J. Math. Kyoto Univ. (JMKYAZ)
35-3 (1995) 495-533

Relations on Pfaffians: number of generators

Dedicated to Mrs. Nishikawa

By

Mitsuyasu HASHIMOTO

Introduction

Let R be a commutative ring with unity, and consider a polynomial ring $S = R[x_{ij}]_{1 \le i < j \le n}$, where n is a positive integer. With letting $x_{ji} = -x_{ij}$ and $x_{ii} = 0$, we can form a generic alternating matrix (x_{ij}). The ideal Pf_{2t} of S generated by all $2t$-subPfaffians of (x_{ij}) is called the *(generic) Pfaffian ideal* of order $2t$.

The main purpose of this article is to determine the number of minimal generators of the relation module, or the fisrt syzygy module of a Pfaffian ideal provided R is a field. In the following cases, the relation module of the Pfaffian ideal is known to be generated by linear relations (i.e., elements of degree $t+1$, because $2t$-Pfaffians are homogeneous of degree t): the case $R \supset \mathbf{Q}$ [10, 12], $t=1$, $n=2t$ (trivial), $n=2t+1$ [5] or $n=2t+2$ [21]. Moreover, Kurano [16] proved that when R is a field of characteristic $p>0$ and if $2p > n-2t$, then the first syzygy of the Pfaffian ideal Pf_{2t} is generated by the linear relations.

On the other hand, Kurano showed that when $n=8$ and $t=2$, we need a new generator of degree $t+2$ of the first syzygy when R is a field of characteristic two [17]. In particular, we see that there is no minimal free resolution of generic Pfaffian ideals over the ring of integers $\mathbf{Z}$ in general.

Our main result is

Theorem 7.8. *Let K be a field of characteristic p.*

1 *If $p \neq 2$, then the first syzygy module of Pf_{2t} as an S-module is generated by linear relations. Or equivalently, we have $\beta^{p}_{2,j} = 0$ for $j \neq t+1$.*

2 *If $p=2$, then we have*

$$\mathrm{Tor}_2^S(S/Pf_{2t}, K) \cong [\mathrm{Tor}_2^S(S/Pf_{2t}, K)]_{t+1} \oplus \bigoplus_{i=1}^{[\log_2 t]} \wedge^{2(t+2^i)} F$$

as $\mathrm{GL}(F)$*-modules so that $\beta^2_{2,j} = 0$ unless $j = t+2^i$ for some $0 \le i \le [\log_2 t]$.*

Communicated by Prof. M. Maruyama, November 29, 1994

3 *We have*

$$\beta_2^p = \begin{cases} n\binom{n}{2t+1} - \binom{n}{2t+2} & (p \neq 2) \\ n\binom{n}{2t+1} - \binom{n}{2t+2} + \sum_{i=1}^{[\log_2 t]} \binom{n}{2^{i+1}+2t} & (p=2), \end{cases}$$

where $\beta_{i,j}^p = \dim_K [\mathrm{Tor}_i^S (K, S/Pf_{2t})]_j$ is the graded Betti number of S/Pf_{2t} over the field K of characteristic p. Note that $\beta_{2,j}^p$ is the number of elements of degree j in a minimal set of homogeneous generators of the first syzygy of the Pfaffian ideal Pf_{2t}.

In particular, the first syzygy is generated by the linear relations when the characteristic is odd. This should be compared with the results on the generic determinantal ideals and the case of generic symmetric matrices due to Kurano [15, 14].

The proof of the theorem heavily depends on Kurano's result: For any p, $\beta_{2,j}^p = 0$ unless $t+1 \leq j \leq 2t$ [16].

For the case $j \leq 2t$, syzygies of Pf_{2t} (at these lower degrees) is closely related to the homology of the t-Schur complexes of the identity map (cf. section 7). To establish this relationship, we need to generalize the plethysm formula [16, 3] to the complex version (section 2, 3).

The t-Schur complex of the identity map was studied to calculate the syzygy of determinantal ideals [23, 7], and its homology is known to be isomorphic to a cohomology group of certain homogeneous vector bundle over a grassmannian variety [23]. For our purpose, we prove a vanishing theorem on the homology of t-Schur complexes (section 4). We also need to utilize Akin-Buchsbaum resolution of Schur modules for skew partitions of length two (section 5). The use of A-B resolution has already appeared in [23]. The strange way of appearing new generators of the first syzygy at characteristic two comes from the strangeness of the homology of arithmetic Koszul complexes (Corollary 5.3).

In section 8, we study the explicit form of the minimal generators of the relation module of the Pfaffian ideals.

Acknowledgement: The contents of section 2 and 3 were worked out when the author was a graduate student at Kyoto University (1987-1988), though proofs of some lemmas are simplified recently. Special thanks are due to Professor J. Nishimura for his kind instruction. The rest of this article was worked out when the author stayed at Brandeis University (1990-1991). Meanwhile, he was partially supported by NSF. My best thanks are due to Professor D. A. Buchsbaum for his kind advice, encouragement and warm hospital-

ity. Special thanks are also due to. J. Weyman, K. Kurano, G. Boffi, M. Artale, and R. Sánchez for valuable discussion on this topic.

1. Preliminaries

Throughout this article, R is a commutaive ring with 1. The symbol $\otimes$ means the tensor product $\otimes_R$ over R. We denote the set of non-negative integers, integers and rational numbers by $\mathbf{N}_0$, $\mathbf{Z}$ and $\mathbf{Q}$, respectively. For a prime number p, we denote the prime field of characteristic p by $\mathbf{F}_p$. The symbol $\mathbf{F}_0$ stands for the field of rational numbers $\mathbf{Q}$. For a set X, the cardinality of X is denoted by $\# X$. For a positive integer n, the n^{th} symmetric group is denoted by $\mathfrak{S}_n$. For a row-sequence (i.e., a sequence of non-negative integers) $\alpha=(\alpha_1,\cdots,\alpha_s)$ of degree n (i.e., $|\alpha| (\overset{def}{=}\sum_i\alpha_i)=n$), we define

$$\mathfrak{S}^{\alpha}\overset{def}{=}\{\sigma\in\mathfrak{S}_n \mid \sigma(i)<\sigma(i+1) \text{ unless } i=\textstyle\sum_{j=1}^{l}\alpha_j \text{ for some } l\}.$$

Let F be a finite free R-module and $i\geq 0$. We denote by S_iF (resp. $\wedge^i F$, D_iF) the i^{th} symmetric power (resp. exterior power, divided power) of F. The symmetric algebra (resp. tensor algebra, exterior algebra, divided power algebra) of F is denoted by SF (resp. TF, $\wedge F$, DF). For a map of finite free R-modules $\varphi: G\to F$, the i^{th} symmetric power (resp. exterior power) of φ is denoted by $S_i\varphi$ (resp. $\wedge^i\varphi$). The symmetric (resp. tensor, exterior) algebra of φ is denoted by $S\varphi$ (resp. $T\varphi$, $\wedge\varphi$). For a finite free R-complex

$$\alpha: 0\to G\overset{\psi}{\to}F\overset{\varphi}{\to}E\to 0$$

of length at most two, we denote the i^{th} symmetric power (resp. the symmetric algebra, the tensor slgebra) of α by $S_i\alpha$ (resp. $S\alpha$, $T\alpha$). For these multi-linear objects, we refer the reader to [2] and [9].

We denote by G_R^2 (resp. $\mathscr{C}$) the category of $\mathbf{Z}^2$-graded R-modules (resp. the category of chain complexes of $\mathbf{Z}$-graded R-modules). These categories are symmetric in the sense of [20] with the tensor product $\otimes$ and the twisiting morphism T (see [9, Chapter I]). With forgetting the boundary map, any object in $\mathscr{C}$ is considered as an object in G_R^2. It is easy to check that the forgetful functor $\mathscr{C}\to G_R^2$ is a faithful exact functor of symmetric categories.

An algebra (resp. coalgebra) in $\mathscr{C}$ or G_R^2 is a monoid (resp. comonoid) in the monoidal category $\mathscr{C}$ or G_R^2, respectively. In other words, an algebra (resp. coalgbra) in G_R^2 is a bigraded R-algrbra (resp. colagebra), and an algebra (resp. coalgebra) in $\mathscr{C}$ is an algebra (resp. coalgebra) in G_R^2 whose structure maps are chain maps.

Tensor products of algebras and coalgebras are defined in these categories using the twisting T. If A and B are algebras in $\mathscr{C}$ or G_R^2, then $A\otimes B$ is again an algebra in $\mathscr{C}$ or G_R^2, respectively, with the structure maps

$$m_{A\otimes B}: A\otimes B\otimes A\otimes B \xrightarrow{1\otimes T\otimes 1} A\otimes A\otimes B\otimes B \xrightarrow{m_A\otimes m_B} A\otimes B$$

and

$$u_{A\otimes B}: R\cong R\otimes R \xrightarrow{u_A\otimes u_B} A\otimes B.$$

Tensor product of two coalgebras are defined similarly.

An algebra-coalegbra is called a bialgebra in $\mathscr{C}$ or G_R^2 when the multiplication and the unit maps are coalgebra maps (i.e., comonoid homomorphisms). As the tensor product of two algebras is different from the usual one, bialgebras in our sense are not bialebras in the usual sense in general.

For a map $\varphi: G\rightarrow F$ of finite free R-modules, we consider that G (resp. F) is of degree (1, 1) (resp. 1, 0) so that $\wedge F$, DG, $S\wedge^2 F$, $\wedge(F\otimes G)$ and $D(D_2G)$ are commutative and cocommutative bialgebras in the category G_R^2, and $\wedge\varphi$ and $S(\wedge^2\varphi)$ are commutative, cocommutative bialgebras in the category $\mathscr{C}$ (see [9, Chapter I)].

Let B be an R-algebra. The multiplication (resp. unit) map $B\otimes B\rightarrow B$ (resp. $R\rightarrow B$) is denoted by m_B (resp. u_B). If there is no danger of confusion, it is simply denoted by m (resp. u).

For an R-coalgebra A, we denote the coproduct (resp. counit) of A by Δ_A (resp. ε_A). For $k\geq 0$, we define $\Delta_A^{(k)}: A\rightarrow A^{\otimes k}$ inductively; $\Delta^{(0)}=\varepsilon_A$, $\Delta^{(1)}=\mathrm{id}_A$, and

$$\Delta^{(k)}=(\Delta_A\otimes \mathrm{id}_{A^{\otimes k-2}})\circ\Delta^{(k-1)}$$

for $k\geq 2$, A map of the from $\Delta^{(k)}$ is called an iterated coproduct.

Let $A=\oplus_{i\geq 0}A_i$ be a graded coalgebra, and $\alpha=(\alpha_1,\cdots,\alpha_s)$ a row-sequence. Then the composite map

$$A_{|\alpha|}\rightarrow A\xrightarrow{\Delta^{(s)}} A\otimes A\otimes\cdots\otimes A \xrightarrow{\text{projection}} A_{\alpha_1}\otimes A_{\alpha_2}\otimes\cdots\otimes A_{\alpha_s}$$

is denoted by Δ_α^A (or simply by Δ_A or Δ if there is no danger of confusion). For $a\in A_{|\alpha|}$, we express as

$$\Delta_\alpha^A(a)=\sum_{(\alpha)} a_{(\alpha_1)}^{(1)}\otimes a_{(\alpha_2)}^{(2)}\otimes\cdots\otimes a_{(\alpha_s)}^{(s)} \tag{1.1}$$

(this is a graded version of Sweedler's sigma notation (cf. [24])).

Let $\varphi: G\rightarrow F$ be a map of finite tree R-modules, and λ/μ a skew partition (see [2] or [9]). We denote by $K_{\lambda/\mu}G$ (resp. $L_{\lambda/\mu}F$, $L_{\lambda/\mu}\varphi$) the Weyl module (or coSchur module) of G (resp. the Schur module of F, the Schur complex of φ) with respect to λ/μ. For $t\geq 0$, the t-Schur complex of φ with respect to λ/μ is denoted by $L_{t,\lambda/\mu}\varphi$ (see [7]). For the result, notation and terminology related to these objects (such as standardness of tableaux, standard basis theorem), we refer the reader to [2] and [7]. However, *we use one different notation*. The complex $\wedge_{t,1,\lambda/\mu}\varphi$ in [7] is denoted simply by $\wedge_{t,\lambda/\mu}\varphi$ in this paper, and $\wedge_{t,\lambda/\mu}\varphi = \sum_i \wedge_{t,i,\lambda/\mu}\varphi$ in [7] will never be used in this paper. Thus, $\wedge_{t,\lambda/\mu}\varphi$ stands for the complex

$$\wedge^{t,\lambda_1-\mu_1}\varphi \otimes \wedge^{\lambda_2-\mu_2}\varphi \otimes \wedge^{\lambda_3-\mu_3}\varphi \otimes \cdots,$$

where $\wedge^{t,r}\varphi$ is the truncated subcomplex

$$0 \to \wedge^t F \otimes D_{r-t}G \to \wedge^{t+1}F \otimes D_{r-t-1}G \to \cdots \to F \otimes D_{r-1}G \to D_r G \to 0$$

of $\wedge^r \varphi$ for $r \geq 0$.

Let $A = (a_{ij})$ be an $n \times n$ matrix over R. We say that A is alternating when $a_{ij} = -a_{ji}$ and $a_{ii} = 0$ for $1 \leq i, j \leq n$. The *Pfaffian* Pfaff A of A is defined to be zero when n is odd, and is defined by

$$\begin{aligned}\mathrm{Pfaff}(a_{ij}) &\overset{def}{=} (2^r \cdot r!)^{-1} \sum_{\sigma \in \mathfrak{S}_n} (-1)^\sigma a_{\sigma 1 \sigma 2} \cdots a_{\sigma(n-1)\sigma n} \\ &= \sum_{\substack{\sigma \in \mathfrak{S}_n,\ \sigma 1 < \sigma 3 < \cdots < \sigma(n-1) \\ \sigma(2i-1) < \sigma(2i)\ (\forall i)}} (-1)^\sigma a_{\sigma 1 \sigma 2} \cdots a_{\sigma(n-1)\sigma n}\end{aligned}$$

when $n = 2r$ is even.

Let $t \geq 1$ and (a_{ij}) an $n \times n$-alternating matrix with coefficient in R. The Pfaffian ideal $\mathrm{P}f_{2t}(a_{ij})$ of the alternating matrix (a_{ij}) is the ideal of R generated by all $2t$-subPfaffians of (a_{ij}) (here by a $2t$-subPfaffian we mean the Pfaffian of a submatrix of the form $(a_{\alpha(i)\alpha(j)})_{1 \leq i, j \leq 2t}$ for some sequence $1 \leq \alpha(1) < \cdots < \alpha(2t) \leq n$).

2. Generalized Pfaffian

Throughout this section, $\varphi : G \to F$ denotes a map of finite free R-modules. We set $m = \mathrm{rank}\ F$ and $n = \mathrm{rank}\ G$. We fix a basis $X = \{x_1, \cdots, x_m\}$ of F and $Y = \{y_1, \cdots, y_n\}$ of G, respectively.

In what follows, we require that any algebra (resp. coalgebra) $A = \oplus_{i,j} A_{ij}$ in G_R^2 satisfies the following conditions:

(2.1) A_{ij} is a finite free R-module for any $(i, j) \in \mathbf{Z}^2$.

(2.2) A is positively graded (i.e., $A_{ij} = 0$ unless $(i, j) \in \mathbf{N}_0^2$), or A is negatively graded (i.e., $A_{ij} = 0$ unless $(-i, -j) \in \mathbf{N}_0^2$).

(2.3) $u : R \to A_{0,0}$ (resp. $\varepsilon : A_{0,0} \to R$) is an isomorphism.

Any bialgebra in G_R^2 is required to satisfy $u \circ \varepsilon = \mathrm{id}$ and $\varepsilon \circ u = \mathrm{id}$. A tensor product of two positively (resp. negatively graded) (co-, bi-) algebras in G_R^2 is again a positively graded (resp. negatively graded) (co-, bi-) algebra in G_R^2.

Let M be an object of G_R^2 (resp. $\mathscr{C}$). Assume that M is finite free as an R-module, and that M is positively graded or negatively graded. Assume moreover that $M_{(0,0)}=0$. Then, TM is an algebra in G_R^2 (resp. $\mathscr{C}$). Moreover, TM is a bialgebra in G_R^2 (resp. $\mathscr{C}$) with letting $\varepsilon(m)=0$ and $\Delta(m)=m\otimes 1+1\otimes m$ for $m\in M$ (see [8, Proposition 4.6]).

Let $A=\oplus_{i,j}A_{i,j}$ be a coalgebra in G_R^2 and $h: A\rightarrow M$ a morphism in G_R^2 with M being a finite free R-module. Assume that $M_{i,j}=0$ unless $(i,j)\in \mathbf{N}_0^2\setminus\{0,0\}$, or $M_{i,j}=0$ unless $(-i,-j)\in\mathbf{N}_0^2\setminus\{0,0\}$ so that the tensor algebra TM is a coalgebra in G_R^2 in our sense. We say that A is *cogenerated* by h (or by M) when the coalgebra map $\tilde{h}: A\rightarrow TM$ is injective, where

$$A \xrightarrow{\tilde{h}} TM \xrightarrow{\text{projection}} M^{\otimes i}$$

is given by

$$A \xrightarrow{\Delta} A^{\otimes i} \xrightarrow{h^{\otimes j}} M^{\otimes j}$$

(this defines a map $\tilde{h}\in \mathrm{Hom}(A, \Pi_{j\geq 0}M^{\otimes j})$, and it is easy to check that the image of $\tilde{h}$ is contained in TM). If B is an algebra in G_R^2 generated by a subobject $N\in G_R^2$ as an R-algebra, N is finite free as an R-module, and if $N_{(0,0)}$, then the graded dual $B^\dagger$ of B is cogenerated by N^*. For example, DG and $\wedge F$ are cogenerated by G and F, respectively. The symmetric (co)-algebra SE of a finite free R-module E is cogenerated by E when R is $\mathbf{Z}$-flat, but not in general.

Let A and B be coalgebras in G_R^2 cogenerated by finite free quotients M and N, respectively. Then, $A\otimes B$ is cogenerated by $M\otimes R\oplus R\otimes N$. In particular, if R is $\mathbf{Z}$-flat, and if α is a finite free R-complex of length at most two (with an appropriate degree), then $S\alpha$ is cogenerated by (the underlying module of) α. For example, $S\wedge^2\varphi$ is cogenerated by $\wedge^2\varphi$ when R is $\mathbf{Z}$-flat.

For a subset S of $\mathbf{Z}^2$ and $M=\oplus_{i,j}M_{i,j}\in G_R^2$, we set $M_S\overset{def}{=}\oplus_{(i,j)\in S}M_{i,j}$. If S is finite and $\pm S\subset \mathbf{N}_0^2\setminus\{(0,0)\}$, then TM_S is a bialgebra in G_R^2.

Lemma 2.4. *Let $A=\oplus_{i,j}A_{i,j}$ and $B=\oplus_{i,j}B_{i,j}$ be coalgebras in G_R^2 and $\phi: A\rightarrow B$ a morphism in G_R^2. Let S_0 be a finite subset of $\mathbf{N}_0$, and s a positive integer. We set $S=\{s\}\times S_0$ and $N=B_S$. If B is cogenerated by N, then the following are equivalent:*

1. *ϕ is a homomorphism of R-coalgebras.*

2. *For each $k\geq 0$, the diagram*

$$\begin{array}{ccc} A_{k\cdot S} & \xrightarrow{\phi} & B_{k\cdot S} \\ \Delta_A\downarrow & & \downarrow\Delta_B \\ A_S^{\otimes k} & \xrightarrow{\phi^{\otimes k}} & N^{\otimes k} \end{array} \tag{2.5}$$

is commutative, where $\phi^{\otimes 0}=\mathrm{id}_R$.

Moreover, if these conditions are satisfied, then ϕ is uniquely determined by $\phi_S: A_S$

$\rightarrow B_S = N$. *If both A and B are coalgebras in C, and if*

$$\phi_S : \mathrm{A}_{S,*} \rightarrow B_{S,*} = N$$

is a chain map, then ϕ is a chain map.

Proof. **1**$\Rightarrow$**2** is obvious. We show **2**$\Rightarrow$**1**. With letting $k=0$ in (2.5), we have that ϕ preserves the counit (remember our convention $\Delta^{(0)}=\varepsilon$). We show that ϕ preserves coproduct. Namely, we show $(\phi\otimes\phi)\circ\Delta_A=\Delta_B\circ\phi$. For (i,j) $\notin \mathbf{N}_0\cdot S=\{(ks, kj')\,|\,k\geq 0, j'\in S_0\}$, we have $B_{i,j}=0$, because $B_{i,j}\rightarrow(TN)_{i,j}=0$ is injective. Hence, it suffices to show that the rectangle (a) in the diagram

$$\begin{array}{ccccc}
A_{(k+k')S} & \xrightarrow{\Delta_A} & A_{kS}\otimes A_{k'S} & \xrightarrow{\Delta_A\otimes\Delta_A} & A_S^{\otimes k}\otimes A_S^{\otimes k'} \\
\phi\downarrow & \text{(a)} & \phi\otimes\phi\downarrow & \text{(b)} & \downarrow\phi^{\otimes k}\otimes\phi^{\otimes k'} \\
B_{(k+k')S} & \xrightarrow{\Delta_B} & B_{kS}\otimes B_{k'S} & \xrightarrow{\Delta_B\otimes\Delta_B} & N^{\otimes k}\otimes N^{\otimes k'}
\end{array}$$

is commutative for $k, k'\geq 0$. But we have (b) and the whole rectangle ((a) + (b)) is commutative by **2**. The map

$$\Delta_B\otimes\Delta_B : B_{kS}\otimes B_{k'S}\longrightarrow N^{\otimes k}\otimes N^{\otimes k'}$$

in the diagram is injective, since N cegenerates B. Hence, (a) is also commutative. So **1** and **2** are equivalent.

Now, we show the uniqueness of ϕ. We set $\tilde{h} : B \rightarrow N$ to be the projection. Then, it is easy to see that $\tilde{h}\circ\phi=\tilde{\phi}_S$. Since $\tilde{h}$ is injective, ϕ is uniquely determined by ϕ_S.

We show the last assertion. Since A is decomposed into a direct sum $A=\oplus_{i\in\mathbf{Z}}A_{i,*}$ as an R-complex, it suffices to show that $\phi_i=\phi|_{A_{i,*}}$ is a chain map for all $i\in\mathbf{Z}$. If i is negative or if $i$is not divisible by s, then $\phi_i=0$ is a chain map. Consider the case $i=ks$ for some $k\geq 0$. Then, the diagram

$$\begin{array}{ccc}
A_{ks,*} & \xrightarrow{\phi_{ks}} & B_{ks,*} \\
\Delta_A\downarrow & & \downarrow\Delta_B \\
A_{S,*}^{\otimes k} & \xrightarrow{\phi_S^{\otimes k}} & N^{\otimes k}
\end{array} \tag{2.6}$$

is commutaive. The map $\phi_S^{\otimes k}\circ\Delta_A : A_{ks,*}\rightarrow N^{\otimes k}$ is a chain map, and $\Delta_B : B_{ks,*}$ $\rightarrow N^{\otimes k}$ is an injective chain map. Hence, $\phi_{ks} : A_{ks,*}\rightarrow B_{ks,*}$ is a chain map.

Definition 2.7. Let $k\geq 1$. We define $\pi_{2k}^S=\pi_{2k}^S(F):\wedge^{2k}F\rightarrow S_k\wedge^2F$ by

$$\pi_{2k}^S(f_1\wedge\cdots\wedge f_{2k})=\mathrm{Pfaff}((f_i\wedge f_j)_{1\leq i,j\leq 2k})$$

for $f_1, \cdots, f_{2k}\in F$ (it is easy to see that this map is well-defined). We define $\pi_0^S(F): R\rightarrow R$ to be id_R. We define $\pi_j^S(F):\wedge^jF\rightarrow S\wedge^2F$ to be the zero map when j is odd or negative. We define $\pi^S:\wedge F\rightarrow S\wedge^2F$ by $\pi^S|_{\wedge^jF}=\pi_j^S(F)$.

Lemma 2.8. *The maps $\pi_j^S(F)$: $\wedge^j F \to S\wedge^2 F$ and $\pi^S(F):\wedge F \to S\wedge^2 F$ are uniquely characterized by the following conditions:*

1 $\pi^S(F)|_{\wedge^j F} = \pi_j^S(F)$.

2 *For each $j\in\mathbf{Z}$, π_j^S is a universal natural transformation (see e.g.,* [9, Definition I.3.10]) *from* $\wedge^j(?)$ to $S\wedge^2(?)$.

3 *The map $\pi^S(F)$ is a homomorphism of coalgebras in G_R^2.*

4 *π_0^S: $R\to R$ and $\pi_2^S:\wedge^2 F \to \wedge^2 F$ are identity.*

Proof. First, we check that π_j^S and π^S defined above satisfy these conditions. The conditions **1**, **2** and **4** are trivial from the definition.

To show **3**, we may assume that $R=\mathbf{Z}$, by virtue of **2**. In this case, $S\wedge^2 F$ is cogenerated by $\wedge^2 F$. By Lemma 2.4 and **4**, it suffices to show that two maps

$$\wedge^{2j}F \xrightarrow{\pi_{2j}^S(F)} S_j\wedge^2 F \xrightarrow{\Delta_{S\wedge^2 F}} (\wedge^2 F)^{\otimes j}$$

and $\Delta_{\wedge F}:\wedge^{2j}F\to(\wedge^2 F)^{\otimes j}$ agree for $j\geq 1$. To verify this, we may assume that $R=\mathbf{Q}$, extending the base ring. We may assume $\dim_{\mathbf{Q}} F\geq 2j$, otherwise both maps are zero. In this case, $\wedge^{2j} F$ is an irreducible polynomial representation of $\mathrm{GL}(F)$. So it suffices to show the image of $x_1\wedge\cdots\wedge x_{2j}$ agree, where $\{x_1, \cdots, x_n\}$ is a basis of F ($n=\dim_{\mathbf{Q}} F$). But a straightforward computation will show that the both images agree with

$$2^{-j}\cdot\sum_{\sigma\in\mathfrak{S}_{2j}}(-1)^{\sigma}(x_{\sigma 1}\wedge x_{\sigma 2})\otimes\cdots\otimes(x_{\sigma(2j-1)}\wedge x_{\sigma(2j)}).$$

Hence, π^S is a coalgebra homomorphism.

Now we prove the uniqueness. First, consider the ground ring $\mathbf{Z}$. Then, $S\wedge^2 F$ is cogenerated by $\wedge^2 F$. By Lemma 2.4, $\pi^S(F)$ is uniquely determined by π_2^S. Hence, so is $\pi_j^S(F)$ for any j. By **2**, they are unique for any ground ring R.

Let $k\geq 1$, and consider the image of $\Delta_{DG}: D_{2k}G\to(D_2G)^{\otimes k}$. It is contained in the invariant submodule $((D_2G)^{\otimes k})^{\mathfrak{S}_k}$ under the action of the k^{th} symmetric group $\mathfrak{S}_k$ on $(D_2G)^{\otimes k}$ via the twising, since DG is cocommutative. By [9, Lemma 3.18], $((D_2G)^{\otimes k})^{\mathfrak{S}_k}$ agrees with the image of $\Delta_{D(D_2G)}$: $D_kD_2G\to(D_2G)^{\otimes k}$. Hence, there exists a unique map $\pi_{2k}^D(G): D_{2k}G\to D_kD_2G$ such that $\Delta_{D(D_2G)}\circ\pi_{2k}^D(G)=\Delta_{DG}$. It is clear that π_{2k}^D is a universal natural transformation of universally free functors on G.

Definiton 2.9 We define $\pi_0^D=\mathrm{id}_R$. We define $\pi_j^D(G)$ to be the zero map from D_jG to $D(D_2G)$ when j is odd or negative. We define $\pi^D(G): DG\to D(D_2G)$ by $\pi^D(G)|_{D_jG}=\pi_j^D(G)$.

By definition, π_j^D is universal for all $j \in \mathbf{Z}$. As we have

$$\Delta_{(2^j)}^{DG}(y_1^{(2j)}) = (y_1^{(2)})^{\otimes j} = \Delta_{(1^j)}^{DD_2G}((y_1^{(2)})^{(j)}),$$

we have $\pi_{2j}^D(y_1^{(2j)}) = (y_1^{(2)})^{(j)}$. Henece, π_{2j}^D agrees with φ_j [3] for $j \geq 0$. It is clear that π_2^D is the identity by definition. By Lemma 2.4, π^D is a homomorphism of coalgebras.

Let $\psi : G' \rightarrow F'$ be a map of finite free R-modules. We consider G' and F' are of degree (1,1) and (1,0), respectively. In [9, Chapter, III], a coalgebra homomorphism

$$\theta(\varphi, \psi): \wedge\varphi \otimes \wedge\psi \rightarrow S(\varphi \otimes \psi)$$

is defined. The map θ is uniquely determind by the property:

(2.10) The map $\theta(\varphi, \psi)$ depends only on F, G, F' and G' and is a universal natural transformation of universally free functors on F, G, F' and G', It is a homomorphism of coalgebras in $\mathscr{C}$, and $\theta|_{\wedge^1\varphi\otimes\wedge^1\psi}: \varphi \otimes \psi \rightarrow \varphi \otimes \psi$ is the identity.

We denote $\theta|_{\wedge^j\varphi\otimes\wedge^j\psi}$ by θ_j for $j \geq 0$. With letting $G = 0$ and $F' = 0$, we obtain a universal natural transformation $\phi^\wedge(F, G'): \wedge F \otimes DG' \rightarrow \wedge(F \otimes G')$ (see [9]).

Definition 2.11. We define the *generalized Pfaffian map* $\pi = \pi(\varphi): \wedge\varphi \rightarrow S(\wedge^2\varphi)$ to be the composite map

$$\wedge\varphi = \wedge F \otimes DG \xrightarrow{\Delta\otimes\Delta} \wedge F \otimes \wedge F \otimes DG \otimes DG$$

$$\xrightarrow{\pi^S \otimes \phi^\wedge \otimes \pi^D} S(\wedge^2 F) \otimes \wedge(F \otimes G) \otimes D(D_2G) = S(\wedge^2\varphi)$$

The restriction of π to $\wedge^j\varphi$ is denoted by π_j or $\pi_j(\varphi)$ for $j \in \mathbf{Z}$.

The map $\pi(\varphi)$ depends only on F and G.

Lemma 2.12. *The map $\pi_j(\varphi): \wedge^j\varphi \rightarrow S\wedge^2\varphi$ and $\pi(\varphi): \wedge\varphi \rightarrow S\wedge^2\varphi$ are uniquely characterized by the following properties:*

1. $\pi(\varphi)|_{\wedge^j\varphi} = \pi_j(\varphi)$.
2. *$\pi(\varphi)$ depends only on F and G, and is universal on F and G.*
3. *$\pi(\varphi)$ is a homomorphism of coalgebras in $\mathscr{C}$.*
4. *$\pi_2: \wedge^2\varphi \rightarrow \wedge^2\varphi$ is the identity.*

In particular, we have $\pi_j = 0$ for j odd or negative. The map π_0: $R \rightarrow R$ is the identity. For $k \geq 1$, *the diagram*

(2.13)
$$\begin{array}{ccc} \wedge^{2k}\varphi & \xrightarrow{\pi_{2k}} & S_k\wedge^2\varphi \\ & \Delta\searrow \quad \swarrow\Delta & \\ & (\wedge^2\varphi)^{\otimes k} & \end{array}$$

commutes.

Proof. The uniqueness is proved similarly to Lemma 2.8. The conditions **1**, **2** and **4** are easy. Since $\Delta_{\wedge F}$, $\Delta_{\wedge G}$, π^S, $\psi^{\wedge}$ and π^D are coalgebra maps in G_R^2, π is also a coalgebra map in G_R^2. So the last three sentences in the lemma follow.

It remains to prove that π is a chain map. Let $\mathrm{rank}_R\, F$ and $\mathrm{rank}_R\, G$ be m and n, respectively, By **2**, we may assume that

$$R = \mathbf{Z}[x_{ij} | 1 \leq i \leq m,\ 1 \leq j \leq n]\ ,$$

and that φ is given by matrix (x_{ij}). So $S(\wedge^2\varphi)$ is cogenerated by $\wedge^2\varphi$. Now set $s=2$, $S_0=\{0,1,2\}$, and apply Lemma 2.4. Since $\pi_2=\mathrm{id}$ is a chain map, π is a chain map.

Lemma 2.14. *Let $k \geq 0$. Then, the diagram*

$$(2.15)\qquad \begin{array}{ccc} \wedge^{2k}\varphi & \xrightarrow{\pi_k} & S_k(\wedge^2\varphi) \\ \Delta_{\wedge\varphi}\downarrow & & \downarrow S_k(\Delta_{\wedge\varphi}) \\ \wedge^k\varphi \otimes \wedge^k\varphi & \xrightarrow{\theta_k} & S_k(\varphi\otimes\varphi) \end{array}$$

is commutative.

Proof. Note that the maps

$$(2.16)\qquad \wedge\varphi \xrightarrow{\pi} S(\wedge^2\varphi) \xrightarrow{S(\Delta_{\wedge\varphi})} S(\varphi\otimes\varphi)$$

and

$$(2.17)\qquad \wedge\varphi \xrightarrow{\Delta_{\wedge\varphi}} \wedge\varphi\otimes\wedge\varphi \xrightarrow{\theta} S(\varphi\otimes\varphi)$$

are coalgebra homomorphisms in $\mathscr{C}$. To prove that these two maps agree, we may assume that $R=\mathbf{Q}$ and that φ is the zero map. So it suffcies to show that these maps agree on degree $(2,*)$ component by Lemma 2.4. It is easy to see that degree $(2,*)$ component of (2.16) and (2.17) are $\Delta\colon \wedge^2\varphi \to \varphi\otimes\varphi$. Hence two maps (2.16) and (2.17) agree. Now we take the degree $(2k,*)$ component of (2.16) and (2.17). Since θ is zero on $\wedge^i\varphi\otimes\wedge^j\varphi$ for $i\neq j$, we obtain the commutativity of (2.15).

Lemma 2.18. *Let α be a finite free R-complex of length at most two, and $i, j\geq 0$. Then, the composite map*

$$S_{i+j}\alpha \xrightarrow{\Delta} S_i\alpha\otimes S_j\alpha \xrightarrow{m} S_{i+j}\alpha$$

agrees with $\binom{i+j}{i}\mathrm{id}$.

Proof. We set $\alpha = 0 \to U \to V \to W \to 0$. We may assume that $R=\mathbf{Q}$. In this case, $S\alpha = SW \otimes \wedge V \otimes DU$ is embedded in $T\alpha$ by id_α as a bialgebra in G_R^2, and $S_{i+j}\alpha$ is identified with the invariance $(T_{i+j}\alpha)^{\mathfrak{S}_{i+j}}$ under the action of $\mathfrak{S}_{i+j}$ via the twisting. By [8, Proposition 4.8], the composite map $T_{i+j}\,\alpha \xrightarrow{\Delta} T_i\alpha \otimes T_i\alpha \xrightarrow{m} T_{i+j}\alpha$ agrees with the action of $\sum_{\sigma\in\mathfrak{S}^{i,j}}\sigma^{-1}$. This action agrees with the multiplication by $\binom{i+j}{i}$ on $(T_{i+j}\alpha)^{\mathfrak{S}_{i+j}}$.

3. Generalized Plethysm formula

Using the generalized Pfaffian map defined in the last section, we can generalize the Plethysm formula of $S\wedge^2 F$ [16,3] and DD_2G [3] to the version of complexes.

Definition 3.1. For a sequence of non-negative intergers $\lambda=(\lambda_1, \lambda_2, \cdots, \lambda_l)$, we define $\pi_\lambda : \wedge_\lambda\varphi \rightarrow S(\wedge^2\varphi)$ to be the composite map

$$\wedge_\lambda\varphi = \wedge^{\lambda_1}\varphi\otimes\cdots\otimes\wedge^{\lambda_l}\varphi \xrightarrow{\pi\otimes\cdots\otimes\pi} S(\wedge^2\varphi)\otimes\cdots\otimes S(\wedge^2\varphi) \xrightarrow{m} S(\wedge^2\varphi).$$

By definition, $\pi_\lambda=0$ unless $\lambda=(\lambda_1,\cdots,\lambda_l)$ is *even* (i.e., λ_i is even for any i). When λ is even, then the image of π_λ is contained in $S_{|\lambda|/2}\wedge^2\varphi$.

Lemma 3.2. *For $i, j\geq 0$, the composite map*

$$\wedge^{2(i+j)}\varphi \xrightarrow{\Delta} \wedge^{2i}\varphi\otimes\wedge^{2j}\varphi \xrightarrow{\pi_{(2i,2j)}} S_{i+j}\wedge^2\varphi$$

agrees with $\binom{i+j}{i}\pi_{2(i+j)}$.

Proof. Since π is a graded coalgebra homomorphism, we have

$$\sum_{(\pi a)}(\pi a)^{(1)}_{(2i)}\cdot(\pi a)^{(2)}_{(2j)} = \sum_{(a)}\pi(a^{(1)}_{(2i)})\cdot\pi(a^{(2)}_{(2j)})$$

in $S_i\wedge^2\varphi\otimes S_j\wedge^2\varphi$ for $a\in\wedge^{2(i+j)}\varphi$. On the other hand, we have

$$\sum_{(\pi a)}(\pi a)^{(1)}_{(2i)}\cdot(\pi a)^{(2)}_{(2j)} = \binom{i+j}{i}(\pi a)$$

by Lemma 2.18, and the assertion follows.

We denote the composite map

$$\wedge\varphi\otimes\wedge\varphi \xrightarrow{\theta} S(\varphi\otimes\varphi) \xrightarrow{Sm} S\wedge^2\varphi$$

by $\bar{\theta}$. The restriction of $\bar{\theta}$ on $\wedge^l\varphi\otimes\wedge^l\varphi$ is denoted by $\bar{\theta}_l$ for $l\geq 0$.

Lemma 3.3. *For i, $j\geq 0$ and $k\geq 0$ with $i+j=2k$, the composite map*

$$\wedge^i\varphi\otimes\wedge^j\varphi\xrightarrow{m}\wedge^{2k}\varphi\xrightarrow{\pi_{2k}}S_k\wedge^2\varphi$$

agrees with the composite map

$$\wedge^i\varphi\otimes\wedge^j\varphi\xrightarrow{\Delta\otimes\Delta}\sum_{\substack{0\leq l\leq\min(i,j)\\ i-l:\text{even}}}\wedge^{i-l}\varphi\otimes\wedge^l\varphi\otimes\wedge^l\varphi\otimes\wedge^{j-l}\varphi$$

$$\xrightarrow{\sum\pi_{i-j}\otimes\bar{\theta}_l\otimes\pi_{j-l}}\sum S_{(i-l)/2}\wedge^2\varphi\otimes S_l\wedge^2\varphi\otimes S_{(j-l)/2}\wedge^2\varphi\xrightarrow{m}S_k\wedge^2\varphi.$$

Or equivaletly, we have

$$\pi_{2k}(a\wedge b)=\sum_{\substack{0\leq l\leq\min(i,j)\\ i-l:\text{even}}}\sum_{(a)}\sum_{(b)}\pi_{i-l}(a^{(1)}_{(j-l)})\cdot\bar{\theta}_l(a^{(2)}_{(l)}\otimes b^{(1)}_{(l)})\cdot\pi_{j-l}(b^{(2)}_{(j-l)})$$

for $a\in\wedge^i\varphi$ and $b\in\wedge^j\varphi$.

Proof. We may assume that $R=\mathbf{Q}$ and that φ is the zero map. Consider the two maps

$$\Phi_1:\wedge\varphi\otimes\wedge\varphi\xrightarrow{m}\wedge\varphi\xrightarrow{\pi}S\wedge^2\varphi$$

and

$$\Phi_2:\wedge\varphi\otimes\wedge\varphi\xrightarrow{\Delta\otimes\Delta}\wedge\varphi\otimes\wedge\varphi\otimes\wedge\varphi\otimes\wedge\varphi\otimes\xrightarrow{\pi\otimes\bar{\theta}\otimes\pi}$$

$$S(\wedge^2\varphi)\otimes S(\wedge^2\varphi)\otimes S(\wedge^2\varphi)\xrightarrow{m}S(\wedge^2\varphi).$$

We show that $\Phi_1=\Phi_2$. Because both of them are coalgebra maps, it suffices to show that Φ_1 agrees with Φ_2 on degree two component, to verify this (note that $S\wedge^2\varphi$ is cogenerted by its degree 2 component). But it is easy to check that the restrictions of $\Phi_i (i=1, 2)$ on $\varphi\otimes\varphi$, $\wedge^2\varphi\otimes R$ and $R\otimes\wedge^2\varphi$ are the multiplication of $\wedge\varphi$, respectivey. So we have $\Phi_1=\Phi_2$. Taking an appropriate homogeneous component, we have the assertion of the lemma.

Let $i,j,l\geq 0$. We donote the composite map

$$\wedge_i\varphi\otimes\wedge^j\varphi\xrightarrow{\Delta\otimes\Delta}\wedge^{i-l}\varphi\otimes\wedge^l\varphi\otimes\wedge^l\varphi\otimes\wedge^{j-l}\varphi$$

$$\xrightarrow{\pi\otimes\bar{\theta}\otimes\pi}S\wedge^2\varphi\otimes S\wedge^2\varphi\otimes S\wedge^2\varphi\xrightarrow{m}S\wedge^2\varphi$$

by $p^l_{i,j}$. By Lemma 3.3, we have

(3.4) $$\pi_{i+j}\circ m=\sum_{\substack{0\leq l\leq \min(i,j)\\ i-l:\text{even}}} p^l_{i,j}.$$

We also consider the maps

$$\xi^l_{i,j}\colon \wedge^i\varphi\otimes\wedge^j\varphi \xrightarrow{\tilde{\square}} \wedge^{i+l}\varphi\otimes\wedge^{j-l}\varphi \xrightarrow{\pi_{(i+l,j-l)}} S_{(i+j)/2}\wedge^2\varphi$$

and

$$\eta^l_{i,j}\colon \wedge^i\varphi\otimes\wedge^j\varphi \xrightarrow{\square} \wedge^{i-l}\varphi\otimes\wedge^{j+l}\varphi \xrightarrow{\pi_{(i-l,\,j+l)}} S_{(i+j)/2}\wedge^2\varphi.$$

Lemma 3.5. *Let i, j and s be non-negative integers with $s\leq\min(i,j)$. Then the following holds.*

1. *If $i+j$ is odd or $i+s$ is odd, then $p^s_{i,j}=\xi^s_{i,j}=\eta^s_{i,j}=0$.*

2. *If $i+j$ and is are even, then we have* $\sum_{\substack{0\leq l\leq s\\ s-l:\text{even}}}\binom{(i-l)/2}{(s-l)/2}p^l_{i,j}=\xi^s_{i,j}$.

3. *If $i+j$ and $i+s$ are even, then we have* $\sum_{\substack{0\leq l\leq s\\ s-l:\text{even}}}\binom{(i-l)/2}{(s-l)/2}p^l_{i,j}=\eta^s_{i,j}$.

4. *If $i+j$ is even, then we have*

$$\sum_{\substack{0\leq l\leq s\\ i+l:\text{even}}} R\cdot\xi^l_{i,j}=\sum_{\substack{0\leq l\leq s\\ i+l:\text{even}}} R\cdot p^l_{i,j}=\sum_{\substack{0\leq l\leq s\\ i+l:\text{even}}} R\cdot\eta^l_{i,j}$$

in $\mathrm{Hom}_R(\wedge_{(i,j)}\varphi, S_{(i+j)/2}\wedge^2\varphi)$.

Proof. **1** is obvious. We show **2**. For $a\in\wedge^i\varphi$ and $b\in\wedge^j\varphi$, we have

$$\xi^s_{i,j}(a\otimes b)=\sum_{(b)}\pi_{i+s}(a\wedge b^{(1)}_{(s)})\cdot\pi_{j-s}(b^{(2)}_{(j-s)})$$
$$\sum_l\sum_{(a)}\sum_{(b)}\pi_{i-l}(a^{(1)}_{(i-l)})\cdot\bar{\theta}(a^{(2)}_{(l)}\otimes b^{(1)}_{(l)})\cdot\pi_{s-l}(b^{(2)}_{(s-l)})\cdot\pi_{j-s}(b^{(3)}_{(j-s)})$$

by Lemma 3.3. On the other hand, this equals to $\sum_l\binom{(i-l)/2}{(s-l)/2}p^l_{i,j}(a\otimes b)$ by Lemma 3.2, and **2** is proved. **3** is proved similarly.

We show

$$\sum_{\substack{0\leq l\leq s\\ i+l:\text{even}}} R\cdot\xi^l_{i,j}=\sum_{\substack{0\leq l\leq s\\ i+l:\text{even}}} R\cdot p^l_{i,j}.$$

We may assume that $i+s$ is even. The direction $\subset$ is obvious by **2**. Since we have

$$p_{i,j}^{s}=\xi_{i,j}^{s}-\left(\sum_{\substack{0\leq l\leq s\\ i+l:\text{even}}}\binom{(i-l)/2}{(s-l)/2)}p_{i,j}^{l}\right),$$

the other direction is shown by induction on s. Similarly, we can peove

$$\sum_{\substack{0\leq l\leq s\\ i+l:\text{even}}} R\cdot\eta_{i,j}^{l}=\sum_{\substack{0\leq l\leq s\\ i+l:\text{even}}} R\cdot p_{i,j}^{l}.$$

using **3**, and this completes the proof of **4**.

Definition 3.6. Let $r\geq 0$. For a partition λ of degree r, we define

$$M_{\lambda}(\pi)\overset{def}{=}\sum_{\mu\geq\lambda,\ |\mu|=r}\operatorname{Im}\pi_{2\mu}$$

and

$$\dot{M}_{\lambda}(\pi)\overset{def}{=}\sum_{\mu>\lambda,\ |\mu|=r}\operatorname{Im}\pi_{2\mu}$$

Lemma 3.7. $M_{(1^r)}(\pi)=S_r\wedge^2\varphi$.

Proof. Induction on r. If $r\leq 1$, then the assertion is clear. Since $S\wedge^2\varphi=S\wedge^2F\otimes\wedge(F\otimes G)\otimes DD_2G$, and the R-algebras $S\wedge^2 F$ and $\wedge(F\otimes G)$ are generated by their degree two components, we may assume that $\varphi=(G\rightarrow 0)$ so that $S_r\wedge^2\varphi=D_rD_2G$ by induction assumption. But this case is done in [3, Proposition 1.4].

Lemma 3.8. *Let $r\geq 0$. Then we have*

$$\operatorname{rank}_R(S_r\wedge^2\varphi)=\textstyle\sum_\lambda\operatorname{rank}_R L_{2\lambda}\varphi.$$

where the sum is taken ever all partitions λ of degree r.

Proof. Clearly, we may assume that $R=\mathbf{Q}$ and that φ is a zero map. It suffces to show that

$$S_r\wedge^2\varphi\cong\oplus_\lambda L_{2\lambda}\varphi$$

as polynomial functors on F and G. Thus, we may assume that the ranks of F and G are sufficiently large.

As is well-known, the following Plethysm Formula holds (e.g., [16]).

$$S_r\wedge^2(F\oplus G)\cong\oplus_\lambda L_{2\lambda}(F\oplus G) \tag{3.9}$$

The left hand side is isomorphic to

$$\bigoplus_{i+j+k=r} S_i\wedge^2F\otimes S_j(F\otimes G)\otimes S_k\wedge^2G$$

$$\cong\bigoplus_{i+j+k=r} S_i\wedge^2F\otimes\left(\oplus_{|\mu|=j}K_\mu F\otimes K_\mu G\right)\otimes\left(\oplus_{|\gamma|=k}L_{2\gamma}G\right)$$

by the Plethysm Formula again. So the formal character of the left fand side is

$$\sum_{i+j+k=r}\sum_{|\mu|=j}\sum_{|\gamma|=k}(h_i\circ e_2)(x)s_\mu(x)s_\mu(y)s_{\widetilde{2\gamma}}(y) \tag{3.10}$$

where the set of variables $x=(x_1,x_2,\cdots)$ (resp. $y=(y_1,y_2,\cdots)$) corresponds to the entries of the diagonal matrices in $\mathrm{GL}(F)$ (resp. $\mathrm{GL}(G)$). The symmetric functions h_i, e_i and s_λ denotes the i^{th} complete, i^{th} elementary, and the Schur function, respectively. The $\circ$ symbol in (3.10) denotes the plethysm (see [19, I.8]). The formal character in (3.10) belongs to the ring $\Lambda=\Lambda_x\otimes\Lambda_y$, where Λ_x (resp. Λ_y) is the ring of symmetric function on x (resp. y). Appling the involution $\omega=1\otimes\omega_y$ on (3.10), we obtain

$$\sum_{i+j+k=r}\sum_{|\mu|=j}\sum_{|\gamma|=k}(h_i\circ e_2)(x)s_\mu(x)s_{\tilde\mu}(y)s_{2\gamma}(y),$$

which equals to the formal character of

$$\sum_{i+j+k=r}S_i(\wedge^2F)\otimes\wedge^j(F\otimes G)\otimes D_kD_2G=S_r\wedge^2\varphi$$

by the plethysm formula: $D_kD_2G\cong\sum_{|\gamma|=k}K_{2\gamma}G$ (see e.g., [14]), where $\omega_y:\Lambda_y\rightarrow\Lambda_y$ is the ring automorphism given by $\omega_y(e_i(y))=h_i(y)$ (see [19]).

Now consider the right hand side of (3.9). Its character is

$$\sum_\lambda s_{\widetilde{2\lambda}}(x,y)=\sum_\lambda\sum_{\mu\subset\widetilde{2\lambda}}s_\mu(x)s_{\widetilde{2\lambda}/\mu}(y).$$

Applying ω on this, we obtain

$$\sum_\lambda\sum_{\mu\subset\widetilde{2\lambda}}s_\mu(x)s_{2\lambda/\tilde\mu}(y)=\sum_\lambda\sum_{\mu\subset2\lambda}s_{\tilde\mu}(x)s_{2\lambda/\mu}(y),$$

which is the formal character of $\oplus_\lambda L_{2\lambda}\varphi$.

This shows that $S_r\wedge^2\varphi\cong\oplus_\lambda L_{2\lambda}\varphi$.

Theorem 3.11 (Generalized plethsm formula). *Let $r\geq0$ and λ be a partition of degree r. Then, there exists a unique isomorphism*

$$\gamma_\lambda:L_{2\lambda}\varphi\rightarrow M_\lambda(\pi)/\dot M_\lambda(\pi)$$

such that the diagram

$$\begin{array}{ccc}\wedge_{2\lambda}\varphi & \xrightarrow{\pi_{2\lambda}} & M_\lambda(\pi)\\ d_{2\lambda}\downarrow & & \downarrow\\ L_{2\lambda}\varphi & \xrightarrow{\gamma_\lambda} & M_\lambda(\pi)/\dot M_\lambda(\pi)\end{array}$$

is commutative. So $S_r\wedge^2\varphi$ is isomorphic to $\oplus_{|\lambda|=r}L_{2\lambda}\varphi$ up to filtration.

Proof. To see that γ_λ is certainly induced, it suffices to show that the image of the composite map

$$\wedge_{\mu}\varphi \xrightarrow{\square^{\mu}_{2\lambda}} \wedge_{2\lambda}\varphi \xrightarrow{\pi_{2\lambda}} M_\lambda(\pi)$$

is containd in $\dot{M}_\lambda(\pi)$ by the standard basis theorem [2, Theorem V. 1.10].To prove this, we may assume that $\lambda=(\lambda_1, \lambda_2)$ is of length two, and we set $\mu=(2\lambda_1+t, 2\lambda_2-t)$. But this case is clear from **4** of Lemma 3.5. So γ_λ is induced.

By definition of $M_\lambda(\pi)$ and $\dot{M}_\lambda(\pi)$, γ_λ is cleary surjective. The injectivity follows from Lemma 3.7 and Lemma 3.8.

Corollary 3.12 ([16, Proposition 3.5], [3, Therorem 2.7]). *Let $r\geq 0$. For a partition λ of r, we set*

$$M_\lambda(\pi^S) \overset{def}{=} \sum_{\mu\geq\lambda, |\mu|=r} \mathrm{Im}\pi^S_{2\mu}$$

and

$$\dot{M}_\lambda(\pi^S) \overset{def}{=} \sum_{\mu>\lambda, |\mu|=r} \mathrm{Im}\pi^S_{2\mu}.$$

Then, we have $M_\lambda(\pi^S)/\dot{M}_\lambda(\pi^S)\cong L_{2\lambda}F$. In particular, $\{M_\lambda(\pi^S)\}_{|\lambda|=r}$ is a filtration of $S_r(\wedge^2 F)$ whose associated graded object is $\bigoplus_{|\lambda|=r} L_{2\lambda}F$.

Proof. Set $G=0$ in the theorem so that $\varphi=0\rightarrow F$ and $\pi=\pi^S$. The assertions follow immediately.

Corollary 3.13 ([3, Therorem 1.9]). *Let $r\geq 0$. For a partition λ of r, we set*

$$M_\lambda(\pi^D) \overset{def}{=} \sum_{\mu\geq\lambda, |\mu|=r} \mathrm{Im}\pi^D_{2\mu}$$

and

$$\dot{M}_\lambda(\pi^D) \overset{def}{=} \sum_{\mu>\lambda, |\mu|=r} \mathrm{Im}\pi^D_{2\mu}.$$

Then, we have $M_\lambda(\pi^D)/\dot{M}_\lambda(\pi^D)\cong K_{2\lambda}G$. In particular, $\{M_\lambda(\pi^D)\}_{|\lambda|=r}$ is a filtration of $D_r(D_2G)$ whose associated graded object is $\oplus_{|\lambda|=r} K_{2\lambda}G$.

Proof. Set $F=0$ and apply the theorem.

4. A vanishing theorem

In this section, we prove a vanishing therorem on homology of t-Schur complex of the identity map. For definition and basics on t-Schur complexes, we refer the reader to [7, section 2]. In this section, we consider the identity map $\mathrm{id}_F : F' \to F$ of finite free R-module given by $\mathrm{id}_F(i') = i$, where $X' = \{1' > \cdots > n'\}$ and $X = \{1 < \cdots < n\}$ are ordered bases of F' and F, respectively. We use notation and results from [7, section 2] freely. However, we use one different notation here (as mentioned in section 1). Let φ: $G \to F$ be a map of finite free R-modules. By $\wedge_{t,\lambda/\mu}\varphi$ we mean the tensor product

$$\wedge^{t,\lambda_1-\mu_1}\varphi \otimes \wedge^{\lambda_2-\mu_2}\varphi \otimes \wedge^{\lambda_3-\mu_3}\varphi \cdots,$$

which was denoted by $\wedge_{t,1,\lambda/\mu}\varphi$ in [7]. We never use $\sum_{i\geq 0}\wedge_{t,i,\lambda/\mu}\varphi$ which was denoted by $\wedge_{t,\lambda/\mu}\varphi$ in [7].

Definition 4.1. Let s, l, s', l' and t be non-negative integers, and λ/μ a relative row-sequence. Then we define

$$\widetilde{B}_t^{s,l,s',l'}(\lambda/\mu) \overset{def}{=} \{S \in \widetilde{B}_t^{s,l}(\lambda/\mu) \mid \nu_{\{1,\dots,l\}}(S,1) \geq s'\}$$

where

$$\widetilde{B}_t^{s,l}(\lambda/\mu) \overset{def}{=} \{S \in \mathrm{Row}_{\lambda/\mu}(\mathbf{X}, X') \mid \nu_{\mathbf{N}}(S, \{1,1'\}) = s,\ \nu_1(S) \geq t,\ \nu_i(S,1') = 0\ (\forall i \leq l)\}$$

as in [7]. We denote the R-span of $\widetilde{B}_t^{s,l,s',l'}(\lambda/\mu)$ by $\widetilde{X}_t^{s,l,s',l'}(\lambda/\mu)$, which is a submodule of $\wedge_{t,\lambda/\mu}\mathrm{id}_F$.

It is easy to check that $\widetilde{X}_t^{s,l,s',l'}(\lambda/\mu)$ is a subcomplex of $\wedge_{t,\lambda/\mu}\mathrm{id}_F$. We set $X_t^{s,l,s',l'}(\lambda/\mu) \overset{def}{=} d_{\lambda/\mu}(\widetilde{X}_t^{s,l,s',l'}(\lambda/\mu))$. It is easy to verify that $X_t^{s,l,s',l'}(\lambda/\mu)$ is a free subcomplex of $L_{\lambda/\mu}\,\mathrm{id}_F$ with the basis

$$B_t^{s,l,s',l'}(\lambda/\mu) \overset{def}{=} \mathrm{St}_{\lambda/\mu}(\mathbf{X},\mathrm{X}') \cap \widetilde{B}_t^{s,l,s',l'}(\lambda/\mu).$$

Assume that $\lambda_{l+1} > \lambda_{l+2}$ and $\lambda_{l+1} > \mu_{l+1}$ for the skew partition λ/μ. The map $v = v_t^{s,l}: X_t^{s,l}(\lambda/\mu) \to X_t^{s-1,l}((\lambda-\varepsilon_{l+1})/\mu)[-1]$ (see [7, p.469] for the definition) maps $X_t^{s,l,s',l'}(\lambda/\mu)$ onto $X_t^{s-1,l,s',l'}((\lambda-\varepsilon_{l+1})/\mu)[-1]$ surjectively.

So we have an exact sequence

$$0 \to X_t^{s,l+1,s',l'}(\lambda/\mu) \xrightarrow{i} X_t^{s,l,s',l'}(\lambda/\mu) \xrightarrow{v} X_t^{s-1,l,s',l'}((\lambda-\varepsilon_{l+1})/\mu)[-1] \to 0.$$

When $\lambda_{l+1} = \lambda_{l+2}$ or $\lambda_{l+1} = \mu_{l+1}$, then we have

$$X_t^{s,l,s',l'}(\lambda/\mu) = X_t^{s,l+1,s',l'}(\lambda/\mu).$$

In particular, we have

$$X_t^{s,l,s',l'}(\lambda/\mu)=X_t^{s,l(\lambda/\mu),s',l'}(\lambda/\mu)$$

when $l\geq l(\lambda/\mu)$. We denote $X_t^{s,l(\lambda/\mu),s',l'}(\lambda/\mu)$ by $X_t^{s,\infty,s',l'}(\lambda/\mu)$.

For a skew partition λ/μ, we set

$$\Gamma_s(\lambda/\mu)\overset{def}{=}\{\gamma\text{: partition}\mid\mu\subset\gamma\subset\lambda,\ \gamma/\mu\text{: a vertical } s\text{-strip}\},$$

and

$$\Gamma_s^{s',l'}(\lambda/\mu)\overset{def}{=}\{\gamma\in\Gamma_s(\lambda/\mu)\mid\sum_{i=1}^{l'}(\gamma_i-\mu_i)\geq s'\}.$$

For γ, $\gamma'\in\Gamma_s(\lambda/\mu)$, we say that $\gamma\geq_{s',l'}\gamma'$ when

1 $\gamma\in\Gamma_s^{s',l'}(\lambda/\mu)$ and $\gamma'\notin\Gamma_s^{s',l'}(\lambda/\mu)$ or

2 The condition **1** does not hold, and $\gamma\geq\gamma'$ (with respect to the lexicographic order).

The relation $\geq_{s',l'}$ is a total order, and is compatible with the dominant order $\succeq$ (we asy that $\lambda\succeq\mu$ when $\sum_{j=1}^{i}\lambda_j\geq\sum_{j=1}^{i}\mu_j$ for any $i\geq 1$). Hence, when we set $M_t^{s',l'}(\gamma)=\sum_{\gamma'\geq_{s',l'}\gamma}\mathrm{Im}\,\Xi_{\gamma'}$, we have

$$X_t^{s,\infty,s',l'}(\lambda/\mu)=M_t^{s',l'}(\gamma_0),$$

where γ_0 is the smallest element of $\Gamma_s^{s',l'}(\lambda/\mu)$, and

$$\Xi_{\gamma'}:\wedge_{\gamma'/\mu}R\otimes\wedge_{t-\gamma'_1+\mu_1,\lambda/\gamma'}\mathrm{id}_{F_1}\to X_t^{s,\infty,s',l'}(\lambda/\mu)$$

is as in [7, p.470].

Just as [7, Proposition 2.24], we can prove:

Lamma 4.2. *$X_t^{s,\infty,s',l'}(\lambda/\mu)$ admits a filtration $\{M_t^{s',l'}(\gamma)\}_{\gamma\geq\gamma_0}$ whose associated graded object is*

$$\bigoplus_{\gamma\geq\gamma_0}L_{\gamma/\mu}R\otimes L_{\mu_1-\gamma_1+t,\lambda/\gamma}\mathrm{id}_{F_1},$$

where F_1 is the R-span of the basis elements $\{2,\cdots,n\}$.

Lemma 4.3. *Let λ/μ be a skew-partition with $l=l(\lambda/\mu)\geq 2$. Assume that $\lambda_l-\mu_l=1$, and $s'=\tilde{\mu}_{\lambda_l}$. Then, for any s, $t\geq 0$, the inclusion map*

$$X_t^{s,l-1,s',s+s'-l+1}(\lambda/\mu)\to X_t^{s,l-1}(\lambda/\mu)$$

is a quasi-isomorphism.

Proof. We may assume that $\mu_l=0$ and that $\lambda_l=1$. Consider the quotient complex $Y=X_t^{s,l-1}(\lambda/\mu)/X_t^{s,l-1,s',s+s'-l+1}(\lambda/\mu)$. It suffices to show that Y is exact. Clearly, Y has $B_Y = B_t^{s,l-1}(\lambda/\mu) \setminus B_t^{s,l-1,s',s+s'-l+1}(\lambda/\mu)$ as its basis. For a tableau $S \in B_t^{s,l-1}(\lambda/\mu)$, we have

$$S \in B_Y \Leftrightarrow S(i,1) = 1 \text{ for } i < l \text{ and } S(i, l) \in \{1, l'\}.$$

Thus, Y is isomorphic to the complex $L_{\gamma/\mu}\,\mathrm{id}_R \otimes X_{t-\gamma_1+\mu_1}^{s+s'-l,\infty,s',s+s'-l}(\lambda/\gamma)$, which is exact (because $L_{\gamma/\mu}\,\mathrm{id}_R$ is homotopically trivial), where

$$\gamma = (\mu_1, \cdots, \mu_{s'}, \mu_{s'+1}+1, \cdots, \mu_l+1) = (\mu_1, \cdots, \mu_{s'}, 1, \cdots, 1).$$

This completes the proof of lemma.

For a skew partition λ/μ and $t \geq 0$, we set

$$a(\lambda/\mu, t) \overset{def}{=} l(\lambda/\mu) + \lambda_1 - \mu_1 - t - 2.$$

Therorem 4.4. *Let λ/μ be a skew partition, i, s, l, $t \geq 0$ with $l < l(\lambda/\mu)$. If $i \leq a(\lambda/\mu, t)$, then $H_i(X_t^{s,l}(\lambda/\mu)) = 0$. In particular, we have $H_i(L_{t,\lambda/\mu}\mathrm{id}_F) = 0$.*

Proof. We may assume that $t \geq 1$ by [7, Lemma 2.2.13], $\lambda_1 - \mu_1 \geq t$, and $l(\lambda/\mu) \geq 2$. We proceed by double induction on rank F and $|\lambda/\mu|$. We may assume that λ/μ is connected by [7, Lemma 3.3.2] and its proof. We may also assume that $s \geq 1$ by induction assumption on rank F. We proceed by reverse induction on l. By [7, Lemma 2.2.3(1)], we may assume that $(\lambda - \varepsilon_{l+1})/\mu$ is a skew-partition. Since λ/μ is connected, we have $l((\lambda - \varepsilon_{l+1})/\mu) \geq l(\lambda/\mu) - 1$. Since $l(\lambda/\mu) \geq 2$, this shows that $a((\lambda - \varepsilon_{l+1})/\mu, t) \geq a(\lambda/\mu, t) - 1$.

First, consider the case $l = l(\lambda/\mu) - 1$. We have an exact sequence

$$(4.5) \qquad 0 \to X_t^{s,\infty}(\lambda/\mu) \to X_t^{s,l}(\lambda/\mu) \to X_t^{s-1,l}((\lambda - \varepsilon_{l+1})/\mu)[-1] \to 0.$$

Case **1** $\lambda_{l+1} - \mu_{l+1} \geq 2$. In this case, there is no $\gamma \in \Gamma_s(\lambda/\mu)$ such that $l(\lambda/\gamma) < l(\lambda/\mu)$. We have

$$X_t^{s,\infty}(\lambda/\mu) \cong \bigoplus_{\gamma \in \Gamma_s(\lambda/\mu)} L_{t-\gamma_1+\mu_1,\lambda/\gamma}\mathrm{id}_{F_1}$$

up to filtration [7, Lemma 2.2.12]. By induction assumption, we have

$$H_i(L_{t-\gamma_1+\mu_1,\lambda/\gamma}\,\mathrm{id}_{F_1}) = 0 \quad (i \leq a(\lambda/\gamma, t-\gamma_1+\mu_1) = a(\lambda/\mu, t))$$

for any $\gamma \in \Gamma_s(\lambda/\mu)$. Hence, we have $H_i(X_t^{s,\infty}(\lambda/\mu)) = 0$ for $i \leq a(\lambda/\mu, t)$. On the other hand, since $l((\lambda - \varepsilon_{l+1})/\mu) = l+1$ so that $a(\lambda/\mu, t) = a((\lambda - \varepsilon_{l+1})/\mu, t)$, we have $H_{i-1}(X_t^{s-1,l}((\lambda - \varepsilon_{l+1})/\mu)) = 0$ for $i \leq a(\lambda/\mu, t)$ by induction assumption. Hence we have $H_i(X_t^{s,l}(\lambda/\mu)) = 0$ for $i \leq a(\lambda/\mu, t)$ by the exact sequence (4.5).

Case **2** $\lambda_{l+1} - \mu_{l+1} \leq 1$. In this case, we have $\lambda_{l+1} - \mu_{l+1} = 1$, since $l(\lambda/\mu) = l+$

1. By Lemma 4.3, it suffices to show that $H_i(X_t^{s,l,s',a}(\lambda/\mu))=0$ for $i\leq a(\lambda/\mu,\ t)$, where $a=s+s'-l$. We have an exact sequence

$$0\rightarrow X_t^{s,\infty,s',a}(\lambda/\mu)\rightarrow X_t^{s,l,s',a}(\lambda/\mu)\rightarrow X_t^{s,\infty,s',a}((\lambda-\varepsilon_{l+1})/\mu)[-1]\rightarrow 0$$

For each $\gamma\in\Gamma_s^{s',a}(\lambda/\mu)$, we have $l(\gamma/\mu)\leq l$, since $s-a=l-s'$. This implies that $l(\lambda/\gamma)=l+1$ so that $a(\lambda/\gamma,t-\gamma_1+\mu_1)=a(\lambda/\mu,t)$, and $H_i(X_t^{s,\infty,s',a}(\lambda/\mu))=0$ for $i\leq a(\lambda/\mu,t)$ by Lemma 4.2.

By the same argument, we have

$$H_{i-1}(X_t^{s-1,\infty,s',a}((\lambda-\varepsilon_{l+1})/\mu))=0$$

for any $i\leq a(\lambda/\mu,t)=a((\lambda-\varepsilon_{l+1})/\mu,t)+1$. Thus, we have

$$H_i(X_t^{s,l,s',a}(\lambda/\mu))=0$$

for $i\leq a(\lambda/\mu,t)$, and we have completed this case.

Now consider the case $l\leq l(\lambda/\mu)-2$. Using the long exact sequence obtained by the short exact sequence

$$0\rightarrow x_t^{s,l+1}(\lambda/\mu)\rightarrow X_t^{s,l}(\lambda/\mu)\rightarrow X_t^{s-1,l}((\lambda-\varepsilon_{l+1})/\mu)[-1]\rightarrow 0,$$

the assertion follows immediately by induction assumption.

5. An application of Akin-Buchsbaum resolution

In this section, we calculate the homology $H_i(L_{t,\lambda/\mu}\mathrm{id}_F)$ of the t-Schur complex of the identity map for the case $l(\lambda/\mu)=2$ and $t=\lambda_1-\mu_1$. For this, the arithmetic Koszul complex and the resolution of Schur module (for the two-rowed case) due to Akin-Bucshsbaum [1] play important role.

First, we review the arithmetic Koszul complex. Let u and v be nonnegative intergers, with $u\geq v$. The complex $K[u;v]$ is the free complex whose degree s component $K[u;v]_s$ is the free R-module with the free basis

$$B[u;v]_s\overset{\mathrm{def}}{=}\{[a_0,\cdots,a_s]\mid a_i\geq 1\ (0\leq i<s),\ a_s\geq u-v,\ \sum_{i=0}^{s}a_i=u\}.$$

The boundary map is given by

$$\partial[a_0,\cdots,a_s]=\sum_{i=0}^{s-1}(-1)^i\binom{a_i+a_{i+1}}{a_i}[a_0,\cdots,a_{i-1},a_i+a_{i+1},a_{i+2},\cdots,a_s].$$

Note that the notation is slightly different from that in [7]. The basis element $\varepsilon_{i_1}\wedge\cdots\wedge\varepsilon_{i_s}$ of $K[u;v]_s$ in [7] corresponds to our $[i_1,i_2-i_1,\cdots,i_s-i_{s-1},u-i_s]\in B[u;v]_s$.

Next, we review the Akin-Buchsbaum resolution. Let

$$\lambda/\mu=(\lambda_1,\lambda_2)/(\mu_1,\mu_2)$$

be a skew partition of length two. We set $\alpha=\lambda-\mu$, $r=\mu_1-\mu_2+1$, and $k=\lambda_2-$

μ_1. We assume that $k\geq 0$. The Akin-Buchsbaum resolution $C(\lambda/\mu)(F)$ is the free complex whose degree s component is

$$C_s(\lambda/\mu)(F)=\bigoplus_{l\geq 0} K[r+l;l]_{s-1}\otimes\wedge_{(\alpha_1+r+l,\alpha_2-r-l)}F$$

for $s\geq 1$, while we set $C_0(\lambda/\mu)(F)=\wedge_\alpha F$. We define

$$\delta_0(\lambda/\mu): C_0(\lambda/\mu)(F)\rightarrow L_{\lambda/\mu}F$$

to be the Schur map $d_{\lambda/\mu}(F)$. Since $K[r+l;l]_0\cong R$ for any r and l, we have $C_1(\lambda/\mu)\cong\oplus_{\nu\in S_\square(\lambda/\mu)}\wedge_\nu F$. The boundary map $\delta_1(\lambda/\mu): C_1(\lambda/\mu)\rightarrow C_0(\lambda/\mu)$ is defined to be the box map. To define the maps

$$\delta_{s+1}: C_{s+1}(\lambda/\mu)\rightarrow C_s(\lambda/\mu)$$

for $s\geq 1$, it suffices to define the map

$$K[r+l;l]_s\otimes\wedge^{\alpha_1+r+l}F\otimes\wedge^{\alpha_2-r-l}F\rightarrow C_s(\lambda/\mu).$$

This map is given by, for a basis element $[a_0,\cdots,a_s]_l$ of $K[r+l;l]_s$ and $x\otimes y\in\wedge_{(\alpha_1+r+l,\alpha_2-r-l)}F$,

$$\delta_{s+1}([a_0,\cdots,a_s]_l\otimes x\otimes y)=\partial([a_0,\cdots,a_s]_l)\otimes x\otimes y$$
$$-\sum_{(x)}[a_1,\cdots,a_s]_{l-a_0}\otimes x^{(1)}_{(\alpha_1+r+l-a_0)}\otimes x^{(2)}_{(a_0)}\wedge y.$$

Theorem 5.1 ([1 p.173]). *Let λ/μ be a two-rowed skew partition and $k=\lambda_2-\mu_1\geq 0$. Then the sequence*

$$0\rightarrow C_k(\lambda/\mu)\xrightarrow{\delta_k}C_{k-1}(\lambda/\mu)\xrightarrow{\delta_{k-1}}\cdots\xrightarrow{\delta_1}C_0(\lambda/\mu)\xrightarrow{\delta_0}L_{\lambda/\mu}F\rightarrow 0$$

is exact.

The resolution $C(\lambda/\mu)$ of $L_{\lambda/\mu}F$ is called the Akin-Buchsbaum resolution.

Now we consider the t-Schur complex $L_{t,\lambda/\mu}\mathrm{id}_F$, where

$$\lambda/\mu=(\lambda_1,\lambda_2)/(\mu_1,\mu_2)$$

is a two-rowed skew partiton with $t=\lambda_1-\mu_1$. Without loss of generality, we may assume that $\mu_2=0$. We set $r=\mu_1+1$.

The complex $L_{t,\lambda/\mu}\mathrm{id}_F$ is as follows.

$$0\rightarrow\wedge^t F\otimes D_{\alpha_2}F\rightarrow\cdots\rightarrow L_{(\lambda_1,\lambda_2-i)}F\otimes D_iF\rightarrow\cdots\rightarrow L_{\lambda/\mu}F\rightarrow 0$$

So the ith term $L_{(\lambda_1,\lambda_2-i)/\mu}F\otimes D_iF$ of $L_{t,\lambda/\mu}\mathrm{id}_F$ admits a resolution $B_i=C((\lambda_1,\lambda_2-i)/\mu)\otimes D_iF$. We define a chain map $\partial_i:B_i\rightarrow B_{i-1}$. The map $\partial_{i,s}$ for $s\geq 1$ is given by

$$K[r+l;l]_{s-1}\otimes\wedge^{\lambda_1+1+l}F\otimes\wedge^{\lambda_2-\mu_1-1-l-i}F\otimes D_iF$$
$$\xrightarrow{1\otimes 1\otimes\partial^{\wedge\mathrm{id}_F}}K[r+l;l]_{s-1}\otimes\wedge^{\lambda_1+1+l}F\otimes\wedge^{\lambda_2-\mu_1-l-i}F\otimes D_{i-1}F,$$

where $\partial^{\wedge \mathrm{id}_F}$ is the bounday map of $\wedge \mathrm{id}_F$. The map $\partial_{i,0}$ is defined to be the boundary map of $\wedge_{t,\lambda/\mu}\mathrm{id}_F = \wedge^{\alpha_1}F \otimes \wedge^{\alpha_2}\mathrm{id}_F$. Then, it is easy to see that $(B_{*,*}, \partial_{*,*})$ forms an R-double complex. The following proposition is essentially used in [23].

Proposition 5.2. *With the notation and the assumption as above, we have*

$$H_i(L_{t,\lambda/\mu}\mathrm{id}_F) \cong H_i(\mathrm{tot}(B_{*,*})) \cong \wedge^{|\lambda/\mu|}F \otimes H_{i-1}(K[\lambda_2;\lambda_2-r]).$$

Proof. Since $B_i = B_{i,*}$ is a resolution of $[L_{t,\lambda/\mu}\mathrm{id}_F]_i$ for any i, the spectral sequence for $B_{*,*}$ degenerates, and we have

$$H_i(\mathrm{tot}(B_{*,*})) \cong H_i^?(H_0^*(B_{?,*})) = H_i(L_{t,\lambda/\mu}\mathrm{id}_F).$$

Now we take the spectral sequence in the other way, to prove the second isomorphism. By definition, the complex $B_{*,s}$ is isomorphic to

$$\bigoplus_{i \geq 0} K[r+l;l]_{s-1} \otimes \wedge^{\lambda_1+1+l}F \otimes \wedge^{\lambda_2-r-l}\mathrm{id}_F.$$

Since $\wedge^{\lambda_2-r-l}\mathrm{id}_F$ is homotopically trivial unless $l=\lambda_2-r$, the spectral sequence of this direction also degenerates, and we have

$$H_i(\mathrm{tot}(B_{*,*})) \cong H_i^*(H_0^?(\mathrm{B}_{?,*})) = H_{i-1}(K[\lambda_2;\lambda_2-r]) \otimes \wedge^{|\lambda/\mu|}F.$$

Corollary 5.3. *Let $R=K$ be a field of characteristic p, and λ a partition. For any $t \geq 1$, we have*

$$H_1(L_{t,\lambda}\mathrm{id}_F) = \begin{cases} K_{(2,1^{t-1})}F & (\lambda = (t+1)) \\ \wedge^{|\lambda|}F & (\lambda = (t,q) \text{ for some } q=p^i\,(i \geq 0)), \\ 0 & (\text{otherwise}) \end{cases}$$

where we regard $1=p^0$ for any $p \geq 0$.

Proof. Assume that $H_1(L_{t,\lambda}\mathrm{id}_F) \neq 0$. By Theorem 4.4, we have $l(\lambda)+\lambda_1-t-2<1$. Clearly, we have $\lambda_1 \geq t$. In particular, we have $l(\lambda) \leq 2$. If $l(\lambda)=1$, then it is easy to see that $\lambda=(t+1)$ and that the lemma is true this case.

So we consider the case $l(\lambda)=2$. In this case, we have $\lambda_1=t$. By the proposition, we have

$$H_1(L_{t,\lambda}\mathrm{id}_F) = \wedge^{|\lambda|}F \otimes H_0(K[\lambda_2;\lambda_2-1]).$$

Since the base ring $R=K$ is a field and $K[\lambda_2;\lambda_1-1]_0$ is one-dimensional, $H_0(K[\lambda_2;\lambda_2-1]) \neq 0$ if and only if the boundary map

$$\partial_1: K[\lambda_2;\lambda_2-1]_1 \rightarrow K[\lambda_2;\lambda_2-1]_0$$

is zero. This condition holds if and only if the binoimal coefficient $\binom{\lambda_2}{i}$ is zero (in K) for $0<i<\lambda_2$. This is true if and only if λ_2 is a power of p. This shows that the Corollary is true for the case $l(\lambda)=2$.

Explicit form of the basis of $H_1(L_{t,\lambda}\mathrm{id}_F)$ will be determined in the next section.

6. Explicit calculation

By Theorem 4.4, we have $H_i(L_{t,\lambda/\mu}\mathrm{id}_F)=0$ for $i\leq a(\lambda/\mu, t)$. In this section, we calculate the homology for $i=a(\lambda/\mu, t)+1$. As in section 4, id_F: $F'\longrightarrow F$ has a fixed ordered basis $\{1<\cdots<n<n'<\cdots<1'\}$ with $\mathrm{id}_F(i')=i$.

Obviously, the complex $L_{t,\lambda/\mu}\mathrm{id}_F$ is a complex of polynomial represeentations of $\mathrm{GL}(F)$, while the subcomplex $X_t^{s,l}(\lambda/\mu)$ is $\mathrm{GL}(F_1)$-equivariant, but not necessarily $\mathrm{GL}(F)$-equivariant, where F_1 is the R-span of $2, \cdots, n$.

For a polynomal represetation M of $\mathrm{GL}(F)$ and $\alpha=(\alpha_1, \cdots, \alpha_n)\in\mathbf{N}_0^n$, the weight α component M_α of M is defined to be

$$\{a\in M|\delta(a)=a\otimes t_1^{\alpha_1}\cdots t_n^{\alpha_n}\}\subset M,$$

where $t_1,\cdots, t_n$ are the coordinates of the set of diagonal matrices corresponding to the basis $\{1,\cdots, n\}$, and $\delta: M\longrightarrow M\otimes R[t_1,\cdots, t_n]$ is the coaction of M as an $R[t_1,\cdots,t_n]$-comodule.

In what follows, we use tableau notation as in [7, section 3].

Lemma 6.1. *Let $\lambda/\mu=(\lambda_1, \lambda_2)/(\mu_1)$ be a connected skew partition with $\lambda_1-\mu_1=t$ and $r=|\lambda/\mu|$. Then, $H_1(L_{t,\lambda/\mu}\,\mathrm{id}_F)_\omega$ is cyclic (possibly zero), and is generated by*

$$A_{\lambda/\mu}=\sum_{\sigma\in\mathfrak{S}^{t,\lambda_2}}(-1)^\sigma\begin{matrix} & \sigma 1\ \cdots\cdots\ \sigma t\\ \sigma(t+1)\cdots & \boxed{\sigma r}\end{matrix}$$

where $\omega=(1,\cdots,1)$ (r times 1).

Proof. Induction on r. Frist, observe that $A_{\lambda/\mu}$ is certainly a cycle, which represents a homology. We denote the weight $(1,\cdots,1)=(1^{r-1})$ (of the maximal torus of $\mathrm{GL}(F_1)$) by ρ.

Case **1.** $k=\lambda_2-\mu_1=1$. If $\lambda_2=1$ moreover, then the lemma is obvious. So we may assume that $\lambda_2\geq 2$. There is an isomorphism

$$H_1(X_t^{1,\infty}(\lambda/\mu))_\rho\longrightarrow H_1(L_{t,\lambda/\mu}\mathrm{id}_F)_\omega$$

arising from the fundamental exact sequence, since $(\lambda-\varepsilon_2)/\mu$ is disconnected. On the other hand, since $\lambda/(\mu+\varepsilon_1)$ is disconnected, the projection map

$$\varphi\colon X_t^{1,\infty}(\lambda/\mu)_\rho\longrightarrow(R\otimes L_{t,\lambda/(\mu+\varepsilon_2)}\,\mathrm{id}_{F_1})_\rho$$

is a quasi-isomorphism. Since $A_{\lambda/\mu}$ is mapped to $\pm 1\otimes A_{\lambda/(\mu+\varepsilon_2)}$ by φ, which is a generator of the cyclic module $(R\otimes L_{t,\lambda/(\mu+\varepsilon_2)}\mathrm{id}_{F_1})_\rho$, we have that $H_1(L_{t,\lambda/\mu}\mathrm{id}_F)_\omega$

is generated by $A_{\lambda/\mu}$.

Case **2**. $k\geq 2$. There is an exact sequence

$$H_1(L_{t-1,\lambda/(\mu+\varepsilon_1)})\mathrm{id}_{F_1})_\rho\xrightarrow{i}H_1(X_t^{1,\infty}(\lambda/\mu))_\omega\xrightarrow{p}H_1(L_{t,\lambda/(\mu+\varepsilon_2)}\mathrm{id}_{F_1})_\rho.$$

By induction assumption, $P=1\otimes A_{\lambda/(\mu+\varepsilon_1)}$ and $Q=1\otimes A_{\lambda/(\mu+\varepsilon_2)}$ together generate $H_1(X_t^{1,\infty}(\lambda/\mu))_\rho$ (and hence $H_1(X_t^{1,1}(\lambda/\mu))_\omega=H_1(L_{t,\lambda/\mu}\mathrm{id}_F)_\omega)$. By [7, Lemma 2.3.1], we have $\delta(A_{(\lambda-\varepsilon_2)/\mu})=\pm Q$, where δ is the connected map

$$H_1(X_t^{0,1}((\lambda-\varepsilon_2)/\mu))\to H_1(X_t^{1,\infty}(\lambda/\mu)).$$

This shows that $Q=0$ in $H_1(X_t^{1,1}(\lambda/\mu))$, and we have taht $A_{\lambda/\mu}=P+(-1)^tQ$ generates $H_1(L_{t,\lambda/\mu}\mathrm{id}_F)_\omega$.

The following lemma, which is the main purpose of this section, is proved similarly.

Lemma 6.2. *Let $\lambda/\mu=(\lambda_1,\lambda_2,\lambda_3)/(\mu_1,\mu_2)$ be a connected skew partition with $\lambda_1-\mu_1=t$ and $\lambda_3\neq 0$. We set $\gamma=\lambda-\mu$ and $r=|\gamma|$. Then, the weight $\omega=(1,\cdots,1)$ component of $H_2(L_{t,\lambda/\mu}\mathrm{id}_F)$ is a cyclic module (possibly zero) generated by*

$$A_{\lambda/\mu}\overset{def}{=}\sum_{\substack{\sigma\in\mathfrak{S}_r\\ \sigma(t+\gamma_2)>\sigma(|\lambda/\mu|)}}(-1)^\sigma\begin{array}{rl}\sigma 1\cdots\cdots & \sigma t\\ \sigma(t+1)\cdots\sigma(t+\gamma_2-1) & \boxed{\sigma(t+\gamma_2)}\\ \sigma(t+\gamma_2+1)\cdots\sigma(\gamma+1) & \boxed{\sigma\gamma}\end{array}.$$

Proof. Induction on $r=|\lambda/\mu|$. A straightforward computation will show that $\partial(A_{\lambda/\mu})=0$ in $L_{\lambda/\mu}\mathrm{id}_F$, and $A_{\lambda/\mu}$ is certainly a cycle. We set $k_1=\lambda_2-\mu_1$, and $k_2=\lambda_2-\mu_2$. We denote the weight $(1,\cdots,1)$ $(r-1$ times 1) for $\mathrm{GL}(F_1)$-modules by ρ.

By Theorem 4.4, we have $H_1(L_{t,(\lambda-\varepsilon_2)/\mu}\mathrm{id}_{F_1})=0$ when $(\lambda-\varepsilon_2)/\mu$ is a partition, since $l((\lambda-\varepsilon_2)/\mu)=3$. Hence, $H_2(L_{t,\lambda/\mu}\mathrm{id}_F)_\omega=H_2(X_t^{1,1}(\lambda/\mu))_\rho$ is a homomorphic image of $H_2\ (X_t^{1,2}(\lambda/\mu))_\rho$ by the fundamental exact sequence.

Case **1** $k_1=k_2=1$. Namely, λ/μ is a skew hook. If $\lambda_3=1$, then $X_t^{1,\infty}(\lambda/\mu)$ is exact. Since $(\lambda-\varepsilon_1)/\mu$ and $(\lambda-\varepsilon_2)/\mu$ are disconnected, the map

$$v\colon H_2(X_t^{1,2}(\lambda/\mu))\to H_2(X_t^{0,2}((\lambda-\varepsilon_3)/\mu))=H_1(L_{t,(\lambda-\varepsilon_3)/\mu}\mathrm{id}_{F_1})$$

is an isomorphism. By Lemma 6.1, this case is O. K., since $v(A_{\lambda/\mu})$ generates $H_1(L_{t,(\lambda-\varepsilon_3)/\mu}\mathrm{id}_{F_1})$.

Consider the case $\lambda_3\geq 2$. In this case, $H_2(L_{t,\lambda/\mu}\mathrm{id}_F)_\omega$ is a homomrphic image of

$$H_2(X_t^{1,\infty}(\lambda/\mu))_\rho\cong H_2(L_{t,\lambda/(\mu+\varepsilon_3)}\mathrm{id}_{F_1})_\rho.$$

By induction assumption, this case is O. K., too.

Now assume that $k_2 \geq 2$. In this case, $\delta(A_{(\lambda-\varepsilon_3)/\mu}) = 0$ in $H_2(X_t^{1,2}(\lambda/\mu))$, where δ is the connected map

$$\delta\colon H_2(X_t^{0,2}((\lambda-\varepsilon_3)/\mu))_\rho \to H_2(X_t^{1,\infty}(\lambda/\mu))_\rho.$$

By [7, Lemma 2.3.1], we have

$$\begin{aligned}\delta(A_{(\lambda-\varepsilon_3)/\mu}) &= \pm(1 \otimes A_{\lambda/(\mu+\varepsilon_3)}) \\ &= \pm \sum_{\substack{\sigma\in\mathfrak{S}^{(t+1,\gamma_2,\gamma_3-1)} \\ \sigma 1 = 1 \\ \sigma(t+\gamma_2+1) > \sigma(|\lambda/\mu|)}} (-1)^\sigma \begin{array}{rl} \sigma 2 \cdots\cdots & \quad\sigma(t+1) \\ \sigma(t+2)\cdots\sigma(t+\gamma_2) & \boxed{\sigma(t+\gamma_2+1)} \\ 1\,\sigma(t+\gamma_2+2)\cdots\sigma(r-1) & \boxed{\sigma r} \end{array}.\end{aligned}$$

Now we consider

Case **2** $k_1=1$, $k_2 \geq 2$. By Theorem 4.4,

$$H_2(L_{t,(\lambda-\varepsilon_3)/\mu}\mathrm{id}_{F_1})_\rho \xrightarrow{\delta} H_2(X_t^{1,\infty}(\lambda/\mu))_\rho \to H_2(L_{t,\lambda/\mu}\mathrm{id}_F)_\omega \to 0$$

is exact. On the other hand, by [7, Proposition 2.2.4] and Theorem 4.4, we have an exact sequence

$$H_2(L_{t,\lambda/(\mu+\varepsilon_2)}\mathrm{id}_F)_\rho \xrightarrow{l} H_2(X_t^{l,\infty}(\lambda/\mu))_\rho \xrightarrow{p} H_2(L_{t,\lambda/(\mu+\varepsilon_3)}\mathrm{id}_{F_1})_\rho \to 0.$$

Since $(p \circ \delta)(A_{(\lambda-\varepsilon_3)/\mu})$ genertates $H_2(L_{t,\lambda/(\mu+\varepsilon_3)}\mathrm{id}_{F_1})_\rho$ and the image of $H_2(L_{t,\lambda/(\mu+\varepsilon_2)}\mathrm{id}_{F_1})_\rho$ by the map l is generated by

$$B_{\lambda/\mu} \overset{def}{=} \sum_{\substack{\sigma\in\mathfrak{S}^\alpha \\ \sigma(\gamma_1+\gamma_2) > \sigma(|\gamma|) \\ \sigma(\gamma_1+\gamma_2)+1>1}} (-1)^\sigma \begin{array}{rl} \sigma 1 \cdots\cdots & \quad\sigma t \\ \sigma(t+1)\cdots\sigma(t+\gamma_2-1) & \boxed{\sigma(t+\gamma_2)} \\ \sigma(t+\gamma_2+1)\cdots\sigma(\gamma-1) & \boxed{\sigma\gamma} \end{array}$$

by induction assumption. Since $A_{\lambda/\mu} = B_{\lambda/\mu} \pm \delta(A_{(\lambda-\varepsilon_3)/\mu})$ and $\delta(A_{(\lambda-\varepsilon_3)/\mu})$ is zero in $H_2(L_{t,\lambda/\mu}\mathrm{id}_F)$, $\delta(A_{\lambda/\mu})$ generates $H_2(L_{t,\lambda/\mu}\mathrm{id}_F)_\omega$.

Now assume that $k_1 \geq 2$. In this case, $\delta(A_{(\lambda-\varepsilon_2)/\mu}) = 0$ in $H_2(X_t^{1,1}(\lambda/\mu))$, where δ is the connected map

$$\delta\colon H_2(L_{t,(\lambda-\varepsilon_2)/\mu}\mathrm{id}_{F_1}) \to H_2(X_t^{1,2}(\lambda/\mu)).$$

However, we have to be careful. If $\lambda_3 = \lambda_2$, then the map δ does not make sense, because $\lambda - \varepsilon_2$ is not a partition. In this case, we regaed $\delta(A_{(\lambda-\varepsilon_2)/\mu})$ as the element

$$\partial\left(\sum_{\substack{\sigma\in\mathfrak{S}^r \\ \sigma(t+1)=1 \\ \sigma(t+\gamma_2) > \sigma(r)}} (-1)^\sigma \begin{array}{rl} \sigma 1 \cdots\cdots & \quad\sigma(t-1)\ \ \sigma t \\ \sigma(t+2)\cdots\sigma(t+\gamma_2-1) & \boxed{\sigma(t+\gamma_2)\quad 1} \\ \sigma(t+\gamma_2+1)\cdots\cdots & \sigma(r-1)\ \boxed{\sigma r} \end{array} \right),$$

which is zero in $H_2(X_t^{1,1}(\lambda/\mu))$. A straightforward computation shows that

(6.3) $$\delta(A_{(\lambda-\varepsilon_2)/\mu})=\pm(1\otimes A_{\lambda/(\mu+\varepsilon_2)}+(-1)^{\gamma_2}(1\otimes A_{\lambda/(\mu+\varepsilon_3)}))$$

in $H_2(X_t^{1,2}(\lambda/\mu))$. Here 1 in the second term must be replaced by $1'$ when $\gamma_3=1$. Observe that the second term is zero when $k_2\geq 2$.

Now we consider

Case **3** $k_1\geq 2$, $k_2=1$. Since $\lambda/(\mu+\varepsilon_2)$ is disconnected, $C_{\lambda/\mu}=1\otimes A_{\lambda/(\mu+\varepsilon_1)}$ and $\delta(A_{(\lambda-\varepsilon_2)/\mu})$ together generate $H_2(X_t^{1,\infty}(\lambda/\mu))_\rho$ by induction assumption and (6.3). On the other hand,

$$H_2(X_t^{1,\infty}(\lambda/\mu))_\rho\to H_2(X_t^{1,2}(\lambda/\mu))_\rho\to H_1(X_t^{0,2}((\lambda-\varepsilon_3)/\mu))=0$$

is exact, and $X_t^{1,2}(\lambda/\mu)$ is quasi-isomorphic to $X_t^{1,1}(\lambda/\mu)$ since $(\lambda-\varepsilon_2)/\mu$ is disconnected. So $A_{\lambda/\mu}=C_{\lambda/\mu}\pm\delta(A_{(\lambda-\varepsilon_2)/\mu})$ generates $H_2(L_{t,\lambda/\mu}\mathrm{id}_F)_\omega$.

Case **4**. $k_1\geq 2$, $k_2\geq 2$. By (6.3), the three elements $C_{\lambda/\mu}=1\otimes A_{\lambda/(\mu+\varepsilon_1)}$, $\delta(A_{(\lambda-\varepsilon_3)/\mu})$ and $\delta(A_{(\lambda-\varepsilon_2)/\mu})$ together generate $H_2(X_t^{1,\infty}(\lambda/\mu))_\rho$. Using induction assumption, it is easy to see that the map $H_2(X_t^{1,\infty}(\lambda/\mu))_\rho\to H_2(X_t^{1,1}(\lambda/\mu))_\rho$ is surjective, as in Case **3**. Since $\delta(A_{(\lambda-\varepsilon_3)/\mu})$ and $\delta(A_{(\lambda-\varepsilon_2)/\mu})$ are zero in $H_2(X_t^{1,1}(\lambda/\mu))_\rho$, we have $A_{\lambda/\mu}=C_{\lambda/\mu}\pm\delta(A_{(\lambda-\varepsilon_2/\mu)})$ generates $H_2(L_{t,\lambda/\mu}\mathrm{id}_F)_\omega$.

7. The second Betti number

Now we start studying our main object—the Pfaffian ideal.

Let F be a free R-module of rank n with a fixed basis $X=\{x_1,\cdots,x_n\}$, and $S=S(\wedge^2F)=R[x_i\wedge x_j]$ the polynomial ring over R with $\binom{n}{2}$ variables. Let $t\geq 1$. The *generic Pfaffian ideal* Pf_{2t} is $Pf_{2t}((x_i\wedge x_j))\subset S$ by definition. Or equivalently, $Pf_{2t}=S\cdot\mathrm{Im}\pi_{2t}^S(F)$. With letting each variable $x_i\wedge x_j$ of degree one, S is a graded R-algebra, and Pf_{2t} is a homogeneous ideal generated by its degree t-component. It is easy to see that the degree r-component $Pf_{2t,r}$ of Pf_{2t} agrees with $M_{(t,1^{r-t})}(\pi^S)$ for $r\geq t$, where $M_\lambda(\pi^S)$ is defined as in Corollary 3.12. By Corollary 3.12, we have S/Pf_{2t} is R-free, since $S_r/Pf_{2t,r}$ admits a Schur module filtration, where $S_r=S_r(\wedge^2F)$. By Lemma 3.3, we have the *Laplace expansion formula*

$$\pi_{2(r+1)}^S(a\wedge f_1\wedge\cdots\wedge f_{2r+1})=\sum_{i=1}^{2r+1}(-1)^{i+1}(a\wedge f_i)\pi_{2r}^S(f_1\wedge\cdots\overset{i}{\vee}\cdots\wedge f_{2r+1})$$

for $r\geq 1$ and $f_1,\cdots,f_{2r+2}\in F$. This shows that $Pf_{2t+2}\subset Pf_{2t}$.

It is known that Pf_{2t} is Gorenstein of codimension $h=(n-2t+2)(n-2t+1)/2$ (i.e., $\mathrm{pd}_S S/Pf_{2t}=\mathrm{grade}\ Pf_{2t}=h$ and $\mathrm{Ext}_S^h(S/Pf_{2t},S)\cong S/Pf_{2t}$). In other words, S/Pf_{2t} is Gorenstein if and only if so is R, see [10] or [13].

In [17], Kurano showed that the number of generators of the kernel of the map

$$\varphi: S\otimes\wedge^{2t}F \xrightarrow{1\otimes\pi^S} S\otimes S \xrightarrow{m} S$$

(the relation moldule of Pf_{2t}) depends on the base field $R=K$.

In this section, we will determine the number of generators of Kerφ when $R=K$ is a field.

The graded Betti number of S/Pf_{2t} for the base field K of characteristic p, $\dim_K\ [\mathrm{Tor}_i^S(K, S/Pf_{2t})]_j$ is denoted by $\beta_{i,j}^p$ (this number depends only on p, and is independent of the choice of K of characteristic p, where K is the S-module $S/S_+=S/Pf_2$, and $[\ \ ]_j$ denotes the degree j-component of a graded S-module. The Betti number $\sum_j\beta_{i,j}^p$ is denoted by β_i^p. The number of generators of Ker φ agrees with β_2^p when $R=K$ is a field of characteristic p. In the rest of this section $R=K$ is a field of characteristic p, unless otherwise specified.

Since the Koszul complex $S\mathrm{id}_{\wedge^2F}$ is a minimal free resolution of K, we have an isomorphism of graded S-modules

$$\mathrm{Tor}_{i+1}^S(K, S/Pf_{2t})\cong H_i(Pf_{2t}\otimes_S S\mathrm{id}_{\wedge^2F})$$

for $i\geq 0$. We define a chain map $p:\wedge^2\mathrm{id}_F\to\mathrm{id}_{\wedge^2F}$ by

$$\begin{array}{ccccccccc}
\wedge^2\mathrm{id}_F: 0 & \longrightarrow & D_2F & \xrightarrow{\Delta} & F\otimes F & \xrightarrow{m} & \wedge^2F & \longrightarrow & 0\\
 & & 0\downarrow & & \downarrow m & 1 & \downarrow 1 & & \\
\mathrm{id}_{\wedge^2F}: 0 & \longrightarrow & 0 & \longrightarrow & \wedge^2F & \longrightarrow & \wedge^2F & \longrightarrow & 0.
\end{array}$$

Lemma 7.1. *The map*

$$1\otimes Sp : Pf_{2t}\otimes_S S\wedge^2\mathrm{id}_F \to Pf_{2t}\otimes_S S\mathrm{id}_{\wedge^2F}$$

is a quasi-isomorphism of $GL(F)$-equivariant graded S-complexes.

Proof. It is obvious that $1\otimes Sp$ is a map of $\mathrm{GL}(F)$-equivariant graded S-complexes. It suffces to show that this map induces a bijection of the homology groups.

There is an R-homomorphism $\sigma:\wedge^2F\to F\otimes F$ such that $m\circ\sigma=\mathrm{id}$, where m is the multiplication map m: $F\otimes F\to\wedge^2F$. Using this splitting σ, we have a splitting s: $\mathrm{id}_{\wedge^2F}\to\wedge^2\mathrm{id}_F$ of p defined by

$$\begin{array}{ccccccccc}
\mathrm{id}\wedge^2F: 0 & \longrightarrow & 0 & \longrightarrow & \wedge^2F & \xrightarrow{1} & \wedge^2F & \longrightarrow & 0\\
 & & 0\downarrow & \Delta & \downarrow\sigma & m & \downarrow 1 & & \\
\wedge^2\mathrm{id}_F: 0 & \longrightarrow & D_2F & \longrightarrow & F\otimes F & \longrightarrow & \wedge^2F & \longrightarrow & 0.
\end{array}$$

Since $p\circ s$ is the identity map of $\mathrm{id}_{\wedge^2F}$, we have $H_*(1\otimes Sp)\circ H_*(1\otimes Ss)=1$ on $H_*(Pf_{2t}\otimes_S S\mathrm{id}_{\wedge^2F})$. This shows that $H_*(1\otimes Sp)$ is surjective. Sinece $H_i(1\otimes Sp)$ preserves S-grading and each homogeneous component of $H_i(Pf_{2t}\otimes_S\wedge^2\mathrm{id}_F)$ is a finite dimensional R-vector space, it is enough to show that there is an isomorphism of graded R-vector spaces

$$H_i(Pf_{2t}\otimes s\wedge^2\mathrm{id}_F)\cong H_i(Pf_{2t}\otimes s\mathrm{id}\wedge^2F) \tag{7.2}$$

to show that $H_*(1\otimes Sp)$ is injective.

As we have a splitting s of p, $\wedge^2\mathrm{id}_F$ is decomposed into the direct sum:

$$\wedge^2\mathrm{id}_F\cong\mathrm{id}_{\wedge^2F}\otimes\mathrm{id}_{D_2F}[-1].$$

Hence, we have an isomorphism of graded R-complexes

$$Pf_{2t}\otimes_S S\wedge^2\mathrm{id}_F\cong(Pf_{2t}\otimes_S S\mathrm{id}_{\wedge^2F})\otimes\wedge\mathrm{id}_{D_2F}=\bigoplus_{i\geq 0}(Pf_{2t}\otimes_S S\mathrm{id}_{\wedge^2F})\otimes\wedge^i\mathrm{id}_{D_2F}.$$

Since $\wedge^i\mathrm{id}_{D_2F}$ is homotopically trivial for $i>0$, and since $\wedge^0\mathrm{id}_{D_2F}=R$, we have established the desired isomorphism (7.2).

Now we want to calculate the homology group of $Pf_{2t}\otimes_S S\wedge^2\mathrm{id}_F$ to study the Betti numbers of S/Pf_{2t}. Note that $\mathcal{P}^t\overset{def}{=}Pf_{2t}\otimes_S S\wedge^2\mathrm{id}_F$ is a graded S-sub-complex of $S\otimes_S S\wedge^2\mathrm{id}_F=S\wedge^2\mathrm{id}_F$. The degree j componemt of $\mathcal{P}^t$ is denoted by $\mathcal{P}^{t,j}$ By Lemma 7.1, we have

$$H_i(\mathcal{P}^{t,j})=[\mathrm{Tor}^S_{i+1}(S/Pf_{2t},R)]_j.$$

By definition of $\pi:\wedge\mathrm{id}_F\to S\wedge^2\mathrm{id}_F$, We have $\pi_{2r}(\wedge^{t+r,2r}\mathrm{id}_F)\subset\mathcal{P}^{t,r}$ for $t\geq 1$. Hence, for a partiton λ, we have

$$\pi_{2\lambda}(\wedge_{t+\lambda_1,2\lambda}\mathrm{id}_F)\subset\mathcal{P}^{t,|\lambda|}.$$

Definition 7.3. Let $r\geq 0$ and $t\geq 1$. For a partition λ of degree r, we define

$$M_{t,\lambda}(\pi)\overset{def}{=}\sum_{\mu\geq\lambda,|\mu|=r}\pi_{2\mu}(\wedge_{t+\mu_1,2\mu}\mathrm{id}_F)$$

and

$$\dot{M}_{t,\lambda}(\pi)\overset{def}{=}\sum_{\mu>\lambda,|\mu|=r}\pi_{2\mu}(\wedge_{t+\mu_1,2\mu}\mathrm{id}_F).$$

We have $M_{t,(1^{r-t})}=\mathcal{P}^{t,r}$, so that $\{M_{t,\lambda}\}$ is a filtration of $\mathcal{P}^{t,r}$.

Lemma 7.4. *Let R be an arbitraty commutative ring, and φ: $G\to F$ a map of finite free R-modules. Let $\lambda/\mu=(\lambda_1,\lambda_2)/(\mu_1,\mu_2)$ be a two-rowed skew-partition, and $t\geq\lambda_2-\mu_1$. Then, we have*

$$\mathrm{Im}\square_{\lambda/\mu}\cap\wedge_{t,\lambda/\mu}\varphi=\sum_{k>\mu_1-\mu_2}\square_{\lambda/\mu}(\wedge_{t+k,\,\gamma+k\cdot\alpha_1}\varphi),$$

where $\gamma=\lambda-\mu$, *and* $\alpha_1=\varepsilon_1-\varepsilon_2$.

Proof. It is obvious that the right-hand side is contained in the left-hand side.

We prove that the left-hand side is contained in the right-hand side. Let $X=\{x_1<\cdots<x_m\}$ and $Y=\{y_1>\cdots>y_n\}$ be ordered bases of F and G, respectively. We set $\mathbf{X}=X\cup Y$, and we let $X<Y$ so that $\mathbf{X}$ is a totally ordered set. Then, any element $a\in \mathrm{Im}\,\square_{\lambda/\mu}\cap \wedge_{t,\lambda/\mu}\varphi$ is expressed as $a=\sum c_S S$ $(c_S\in R)$ uniquely, where the the aum is taken over row-standard tableaux mod Y of skew partition λ/μ with valued in $\mathbf{X}$. We set $N(a)=\{S\in \mathrm{Row}_{\lambda/\mu}(\mathbf{X},Y)\ |c_S\neq 0\}$. We prove a is contained in the right-hand side by induction on $\mathrm{ht}\,(a)\overset{def}{=}\sum_{S\in N(a)}2^{\mathrm{ht}(S)}$, where $\mathrm{ht}\,(S)$ is the number of elements in $\mathrm{Row}_{\lambda/\mu}(\mathbf{X},Y)$ which is smaller than S in the lexicographic order. If any element of $N(a)$ is standard mod Y, then $d_{\lambda/u}(a)\neq 0$ unless $N(a)=\emptyset$ by the standard basis theorem. Since $\mathrm{Ker}\ d_{\lambda/\mu}=\mathrm{Im}\,\square_{\lambda/\mu}$, this means $a=0$. So we may assume that there is some $S_0\in N(a)$ which is not standard mod Y. The column-standardness is violated in some column of S_0, say j^{th} column $(\mu_1<j\leq\lambda_2)$. Since $a\in\wedge_{t,\lambda/\mu}\varphi$, we have $S_0(1,\mu_1+1),\cdots,S_0(1,\mu_1+t)\in X$. This shows that $X\ni S(1,j)>S(2,j)$. We put $u=j-\mu_1-1$ and $v=\lambda_2-j$. When we define $S_1\in\mathrm{Row}_{(u,|\gamma|-u-v,v)}(\mathbf{X},Y)$ by $S_1(1,l)=S_0(1,l+\mu_1)$ for $1\leq l\leq u$, $S_1(2,l)=S_0(2,l+\mu_2)$ for $1\leq l\leq j-\mu_2$, $S_1(2,l)=S_0(1,l+\mu_2-1)$ for $j-\mu_2<l\leq|\gamma|-u-v$, and $S_1(3,l)=S_0(2,l+j)$ for $1\leq l\leq v$. Then we have

$$\overline{\square}_u(S_1)=\pm S_0+\text{(lower terms)}$$

as in the proof of Lemma II.2.15 of [2], where $\overline{\square}_u\colon\wedge_{(u,|\gamma|-u-v,v)}\varphi\longrightarrow\wedge_{\lambda/\mu}\varphi$ is the composite map

$$\wedge^u\varphi\otimes\wedge^{\gamma_1+\gamma_2-u-v}\varphi\otimes\wedge^v\varphi\xrightarrow{1\otimes\Delta\otimes 1}\wedge^u\varphi\otimes\wedge^{\gamma_1-u}\varphi\otimes\wedge^{\gamma_2-v}\varphi\otimes\wedge^{\gamma_2}\varphi$$
$$\xrightarrow{m\otimes m}\wedge^{\gamma_1}\varphi\otimes\wedge^{\gamma_2}\varphi.$$

It suffices to show that $\overline{\square}_u(S_1)$ is contained in the right-hand side so that we can apply the induction assumption on $a\mp c_{S_0}\overline{\square}_u(S_1)$. But this is clear by Lemma I.3.9 in [9] (the special case $i_1=0$).

Lemma 7.5. *Let R and φ be as in Lemma* 7.4, *λ/μ a skew partition, and M a submodule of $\wedge_{\lambda/u}\varphi$ generated by a poset ideal B of* Row $(\mathbf{X},Y)$, *where X, Y and $\mathbf{X}$ are as in the proof of Lemma 7.4. Then,*

$$\text{(7.6)}\qquad \mathrm{Ker}\ d_{\lambda/\mu}\cap M=\sum_{i=1}^{l(\lambda/\mu)}\Big(\Big(\sum_{k>\mu_i-\mu_{i+1}}\mathrm{Im}\,\square_{\lambda/\mu}^{\lambda-\mu+k\alpha_i}\Big)\cap M\Big).$$

Proof. The direction $\supset$ is obvious by the standard basis therorem [2, Theorem V.1.10].

We prove the opposite direction $\subset$. Let $a=\sum_{S\in B} c_S S$ $(c_S\in R)$ be an element in the left-hand side.

As in the proof of Lemma 7.4, we proceed by induction on $\mathrm{ht}(a)$. As in the proof of Lemma 7.4, we may assume that there is some $S_0\in N(a)$ which is not standard mod Y. When the column-standardness is violated between ith and the $(i+1)$th rows, then there exists some $T_1, \cdots, T_u < S_0$ and $a_1, \cdots, a_u\in R$ such that

$$S_0-\sum_j a_j T_j \in \sum_{k>\mu_i-\mu_{i+1}} \mathrm{Im}\square_{\lambda/\mu}^{\lambda-\mu+k\alpha_i}.$$

As B is a poset ideal, we have that $S_0-\sum_j a_j T_j$ is contained in

$$\left(\sum_{k>\mu_i-\mu_{i+1}} \mathrm{Im}\square_{\lambda/\mu}^{\lambda-\mu+k\alpha_i}\right)\cap M.$$

Since we have $a-c_{S_0}(S_0-\sum_j a_j T_j)$ is contained in the right-hand side of (7.6) by induction assumption, we have that a is also contained in the right-hand side.

The following lemma shows that the homology of t-Schur complexes of the identity map is closely related to the Betti numbers of Pfaffian ideals.

Lemma 7.7 *If $|\lambda|\leq 2t$, then we have*

$$\dot{M}_{t,\lambda}=\dot{M}_\lambda\cap M_{t,\lambda}.$$

In particular, we have a unique isomorphism

$$\gamma_{t,\lambda}\colon L_{t+\lambda_1,2\lambda}\mathrm{id}_F \longrightarrow M_{t,\lambda}/\dot{M}_{t,\lambda}$$

which makes the following diagram commutative.

$$\begin{array}{ccc} \wedge_{t+\lambda_1,2\lambda}\mathrm{id}_F & \xrightarrow{\pi_{2\lambda}} & M_{t,\lambda} \\ d_{2\lambda}\downarrow & & \downarrow \\ L_{t+\lambda_1,2\lambda}\mathrm{id}_F & \xrightarrow{\gamma_{t,\lambda}} & M_{t,\lambda}/\dot{M}_{t,\lambda} \end{array}$$

In particular, if $r\leq 2t$, then $\mathcal{P}^{t,r}$ is isomorphic to $\oplus_{|\lambda|=r} L_{t+\lambda_1,2\lambda}\,\mathrm{id}_F$ up to filtration.

Proof. Note that

$$\gamma_\lambda\colon L_{2\lambda}\mathrm{id}_F \to M_\lambda/\dot{M}_\lambda$$

maps $L_{t+\lambda_1,2\lambda}\mathrm{id}_F$ isomorphically onto

$$(\pi_{2\lambda}(\wedge_{t+\lambda_1,2\lambda}\mathrm{id}_F)+\dot{M}_\lambda)/\dot{M}_\lambda=(M_{t,\lambda}+\dot{M}_\lambda)/\dot{M}_\lambda.$$

If $\dot{M}_{t,\lambda}=\dot{M}_\lambda\cap M_{t,\lambda}$, then we define $\gamma_{t,\lambda}$ to be the composite map

$$L_{t+\lambda_1,2\lambda}\mathrm{id}_F \xrightarrow{\gamma_\lambda} (M_{t,\lambda}+\dot{M}_\lambda)/\dot{M}_\lambda \cong M_{t,\lambda}/(\dot{M}_\lambda \cap M_{t,\lambda}) = M_{t,\lambda}/\dot{M}_{t,\lambda},$$

and it is easy to prove the rest of the assertions in the lemma.

So it suffices to prove $\dot{M}_{t,\lambda}=\dot{M}_\lambda \cap M_{t,\lambda}$. The direction $\subset$ is clear. For the opposite direction, it suffices to show that Ker $d_{2\lambda} \cap \wedge_{t+\lambda,2\lambda}\mathrm{id}_F$ is mapped to $\dot{M}_{t,\lambda}$ by $\pi_{2\lambda}$. By Lemma 7.5, we have

$$\mathrm{Ker}\ d_{2\lambda} \cap \wedge_{t+\lambda,2\lambda}\mathrm{id}_F = \textstyle\sum_i ((\sum_{k>0} \mathrm{Im}\,\square_{2\lambda}^{2\lambda+k\alpha_i}) \cap \wedge_{t+\lambda,2\lambda}\mathrm{id}_F).$$

We may assume that $t+\lambda_1 \geq 2\lambda_1$. So we have $\lambda_1 \geq \lambda_2+\cdots+\lambda_{l(\lambda)}$ by our assumption $|\lambda| \leq 2t$. So it is easy to see that

$$\pi_{2\lambda}((\sum_{k>0}\mathrm{Im}\,\square_{2\lambda}^{2\lambda+k\alpha_i}) \cap \wedge_{t+\lambda,2\lambda}\mathrm{id}_F) \subset \dot{M}_{t,\lambda}.$$

for $i \geq 2$.

Hence, we may assume that $l(\lambda)=2$. By Lemma 7.4, Ker $d_\lambda \cap \wedge_{t+\lambda_1,2\lambda}\mathrm{id}_F$ is the sum of the images of the maps

$$\square_h \colon \wedge_{t+\lambda_1+h,2\lambda+h\cdot(\varepsilon_1-\varepsilon_2)}\mathrm{id}_F \to \wedge_{t+\lambda_1,2\lambda}\mathrm{id}_F$$

for $h=1,\cdots,2\lambda_2$. For each h we have, $\eta^h_{2\lambda_1+h,2\lambda_2-h}=\pi\circ\square_h$ is a linear combination of $\xi^l_{2\lambda_1+h,2\lambda_2-h}=\pi\circ\tilde{\square}_l$ for intergers l such that $0\leq l\leq h$ and $h-l$ even by Lemma 3.5, where $\tilde{\square}_l\colon \wedge_{(2\lambda_1+h,2\lambda_2-h)}\mathrm{id}_F \to \wedge_{(2\lambda_1+h+l,2\lambda_2-h-l)}\mathrm{id}_F$ is the boxtilde map. Note that $\tilde{\square}_l(\wedge_{t+\lambda_1+h,(2\lambda_1+h,2\lambda_2-h)}\mathrm{id}_F) \subset \wedge_{t+\lambda_1+h,(2\lambda_1+h+l,2\lambda_2-h-l)}\mathrm{id}_F$.

Since $t+\lambda_1+h \geq t+\lambda_1+(h+l)/2$ when $l\leq h$, we have

$$\xi^l_{2\lambda_1+h,2\lambda_2-h}(\wedge_{t+\lambda_1+h,(2\lambda_1+h,2\lambda_2-h)}\mathrm{id}_F) \subset \dot{M}_{t,\lambda}$$

when $l\leq h$. Hence, we have $\eta^h_{2\lambda_1+h,2\lambda_2-h}(\wedge_{t+\lambda_1+h,(2\lambda_1+h,2\lambda_2-h)}\mathrm{id}_F) \subset \dot{M}_{t,\lambda}$ for $h=1,\cdots,2\lambda_2$. This shows that $\pi_{2\lambda}(\mathrm{Ker}\ d_\lambda \cap \wedge_{t+\lambda_1,2\lambda}\mathrm{id}_F) \subset \dot{M}_{t,\lambda}$.

Using the lemma and the results on the homology of t-Schur complexes so far we developed, we can prove the following theorem, which is a refinement of [16,Theorem 5.3].

Theorem 7.8. *Let K be a field of characteristic p.*

1 *If $p\neq 2$, then the first syzygy module of Pf_{2t} as an S-module is generated by linear relations. Or equivalently, we have $\beta^p_{2,j}=0$ for $j\neq t+1$.*

2 *If $p=2$, then we have*

$$\mathrm{Tor}_2^S(S/Pf_{2t},K) \cong [\mathrm{Tor}_2^S(S/Pf_{2t},K)]_{t+1} \otimes \bigoplus_{i=1}^{[\log_2 t]} \wedge^{2(t+2^i)}F$$

as $GL(F)$-modules so that $\beta^2_{2,j}=0$ unless $j=t+2^i$ for some $0\leq i\leq [\log_2 t]$.

3 *We have*

$$\beta_2^p = \begin{cases} n\binom{n}{2t+1} - \binom{n}{2t+2} & (p \neq 2) \\ n\binom{n}{2t+1} - \binom{n}{2t+2} + \sum_{i=1}^{[\log_2 t]} \binom{n}{2^{i+1}+2t} & (p=2). \end{cases}$$

Proof. First we recall Kurano's result [16, Theorem 5.3] which says that for any p, we have $\beta_{2,j}^p = 0$ unless $t+1 \leq j \leq 2t$. By Lemma 7.7, for any $r \leq 2t$, we have a spectral sequence of polynomial representation of GL (F) associated with the filtartoion $\{M_{t,\lambda}\}_{|\lambda|=r}$, with E^1-terms $H_i(L_{t+\lambda_1,2\lambda}\mathrm{id}_F)$, and converges to $[\mathrm{Tor}_{i+1}^S(S/Pf_{2t}, K)]_r$.

If $t+2 \leq r \leq 2t$ and $p \neq 2$, then $H_1(L_{t+\lambda_1,2\lambda}\mathrm{id}_F) = 0$ for any partition λ of degree r by Corollary 5.3 (when $\lambda = (t, a)$ $(2 \leq a \leq t)$, $2a$ is not a power of p when $p \neq 2$). This shows that $\beta_{2,j}^p = 0$ when $p \neq 2$ and $j \neq t+1$, and **1** is proved.

Consider the case $t+2 \leq r \leq 2t$ and $p=2$. By Corollary 5.3 again, $\beta_{2,j}^p = 0$ unless j is of the form $t+2^l$ for some $l \geq 1$. Consider the case $j = t+2^l$ for some $1 \leq l \leq [\log_2 t]$. Then, we have $H_1(L_{t+\lambda_1,2\lambda}\mathrm{id}_F) = 0$ unless $\lambda = (t, 2^l)$, and $H_1(L_{2t,(2t,2^{l+1})}\mathrm{id}_F) \cong \wedge^{2t+2^{l+1}}F$. Hence, to prove the assertion **2**, it suffices to show that the E^1-term $H_1(L_{2t,(2t,2^{l+1})}\mathrm{id}_F) \cong \wedge^{2t+2^{l+1}}F$ is canonically isomorphic to the E^∞-term. We may assume that $n \geq 2(t+2^l)$.

By Theorem 4.4, $H_0(L_{t+\mu_1,2\mu}\mathrm{id}_F) = 0$ for any $\mu > \lambda = (t, 2^l)$ ($>$ is the lexicographic order). So the E^∞-term in question is a homomorphic image of the E^1-term. By Theorem 4.4, $\mu < \lambda$ and $H_2(L_{t+\mu_1,2\mu_2}\mathrm{id}_F) \neq 0$ imply μ is of the form (t, a, b) $(a+b=2^l, a \geq b > 0)$.

Assume that $E^1 \not\cong E^\infty$. Then we can take a minimum number b such that the map of the spectral sequence

$$d_b\colon H_2(L_{2t,(2t,2a,2b)}\mathrm{id}_F) \to H_1(L_{2t,(2t,2^{l+1})}\mathrm{id}_F) = \wedge^{2t+2^{l+1}}F$$

is non-zero (or equivalently surjective, since $\wedge^{2t+2^{l+1}}F$ is an irreducible representation). This means that the weight $\omega = (1,\cdots,1)$ $(2|\lambda|$ times $1)$ component $(d_b)_\omega$ of d_b is non-zero. On the other hand, by Lemma 6.2, $H_2(L_{2t,(2t,2a,2b)}\mathrm{id}_F)_\omega$ is generated by $A_{(2t,2a,2b)}$. A straightforward computation shows that

$$d_b(A_{(2t,2a,2b)}) = \partial(\pi_{(2t,2a,2b)}A_{(2t,2a,2b)}) = \pi_{(2t,2a,2b)}(\partial A_{(2t,2a,2b)})$$

equals to

$$\pi_{(2t,2a,2b)}\left(\square \sum_{\sigma \in \mathfrak{S}^{(2t,2^{l+1})}} (-1)^\sigma \begin{matrix} \sigma 1 & \cdots\cdots & \sigma(2t) \\ \sigma(2t+1) & \cdots\cdots & \boxed{\sigma(2^{l+1})} \end{matrix}\right)$$

modulo $\dot{M}_{t,\lambda}$, where $\square$ is the box map $\wedge_{2\lambda}\mathrm{id}_F \to \wedge_{(2t,2a,2b)}\mathrm{id}_F$, which is nothing but a diagonalization of the second row of 2λ. By Lemma 3.2, we have

$$d_b(A_{(2t,2a,2b)}) = \binom{2^{l+1}}{2b} \pi_{2\lambda}\left(\sum_{\sigma \in \mathfrak{S}^{(2t,2^{l+1})}} (-1)^{\sigma} \begin{matrix} \sigma 1 & \cdots\cdots & \sigma(2t) \\ \sigma(2t+1) & \cdots\cdots & \boxed{\sigma(2^{l+1})} \end{matrix} \right),$$

which is zero since $\binom{2^{l+1}}{2b}$ is an even number. This is a contradiction, and the proof of **2** is complete.

We prove **3**. By **1** and **2** proved above, it suffices to show that

$$\beta^p_{2,t+1} = n\binom{n}{2t+1} - \binom{n}{2t+2} \tag{7.9}$$

for any p.

As S/Pf_{2t} is R-free for any commutative ring R by the Plethysm formula [16, Proposition 3.5], it is esay to see that the alternating sum $\sum_{i=0}^{\infty}(-1)^i \beta^p_{i,,j}$ is independent of p. Since $\beta^p_{i,t+1}=0$ unless $i=2$, we have that $\beta^p_{2,t+1}$ is independent of p. To prove (7.9), we may assume that K is of characteristic zero. But this case is done by Józefiak, Pragacz and Weyman [12].

Let us consider an arbitrary commutative ring R as the base ring. A finite free graded S-complex $\mathbf{F}$ is said to be *minimal* when the boundary map of $S/S_+ \otimes_S \mathbf{F}$ is zero, where $S_+ = (x_{ij}) \cdot S$. When R is a filed of characteristic p, then a finite free resolution

$$\mathbf{F}: \cdots \rightarrow F_i \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow S/Pf_{2t} \rightarrow 0$$

of S/Pf_{2t} is minimal if and only if rank $F_i = \beta^p_i$. A base change $R' \otimes_R \mathbf{F}$ of a minimal free resolution $\mathbf{F}$ is still a minimal free resolution.

Corollary 7.10 *If $4 \leq 2t \leq n-4$, then there is no minimal free resolution of S/Pf_{2t} over the base ring $\mathbf{Z}$, the ring of integers.*

Proof. By the theorem, $\beta^p_{2,t+2}$ depends on the characteristic.

Remark 7.11. When $2t \geq n-2$, there is a minimal free resolution of S/Pf_{2t} [21]. What about the case $2t = n-3$? This case is open. Is there any example of a Pfaffian ideal which does not have any minimal free resolution over $\mathbf{Z}[1/2]$? This question is still open, too.

8. Generators of the relation

In this section, the base ring $R = K$ is assumed to be a field. In the last section, we calculated the number of generators β^p_2 of the relation module Ker φ of Pf_{2t}, where p is the characteristic of K, and φ is as in the last section. We fix a basis $X=\{x_1,\cdots,x_n\}$ of F, and denote the element $x_{i_1}\wedge\cdots\wedge x_{i_k}$ of $\wedge^k F$ by $[i_1,\cdots,i_k]$, and $\pi^S_{2r}([i_1,\cdots,i_k])\in S_r$ by $\langle i_1,\cdots,i_{2r}\rangle$. For example, the Laplace expansion formla (see section 7) is rewrited as

$$\sum_{\alpha=1}^{2r+1}(-1)^{\alpha+1}x_{ai_\alpha}\langle i_1,\cdots\overset{\alpha}{\vee}\cdots,i_{2r+1}\rangle=\langle a,i_1,\cdots,i_{2r+1}\rangle \tag{8.1}$$

for $r\geq 0$, $1\leq a,i_1,\ldots,i_{2r+1}\leq n$ (see [16, Lemma 3.3], [11]).

From Lemma 3.2, we have

$$\sum_{\sigma\in\mathfrak{S}^{a,b}}(-1)^\sigma\langle i_{\sigma(1)},\cdots,i_{\sigma(2a)}\rangle\cdot\langle i_{\sigma(2a+1)},\cdots,i_{\sigma(2(a+b))}\rangle=\binom{a+b}{b}\langle i_1,\cdots,i_{2(a+b)}\rangle$$

for a, $b\geq 0$ and $1\leq i_1,\cdots,i_{2(a+b)}\leq n$ (see [16, Lemma 3.3]). We call this formula the *second Laplace expansion*.

Let $1\leq a,b,i_1,\ldots,i_{2t}\leq n$. We define $T(a,b;i_1,\ldots,i_{2t})\in S\otimes\wedge^{2t}F$ by

$$T(a,b;i_1,\ldots,i_{2t})\overset{def}{=}\sum_{\alpha=1}^{2t}(-1)^\alpha x_{ai_\alpha}\otimes[b,i_1,\cdots\overset{\alpha}{\vee}\cdots,i_{2t}]+\sum_{\alpha=1}^{2t}(-1)^\alpha x_{bi_\alpha}\otimes[a,i_1,\cdots\overset{\alpha}{\vee}\cdots,i_{2t}].$$

It is easy to see that $T(a,b;i_1,\ldots,i_{2t})\in\operatorname{Ker}\varphi$.

Proposition 8.2. *Let K be a noetherian commutative ring which contains 1/2. Then, we have*

1 *We have*

$$\operatorname{Tor}_2^S(S/Pf_{2t},K)\cong K_{(2,1^{2t})}F.$$

2 *The first syzygy module Ker φ is minimally generated by the set*

$$X=\{T(a,b;i_1,\ldots,i_{2t})\mid a<i_1<\cdots<i_{2t}\leq n,\ 1\leq a\leq b\leq n\}$$

as an S-module.

Proof. We may assume that K is a field of characteristic different from two, because whether or not a $\mathbf{Z}[1/2]$-form M of $K_{(2,1^{2t})}F$ is isomorphic to $K_{(2,1^{2t})}F$ is determined by whether it is generated by a weight $(2,1^{2t})$-vector (see e.g., [8, Theorem 7.2]), and this is checked by the specializations at fields. As in the proof of Theorem 7.8, we have that the only partition λ of degree $t+1$ such that $H_1(L_{t+\lambda_1,2\lambda}\mathrm{id}_F)$ is not zero is $\lambda=(t+1)$ by Corollary 5.3, and we have $H_1(L_{2t+1,(2t+2)}\mathrm{id}_F)\cong K_{(2,1^{2t})}F$. Hence, by Theorem 7.8, $\operatorname{Tor}_2^S(S/Pf_{2t},K)$ is a subquotient of $K_{(2,1^{2t})}F$ whose dimension is equal to $\dim_K K_{(2,1^{2t})}F$. So **1** is proved.

By definition, $T(a,b;i_1,\ldots,i_{2t})$ agrees with the image of $x_ax_b\otimes[i_1,\ldots,i_{2t}]$ by the *first Laplace exansion* map

$$\Psi:D_2F\otimes\wedge^{2t}F\xrightarrow{\Delta\otimes 1}F\otimes F\otimes\wedge^{2t}F\xrightarrow{1\otimes m}F\otimes\wedge^{2t+1}F\xrightarrow{\tilde{\square}}\wedge^{2t}F\otimes\wedge^{2t}F.$$

By Standard Basis Theorem on Weyl modules [2, Theorem II,3. 16], the set X spans Im Ψ (note that $x_a^2 = 2x_a^{(2)}$ and we assume that 2 is invertible in K). So the S-span of X, which we denote SX, is a GL(F)-submodule of $[\mathrm{Ker}\ \varphi]_{t+1} \cong K_{(2,1^{2t})}F$, and SX contains a non-zero weight $(2,1^{2t})$ vector $T(1,1;\ 2,\ldots,\ 2t+1)$. Since the Weyl module $K_{(2,1^{2t})}F$ is generated by its weight $(2,\ 1^{2t})$-component as a GL (F)-module, we have $SX = [\mathrm{Ker}\ \varphi]_{t+1}$, and it generates Ker φ by Theorem 7.8.

Corollary 8.3. *Let R be a noetherian commutative ring which contains 1/2. Let ψ: $S \otimes K_{(2,1^{2t})}F \to S \otimes \wedge^{2t} F$ be the unique* GL (F)*-equivatiant S-linear map given by*

$$1 \otimes ([1,\cdots,\ 2t+1] \otimes 1) \to 1/2 \cdot T(1,1;2,\cdots,\ 2t+1).$$

Then we have

1 *The sequence*

$$\mathbf{F}:\ S \otimes K_{(2,1^{2t})}F \xrightarrow{\psi} S \otimes \wedge^{2t}F \xrightarrow{\varphi} S \to S/Pf_{2t} \to 0$$

is exact.

2 *Let (a_{ij}) be an $n \times n$-alterating matrix with coefficients in R. We denote by I the ideal of S generated by $(x_{ij} - a_{ij})$ so that $S \to S/I$ is identified with the substitution $S \to R$ $(x_{ij} \mapsto a_{ij})$. The following are equivalent.*

a) *$S/I \otimes_S \mathbf{F}$ is exact.*

b) *$Pf_{2t}(a_{ij}) = R$ or* grade $Pf_{2t}(a_{ij}) = \binom{n-2t+2}{2}$.

Proof. **1** is nothing but a reformulation of the proposition. We prove **2**. We may assume that R is complete local so that R is a homomorphic image of a regular local ring $\widetilde{R}$. We denote the polynomial ring $\widetilde{R} \otimes_{\mathbf{Z}} S^{\mathbf{Z}}$ by $\widetilde{S}$ so that $S = R \otimes_{\widetilde{R}} \widetilde{S}$. We denote the kernel of the composite $\widetilde{S} \to S \to S/\mathrm{I}$ by J. As we can construct a finite free resolution of $\widetilde{S}/Pf_{2t}^{\mathbf{Z}}\widetilde{S}$ of the from

$$0 \to \mathbf{P}_q \to \cdots \to \mathbf{P}_3 \to \widetilde{S} \otimes K_{(2,1^{2t})}F \xrightarrow{\psi} \widetilde{S} \otimes \wedge^{2t}F \xrightarrow{\varphi} \widetilde{S} \to \widetilde{S}/Pf_{2t}^{\mathbf{Z}} \cdot \widetilde{S} \to 0,$$

the condition a) is equivalent to $\mathrm{Tor}_1^{\widetilde{S}}(\widetilde{S}/Pf_{2t}^{\mathbf{Z}} \cdot \widetilde{S},\ \widetilde{S}/J) = 0$. This condition is equivalent to $\mathrm{Tor}_i^{\widetilde{S}}(\widetilde{S}/Pf_{2t}^{\mathbf{Z}} \cdot \widetilde{S}, \widetilde{S}/J) = 0$ for $i > 0$ by Lichtenbaum's theorem [18]. On the other hand, since $(Pf_{2t}^{\mathbf{Z}}) \cdot \widetilde{S}$ is perfect of codimension $h = \binom{n-2t+2}{2}$, this condition is equivalent to depth$(Pf_{2t}^{\mathbf{Z}} \cdot \widetilde{S}, \widetilde{S}/J)$ = depth $(Pf_{2t}(a_{ij}), K) \geq h$ by the lemma below.

Lemma 8.4 (depth senitivity). *Let R be a noetherian ring, I a perfect ideal of R of codimension g, and M a finitely generated R-module such that $M \neq IM$. Then, we have* depth $(I,\ M) = g - \max\{i \mid \mathrm{Tor}_i^R(R/I,\ M) \neq 0\}$.

Proof. This is well-known (see [6, Proposition 2.1]), and follows easily from the criterion of the exactness of complexes [4].

When the characteristic is two, the description of a minimal set of generators of Ker φ is more complicated. We consider the following three types of relations (i.e., elements of Ker φ).

type I $\sum_\alpha (-1)^\alpha x_{ai_\alpha} \otimes [a, i_1, \cdots, \overset{\alpha}{\vee}, \cdots, i_{2t}]$ $(1 \leq a < i_1 < \cdots < i_{2t} \leq n)$.

type II $T(a, b; i_1, \cdots, i_{2t})$ $(1 \leq a < i_1 < \cdots < i_{2t} \leq n,\ 1 \leq i_1 \leq b \leq n)$, where $T(a, b; i_1, \cdots, i_{2t})$ is as in Proposition 8.2.

type III Let $0 \leq a \leq [\log_2 t]$. For any $\underline{i} = (i(1), \dots, i(2t+2^{a+1}))$ such that $1 \leq i(1) < \cdots < i(2t+2^{a+1}) \leq n$, we have

$$\varphi\left(\sum_{\sigma \in \mathfrak{S}_{2t,2^{a+1}}} (-1)^\sigma \langle i(\sigma(2t+1)), \cdots, i(\sigma(2t+2^{a+1}))\rangle \otimes [\sigma 1, \cdots, \sigma(2t)]\right)$$
$$= \binom{t+2^a}{t} \langle i(1), \cdots, i(2t+2^{a+1})\rangle$$

by the second Laplace expansion rule. On the other hand, we have an expansion

$$\langle i(1), \dots, i(2t+2^{a+1})\rangle = \sum_{\underline{j}} A(\underline{j}) \cdot \langle i(j_1), \dots, i(j_{2t})\rangle$$

($A(\underline{j}) \in S$) obtained by successive use of (first) Laplace expansion. When we fix such an expansion, then

$$W(\underline{i}) \overset{def}{=} \sum_{\sigma \in \mathfrak{S}_{2t,2^{a+1}}} (-1)^\sigma \langle i(\sigma(2t+1)), \dots, i(\sigma(2t+2^{a+1}))\rangle \otimes [\sigma 1, \dots, \sigma(2t)]$$
$$- \binom{t+2^a}{t} \sum_{\underline{j}} A(\underline{j}) \otimes [i(j_1), \dots, i(j_{2t})]$$

is an element of Ker φ.

Note that these elements are defined over $\mathbf{Z}$, and are relations over any commutative ring.

Proposition 8.5 *Let K be a field of characteristic two. Then, Ker φ is minimally generated by the relations of* **type I-III**.

Proof. First note that the homology $H_1(Pf_{2t} \otimes_S S \wedge^2 \mathrm{id}_F)$, which we calculated explicitly in the last section, is isomorphic to $K \otimes_S$ Ker φ by the connected morphism

$$\delta: H_1(Pf_{2t} \otimes_S S \wedge^2 \mathrm{id}_F) \to H_0(\mathrm{Ker}\varphi \otimes_S S \wedge^2 \mathrm{id}_F) = \mathrm{Ker}\varphi \otimes_S K$$

of the long exact sequence coming from the short exact sequence

$$0 \to \mathrm{Ker}\varphi \otimes_S S\wedge^2 \mathrm{id}_F \to (S\otimes \wedge^{2t}F)\otimes_S S\wedge^2\mathrm{id}_F \to Pf_{2t}\otimes_S S\wedge^2\mathrm{id}_F \to 0.$$

First we consider the degree $t+1$ generators of the relation. By Lemma 7.7, there is an exact sequence

$$0 \to \wedge^{2t+1,(2t+2)}\mathrm{id}_F \to [Pf_{2t}\otimes_S S\wedge^2\mathrm{id}_F]_{t+1} \to L_{2t,(2t,2)}\mathrm{id}_F \to 0$$

which induces a long exact sequence of homology groups

$$0 \to H_2(L_{2t,(2t+2)}\mathrm{id}_F) \to H_1(\wedge^{2t+1,2t+2}\mathrm{id}_F) \xrightarrow{\iota} H_1([Pf_{2t}\otimes_S S\wedge^2\mathrm{id}_F]_{t+1}) \to H_1(\wedge^{2t+1,2t+2}\mathrm{id}_F) \to 0.$$

With the map δ, the image of

$$\iota\colon H_1(\wedge^{2t+1,2t+2}\mathrm{id}_F) \to H_1([Pf_{2t}\otimes_S S\wedge^2\mathrm{id}_F]_{t+1})$$

is mapped to the first Laplace relation Im Ψ, where Ψ is as in Proposition 8.2. The set X in Proposition 8.2 and the elements of **type I** together generate the relations of the fisrt Laplace type (i.e., Im Ψ). But it is easy to see that the elements of **type I** together with **type II** also generate Im Ψ because we heve relations

$$T(a,b;c,i_2,\dots,i_{2t}) + T(b,c;a,i_2,\dots,i_{2t}) + T(a,c;b,i_2,\dots,i_{2t}) = 0$$

and

$$\sum_{\sigma\in\mathfrak{S}^{(2,2t)}} T(\sigma(1),\sigma(2);\sigma(3),\cdots,\sigma(2t+2)) = 0$$

(the second relation is only for characteristic two). Since the number of elements of **type I**, **type II** and **type III** of degree $t+1$ agrees with the number of standard tableaux of shape $(2t+1,1)$, it suffices to show that the image by δ^{-1} of the elements of **type III** of degree $t+1$ generates $H_1(L_{2t,(2t,2)}\mathrm{id}_F) \cong \wedge^{2t+2}F$. But this is obvious by Lemma 6.1. Now we have calculated the linear relations of Pfaffians as remarked in [21, Remark 2.1].

Next we calculate the minimal generators of higher degree. Consider the relation $W(\underline{i})$ of **type III**. As the expansion

$$\langle i(1),\dots, i(2t+2^{a+1})\rangle = \sum_{\underline{j}} A(\underline{j}) \cdot \langle i(j_1),\dots, i(j_{2t})\rangle$$

is obtained by the successive use of the Laplace expansion, we can choose $\widetilde{A}(\underline{j})$ from $S_{2^a-1}\wedge^2F\otimes F\otimes F$ such that the image of $\widetilde{A}(\underline{j})$ by the map

$$S_{2^a-1}\wedge^2F\otimes F\otimes F \xrightarrow{1\otimes m} S_{2^a-1}\wedge^2F\otimes\wedge^2F \xrightarrow{m} S_{2^a}\wedge^2F$$

is $A(\underline{j})$ and that the image of $\sum_{\underline{j}}\langle i(j_1),\dots, i(j_{2t})\rangle\otimes\widetilde{A}(\underline{j})$ by the map

$$S_t\wedge^2F\otimes S_{2^a-1}\wedge^2F\otimes F\otimes F\xrightarrow{m\otimes 1}S_{t+2^a-1}\wedge^2F\otimes(F\otimes F)\subset S\wedge^2\mathrm{id}_F$$

is equals to $\pm\pi([i(1),\cdots\overset{l}{\vee}\cdots,i(2t+2^{a+1})]\otimes x_{i(l)})$ so that it is contained in $\pi_{2(t+2^a)}(\wedge^{2t+2^{a+1}-1,2t+2^{a+1}}\mathrm{id}_F)$, where l is the index used in the starting of the successive Laplace expansion of $\langle i(1),\ldots,i(2t+2^{a+1})\rangle$.

This shows that $W(1,2,\ldots,2t+2^{a+1})$ is the image of

$$A_{(2t,2^{a+1})}\mp\pi([1,2,\cdots\overset{l}{\vee}\cdots,2t+2^{a+1}]\otimes x_l)$$

by the connected map δ, where $A_{(2t,2^{a+1})}$ is as in Lemma 6.1. By Lemma 6.1, the one-dimensional vector space

$$H_1(\mathcal{P}^{t,t+2^a})_\omega\cong H_1(L_{2t,(2t,2^{a+1})}\mathrm{id}_F)_\omega$$

is generated by $A_{(2t,2^{a+1})}\mp\pi([1,2,\cdots\overset{l}{\vee}\cdots,2t+2^{a+1}]\otimes x_l)$. This shows that the element $W(1,2,\ldots,2t+2^{a+1})$ generates $[S/S_+\otimes_S\mathrm{Ker}\varphi]_\omega$, where ω is the weight $(1,1,\ldots,1)$ ($(2t+2^{a+1})$-times 1). Since we know that $[\mathrm{Tor}_2^S(S/Pf_{2t},K)]_{t+2^a}$ is isomorphic to the exterior power $\wedge^{2t+2^{a+1}}F$, $W(\underline{i})$ is a basis element of $S/S_+\otimes\mathrm{Ker}\varphi$ for other weight, in a similar way. Thus, the generators of the higher degrees are exhausted by the elements of **type III**.

NAGOYA UNIVERSITY
COLLEGE OF MEDICAL TECHNOLOGY

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