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On Beauville's conjecture and related topics

By

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The main purpose of this paper is to discuss a fifteen years old conjecture proposed by A. Beauville on the number of singular fibres of a semi-stable fibration over $\mathbf{P}^1$ ([B1]). The departure point is the so-called function field analog of second Shafarevich's conjecture. Precisely, let $f: X \to C$ be a non-isotrivial fibration over a complex algebraic curve C whose generic fibre is a smooth projective irreducible curve F of genus $g \geq 1$. Put

$$s = \text{the number of } S, \ S = \{t \in C: X_t = f^{-1}(t) \text{ is singular}\}.$$

Shafarevich's conjecture (the function field case). $s > 0$ if $C \simeq \mathbf{P}^1$.

I. Shafarevich proved this statement in [Sh]. By using the action of automorphism group on $\mathbf{P}^1$ A. Parshin ([Par1]) has established that $s \geq 3$ (see also [B1]). Note that in any characteristic (but with a semi-stability condition) the same result was obtained by L. Szpiro ([Sz]). In fact in the semi-stable case over $\mathbf{C}$ a more precise bound was given by A. Beauville ([B1]). Let $g(\tilde{X}_t)$ denote the genus of the normalization of X_t, ρ_2 "the number of transcendental cycles" of X. Let r be the defect relating the Picard number ρ of X and numbers of components of singular fibres (see A.2.2, Appendix A). There is a necessary and sufficient condition for s to be 4 ([B1], cf. also Appendix A).

Theorem (A. Beauville). Let $f: X \to \mathbf{P}^1$ be a semi-stable non-isotrivial fibration. Assume that $g \geq 1$ then $s \geq 4$.

Moreover $s = 4$ if and only if the folowing conditions hold

1) $\rho_2 = 0$,
2) $g(\tilde{X}_t) = g_0$ $\forall t \in S$, where $g_0 = \dim$ of the fixed part of $\mathrm{Pic}^0(X/\mathbf{P}^1)$,
3) $r = 0$,
4) $g_0 = 0$.

Furthermore A. Beauville (*loc. cit.*) constructed some examples with $s = 4$: all those fibrations are elliptic (see also [B2], where he has given a complete classification of all such elliptic fibrations - six cases). In fact A. Beauville was tending to conjecture the following

Beauville's conjecture ([B1]). $s \geq 5$ if $g > 1$.

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As communicated to the author by A. Beauville ([B3]) G. Xiao made an important step towards the understanding of the conjecture. He remarked that under the condition $s = 4$ the canonical class inequality together with the inequality of Xiao for the slope of f (see §§1–2 below) implies

$$\omega^2 = 4(g-1). \tag{$*$}$$

Recall that ω usually denotes the relative canonical class of f. Equivalently one has $(K_X + F)^2 = 0$, where F is the fibre class on X. Therefore the linear system $|K_X + F|$ is composed with a pencil so that $K_X + F \equiv (g-1)P$, where P is a rational curve with $(P, F) = 2$. In particular the generic fibre is a hyperelliptic curve. The pencils $|F|$ and $|P|$ define a 2-to-1 map from X to $\mathbf{P}^1 \times \mathbf{P}^1$. The idea of A. Beauville and G. Xiao is to study singularities of the branch locus of this mapping.

After an interruption of time this line of ideas was continued by S.-L. Tan in the recent preprint [Tn]. First in the situation due to the arguments above S.-L. Tan gave a quite satisfactory analysis of singularities of the branch locus. Further according to the general theory of double coverings of Horikawa and others S.-L. Tan considered a new branch locus by adding to the original one four singular fibres. The Sakai-Miyaoka inequality applied to the corresponding double covering implies $g \leq 5$. In other words Beauville's conjecture has been proved for $g > 5$.

It should be emphasized that there are two different approaches using the equality ($*$) in direction of the conjecture.

1) Tan's approach (well as $g > 5$) consists of applying the Sakai-Miyaoka inequality again to obtained double coverings after a geometric study.

2) Our idea which is the major gist of this paper is to use higher covers of the base as in the way of getting ($*$). This means that the equality ($*$) actually is sufficient to obtain a complete proof of Beauville's conjecture by taking into account the negligible part in the Sakai-Miyaoka inequality (see below and §2 for details). Moreover this important remark enables us to get an improvement of the canonical class inequality.

Theorem (Theorem 2.5.1). *Let $f: X \to C$ be a non-isotrivial semi-stable fibration with $g > 1$. Denote by ω the relative canonical class $K_{X/C}$. Then*

1) $\omega^2 < (2q - 2 + s)(2g - 2)$.

2) *Moreover there exists a universal constant $A = A(s, q) < 2q - 2 + s$ depending on s and $q =$ genus (C) such that*

$$\omega^2 \leq A(2g-2).$$

Let us explain the idea of the proof. Take a cyclic covering of degree $n: C_n \to C$ (totally) branched over S and let X_n be the relatively minimal resolution of $X \times_C C_n$ (herein to be more precise in the case with arbitrary s one considers, for example, coverings $C_n \to C$ of odd degrees). Then applying the Sakai-Miyaoka inequality to X_n and Xiao's inequality for the slope of f one finds the desired constant. In case $C \simeq \mathbf{P}^1$ we should actually put $n = 7$. A computation in this case shows that one can take, e.g., $A(5, 0) = 2.84$, $A(6, 0) = 3.85$.

Thus essentially to obtain a complete proof of Beauville's conjecture it suffices to apply the Sakai-Miyaoka inequality once to higher coverings of the base. However it should be noted that Tan's method ([Tn]) in fact is applicable for a more general situation as will be shown in what follows. From the proof of Beauville's theorem using the explicit formula for ρ_2 (cf. Appendix A) one has seen that the first three conditions imply the fourth one. So the situation in Beauville's theorem can be generalized in several ways related to the number s_0, where

$$s_0 = \text{the number of } \{t \in \mathbf{P}^1 : g(\tilde{X}_t) = g_0\}.$$

The following theorem shows the applicability of Tan's method for a larger class of fibrations.

Theorem (Theorem 3.1). *Let $f: X \to \mathbf{P}^1$ be a relatively minimal fibration with $g > 1$ and*

1) $\omega^2 = 4(g-1)$,
2) $\chi(\mathcal{O}_X) \geq 3 - g$.

Then

1) *X is a ruled surface in the sense of* [B4], *i.e., one has a surjective morphism $\pi: X \to E$, $g(E) = g_0$ whose generic fibre is a nonsingular rational curve. Families f and π define a 2-to-1 map from X to $Y = E \times \mathbf{P}^1$ with branch locus, say B and a divisor γ such that $2\gamma \equiv B$. In particular $g_0 = 0$ implies that X is rational and that f is hyperelliptic. Moreover X is the canonical resolution of double covering $X(Y, \gamma, B)$ with data (Y, γ, B) having at most rational double points as its singularities (for the definition of $X(Y, \gamma, B)$ see* § 1),

2) *if f is regular, i.e., $s_0 > 0$ then $g_0 \leq 1$,*

Assume in addition that f is semi-stable then

3) *the last statement in* 1) *is also true, i.e., X is the canonical resolution of $X(Y, \gamma, B)$ with rational double singularities. The second projection of Y induces f and B has numerical type* $(2a, 2)$ *on Y, where $a = g + 1 - 2g_0$.*

4) *If we denote by F_0 the fibre of type* $(0, 1)$ *on Y then for any point $p \in B \cap F_0$ the intersection number $(B, F_0)_p \leq 2$.*

5) *We have the following estimates for the case $g_0 \leq 1$ (in particular if f is regular)*

i) $s_0 \leq 8$ (for $g > 4$ if $g_0 = 0$),

ii) *for each value* $s_0 \geq 5$, $g \leq g_0 + \dfrac{r}{s_0 - 4}$,

iii) *if $r \leq 1$ then $s_0 \leq 4$. Moreover case $s_0 = 4$, $r = 0$ is not realized, case $s_0 = 4$, $r = 1$ implies $g \leq 10$, if $g_0 = 0$ and $g \leq 16$, if $g_0 = 1$.*

It is interesting to note that the equality $g = g_0 + \dfrac{r}{s_0 - 4}$ holds for the series of examples constructed in [B1]. Elliptic examples with $s = s_0 = 5$, $r = 1$ are relatively easy to be constructed (cf. [Vt1], Appendix C). In fact if $g > 1$ the number r can be arbitrarily large as shown by an explicit calculation in those examples of [B1].

The paper is organized as follows. In § 1 we summarise some preliminary well-known facts, e.g., Xiao's inequality, Horikawa's canonical resolutions of double coverings. In particular using Matsusaka's inequalities ([Ma]) one obtains a nice explanation of Persson's well-known result on hyperelliptic fibrations with lowest slope. In § 2 we give a refinement of the canonical class inequality combining inequalities of Xiao and Sakai-Miyaoka. This leads us to a complete proof of Beauville's conjecture. § 3 is devoted to the proof of Theorem 3.1 formulated above. Since the proof is essentially using the idea of A. Beauville and G. Xiao mentioned at the beginning and Tan's method ([Tn]) we shall describe it briefly. The fact that we are really dealing with a larger class of fibrations can be illustrated by Beauville's series of examples in [B1]. We shall make a detailed calculation for this series after proving Theorem 3.1. At this point a new interesting example of genus two fibration with 5 singular fibres constructed by M.-H. Saito can serve also for the illustration purpose. Originally the motivation of constructing that example grew out of an attempt to find a counterexample to Beauville's conjecture. In fact our discussions with M.-H. Saito on the topic here played a role for understanding the essence of Beauville's conjecture and the importance of using the Sakai-Miyaoka inequality in our context expounded above (df. also [B3],[Vt0]). With M.-H. Saito's kind permission we reproduce his example in closing § 3.

For the sake of completeness we conclude the paper with several appendices. Appendix A gives a summary of general well-known facts in the semi-stable case from which we shall make a free use in the paper, in particular Arakelov's estimate for the degree of $f_* \omega_{X/C}$, the formula for "the number of transcendental cycles" ρ_2 of X and Beauville's theorem. As a consequence in the case $C \simeq \mathbf{P}^1$ we should infer from the condition $s = 4$ that $\omega - F$ is numerically effective which implies the inequality $\omega^2 \geq 4(g-1)$ (cf. also § 2). We shall give a detailed exposition of these facts. In view of the case $s = 4$ in Beauville's theorem and its generalization (Theorem 3.1) the case of hyperelliptic fibrations with at most two components in each fibre remaining after a contraction of all (-2)-curves in fibres represents some interest. Appendix B describes degenerate configurations in this case following the ideas of [Par2]. The elliptic case with $s = 3$ is more or less well-known. There is a beautiful description using results of D. Cox ([C]), D. Cox and S. Zucker ([C-Z]). In general these arguments can serve as a supplement to various numerical criteria for the problem of classifying all possible Kodaira's configurations on rational elliptic surfaces completely solved by U. Persson ([Per2], cf. also [Mir]). It will appear somewhere else (cf. [Vt1], Appendix C).

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Results of §2 were partly published in [Vt2], [Vt3].

Note Added in Proof. The author would like to thank S.-L. Tan for sending preprint [Tn] dated January 8, 1994 which unfortunately arrived in Hanoi just on June 26, 1994. In May, 1994 A. Beauville informed me that in the meantime S.-L. Tan has also obtained a complete proof of his conjecture. At the end of June, 1994 I received preprint [Tn']* kindly sent by H. Esnault and E. Viehweg. In [Tn'] S.-L. Tan has combined two approaches expounded above. However in this way the author of [Tn'] was able to give a weaker variant of the improved canonical class inequality (cf. also [Vt2], [Vt3]).

§1. Preliminaries

1.1. Let $f: X \to C$ a non-isotrivial relatively minimal fibration having the following invariants:

$g =$ genus $(X_\eta) \geq 1$, where the generic fibre X_η is a smooth irreducible projective curve F,

$q =$ genus (C), the base C is a connected nonsingular projective curve,

$s =$ the number of S, $S = \{t \in C: X_t = f^{-1}(t) \text{ is singular}\}$.

Recall that the ground field k is the field of complex numbers $\mathbf{C}$.

Let $\omega_{X/C}$ be the relative dualizing sheaf and by ω we denote the relative canonical class $K_{X/C}$. Define

$$d := \deg(f_* \omega_{X/C}) = -\deg(R^1 f_* \mathcal{O}_X).$$

Then since f is non-isotrivial one has $d > 0$ and so we define the slope λ_f of f

* [Tn'] Tan, S.-L., The minimal number of singular fibers of a semi-stable curve over $\mathbf{P}^1$, Preprint of MPI/94-45.

([X]) as the following ratio

$$\lambda_f := \frac{\omega^2}{d}.$$

1.1.1. Xiao's inequality ([X]). With the above notation we have

$$\lambda_f \geq 4 - \frac{4}{g} \tag{1}$$

Furthermore G. Xiao (*loc. cit.*) has conjectured the following

1.1.2. Xiao's conjecture. Fibrations with the lowest slope are hyperelliptic, i.e., the generic fibre is hyperlliptic.

This conjecture has been proved in the semi-stable case by M. Cornalba and J. Harris ([C-H]). Very recently K. Konno ([K]) has obtained a complete proof of Xiao's conjecture.

1.2. On Matsusaka's inequalities for hyperelliptic fibrations. We recall some well-known facts from the canonical resolution theory of double conerings ([H], [Per1], [Ma]). Let X be a normal surface and Y -a smooth surface. By a double covering $\pi: X \to Y$ we mean a finite surjective morphism of degree 2, an involution on X with no isolated fixed points. This covering has the following data: an even branch curve B and a line bundle L on Y such that $L^{\otimes 2} \simeq \mathcal{O}_Y(B)$. We denote this covering by $X(Y, L, B)$, or sometimes by $X(Y, \gamma, B)$, where γ is a divisor such that $L \simeq \mathcal{O}_Y(\gamma)$.

1.2.1. Horikawa's canonical resolution. Put $Y_0 = Y$, $L_0 = L$, $B_0 = B$ and let

$$\begin{array}{ccccccccccc}
X_{CR} & = & X_n & \longrightarrow & X_{n-1} & \longrightarrow \cdots \longrightarrow & X_1 & \longrightarrow & X_0 & = & X \\
\Big\downarrow{\scriptstyle \pi_{CR}} & & \Big\downarrow{\scriptstyle \pi_n} & & \Big\downarrow{\scriptstyle \pi_{n-1}} & & \Big\downarrow{\scriptstyle \pi_1} & & \Big\downarrow{\scriptstyle \pi_0} & & \Big\downarrow{\scriptstyle \pi} \\
Y_{CR} & = & Y_n & \xrightarrow{\sigma_n} & Y_{n-1} & \xrightarrow{\sigma_{n-1}} \cdots \longrightarrow & Y_1 & \xrightarrow{\sigma_1} & Y_0 & = & Y
\end{array}$$

be the tower of Horikawa's canonical resolution ([H], cf. also [Per1], [Ma]), i.e., such that

1) $X_i = X(Y_i, L_i, B_i)$ is the double covering with data (Y_i, L_i, B_i),
2) σ_i: $Y_i \to Y_{i-1}$ is a blowing-up at a singular point p_i of B_{i-1},
3) $B_i = \sigma_i^*(B_{i-1}) - 2\left[\dfrac{m_i}{2}\right]E_i$,

$$L_i = \sigma_i^* L_{i-1} \otimes \mathcal{O}_{Y_{i-1}}\left(-\left[\frac{m_i}{2}\right]E_i\right),$$

where $m_i = \operatorname{mult}_{p_i}(B_{i-1})$ and E_i is the exceptional curve of σ_i over p_i,

4) B_n is nonsingular.

The conditions 1)-4) imply that X_{CR} is smooth and it is called the canonical resolution of $X = X(Y, L, B)$ over Y having the universality property, i.e., if σ': $Y' \to Y$ is a proper surjective birational morphism between smooth surfaces such that divisor $B_{Y'} = \sigma'^* B - 2\sum\left[\frac{m_i'}{2}\right] E_i'$ is nonsingular, where E_i' are exceptional curves of σ' (cf. 1)–4) above), then there exists a unique morphism $\sigma''\colon Y' \to Y_{CR}$ such that the following diaram

$$\begin{tikzcd} Y' \arrow[r, "\sigma''"] \arrow[dr, "\sigma'"'] & Y_{CR} \arrow[d, "\sigma_{CR}"] \\ & Y \end{tikzcd}$$

is commutative, i.e., $\sigma' = \sigma_{CR} \circ \sigma''$.

Note that the canonical resolution is not necessarily the minimal resolution. Further to each relatively minimal hyperelliptic fibration $f\colon X \to C$ one can associate a triple (Y, L, B) such that $X(Y, L, B)$ is birationally equivalent to X.

1.2.2. Lemma ([Ma]). *With the same notation as in* 1.2.1 *the natural morphism* $h\colon X_{CR} \to X$ *is a contraction of all* (-1)*-curves on* X_{CR} *and for every* $i (1 \le i \le n)$ *we have*

$$2 \le m_i \le g + 1.$$

1.2.3. Local canonical degrees $d_t(X)$. Let $f\colon X \to C$ be a hyperelliptic fibration with triple (Y, L, B) as above and $p\colon Y \to C$ the structure morphism. Then there exists an open subset $C^0 \subset C$ such that

i) $p^{-1}(C^0) \simeq \mathbf{P}^1 \times C^0$,

ii) $f^{-1}(C^0)$ can be identified with the closure of the set $\{(x, t, y) \in \mathbf{C} \times C \times \mathbf{C}\colon y^2 = \varphi(x, t)\}$, where x is an inhomogeneous coordinate of $\mathbf{P}^1$ and φ is a polynomial of degree $2g + 1$ (or $2g + 2$) in x with coefficients in the rational function field of C^0.

We put

$$D := \Delta(\varphi)^g \left(\frac{dx}{y} \wedge \cdots \wedge x^{g-1} \frac{dx}{y}\right)^{\otimes 4(2g+1)}$$

where $\Delta(\varphi)$ is the discriminant of φ as a polynomial in x. In fact D can be naturally extended to a rational section of $(\wedge^g f_* \omega_{X/C})^{\otimes 4(2g+1)}$ and is independent in the choice of (Y, L, B), C^0, x, y and φ (cf. [U]). We can now define a local degree $d_t(X)$ at each point $t \in C$ by setting

$$d_t(X) := \frac{1}{4(2g+1)} \operatorname{ord}_t D \tag{2}$$

1.2.4. Matsusaka's inequalities ([Ma]). In the above notation we have

$$d_t(X) \ge \frac{g}{4(2g+1)} e_t(X)$$

for every point $t \in C$, where $e_t(X) = \chi(X_t) - (2 - 2g)$ is the local Euler number over t.

1.2.5. Corollary. *For hyperelliptic fibrations we have* $\lambda_f \geq 4 - \frac{4}{g}$. *Moreover the condition* $\lambda_f = 4 - \frac{4}{g}$ *is equivalent to the equality in all Matsusaka's inequalities.*

Proof. From the definition (2) it is easy to see that

$$\sum_{t \in C} d_t(X) = d.$$

On the other hand one has

$$12d = \omega^2 + \sum_{t \in C} e_t(X).$$

This formula is an easy consequence of the following formulae

$$c_2(X) = \sum_{t \in C} e_t(X) + 4(g-1)(q-1),$$
$$c_1^2(X) = \omega^2 + 8(g-1)(q-1),$$
$$\chi(\mathcal{O}_X) = d + (g-1)(q-1)$$

and Noether's classical formula (cf. Appendix A).

Now the assertion follows immediately from 1.2.4.

Furthermore one can get another explanation of the following well-known fact.

1.2.6. Corollary ([Per1], Prop. 2.12, cf. also [X]). *For hyperelliptic fibrations one has* $\lambda_f = 4 - \frac{4}{g}$ *if and only if* X *is the canonical resolution of a double covering of a geometrically ruled surface over* C *with only rational double points as its singularities.*

Proof. With the notation of 1.2.1 let $E_i^{(j)}$ denote the proper transform of the exceptional curve of σ_i to Y_j for $i \leq j$. Then from Lemma 4.3.1 ([Ma]) it follows that

$$4(2g+1)d_t(X) - g \cdot e_t(X_{CR}) = -2 \sum_{p_{CR}(E_i^{(n)}) = t} \left(\left[\frac{m_i}{2} \right] - 1 \right) \left(\left[\frac{m_i}{2} \right] - g \right)$$

where $p_{CR}\colon Y_{CR} \to C$ is the structure morphism.

Hence by virtue of Lemma 1.2.2 and Corollary 1.2.5 one sees that $X = X_{CR}$ and all m_i are less than 4. Consequently we infer (cf. [H], Lemma 5) that the double covering $X(Y, L, B)$ has at most rational double points as its singularities.

§2. A refinement of the canonical class inequality

2.1. The Sakai-Miyaoka inequality ([Sa], [Miy]). Let X be a minimal nonsingular surface of general type. Assume that $\mathscr{E}_1,\dots,\mathscr{E}_k$ are disjoint configurations of (-2)-curves which should be contracted to the rational double points $p_1,\dots,p_k$. Then we have the so-called Sakai-Miyaoka inequality for the Chern classes of X

$$c_1^2(X) \le 3c_2(X) - \sum_i \mu(\mathscr{E}_i). \tag{3}$$

Here the correction terms $\mu(\mathscr{E}_i)$ are defined as

$$\mu(\mathscr{E}_i) = 3\left[e(\mathscr{E}_i) - \frac{1}{|G_i|} \right],$$

where $e(\mathscr{E}_i)$ is the Euler number of $\mathscr{E}_i$ and p_i is a quotient singularity with a finite group, say G_i. Thus one has the following possibilities for ADE-types of singularities

$$\begin{aligned} &e(A_r) = r + 1, \quad |G_{A_r}| = r + 1,\\ &e(D_r) = r + 1, \quad |G_{D_r}| = 4(r - 2), \quad \text{for } r \ge 4,\\ &e(E_6) = 7, \quad |G_{E_6}| = 24,\\ &e(E_7) = 8, \quad |G_{E_7}| = 48,\\ &e(E_8) = 9, \quad |G_{E_8}| = 120. \end{aligned}$$

2.2. Now let $f\colon X \to C$ be a non-isotrivial semi-stable fibration with $g > 1$ and $q = g(C) \ge 1$, then in this case we have only the type A_{r_i} for all $\mathscr{E}_i$. Thus by neglecting the non-positive part $-3\sum_i \left(1 - \dfrac{1}{r_i + 1}\right)$ from the right-hand side of (3) and using formulae (A3), (A4) of Appendix A one gets ([Par3])

$$\omega^2 \le 3\delta' + (2g - 2)(2q - 2), \tag{4}$$

where δ' is the number of double points after contracting all (-2)-curves in fibres.

2.3. Furthermore if we take a cyclic covering $C_n \to C$ of degree n (totally) branched only over S (to be more precise n odd if s odd) so that $2q_n - 2 = n(2q - 2) + (n - 1)s$ by the Riemann-Hurwitz formula, where $q_n =$ genus(C_n). Let X_n be the relatively minimal resolution of $X \times_C C_n$ then applying (4) to X_n one obtains

$$\omega_n^2 \le 3\delta' + n(2g - 2)(2q - 2 + s), \tag{5}$$

where ω_n denotes the relative canonical class of $X_n \to C_n$. Since it is well-known ([A], [Sz]) that $\omega_n^2 = n\omega^2$ and an $n \to \infty$ we infer from (5) the so-called canonical class inequality ([Va], [E-V])

$$\omega^2 \le (2g - 2)(2q - 2 + s). \tag{6}$$

A key point further is to use the above negligible part as will be shown in the following theorem.

2.4. Theorem (Beauville's conjecture). *Let $f: X \to \mathbf{P}^1$ be a non-isotrivial semi-stable fibration with $g > 1$. Then $s \geq 5$.*

Proof. First note that X_n is minimal of general type as well as $n \geq 2$. Now taking into account the negligible part from 2.2 we rewrite the Sakai-Miyaoka inequality for X_n

$$n\omega^2 \leq n(2g-2)(s-2) - s(2g-2) + \sum_i \frac{3}{\tilde{r}_i + 1} \tag{7}$$

where p_i has the type $A_{\tilde{r}_i}$ on X_n. Since $\tilde{r}_i + 1 = n(r_i + 1) \geq n$ so that

$$\sum_i \frac{3}{\tilde{r}_i + 1} \leq \frac{3\delta'}{n}.$$

Therefore we infer from (7) that

$$s(2g-2) - \frac{3\delta'}{n} \leq n[(2g-2)(s-2) - \omega^2] \tag{8}$$

Further by virtue of Beauville's Theorem (cf. Appendix A) one can assume that $s = 4$. From (8) (or (6)) as $n \to \infty$ it is clear that

$$\omega^2 \leq 4(g-1) \tag{9}$$

On the other hand from Beauville's Theorem it follows that $d = g$. Then using Xiao's inequality (1) we have

$$\omega^2 \geq 4(g-1) \tag{10}$$

(see also Corollary A.4.4)

Combining (9), (10) one gets $\omega^2 = 4(g-1)$. As emphasized in the Introduction this is a key point essentially due to G. Xiao ([B3], cf. also [Tn]). Taking into account this equality it is easy to see that (8) can be rewritten as

$$8(g-1) - \frac{3\delta'}{n} \leq 0$$

The latter inequality is impossible as $g > 1$ and $n \gg 1$. This completes the proof of the theorem.

2.5. In closing this section we give an improvement of the canonical class inequality using the trick above and inequality (1).

2.5.1. Theorem. *Let $f: X \to C$ be a non-isotrivial semi-stable fibration with $g > 1$. Then there exists a constant $A = A(s, q) < 2q - 2 + s$ depending on s and q such that*

$$\omega^2 \leq A(2g-2).$$

Proof. First we note that $2q - 2 + s > 0$ because f is non-isotrivial. Now we rewrite (8) for this general case as

$$s(2g-2) - \frac{3\delta'}{n} \leq n[(2g-2)(2q-2+s) - \omega^2] \tag{11}$$

From (6) and (11) it is easy to see that $\omega^2 < (2g-2)(2q-2+s)$ as $g > 1$. So if we write $\omega^2 = a(2g-2)$ then (11) is equivalent to

$$c(a, n)\omega^2 \leq \frac{3\delta'}{n},$$

where $c(a, n) = \dfrac{s - n(2q-2+s)}{a} + n.$

Obviously $\delta' \leq \delta =$ the number of double points in fibres of X. Now the standard formula $\delta = 12d - \omega^2$ (cf. Appendix A) together with inequality (1) enables us to deduce

$$\left(1 - \frac{1}{g}\right)\left[c(a, n) + \frac{3}{n}\right] \leq \frac{9}{n} \tag{12}$$

The idea here is to choose an optimal value n and $a(n) < 2q - 2 + s$ such that $c(a, n) > \dfrac{6}{n}$ for all $a > a(n)$, since in this case the inequality (12) holds only for finitely many values of g. Thus we can find the desired constant as will be shown below.

One firstly sees easily that the inequality $c(a, n) > \dfrac{6}{n}$ is equivalent to the inequality $a > a(n)$, where

$$a(n) = \frac{n[n(2q-2+s) - s]}{n^2 - 6} \tag{13}$$

Thus we need to choose $n = n_1$ such that $a(n_1) < 2q - 2 + s$. A computation using (13) shows that it suffices to choose $n_1 > 6 + \dfrac{12(q-1)}{s}$. Now putting $a_1 = a(n_1) + \varepsilon$ for a sufficiently small ε we see that (12) holds only for $g \leq g_1$, where

$$g_1 = \frac{n_1 c(a_1, n_1) + 3}{n_1 c(a_1, n_1) - 6}.$$

Consequently the desired constant should be chosen as follows

$$A = \max\left\{a_1, 2q - 2 + s - \frac{1}{2g-2}\right\}_{g \leq g_1}.$$

2.5.2. Remark. In case the $C \simeq \mathbf{P}^1$ the optimal value for n is $n_1 = 7$. A computation in this case shows that $A(5, 0) = 2.84$ and $A(6, 0) = 3.85$.

§3. On a class of semi-stable fibrations

3.1. This paragraph is devoted to the proof of Theorem 3.1 formulated in the Introduction. For convenience it will be given in a series of several claims below. We keep the same notation as before. Recall that we have defined the number s_0 for a fibration $f\colon X \to C$ such that

$$s_0 = \text{the number of } S_0,\ S_0 = \{t \in C\colon g(\tilde{X}_t) = g_0\}.$$

3.1.1. Definition. A fibration f is called regular iff there is at least a fibre X_t such that $g(\tilde{X}_t) = g_0$, i.e., $s_0 > 0$. Otherwise we say that f is irregular.

The following lemma is essentially due to [B3].

3.1.2. Lemma. *Let $f\colon X \to \mathbf{P}^1$ be a relatively minimal fibration with $g > 1$ and*

1) $\omega^2 = 4(g-1)$,
2) $\chi(\mathcal{O}_X) \geq 3 - g$.

Then

1) *X is a ruled surface (in the sense of* [B4]), *defined as a family $\pi\colon X \to E$, $g(E) = g_0$. Families f and π define a 2-to-1 map from X to $Y = E \times \mathbf{P}^1$ with branch locus, say B.*

2) *In particular if $g_0 = 0$ then X is a rational surface and f is hyperelliptic. Moreover X is the canonical resolution of $X(Y, \gamma, B)$, $2\gamma \equiv B$ with double rational singularities. The second projection of Y induces f and B is of numerical type* $(2a, 2)$, *where $a = g + 1 - 2g_0$.*

3) *If f is regular then $g_0 \leq 1$.*

Proof. 1) The Riemann-Roch inequality applied to $K_X + F$ gives us

$$h^0(K_X + F) \geq \chi(\mathcal{O}_X) + g - 1.$$

On the other hand the first condition implies that $(K_X + F)^2 = 0$. Therefore the linear system $|K_X + F|$ is composed with a pencil without base points. So writing the general member of this pencil as the sum of irreducible components $D_\lambda = \sum_i P_{\lambda,i}$ we infer from the theorem of Bertini that they mutually do not intersect and by the same token

$$(P_{\lambda,i}, P_{\lambda,j}) = 0 \qquad \forall i, j \tag{14}$$

Now we shall use the familiar construction expounded in the proof of theorem of Castelnuovo-de Franchis ([G-H], Chap. IV, §5). Namely let $E = \{P_{\lambda,i}\}_{\lambda,i}$ be the set of connected components of curves in the pencil $\{D_\lambda\}$ which is a (ramified) covering of $\mathbf{P}^1$. Thus one obtains the fibering $\pi\colon X \to E$ by sending each point p to the pair (λ, i) such that $p \in P_{\lambda,i}$. Since it is easy to see from the adjunction formula and (14) that $(K_X, P_{\lambda,i})$ are even integers ≥ -2 and $\sum_i (K_X, P_{\lambda,i}) = 2 - 2g < 0$. On the other hand all $P_{\lambda,i}$ are homologically equivalent, so that

$(K_X, P_{\lambda,i}) = -2\ \forall i$. Thus one gets $P_{\lambda,i} \simeq \mathbf{P}^1$, $(F.\ P_{\lambda,i}) = 2$ and the first assertion.

2) Evidently then $g_0 = g(E) = 0$ implies the rationality of X and that f is hyperelliptic. In particular, $\chi(\mathcal{O}_X) = 1$, so that $d = g$ by standard formulae using the Leray spectral sequence (cf. the proof of 1.2.5 or Appendix A). Hence $\lambda_f := \omega^2/d = 4 - 4/g$. So the assertion follows direcly from 1.2.6 and standard calculations for double coverings with rational singularities (cf. [H], [Per1]).

3) If there is a singular fibre say X_t such that $g(\tilde{X}_t) = g_0$ and if $g_0 \geq 2$ then the Riemann-Hurwitz formula shows that X_t is the union of a section and several fibres of π (cf. [B1], C). This contradicts the fact that $(X_t, P_{\lambda,i}) = 2$.

3.1.3. From now on we assume the semi-stability condition for f. We always assume that conditions 1)-2) in Lemma 3.1.2 hold. It is easy to see that in this case $X(Y, \gamma, B)$ is the stable model of X, i.e., can be obtained from X by a contraction of all (-2)-curves in fibres. The analysis of singularities of branch locus B in the above situation is almost the same as in [Tn]. If we denote by $F_0(\simeq E)$ the fibre of type (0, 1) on Y then

For any point $p \in B \cap F_0$ *the intersection number* $(B, F_0)_p \leq 2$.

Further as in [Tn] we denote by F_{0t} the images of singular fibres X_t in Y. Let a_t (resp. b_t, c_t) be the number of points p of type A (resp. B, C) on F_{0t}

A: $(B, F_0)_p = 1$,

B: $(B, F_0)_p = 2$, (B, p) is smooth,

C: $(B, F_0)_p = 2$, (B, p) is singular.

By virtue of Lemma 3.1.2 one has the following

3.1.4. Claim. $a_t + 2b_t + 2c_t = 2g + 2 - 4g_0$.

Let us consider the case of $g_0 \leq 1$ (in particular if f is regular)

3.1.5. Claim. We have

i) $s_0 \leq 8$ (well as $g > 4$ if $g_0 = 0$),

ii) for each value $s_0 \geq 5$ one has $g \leq g_0 + \dfrac{r}{s_0 - 4}$,

iii) if $r \leq 1$ then $s_0 \leq 4$.

For the proof it suffices to use the formula for ρ_2 (cf. Appendix A) and the fact that the number of double points in fibres δ is $8g + 4 - 12g_0$ (since for ruled surfaces one has $p_g = 0$ and then it remains to use the first condition in Lemma 3.1.2 together with formulae (A7), (A8) Appendix A). Note that obviously $s_0 = 8$ implies $s = s_0$ and, as will be shown below (3.2.1) the case $g_0 = 1$, $n = 3$ gives an example with $s = s_0 = 8$.

3.1.6. Case $s_0 = 4$, $r = 0$. Since in this case again by the formula for ρ_2 one sees that $g(\tilde{X}_t) = g\ \forall t \notin S_0$. The Riemann-Hurwitz formula shows that the image of X_t in the stable model $X(Y, \gamma, B)$ is irreducible and therefore, smooth, except for the degeneracies at the cross of two elliptic curves ($g = 2$, $g_0 = 1$). But in this case each elliptic curve is isomorphically mapped onto E that contradicts Persson's result (cf. Corollary 1.2.6 of §1). Thus one obtains that all X_t are smooth $\forall t \notin S_0$ so that $s = s_0 = 4$ and by virtue of Theorem 2.4 this case is not realized.

3.1.7. Case $s_0 = 4$, $r = 1$, $g_0 \leq 1$. Following [Tn] we take a new branch curve $\tilde{B} = B + \sum_{t \in S_0} F_{0t}$. Then $\tilde{B}$ has the following types of double points on F_{0t}

$$a_t A_1 + b_t A_3 + \sum_{\mu_p \geq 1, p \in F_{0t}} D_{\mu_p + 3},$$

where μ_p is the Milnor number at (B, p), i.e., the length of corresponding chain of (-2)-curves.

3.1.8. Claim. Let $\tilde{X}$ be the canonical resolution of $X(Y, \tilde{\gamma}, \tilde{B})$, where $2\tilde{\gamma} \equiv \tilde{B}$. Then $\tilde{X}$ is of general type with the following invariants

$$c_1^2(\tilde{X}) = 4(g - 1),$$

$$\chi(\mathcal{O}_{\tilde{X}}) = 2g + 1 - 3g_0.$$

The proof is a direct calculation on double coverings with rational singularities.

3.19. Claim. $g \leq 6g_0 + 3r + 7$.

Proof. It suffices to remark that

$$\sum_{t \in S_0} \left(\frac{a_t}{2} + b_t + c_t \right) = 4g + 4 - 8g_0,$$

$$\sum_{t \in S} \mu_p \geq 4g - 8g_0 - r + \frac{1}{2} \sum_{t \in S_0} a_t.$$

Further applying the Sakai-Miyaoka inequality to $\tilde{X}$ as in [Tn] we are done.

3.2. Examples

3.2.1. Explicit calculations for the series in [B1]. First recall the construction in [B1]. Consider a morphism $\varphi\colon E \to \mathbf{P}^1$ of degree n and an automorphism u of $\mathbf{P}^1$ such that

(i) all ramification points of φ have index 2,

(ii) the set $R \subset \mathbf{P}^1$ of branch points of φ is stable under the action of u and contains no fixed points of u.

Put $Y = E \times \mathbf{P}^1$, $B = \Gamma_\varphi \cup \Gamma_{u \circ \varphi}$, $\gamma = \Gamma_\varphi$, where Γ_φ, $\Gamma_{u \circ \varphi}$ are graphs of φ and $u \circ \varphi$ respectively. Then the double covering $X(Y, \gamma, B)$ has singularities of type A_1 at the intersection points of $\Gamma_\varphi \cap \Gamma_{u \circ \varphi}$. Let X be its canonical resolution then $f\colon X \to \mathbf{P}^1$ induced by the second projection of Y is semi-stable. Under hypothesis (i), (ii) above we have

1) $S = R \cup \{\text{two fixed points of } u\}$.

2) $g(F) = n - 1 + 2g_0$, where $g_0 = g(E)$.

3) $B \equiv 2n\mathbf{P}^1 + 2E$, $\gamma \equiv n\mathbf{P}^1 + E$, where $\mathbf{P}^1$ and E denote fibres of type (1, 0) and (0, 1) respectively (an abuse of notation).

4) $K_X^2 = 2(K_Y + \gamma)^2 = -4(g(F) - 1)$, $\rho = b_2 = 4n + 2$.

So we have the situation in Theorem 3.1. Now for simplicity we assume that the number of R is smallest as n and g_0 are given.

5)
$$s = \begin{cases} 6 + \dfrac{4(g_0 - 1)}{n} & n \text{ even } | 4(g_0 - 1) \\ 6 + \dfrac{4g_0}{n - 1} & n - 1 \text{ even } | 4g_0 \end{cases}$$

6) (The case of n even and $g_0 = 0$ will be considered separately in 3.3.2 below)

$$r = \begin{cases} 2n & g_0 \geq 1 \\ 2(n - 1) & n \text{ odd}, g_0 = 0 \end{cases} \qquad g(\tilde{X}_t) = \begin{cases} 2g_0 - 1 & \text{either } n \text{ even, or } n \text{ odd and } t \in S \setminus R \\ 2g_0 & n \text{ odd}, t \in R \end{cases}$$

In particular f is regular in the sense of Definition 3.1.1 iff $g_0 \leq 1$. In fact such fibrations with $s = 6$ further are constructed explicitly in [B1].

3.2.2. Example of genus 3 fibration with $s = 5$ **([B1]).** As one has seen in 3.3.1 case n even and $g_0 = 0$ is possible iff $n = 2, 4$. The first one ($n = 2$, $s = 4$) gives an elliptic example with configuration $(2I_4, 2I_2)$ (cf. [B1]). Another one ($n = 4$, $s = 5$) can be defined, e.g., by the folowing equation

$$y^2 = (x^4 - tx^2 + 1)[(t + 2)x^4 - 2(t - 6)x^2 + (t + 2)]$$

with 5 singular fibres over $\{\pm 2\sqrt{-3}, \pm 2, \infty\}$. So that $s = s_0 = 5$, $g_0 = 0$, $r = 3$. In fact it is namely the example obtained from the above construction by putting $\varphi(x) = x^2 + \dfrac{1}{x^2}$, $E \simeq \mathbf{P}^1$, $u(t) = \dfrac{2t + 12}{2 - t}$.

3.2.3. Example of genus 2 fibration with $s = 5$. M.-H. Saito has constructed the following interesting example which I reproduce below with his kind permission.

Take the Hirzebruch surface $\mathbf{F}_3 = \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(3)) \to \mathbf{P}^1$ and let $\mathscr{Y}$ and $\mathscr{X}$ be homogeneous coordinates of fibres so that $S_0 = \{\mathscr{Y} = 0\}$, $S_\infty = \{\mathscr{X} = 0\}$ are sections such that $S_0^2 = 3$, $S_\infty^2 = -3$ and $(S_0, S_\infty) = 0$. In fact we have $S_0 \equiv S_\infty + 3F_0$, where F_0 is the fibre of type (0, 1). Consider the pencil in $|2S_0|$ defined by the following two elements

$$C_0 : \mathscr{Y}\{\mathscr{Y} + x(x - 1)\mathscr{X}\} = 0,$$
$$C_\infty : \mathscr{X}\{[x^4 - (x - 1)^4]\mathscr{Y} - (x - 1)^5 x\mathscr{X}\} = 0,$$

where x is an inhomogeneous coordinate of the base $\mathbf{P}^1$.

Write $C_0 = S_0 \cup D_0$, $C_\infty = S_\infty \cup D_\infty$. Then the base points of the pencil are the set of points

$$p_0 = \{x = 0\} \cap \{\mathcal{Y} = 0\},$$
$$p_1 = \{x = 1\} \cap \{\mathcal{Y} = 0\}.$$

Moreover the intersection multiplicities of D_∞ with S_0 and D_0 are given as follows

$$\operatorname{mult}_{p_0}(D_\infty, S_0) = \operatorname{mult}_{p_1}(D_\infty, D_0) = 1,$$
$$\operatorname{mult}_{p_1}(D_\infty, S_0) = \operatorname{mult}_{p_0}(D_\infty, D_0) = 5.$$

So blowing up $\mathbf{F}_3$ 6 times each at p_0, p_1 and their infinitely near points one obtains a genus 2 fibration which can be written in the following equation

$$C_t \colon y^2 = P(x, t),$$

where

$$P(x, t) = 4t(x-1)^5 x + \{x(x-1) + t[x^4 - (x-1)^4]\}^2.$$

The discriminant of $P(x, t)$ as a polynomial in x can be calculated by a computer program as

$$\Delta(t) = \text{const.}\, t^{14}(4t+1)^2(8t+1)^2(t-1).$$

Thus the pencil has the following configuration of singular fibres (in the notation of [N-U]): $(I_{6-6-1}, I_{1-0-0}, I_{1-0-0}, I_{1-1-0}, I_{1-1-1})$ resp. over $\{0, 1, -1/4, -1/8, \infty\}$. So that $s_0 = 3$, $r = 0$, $g_0 = 0$. The fact that we have types I_{6-6-1} and I_{1-1-1} over 0 and ∞ respectively is clear from the geometric construction above. Further since the leading coefficient and that in x^5 of $P(x, t)$ are $t(4t+1)$ we have the type I_{1-0-0} (over $t = -\frac{1}{4}$ and $t = 1$) and the type I_{1-1-0} over $t = -\frac{1}{8}$ according to [N-U]. Thus the correction terms coming from $\left(\dfrac{dx}{y} \wedge x\dfrac{dx}{y}\right)^{\otimes 10}$ are given as follows (cf. [U], and 1.2.3 and 1.2.5).

$$\operatorname{ord}_t\left(\frac{dx}{y} \wedge x\frac{dx}{y}\right)^{\otimes 10} = \begin{cases} -1 & \text{over } t = 0, -\dfrac{1}{4} \\ 2 & \text{over } t = \infty \\ 0 & \text{otherwise} \end{cases}$$

3.2.4. Remarks. 1) As noted in the Introduction this example is enough close to a "counterexample" to Beauville's conjecture in the sense that we were able to obtain it if two fibres of the type I_{1-0-0} (over 1 and $-1/4$) could be deformed to a fibre of the type I_{1-1-0}.

2) It makes sense in another connection to generalize the construction in 3.2.3 for any $\mathbf{F}_n (n \geq 2)$, e.g., the pencil

$$C_t \colon \mathcal{Y}^2 + \{x(x-1) + t[x^2 - (x-1)^2]\}\mathcal{Y}\mathcal{X} - t(x-1)^3 x \mathcal{X}^2 = 0$$

on $\mathbf{F}_2$ gives us familiar configuration (I_8, I_2, I_1, I_1) over $\{0, \infty, 1/4, -1/4\}$ respectively.

Appendix A: semi-stable fibrations and their integer invariants

This appendix is essentially the major part of [Vt0]. Our summary will be based on [A], [B1], [Par2], [Sz], [Vg] and [Vt4].

A.1. For a semi-stable fibration $f\colon X \to C$ there exist two fundamental exact sequences

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & f^*\Omega^1_C & \longrightarrow & \Omega^1_X & \longrightarrow & \Omega^1_{X/C} & \longrightarrow & 0 \\ & & & & & & \downarrow & & \\ & & & & & & \omega_{X/C} & & \\ & & & & & & \downarrow & & \\ & & & & & & N & & \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array} \tag{A1}$$

where $N = \bigoplus_i \mathbf{C}(P_i)$-the skyscraper sheaf sitting at the double points of fibres.

The dual to (A1) exact sequence is the following:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & T_{X/C} & \longrightarrow & T_X & \longrightarrow & f^*T_C & \longrightarrow & N & \longrightarrow & 0 \\ & & \wr\| & & & & & & & & \\ & & \omega^{-1}_{X/C} & & & & & & & & \end{array} \tag{A2}$$

The following lemma is a direct consequence of (A1) and (A2).

A.1.1. Lemma. *The following formulae are valid*

1) $\chi(X) = c_2(X) = 4(q-1)(g-1) + \delta$ (A3)

where $\delta = \sum_{t \in C} \delta_t$, δ_t *is the number of double points in* X_t,

2) $c_1^2(X) = \omega^2 + 8(q-1)(g-1)$ (A4)

We denote by n_t the number of components of X_t.

A.1.2. Lemma ([B1]). *The following formula holds*

$$\delta_t - n_t + 1 = g - g(\tilde{X}_t) \tag{A5}$$

where $g(\tilde{X}_t)$ *denotes the genus of the normalization of* X_t.

Proof. We give here an elementary proof of (A5). First one remarks that double points of X_t are divided into two types:

— the singular points of components each of these lowers the genus of corresponding component onto one,

— the intersection points of components.

If we put $X_t = \sum C_i$ then from $X_t^2 = 0$ it follows that

$$\sum C_i^2 + 2\sum_{i<j} C_i \cdot C_j = 0.$$

By the adjunction formula we have

$$\sum_i (2p_a(C_i) - 2 - K_X \cdot C_i) + 2\sum_{i<j} C_i \cdot C_j = 0.$$

Since $\sum_i K_X \cdot C_i = K_X \cdot X_t = 2g - 2$ then

$$\sum p_a(C_i) + \sum_{i<j} C_i \cdot C_j - n_t + 1 = g.$$

Now remark that the quantity of points of the first type is $\sum p_a(C_i) - g(\tilde{X}_t)$ and the one of the second type is $\sum_{i<j} C_i \cdot C_j$. We obtain the assertion of the lemma.

A.1.3. Corollary ([A]). *On the stable model X the number of double points in each fibre is not greater than $3g - 3$.*

Proof. Indeed, in view of the absence of rational (-2)-curves it follows that $n_t \leq 2g - 2$. Applying the formula in Lemma A.1.2 one obtains: $\delta_t \leq 3g - 3$.

A.1.4. Lemma. *We have the following formulae*

1) $\chi(\mathcal{O}_X) = d + (1 - q)(1 - g)$ (A6)

2) $\omega^2 = 12d - \delta$ (A7)

3) $p_g = d + (g - g_0)(q - 1) + g_0 q$ (A8)

where recall that $g_0 = \dim$ of the fixed part of $\mathrm{Pic}^0(X/C)$.

The proof is standard by using the Leray spectral sequence for f, the Riemann-Roch theorem on C and Noether's classical formula.

A.2. Let ρ denote the Picard number of X that is $\rho = \mathrm{rk}_{\mathbf{Q}}\, NS(X)$, where $NS(X)$ is the Néron-Severi group of X. In order to get Arakelov's estimate for d we shall need the following obvious lemma.

A.2.1. Lemma ([Par2], [B1]). *One has the following inequality*

$$\rho \geq 2 + \sum_{t \in C} (n_t - 1) \tag{A9}$$

Recall that n_t denotes the number of components of X_t.

A.2.2. Definition. The defect r is to be defined as in the following formula

$$\rho = 2 + \sum_{t\in C} (n_t - 1) + r, \tag{A10}$$

(cf. [Tt]).

A.2.3. Lemma. $g(\tilde{X}_t) \geq g_0$, $\forall t \in C$.

The proof can be found in [A] and [B1]. There are two ways to get it: the first one ([A]) uses the Néron model of J, the second one ([B1]) uses Albanese's morphism of X. Here note a nice version of the proof by M.-H. Saito and S. Ishii using Mixed Hodge Structures (personal communication). We remark that there exist fibrations (so-called "irregular") such that $g(\tilde{X}_t) > g_0$, $\forall t \in S$ (cf. [B1] or § 3).

A.2.4. Proposition ([A]). *In the same notation the following inequality is valid*

$$d \leq (g - g_0)\left(q - 1 + \frac{s}{2}\right) + g_0 q \tag{A11}$$

Proof. By definition we have

$$\chi(X) = 2 - 4q(X) + h^{1,1} + 2p_g.$$

Further by (A3): $\chi(X) = 4(1 - q)(1 - g) + \delta$ and using formula (A8) for p_g one gets

$$d = (g - g_0)(q - 1) + \frac{1}{2}(\delta - h^{1,1}) + 1 + g_0 q.$$

From the Lefschetz (1, 1)-theorem it follows that $\rho \leq h^{1,1}$. This fact, together with (A9) and (A5), enables us to deduce

$$d \leq (g - g_0)\left(q - 1 + \frac{s}{2}\right) + g_0 q - \frac{1}{2}\sum_{t\in S} (g(\tilde{X}_t) - g_0).$$

Now estimate (A11) already can be obtained if remark that $g(\tilde{X}_t) \geq g_0$ (Lemma A.2.3) $\forall t \in C$.

A.2.4. Corollary.

$$\sum_{t\in S} n_t \leq [5(g - g_0) + 1]s + 12[(g - g_0)(q - 1) + g_0 q].$$

Proof. Indeed by (A5) one gets

$$\sum_{t\in S} n_t \leq \delta - (g - g_0 - 1)s.$$

On the other hand it is known that $\omega^2 \geq 0$ (see, e.g., [Par1], [A], [Sz]).

This means that $\delta \leq 12d$. Now the corollary immediately follows from (A11).

A.2.6. Remark. In the elliptic case (i.e., $g = 1$) we have

$$\sum_{t \in S} n_t \leq 6s + 12(q - 1) \tag{A12}$$

Essentially this inequality belongs to L. Szpiro. It is well known that a number field analog of (A12) enables us to deduce Fermat's last theorem due to G. Frey. Concerning the connection with Frey's result, see [Par3].

A.3. Now let us denote "the number of transcendental cycles" of X (or the number of independent 2-differentials of the second kind) by ρ_2.

A.3.1. Proposition. *In the previous notation the following formula holds*

$$\rho_2 = (g - g_0)(4q - 4 + s) - \sum_{t \in S} (g(\tilde{X}_t) - g_0) - r + 4g_0 q \tag{A13}$$

A.3.2. Corollary. $\rho_2 \leq (g - g_0)(4q - 4 + s) + 4g_0 q$.

Proof. It is well known that (see, for example, [G-H], Chap. III, §5)

$$h^{1,1} = \rho + \rho_2 - 2p_g.$$

By the definition of $\chi(X)$ and from (A3) it is easy to see that

$$\rho_2 = 4(1 - q)(1 - g) + (\delta - \rho) + 4(q + g_0) - 2.$$

Now applying (A5) and (A10) we obtain the required result.

A.4. Now let us consider semi-stable fibrations over the projective line $\mathbf{P}^1$. Since $q = 0$ we rewrite the basic relations of previous parts in the following form.

A.4.1. Lemma. *In the notation of previous parts the following relations are valid*

1) $\chi(\mathcal{O}_X) = d - (g - 1)$ (A14)

2) $p_g = d - (g - g_0)$ (A15)

3) $\rho_2 = (g - g_0)(s - 4) - \sum_{t \in S} (g(\tilde{X}_t) - g_0) - r$ (A16)

Recall that we consider the case with $g \geq 1$. As an immediate consequence one gets the following

A.4.2. Theorem ([B1]). *Let $f\colon X \to \mathbf{P}^1$ be a non-isotrivial semi-stable fibration. Then $s \geq 4$. Moreover, $s = 4$ is equivalent to the following three conditions*

1) $\rho_2 = 0$,

2) $g(\tilde{X}_t) = g_0 \ \forall t \in S$,

3) $r = 0$.

Proof. It is obvious by virtue of (A16).

A.4.3. Remarks. 1) It should be noted that the author learned from M.-H. Saito recently a nice proof of Beauville's estimate $s \geq 4$ using Variation of Hodge Structure (unpublished).

2) Using the theory of algebraic surfaces A. Beauville (*loc. cit*) has proved a stronger statement that $g_0 = 0$ if $s = 4$.

A.4.4. Corollary. *The notation being as above, if* $s = 4$ *then*

1)
$$f_* \omega_{X/\mathbf{P}^1} \simeq \bigoplus_{g \text{ copies}} \mathcal{O}_{\mathbf{P}^1}(1),$$

2) $\omega - F$ *is numerically effective,*

3) $\omega^2 \geq 4(g-1)$.

Proof. 1) By the second remark of A.4.3 one has: $d = g$ and $g_0 = 0$. On the other hand, according to [A],

$$g_0 = \dim H^0(\mathbf{P}^1, R^1 f_* \mathcal{O}_X).$$

Consequently it follows from applying Grothendieck's theorem on vector bundles over $\mathbf{P}^1$ (see, e.g., [G-H], Chap. IV, § 3) that

$$R^1 f_* \mathcal{O}_X \simeq \bigoplus_{g \text{ copies}} \mathcal{O}_{\mathbf{P}^1}(-1).$$

So that by the duality we obtain the required assertion.

2) From Lemma 3 ([X]) we have known that $\omega - \mu_f(f_* \omega_{X/\mathbf{P}^1}) \cdot F$ is nef. It remains to use the definition of μ_f (cf. [X]) and the first assertion.

3) Evident by 2).

Appendix B. Degenerate configurations for a class of hyperelliptic semistable fibrations

B.1. Assume that there are at most two components in each fibre on the stable model and all the Weierstrass points P_i of the generic fibre are rational. Recall in this case Arakelov's formula ([A])

$$\frac{g(g+1)}{2} \omega \equiv \frac{g(g-1)}{2} \sum_i \bar{P}_i + \sum_j C_j + f^* \bar{d}$$

where $\bar{P}_i$ denote the Weierstrass sections corresponding to P_i, C_j (if any) are components of fibres and $\bar{d}$ denotes the class of $\det(f_* \omega_{X/C})$. It is not difficult to show that the contribution of each chain $\{C_1, \ldots, C_r\}$ of type A_r of (-2)-curves intersected with Weierstrass sections is given as follows (r odd)

$$\frac{g(g-1)}{2}\left(C_1 + 2C_2 + \cdots + \frac{r+1}{2} C_{(r+1)/2} + \cdots + 2C_{r-1} + C_r\right).$$

B.2. The following fact is well-known ([A], [Par2]): we have an exact sequence

$$0 \to T \simeq \mathbf{G}_m^{g-g(\tilde{X}_t)} \to \mathrm{Pic}^0(X_t) \to \mathrm{Pic}^0(\tilde{X}_t) \to 0$$

where $\tilde{X}_t$ is the normalization of X_t, $\mathbf{G}_m \simeq \mathbf{C}^*$. So $\mathrm{Pic}^0(\tilde{X}_t)$ is an abelian variety of dim $g(\tilde{X}_t)$. Below we give all possible configurations together with description of $\mathrm{Pic}^0(X_t)$ respectively. As far as a general rule of computing $\mathrm{Pic}^0(X_t)$ is concerned and for the local behavior of Weierstrass sections we refer to [Par2].

$\boldsymbol{I}_{g(\tilde{X}_t)}$ (analogous to types ***II, III*** of [Par2])

Here C is a hyperelliptic or rational curve of genus $g(\tilde{X}_t)$, $\{C_j^i\}_{j=1}^{r_i}$ is a chain of (-2)-curves, r_i odd, the number of such chains $= g - g(\tilde{X}_t)$,

$$\mathrm{Pic}^0(\tilde{X}_t) \simeq \mathrm{Pic}^0(C),\ T \simeq \bigoplus_{i=1}^{g-g(\tilde{X}_t)} [(\mathbf{C}^*)_{c_1^i} \oplus (\mathbf{C}^*)_{c_2^i}]/\Delta,$$

and c_1^i, c_2^i are intersection points of the chain $\{C_j^i\}$ with C, Δ the diagonal subgroup of a direct sum.

$\boldsymbol{II}_{g(\tilde{X}_t), g_1, m_1, m}$ $(m = 1, 2)$ (analogous to types ***IV − VI*** of [Par2])

Here C, D are hyperelliptic or rational curves of genera g_1, g_2, $\{C_k^i\}_{k=1}^{r_i}$, $\{D_k^j\}_{k=1}^{r_j}$, $\{Z_k^{i'}\}$ are chains of (-2)-curves, r_i, r_j odd. We denote by m_1, m_2, m the number of chains $\{C_k^i\}$, $\{D_k^j\}$, $\{Z_k^{i'}\}$ respectively and take a lexicographical order: $g_1 \geq g_2$ (if $g_1 = g_2$ then $m_1 \geq m_2$). The curves C, D are joined by m chains $\{Z_k^{i'}\}$. We have the following relations

$$g_1 + g_2 = g(\tilde{X}_t),\ m_1 + m_2 = g - g(\tilde{X}_t) - m + 1 \tag{B1}$$

$$m_i + g_i \leq g - 1 \ (\text{if } m = 2 \text{ then } m_i + g_i < g - 1) \qquad i = 1, 2,$$

$$\mathrm{Pic}^0(\tilde{X}_t) \simeq \mathrm{Pic}^0(C) \times \mathrm{Pic}^0(D),$$

$$T \simeq \bigoplus_{i=1}^{m_1} [(\mathbf{C}^*)_{c_1^i} \oplus (\mathbf{C}^*)_{c_2^i}]/\Delta \oplus \bigoplus_{j=1}^{m_2} [(\mathbf{C}^*)_{d_1^j} \oplus (\mathbf{C}^*)_{d_2^j}]/\Delta$$

$$\oplus \Big[\bigoplus_{i'=1}^{m} [(\mathbf{C}^*)_{a_{i'}} \oplus (\mathbf{C}^*)_{b_{i'}}]/\Delta\Big]/\Delta,$$

where $c_1^i, c_2^i, a_{i'}$ (resp. $d_1^j, d_2^j, b_{i'}$) are intersection points of the chain $\{C_k^i\}$, $\{Z_k^{i'}\}$ (resp. $\{D_k^j\}$, $\{Z_k^{i'}\}$) with C (resp. D). We show why m is 1 or 2. In this case the counting of Weierstrass sections implies $m_1 + m_2 + g_1 + g_2 \geq g - 1$. Comparing with (B1) one obtains $m \leq 2$ as required.

III (analogous to the type ***VII*** of [Par2])

Here C, D are curves of genus 0 joined by $g + 1$ chains $\{C_j^i\}_{j=1}^{r_i}$ of (-2)-curves, r_i odd, $i = 1, \ldots, g + 1$,

$$\mathrm{Pic}^0(X_t) \simeq T \simeq [\bigoplus_{i=1}^{g+1} [(\mathbf{C}^*)_{c_i} \oplus (\mathbf{C}^*)_{d_i}]/\Delta]/\Delta$$

and c_i (resp. d_i) are intersection points of the chains $\{C_j^i\}$ with C (resp. D).

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