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Equivariant Hopf structures on a sphere

By

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1. Introduction

Let G be a compact Lie group. For a G-space X with a base point a pointed G-map

$$\mu: X \times X \longrightarrow X$$

is said to be an equivariant Hopf structure of X if the restriction of μ to $X \vee X$ is equivariantly homotopic to $id \vee id$. A G-space with an equivariant Hopf structure is called a Hopf G-space. If $G = \{id\}$, the trivial group, then a Hopf G-space is a usual Hopf space (H-sapce). According to a theorem of Adams the sphere S^n admits a Hopf structure precisely when n = 0, 1, 3, or 7. Equivariant Hopf structures on a sphere were considered in [Br], [Ir], [Is], [CC] and [H].

The multiplication in $\mathbf{C}$, the complex numbers, $\mathbf{H}$, the quaternions, and $\mathbf{O}$, the Cayley numbers define the Hopf structures on S^1, S^3 and S^7, respectively. These Hopf structures are equivariant with respect to the action of the automorphism groups $O(1)$ of $\mathbf{C}$, $SO(3)$ of $\mathbf{H}$ and G_2 of $\mathbf{O}$.

Let H and G be compact Lie groups and V a real representation space of H. We say that the representation V factors through G if there is a homomorphism f: $H \to G$ and a G-module W such that V is isomorphic to f^*W. In this paper we assume that every representation has an invariant metric. We denote by $S(V)$ the unit sphere of V. Then by [CC, Theorem 1.3(ii)] or by using the result about the $\mathbf{Z}/2$-equivariant Hopf structure on the spheres, [Br], [Ir], we have

Proposition 1.1. *Let G be a compact Lie group and V a real G-module of dimension 1 or 3. Then $S(\mathbf{R} \oplus V)$ has a G-equivariant Hopf structure if and only if V factors through $O(1)$ or $SO(3)$, respectively.*

On the otherhand as for equivariant Hopf structures on S^7 Cook and Crabb [CC] showed

Theorem 1.2. *We take $G = \mathbf{Z}/p^r$, where $p > 1$ is prime. Write E for the standard one-dimensional complex representation of G and E^k for its k-th tensor power. Let V be a non-trivial 7-dimensional real representation of G.*

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Then $S(\mathbf{R}\oplus V)$ admits a G-equivariant Hopf structure if and only if V is isomorphic to $\mathbf{R}\oplus E^a\oplus E^b\oplus E^c$, for some non-zero integers a, b, c with $\nu_2(a)=\nu_2(b)<\nu_2(c)\leq r$ if $p=2$, $\nu_p(a)=\nu_p(b)\leq\nu_p(c)\leq r$ if p is odd (and $\nu_p(a)<r$ since V is non-trivial).

This theorem implies that Proposition 1.1 does not hold when V is a 7-dimensional real representation. In contrast to this fact we prove that Proposition 1.1 holds when V is a 7-dimensional real representation of S^1.

Theoren 1.3. *Write E for the standard one-dimensional complex representation of S^1 and E^k for its k-th tensor power. Let V be a nontrivial 7-dimensional real representation of S^1. Then $S(\mathbf{R}\oplus V)$ admits an S^1-equivariant Hopf structure if and only if V is isomorphic to $\mathbf{R}\oplus E^a\oplus E^b\oplus E^c$, for some integers a, b, c with $c=\pm a\pm b$.*

To prove this theorem we make use of the same method as in [Is] and this shall be done in the next section. In Section 3 we consider G-equivariant Hopf structures on a sphere where G is an elementary abelian p-group.

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2. S^1-Equivariant Hope structure

First we recall the Ishikawa's work [Is]. Throughout this section G denotes S^1.

Let V be a complex G-module with a trivial G-module $\mathbf{C}^n$ $(n>0)$ as the direct summand. If $S(V)$ is equipped with an equivariant Hopf structure, the equivariant projective plane XP_2 for $X=S(V)$ is constructed by the ordinary way. The equivariant complex K-group $K_G(X)$ is an $R(G)$-module and the complex representation ring $R(G)$ of G is isomorphic to $\mathbf{Z}[t, t^{-1}]$, where t is the standard one-dimensional complex representation of G. Then

Theorem 2.1. *$\widetilde{K}_G(XP_2)$, for $X=S(V)$, is isomorphic to a free $R(G)$-module with a basis $\{u, v\}$ and the elements u and v satisfy*

$$u^2=v\ ,\quad uv=0\quad \text{and}\quad v^2=0\ .$$

Let $V=E^{a_1}\oplus\cdots\oplus E^{a_s}\oplus\mathbf{C}^n$, where each a_i is non zero integer. We put

$$r=r(t)=(t^{a_1}-1)\cdots(t^{a_s}-1)\quad \in R(S^1)$$

and

$$r_k=r_k(t)=(t^{ka_1}-1)\cdots(t^{ka_s}-1)/\mathrm{r}\quad \in R(S^1)\ .$$

Let φ: $X^GP_2\rightarrow XP_2$ be the natural inclusion map. $\widetilde{K}_G(X^GP_2)\cong R(G)\otimes\widetilde{K}(X^GP_2)$ is also a free $R(G)$-module with a basis $\{\tilde{u}, \tilde{v}\}$ and the elements $\tilde{u}$ and $\tilde{v}$ satisfy

$$\tilde{u}^2=\tilde{v}\ ,\quad \tilde{u}\tilde{v}=0 \quad \text{and} \quad \tilde{v}^2=0\ .$$

Then we have

$$\tilde{\varphi}^*(u)=r\tilde{u}+\beta\tilde{v}\ ,\quad \beta\in R(S^1)\ .$$

Now as for the Adams operation we obtain

Theorem 2.2. *In $\tilde{K}_G(XP_2)$, $X=S(E^{a_1}\oplus\cdots\oplus E^{a_s}\oplus\mathbf{C}^n)$, for each $k\in\mathbf{Z}$ we have*

$$\psi^k(u)=k^n r_k u+\alpha_k v\ ,$$

where $\alpha_k=\{k^{2n}\psi^k(\beta)-k^n r_k\beta+\sigma_k r r_k\}/r^2$ and $\psi^k(\tilde{u})=k^n\tilde{u}+\sigma_k\tilde{v}$.

With these preparation we prove Theorem 1.3.

If the representation V is isomorphic to $\mathbf{R}\oplus E^a\oplus E^b\oplus E^c$ with $c=\pm a\pm b$, the multiplication of the Cayley numbers define the equivariant Hopf structure on $S(\mathbf{R}\oplus V)$ (see [CC], [Is]).

Conversely we assume that $X=S(\mathbf{R}\oplus V)$, $V=\mathbf{R}\oplus E^a\oplus E^b\oplus E^c$, has an equivariant Hopf structure. If one of a, b and c is 0, then considering the subspaces of fixed points of some subgroups of G on X, which must be S^1 or S^3, we easily obtain the desired result. Since there is an equivariant diffeomorphism between $S(\mathbf{C}\oplus E^a\oplus E^b\oplus E^c)$ and $S(\mathbf{C}\oplus E^{\pm a}\oplus E^{\pm b}\oplus E^{\pm c})$, we can assume that $0<a\leq b\leq c$.

We recall some notation:

$$\begin{aligned} r=r(t)&=(t^a-1)(t^b-1)(t^c-1) &&\in \mathbf{Z}[t,t^{-1}]\ ,\\ r_k=r_k(t)&=(t^{ka}-1)(t^{kb}-1)(t^{kc}-1)/r &&\in \mathbf{Z}[t,t^{-1}]\ .\end{aligned}$$

Then by Theorem 2.2 there are elements $\beta(t)$ and $\alpha_k(t)$, $k\in\mathbf{Z}$, in $\mathbf{Z}[t,t^{-1}]$ which satisfy the relation

$$r^2\alpha_k=k^2\psi^k(\beta)-kr_k\beta+\binom{k}{2}rr_k \tag{2.3}$$

for each $k\in\mathbf{Z}$. Here we use the following fact. Since $X^G=S^1$, under a suitable choice of the genenators $\{\tilde{u},\tilde{v}\}$ we have $\sigma_k=\binom{k}{2}$.

A generator u of $\tilde{K}_G(XP_2)$ can be changed to $u+\gamma v$ ($\gamma\in\mathbf{Z}[t,t^{-1}]$), then β changes to $\beta+r^2\gamma$. Since the leading coefficient of r^2 and its constant term are 1, it is possible to choose the generators of $\tilde{K}_G(XP_2)$ so that $\beta(t)$ is a polynomial of degree at most $2(a+b+c)-1$. Then we consider the case when $k=-1$ in (2.3).

$$\begin{aligned} r^2(t)\alpha_{-1}(t)&=\beta(t^{-1})+r_{-1}(t)\beta(t)+r_{-1}(t)r(t)\\ &=\beta(t^{-1})-\{\beta(t)+r(t)\}\,t^{-(a+b+c)}\end{aligned}$$

Thus we have

$$(2.4)\qquad \beta(t^{-1})t^{a+b+c}-\beta(t)=r^2(t)\tilde{\alpha}_{-1}(t)+r(t)\ ,$$

where $\tilde{\alpha}_{-1}(t)=\alpha_{-1}(t)t^{a+b+c}$. If in this formula we substitute t^{-1} for t, then we obtain

$$\beta(t)t^{-(a+b+c)}-\beta(t^{-1})=r^2(t)t^{-2(a+b+c)}\tilde{\alpha}_{-1}(t^{-1})-r(t)t^{-(a+b+c)}\ ,$$

that is,

$$(2.5)\qquad \beta(t^{-1})t^{a+b+c}-\beta(t)=-r^2(t)\tilde{\alpha}_{-1}(t^{-1})t^{-(a+b+c)}+r(t)\ .$$

(2.4) and (2.5) imply

$$\tilde{\alpha}_{-1}(t^{-1})=-\tilde{\alpha}_{-1}(t)t^{a+b+c}\ ,$$

which means that $\tilde{\alpha}_{-1}(t)=0$. Thus we have

$$(2.6)\qquad \beta(t^{-1})t^{a+b+c}=\beta(t)+r(t)\ .$$

Especially by this formula we know that degree of $\beta(t)$ is at most $a+b+c$. If we put

$$\beta(t)=dr(t)+\beta_1(t)\ ,\quad d\in\mathbf{Z}\ ,\quad \deg\beta_1(t)<a+b+c\ ,$$

by (2.3) we have

$$\begin{aligned} r^2\alpha_k&=k^2(drr_k+\beta_1(t^k))-kr_k(dr(t)+\beta_1(t))+\binom{k}{2}rr_k\ ,\\ &=(2d+1)\binom{k}{2}rr_k+k^2\beta_1(t^k)-kr_k\beta_1(t)\ ,\end{aligned}$$

that is,

$$k^2\beta_1(t^k)-kr_k\beta_1(t)=r\left\{r\alpha_k-(2d+1)\binom{k}{2}r_k\right\}\ .$$

Especially in the case $k=2$ we have

$$4\beta_1(t^2)-2r_2\beta_1(t)=r\{r\alpha_2-(2d+1)r_2\}\ .$$

Comparing the degree of the both-hand sides in this formula, we see that $\alpha_2=2d+1$, and it follows that

$$4\beta_1(t^2)-2r_2\beta_1(t)=(2d+1)r(r-r_2)=-2(2d+1)(t^{a+b}+t^{b+c}+t^{c+a}+1)r(t)\ ,$$

that is,

$$(2.7)\qquad r_2(t)\beta_1(t)-2\beta_1(t^2)=(2d+1)(t^{a+b}+t^{b+c}+t^{c+a}+1)r(t)\ .$$

Similarly by the formula (2.6) we have

$$\beta_1(t^{-1})\, t^{a+b+c} = (2d+1)\, r(t) + \beta_1(t) \quad . \tag{2.8}$$

By (2.7) we know that the degree of $\beta_1(t)$ is $b+c$ and its coefficient is $2d+1$. Then we let

$$\beta_1(t) = (2d+1)\,(t^{a+b}+t^{b+c}+t^{c+a}+1)\,\beta_2(t)$$

with $\deg \beta_2(t) < b+c$. Substituting this formula for (2.7) and (2.8) we obtain

$$r_2(t)\beta_2(t) - 2\beta_2(t^2) = -4(2d+1)\,(t^{a+b}+t^{b+c}+t^{c+a}+t^{2a+b+c}+t^{a+2b+c}+t^{a+b+2c}) \ ,$$
$$\beta_2(t^{-1})\, t^{a+b+c} = \beta_2(t) \ .$$

It is easy to see that each coefficient of $\beta_2(t)$ is divisible by $4(2d+1)$. We let

$$\beta_3(t) = -\beta_2(t)/4(2d+1) \ ,$$

then

$$r_2(t)\beta_3(t) - 2\beta_3(t^2) = (t^{a+b}+t^{b+c}+t^{c+a}+t^{2a+b+c}+t^{a+2b+c}+t^{a+b+2c}) \ ,$$
$$\beta_3(t^{-1})\, t^{a+b+c} = \beta_3(t) \ .$$

Now Theorem 1.3 follows from the following proposition.

Proposition 2.9. *Let a, b and c be integers such that $0<a\le b\le c$. $f(t)$ is a polynomial of t which satisfies*

$$\begin{aligned}(t^a+1)\,(t^b+1)\,(t^c+1)f(t) - 2f(t^2) \\ = (t^{a+b}+t^{b+c}+t^{c+a}+t^{2a+b+c}+t^{a+2b+c}+t^{a+b+2c}) \ ,\end{aligned} \tag{2.10}$$

$$f(t^{-1})\, t^{a+b+c} = f(t) \ . \tag{2.11}$$

Then we have $c=a+b$.

To prove this proposition we use the following lemma. Since its proof is easy, we omit the proof.

Lemma 2.12.

$$\gcd(1+t^m, 1+t^n) = \begin{cases} 1, & \text{if } \nu_2(n) \neq \nu_2(m), \\ 1+t^d, & \text{if } \nu_2(n) = \nu_2(m), \end{cases}$$

where $d=\gcd(m, n)$.

Proof of Proposition 2.9. First we remark that $\deg f = c \ge a+b$. This follows easily from (2.10) and (2.11).

By (2.10) we have $f(1)=1$. Differentiate the both sides of (2.10) at $t=1$ and we get $4f'(1) = 2(a+b+c)$. Therefore $a+b+c$ must be even. If we let $d=\gcd(a, b, c)$, it is easy to see that $f(t)$ is a polynomial of t^d. Thus we can assume $\gcd(a, b, c)=1$. Since $a+b+c$ is even, one of a, b and c is even and the others are odd. Here we prove the proposition only when a is even and b and c are odd. In the other two cases it is proved similarly or easierly.

Now we let a be even and b and c odd. Write

$$f(t)=g(t)+th(t)$$

where $g(t)$ and $h(t)$ are polynomials of t^2. Comparing odd powers part of the polynomials in (2.10) we have

$$\begin{aligned}(2.13)\qquad &(1+t^a)(1+t^{b+c})th(t)+(1+t^a)(t^b+t^c)g(t)\\ &=t^{a+b}+t^{a+c}+t^{a+b+c}(t^b+t^c)=t^{a+b}(1+t^{c-b})(1+t^{b+c})\ .\end{aligned}$$

Since the left-hand side of this equation is divisible by $1+t^a$, $1+t^{c-b}$ or $1+t^{b+c}$ must be divisible by $1+t^a$.

Case 1. $1+t^{c-b}$ is divisible by $1+t^a$.

In this case by (2.13) $(t^b+t^c)g(t)$ must be divisible by $1+t^{b+c}$.

$$\gcd(t^b+t^c,\ 1+t^{b+c})=\gcd(1+t^{c-b},\ 1+t^{b+c})=1$$

by Lemma 2.12. Thus $1+t^{b+c}$ divides $g(t)$. But because of the deree of the polynomials we have that $g(t)=0$. Then by (2.13)

$$th(t)=t^{a+b}\frac{1+t^{c-b}}{1+t^a}\ .$$

Take the even powers part of the polynomials in (2.10) we get

$$(1+t^a)(t^b+t^c)th(t)=t^{b+c}+t^{2a+b+c}+2t^2h(t^2)\ ,$$

which implies that

$$(1+t^a)(1+t^{c-b})\frac{1+t^{c-b}}{1+t^a}-t^a\frac{1+t^{2c-2b}}{1+t^{2a}}=t^{c-b-a}(1+t^{2a})\ .$$

If we compare the constant term, then we have $c=a+b$.

Case 2. $1+t^{b+c}$ is divisible by $1+t^a$.

By (2.13) we have

$$(1+t^{b+c})th(t)+t^b(1+t^{c-b})g(t)=t^{a+b}(1+t^{c-b})\frac{1+t^{b+c}}{1+t^a}\ .$$

Thus $1+t^{c-b}$ must divide $(1+t^{b+c})th(t)$. Since by Lemma 2.12 $1+t^{c-b}$ is prime to $1+t^{b+c}$, $1+t^{c-b}$ must divide $th(t)$. But this is impossible. For the deree of $th(t)$ is c and it is divisible by t^{a+b}.

Corollary 2.14. *Let V be a 7-dimensional real T^2-module. Then $S(\mathbf{R}\oplus V)$ has a T^2-equivariant Hopf structure if and only if V factors through G_2.*

Proof. By E^aF^b we denote the one dimensional complex T^2-module with the T^2-action:

$$z \cdot v = z^a v \ , \quad w \cdot v = w^b v \ ,$$

for $(z, w) \in T^2$, $v \in \mathbf{C}$. Then the representation V is isomorphic to $\mathbf{R} \oplus E^{a_1}F^{a_2} \oplus E^{b_1}F^{b_2} \oplus E^{c_1}F^{c_2}$ for some integers $a_1, a_2, b_1, b_2, c_1, c_2$. For various homomorphims $f\colon S^1 \to T^2$ f^*V must satisfy the condition of Theorem 1.3. Making use of this fact it is easy to prove Corollay 2.14.

3. Elementary Abelian Groups Acting on Sphers

In this section we always assume that a G-space has the equivariant homotopy type of a G-CW complex with finite skeletons. A G-space X is finite if X has the equivariant homotopy type of a finite G-CW complex.

Now we consider the maximum elementary abelian p-group which acts on a finite mod p homology sphere preserving its Hopf structure. First we consider an example. Let D be the unit disk of $\mathbf{C}$ with the standard $\mathbf{Z}/p$-action. Then

$$S^7 \times D^k \ ,$$

is a $(\mathbf{Z}/p)^k$-Hopf space while $(\mathbf{Z}/p)^k$-action is effective. This example shows that the following definition is reasonable when we consider the above problem.

Definition. Let G be a p-group where p is prime. X is a mod p homology n-shpere on which G acts. The action is said to be strongly effective if for every element g of G which is not the identity the fixed point set X^g is a mod p homology r-sphere with $r<n$.

We note that it is well known that X^g is a mod p homology r-sphere with $r \leq n$ (see [Br]). Moreover if p is odd, then $n-r$ is even.

Theorem 3.1. *The group $\mathbf{Z}/2$ is the only non-trivial 2-group which acts strongly effectively on a finite mod 2 homology 1-sphere preserving its Hopf structure. The group $\mathbf{Z}/p$, p odd, cannot act strongly effectively on a finite mod p homology 1-sphere preserving its Hopf structure.*

Theorem 3.2. *The group $(\mathbf{Z}/2)^3$ cannot act strongly effectively on a finite mod 2 homology 3-sphere. The group $(\mathbf{Z}/p)^2$, p odd, cannot act strongly effectively on a finite mod p homology 3-sphere.*

Theorem 3.3. *The group $(\mathbf{Z}/2)^4$ cannot act strongly effectively on a finite mod 2 homology 7-sphere. The group $(\mathbf{Z}/p)^3$, p odd, cannot act strongly effectively on a finite mod p homology 7-sphere.*

To prove these theorems we use

Theorem 3.4. [Bo]. *Let p be a prime, G an elementary abelian p-group* *and X be a finite mod p homology n-shere. Let $n(H)$ be the integer such that X^H is a mod p homolrgy $n(H)$-sphere, where H is a subgroup of G, and let $r=n(G)$. Then*

$$n-r=\sum_H (n(H)-r)$$

where H runs through the subgroups of index p.

Proof of Theorem 3.1. Let $G=(\mathbf{Z}/2)^2$ and X be a Hopf G-space which is a mod 2 homology 1-sphere. Then X^G must be a mod 2 homology 0-sphere. Moreover by the strong effectivity of the action for a subgroup H of G of index 2 X^H is also a mod 2 homology 0-sphere. These facts contradict Theorem 3.4.

Let $G=\mathbf{Z}/4$ and X be a Hopf G-space which is a mod 2 homology 1-sphere. Let γ be a generator of G, then X^γ is a mod 2 homology 0-sphere and X/X^γ is mod 2 homology equivalent to $S^1 \vee S^1$. Since $X/X^\gamma - pt$ has two components and γ^2 induces the identity map on $H^*(X/X^\gamma; \mathbf{Z}/2)$, there is a γ^2 invariant closed subspace Y in X/X^γ which is mod 2 homology 1-sphere. But Y^{γ^2} is mod 2 isomorphic to point, which is impossible.

The last statement is proved by using the strong effectivity and the fact stated just after the definition of the strong effectivity.

Proof of Theorem 3.2. Let $H=(\mathbf{Z}/2)^2$ and X be a Hopf H-space which is a mod 2 homology 3-sphere. For every element h of order 2 in H X^h is mod 2 homology 1-sphere by the strong effectivity of the action and a result of Hamanaka [H]. Then X^H must be a mod 2 homology 0-sphere or 1-sphere since X^H is a Hopf space. If it were a mod 2 homology 1-sphere, we would have the following equation by Theorem 3.4

$$3-1=\Sigma(1-1)=0 .$$

Therefore X^H is a mod 2 homology 0-sphere.

Now let $G=(\mathbf{Z}/2)^3$ and X be a Hopf G-space which is a mod 2 homology 3-sphere. By the above result for every subgroup H in G of index 2 X^H is a mod 2 homology 0-sphere. Then by Theorem 3.4 we have

$$3-0=\Sigma(0-0)=0 ,$$

which is clearly impossible.

For the case of odd prime we can prove the theorem similarly.

To prove Theorem 3.3 we need the following Theorem 3.6, which we owe to A. Kono. Since we can prove Theorem 3.3 similarly as above by using Theorem 3.6, we omit the proof of Theorem 3.3.

Let p be a prime. For a space X we define

$$p\text{-}\dim X=\max\{i;\ H^i(X;\mathbf{Z}/p)\neq 0\}$$

and

$$\mathbf{Q}\text{-}\dim X = \max\{i;\ H^i(X;\mathbf{Q}) \neq 0\} \quad .$$

Then

Lemma 3.5. *If X is a finite, simply connected Hopf space, then $\mathbf{Q}$-dim$X$= p-dim$X$ for any prime p.*

Proof. Since X is a Hopf space, $H^*(X;\mathbf{Z}/p)$ is a commutative associative Hopf algebra. It follows that $H^*(X;\mathbf{Z}/p)$ has an algebra decomposition

$$H^*(X;\mathbf{Z}/p) \cong \otimes_{i=1}^{k} A_i \qquad \text{where each} \quad A_i = \frac{\mathbf{Z}/p[a_i]}{a_i^{r_i+1}} \ .$$

Since X is simply connected, each a_i has degree greater than or equal to 2. Then in the Bockstein spectral sequence the element $a_1^{r_1} \cdots a_k^{r_k}$ survives to E_∞-term by the reason of degree. Thus we have p-dim$X$=$\mathbf{Q}$-dim$X$.

Theorem 3.6. *Let X be a finite Hopf space. If X is a mod p n-sphere, $n>0$, then $n=1$, 3 or 7.*

Proof. First we assume that X is simply connected. By the loop space theorem [L], [K], $H^*(\Omega X;\mathbf{Z})$ has no torsion. Since X is a rational n-sphere, we have

$$H^j(X;\mathbf{Z}) = \begin{cases} 0, & \text{for } 0<j<n-1, \\ \mathbf{Z}, & j=n-1. \end{cases}$$

Thus there is a map $S^n \to X$, which induces isomorphisms on the integral homology groups since by Lemma 3.5 $H^j(X;\mathbf{Z})=0$ for $j>n$. This implies that S^n is homotopy equivalent to a Hopf space X. Therefore we have $n=3$ or 7.

Now we assume that X is not necessarily simply connected and that $n>1$. Then $\pi_1(X)$ is a finite group and its order is prime to p. Let $\widetilde{X}$ be the universal covering sapce of X, then $\widetilde{X}$ is a compact Hopf space and a mod p n-sphere. By the above result we have that $n=3$ or 7.

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