_k(x))d\lambda(x)\right|^2$$

$$=\frac{1}{2}\sum_k\left|\int_R e^{i\delta x}d\lambda(x)-\int_R e^{i\delta x}d\mu_{X_k+Y_k}(x)\right|^2$$

$$=\frac{1}{2}\sum_k|\mathbf{E}[e^{i\delta X_1}](1-\mathbf{E}[e^{i\delta Y_k}])|^2$$

$$=\frac{1}{2}|\mathbf{E}[e^{i\delta X_1}]|^2\sum_k(\mathbf{E}[1-\cos\delta Y_k]^2+\mathbf{E}[\sin\delta Y_k]^2).$$

Since $\mathbf{E}[e^{i\delta X_1}]\neq 0$, we have

$$\sum_k \mathbf{E}[1-\cos\delta Y_k]^2<\infty \tag{2.4}$$

and

$$\sum_k \mathbf{E}[\sin\delta Y_k]^2<\infty. \tag{2.5}$$

By using the inequality $1-\cos\theta\geq\frac{1}{4}\theta^2$ on $\{|\theta|\leq 1\}$, (2.4) implies that

$$\infty>\sum_k \mathbf{E}[\frac{1}{4}(\delta Y_k)^2:\ |\delta Y_k|\leq 1]^2$$

$$\geq\sum_k \mathbf{E}[\frac{1}{4}\delta^2 Y_k^2:\ |Y_k|\leq R]^2.$$

So we conclude that $(b)_R$ holds. From the inequality $|\theta-\sin\theta|\leq\theta^2(\theta\in\boldsymbol{R})$ and $(b)_R$, we have

$$\begin{aligned}\sum_k \mathbf{E}[\delta Y_k-\sin\delta Y_k : |Y_k|\leq R]^2 &\leq \sum_k \mathbf{E}[|\delta Y_k-\sin\delta Y_k| : |Y_k|\leq R]^2\\ &\leq \sum_k \mathbf{E}[\delta^2 Y_k^2 : |Y_k|\leq R]^2\\ &<\infty.\end{aligned}$$

From (2.5) and $(e)_R$, we also have

$$\begin{aligned}\sum_k \mathbf{E}[\sin\delta Y_k : |Y_k|\leq R]^2 &\leq 2\sum \mathbf{E}[\sin\delta Y_k]^2+2\sum_k \mathbf{E}[\sin\delta Y_k : |Y_k|>R]^2\\ &\leq 2\sum_k \mathbf{E}[\sin\delta Y_k]^2+2\sum_k \mathbf{P}[|Y_k|>R]^2\\ &<\infty.\end{aligned}$$

Combining these results, we obtain that

$$\begin{aligned}&\sum_k \mathbf{E}[\delta Y_k : |Y_k|\leq R]^2\\ &\quad\leq 2\sum_k \mathbf{E}[\delta Y_k-\sin\delta Y_k : |Y_k|\leq R]^2+2\sum_k \mathbf{E}[\sin\delta Y_k : |Y_k|\leq R]^2\\ &\quad<\infty.\end{aligned}$$

This implies $(c)_R$ and completes the proof of (i).

(ii) The proof is almost the same as that of Theorem 1.2 (ii), which is Theorem 3 of Sato-Watari [8], but we would give it to make necessary modifications clear.

We may assume that $\epsilon=1$. Let f be the density function of μ_{X_1}. Then we have $\frac{d\mu_{X_k+Y_k}}{dm}(x)=\mathbf{E}[f(x-Y_k)]$, where m is the Lebesgue measure. From Kakutani's theorem [1] we see $\mu_{\mathbf{X}+\mathbf{Y}}\sim\mu_{\mathbf{X}}$ if and only if

$$\sum_k\int_{-\infty}^{+\infty}\left(\sqrt{\mathbf{E}[f(x-Y_k)]}-\sqrt{f(x)}\right)^2dx<\infty.$$

By considering $-\mathbf{Y}$ instead of $\mathbf{Y}$, it suffices to prove $\mu_{\mathbf{X}-\mathbf{Y}}\sim\mu_{\mathbf{X}}$, so we shall show

$$\sum_k\int_{-\infty}^{+\infty}\left(\sqrt{\mathbf{E}[f(x+Y_k)]}-\sqrt{f(x)}\right)^2dx<\infty$$

as in [8, Theorem 3].

Using the inequality

$$(\sqrt{a+b}-\sqrt{c+d})^2\leq(\sqrt{a}-\sqrt{c})^2+(\sqrt{b}-\sqrt{d})^2$$

for every $a,\ b,\ c,\ d\geq 0$, we have

$$(\sqrt{\mathbf{E}[f(x+Y_k)]}-\sqrt{f(x)})^2 \leq (\sqrt{\mathbf{E}[f(x+Y_k): |Y_k|\leq 1]}-\sqrt{f(x)\mathbf{P}[|Y_k|\leq 1]})^2$$
$$+(\sqrt{\mathbf{E}[f(x+Y_k): |Y_k|>1]}-\sqrt{f(x)\mathbf{P}[|Y_k|>1]})^2.$$

Since $(d)_1$ implies $\sum_k \int_{-\infty}^{+\infty}(\sqrt{\mathbf{E}[f(x+Y_k): |Y_k|>1]}-\sqrt{f(x)\mathbf{P}[|Y_k|>1]})^2 dx<\infty$ (see the proof of [8, Theorem 3]), it is enough to show

$$\sum_k \int_{-\infty}^{+\infty} \Phi_k(x)^2 dx<\infty,$$

where $\Phi_k(x)=\sqrt{\mathbf{E}[f(x+Y_k): |Y_k|\leq 1]}-\sqrt{f(x)\mathbf{P}[|Y_k|\leq 1]}$.

Define $\phi_{k,x}(t)=\sqrt{\mathbf{E}[f(x+tY_k): |Y_k|\leq 1]}$, $0\leq t\leq 1$. Then

$$\phi'_{k,x}(0)=\frac{\mathbf{E}[f'(x)Y_k: |Y_k|\leq 1]}{2\sqrt{\mathbf{E}[f(x): |Y_k|\leq 1]}}=\frac{f'(x)\mathbf{E}[Y_k: |Y_k|\leq 1]}{2\sqrt{f(x)\mathbf{P}[|Y_k|\leq 1]}}.$$

Since

$$\Phi_k(x)=\phi_{k,x}(1)-\phi_{k,x}(0)=\phi'_{k,x}(0)+\int_0^1(1-t)\phi''_{k,x}(t)dt,$$

we have

$$\sum_k \int_{-\infty}^{+\infty} \Phi_k(x)^2 dx \leq 2\sum_k \int_{-\infty}^{+\infty} \phi'_{k,x}(0)^2 dx+2\sum_k \int_{-\infty}^{+\infty}\left\{\int_0^1(1-t)\phi''_{k,x}(t)dt\right\}^2 dx.$$

The first term of the right-hand side is equal to

$$\frac{1}{2}\int_{-\infty}^{+\infty}\frac{f'(x)^2}{f(x)}dx \sum_k \frac{\mathbf{E}[Y_k: |Y_k|\leq 1]^2}{\mathbf{P}[|Y_k|\leq 1]}.$$

This is finite as we see from the facts that $\mathbf{P}[|Y_k|\leq 1]\geq \frac{1}{2}$ for sufficiently large k, and that (A) and $(c)_1$ hold. The second term is also finite by the proof of [8, Theorem 3].

(iii) Define for each k,

$$V_k(x)=\mathbf{E}[f(x-Y_k): |Y_k|>\epsilon]/f(x)-\mathbf{P}[|Y_k|>\epsilon],$$

$$W_k(x)=\mathbf{E}[f(x-Y_k)-f(x): |Y_k|\leq \epsilon]/f(x).$$

Then $p_k(x)-1=V_k(x)+W_k(x)$ and by Theorem 2.1 it is enough to prove that $\sum_k V_k(X_k)$ and $\sum_k W_k(X_k)$ converge a.s. By Lemma 1 of Kitada-Sato [2], $\sum_k V_k(X_k)$ converges absolutely a.s. Since $\mathbf{E}[W_k(X_k)]=0$ for every k and $W_k(X_k)$'s are independent, it is enough to show $\sum_k \mathbf{E}[W_k(X_k)^2]<\infty$ in order to prove the a.s. convergence of $\sum_k W_k(X_k)$. We have

$$\sum_k \mathbf{E}[W_k(X_k)^2]$$

$$=\sum_k\int_{-\infty}^{+\infty}\mathbf{E}[f(x-Y_k)-f(x):|Y_k|\leq\epsilon]^2/f(x)dx$$

$$=\sum_k\int_{-\infty}^{+\infty}\mathbf{E}\Big[-f'(x)Y_k+\int_0^1(1-t)f''(x-tY_k)Y_k^2dt:|Y_k|\leq\epsilon\Big]^2/f(x)dx$$

$$\leq 2\sum_k\int_{-\infty}^{+\infty}\mathbf{E}[f'(x)Y_k:|Y_k|\leq\epsilon]^2/f(x)dx$$

$$+2\sum_k\int_{-\infty}^{+\infty}\mathbf{E}\Big[\int_0^1(1-t)f''(x-tY_k)Y_k^2dt:|Y_k|\leq\epsilon\Big]^2\Big/f(x)dx.$$

The first term is equal to

$$2\int_{-\infty}^{+\infty}\frac{f'(x)^2}{f(x)}dx\sum_k\mathbf{E}[Y_k:|Y_k|\leq\epsilon]^2,$$

which is finite. The second term is not greater than

$$2\sum_k\left\|\mathbf{E}\Big[\int_0^1\Big(1-t\cdot\frac{|Y_k|}{\epsilon}\Big)\Big|\frac{f''(x-tY_k)}{\sqrt{f(x)}}\Big|Y_k^2dt:|Y_k|\leq\epsilon\Big]\right\|_2^2,$$

where $\|\cdot\|_2$ means the norm of $L^2(\boldsymbol{R}, dx)$. By using Minkowski's inequality for integrals, this is further dominated by

$$2\sum_k\mathbf{E}\Big[\int_0^1\Big\|\Big(1-t\cdot\frac{|Y_k|}{\epsilon}\Big)\Big|\frac{f''(x-tY_k)}{\sqrt{f(x)}}\Big|Y_k^2\Big\|_2dt:|Y_k|\leq\epsilon\Big]^2$$

$$\leq 2\sup_{|z|<\epsilon}\Big(1-\frac{|z|}{\epsilon}\Big)^2\Big\|\frac{f''(x+z)}{\sqrt{f(x)}}\Big\|_2^2\sum_k\mathbf{E}\Big[\int_0^1Y_k^2dt:|Y_k|\leq\epsilon\Big]^2$$

$$=2\epsilon^{-2}\sup_{|z|<\epsilon}(\epsilon-|z|)^2\int_{-\infty}^{+\infty}\frac{f''(x+z)^2}{f(x)}dx\sum_k\mathbf{E}[Y_k^2:|Y_k|\leq\epsilon]^2$$

$$<\infty,$$

which completes the proof of (iii).

3. Concluding remarks

1. The three indices in the exponents of the conclusion $(b)_\epsilon(c)_\epsilon(e)_\epsilon$ of Theorem 1.8 (i) are best possible. To be more precise, for any $r\in(1, 2)$, there exist $\mathbf{X}$ and $\mathbf{Y}$ such that

$$\mu_{\mathbf{X}+\mathbf{Y}}\sim\mu_{\mathbf{X}},\qquad \sum_k\mathbf{E}[Y_k^2:|Y_k|\leq\epsilon]^{\frac{2}{r}}=\infty,$$

$$\sum_k|\mathbf{E}[Y_k:|Y_k|\leq\epsilon]|^{\frac{2}{r}}=\infty,\qquad \sum_k\mathbf{P}[|Y_k|>\epsilon]^{\frac{2}{r}}=\infty\quad\text{for every }\epsilon>0.$$

In particular, we can take $\mathbf{X}$ as a standard Gaussian sequence. For the proof, we need the following lemma for a standard Gaussian sequence $\mathbf{X}$.

Lemma 3.1 ([2, *Lemma* 2]). *Let* $\mathbf{Y}=\{Y_k\}$ *have the distributions*

$$\mathbf{P}[Y_k=a_k]=\mathbf{P}[Y_k=-a_k]=\frac{1}{2}p_k,\ \mathbf{P}[Y_k=0]=1-p_k$$

for each k, where $\{a_k\}$ *and* $\{p_k\}$ *are sequences of positive numbers such that* $\lim_k a_k=\infty$, $\sum_k p_k=\infty$, *and* $\sum_k p_k^2<\infty$. *Set* $\gamma_k=\frac{1}{a_k}\log\left\{2\left(1+\frac{1}{p_k}\right)\right\}+\frac{1}{2}a_k$. *Then* $\mu_{X+Y}\sim\mu_X$ *if and only if the following two series are convergent ;*

$$\sum_k p_k\int_{\gamma_k-a_k}^{\gamma_k}\exp\left(-\frac{1}{2}u^2\right)du<\infty,$$

$$\sum_k p_k^2\int_{-2a_k}^{\gamma_k-2a_k}\exp\left(a_k^2-\frac{1}{2}u^2\right)du<\infty.$$

Let $\mathbf{Y}^{(j)}=\{Y_k^{(j)}\}$, $j=1, 2, 3$ be independent random sequences with distributions

$$\mathbf{P}[Y_k^{(1)}=k^{-\frac{2-r}{4}}]=\mathbf{P}[Y_k^{(1)}=-k^{-\frac{2-r}{4}}]=\frac{1}{2}k^{1-r},\ \mathbf{P}[Y_k^{(1)}=0]=1-k^{1-r},$$

$$\mathbf{P}[Y_k^{(2)}=k^{-\frac{r}{2}}]=1,$$

$$\mathbf{P}[Y_k^{(3)}=a_k]=\mathbf{P}[Y_k^{(3)}=-a_k]=\frac{1}{2}p_k,\ \mathbf{P}[Y_k^{(3)}=0]=1-p_k,$$

where $a_k=\sqrt{\log\log k}$, $p_k=\left(\frac{1}{2}k^{\frac{r}{2}}-1\right)^{-1}$ for $k\geq 16$.

Then, Theorem 1.8(iii) implies $\mu_{X+Y^{(1)}}\sim\mu_X$ and $\mu_{X+Y^{(2)}}\sim\mu_X$. We shall check that Lemma 3.1 is applicable to $\mathbf{Y}^{(3)}$. It is easy to see that $\sum_k p_k=\infty$ and $\sum_k p_k^2<\infty$. Set $\gamma_k=\frac{1}{a_k}\log\left\{2\left(1+\frac{1}{p_k}\right)\right\}+\frac{1}{2}a_k=\frac{1}{a_k}\cdot\frac{r}{2}\log k+\frac{1}{2}a_k$. There exists $N\in\boldsymbol{N}$ such that $\frac{\log k}{\log\log k}\geq 32$ for all $k\geq N$. Then, $\gamma_k-a_k>\frac{1}{2}\cdot\frac{1}{a_k}\cdot\frac{r}{2}\log k$ for $k\geq N$ and

$$\begin{aligned}\sum_{k\geq N}p_k\int_{\gamma_k-a_k}^{\gamma_k}\exp\left(-\frac{1}{2}u^2\right)du &\leq \sum_{k\geq N}p_ka_k\exp\left\{-\frac{1}{2}(\gamma_k-a_k)^2\right\}\\ &\leq \sum_{k\geq N}p_ka_k\exp\left\{-\frac{1}{2}\left(\frac{1}{2}\cdot\frac{1}{a_k}\cdot\frac{r}{2}\log k\right)^2\right\}\\ &\leq \sum_{k\geq N}p_ka_k\exp(-\log k)\\ &=\sum_{k\geq N}\left(\frac{1}{2}k^{\frac{r}{2}}-1\right)^{-1}\sqrt{\log\log k}\cdot k^{-1}\\ &<\infty,\end{aligned}$$

and

$$\sum_{k\geq N} p_k^2 \int_{-2a_k}^{\gamma_k-2a_k} \exp\left(a_k^2-\frac{1}{2}u^2\right)du \leq \sum_{k\geq N} p_k^2\sqrt{2\pi}\exp(a_k^2)$$

$$=\sum_{k\geq N}\left(\frac{1}{2}k^{\frac{r}{2}}-1\right)^{-2}\sqrt{2\pi}\log k$$

$$<\infty.$$

Therefore we deduce from Lemma 3.1 that $\mu_{\mathbf{X}+\mathbf{Y}^{(3)}}\sim\mu_{\mathbf{X}}$. Now we define an independent random sequence $\mathbf{Y}=(Y_k)$ by $\mu_{Y_k}=\frac{1}{3}\sum_{j=1}^3\mu_{Y_k^{(j)}}$. Since $\mu_{X_k+Y_k}=\frac{1}{3}\sum_{j=1}^3\mu_{X_k+Y_k^{(j)}}$, we have

$$\frac{d\mu_{X_k+Y_k}}{d\mu_{X_k}}-1=\frac{1}{3}\sum_{j=1}^3\left(\frac{d\mu_{X_k+Y_k^{(j)}}}{d\mu_{X_k}}-1\right).$$

By Theorem 2.1, we conclude that $\mu_{\mathbf{X}+\mathbf{Y}}\sim\mu_{\mathbf{X}}$. Moreover we have

$$\sum_k \mathbf{E}[Y_k^2 : |Y_k|\leq\epsilon]^{\frac{2}{r}}=\infty,\ \sum_k |\mathbf{E}[Y_k : |Y_k|\leq\epsilon]|^{\frac{2}{r}}=\infty,$$

$$\sum_k \mathbf{P}[|Y_k|>\epsilon]^{\frac{2}{r}}=\infty \qquad \text{for every } \epsilon>0.$$

We also see $(a)_\epsilon$ does not hold for any $\epsilon>0$.

2. We might replace the condition $(a)_\epsilon$ with $(b)_\epsilon$ in Theorem 1.8 (ii), which is yet to be investigated. So far, we need an additional assumption of $\mathbf{X}$ such as (iii). However we cannot replace $(d)_\epsilon$ with $\sum_k \mathbf{P}[|Y_k|>\epsilon]^r<\infty$ for any $r>1$ even if $\mathbf{X}$ is standard Gaussian. For $1<s<2$, we shall give an example that $\mathbf{X}$ is standard Gaussian and for every $\epsilon>0$, $\mathbf{E}[Y_k^2 : |Y_k|\leq\epsilon]=\mathbf{E}[Y_k : |Y_k|\leq\epsilon]=0$ all but for finitely many k, $\sum_k\mathbf{P}[|Y_k|>\epsilon]^{s^2}<\infty$, and $\mu_{\mathbf{X}+\mathbf{Y}}\perp\mu_{\mathbf{X}}$.

Take any $s\in(1, 2)$. Define $\mathbf{Y}$ by

$$\mathbf{P}[Y_k=a_k]=\mathbf{P}[Y_k=-a_k]=\frac{1}{2}p_k,\ \mathbf{P}[Y_k=0]=1-p_k,$$

$$\text{where } a_k=\sqrt{\frac{2}{s}\log k},\ p_k=\left(\frac{1}{2}k^{\frac{1}{s}}-1\right)^{-1} \text{ for } k\geq 16.$$

Then $\sum_k p_k=\infty$, $\sum_k p_k^2<\infty$ and $\mathbf{Y}$ satisfies the conditions mentioned above.

Set $\gamma_k=\frac{1}{a_k}\log\{2(1+\frac{1}{p_k})\}+\frac{1}{2}a_k$. Then $\gamma_k=a_k$, so

$$\sum_k p_k\int_{\gamma_k-a_k}^{\gamma_k}\exp\left(-\frac{1}{2}u^2\right)du\geq\sum_k p_k\int_0^1\exp\left(-\frac{1}{2}u^2\right)du=\infty.$$

Therefore we conclude $\mu_{\mathbf{X}+\mathbf{Y}}\perp\mu_{\mathbf{X}}$ by Lemma 3.1.

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