J. Math. Kyoto Univ. (JMKYAZ)
36-1 (1996) 199-209

Characterization of harmonic functions with singularity in hyperplane

By

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1. Introduction

A real valued function $u(x)$ defined on an open subset Ω of $\mathbf{R}^n$ is called harmonic in Ω if it is twice differentiable and satisfies the Laplace equation

$$\Delta u=\left(\frac{\partial^2}{\partial x^2_1}+\frac{\partial^2}{\partial x^2_2}+\cdots+\frac{\partial^2}{\partial x^2_n}\right)u(x)=0, \quad x\in\Omega.$$

We denote by $E(x)$ the fundamental solution of the Laplace operator Δ. That is,

$$E(x)=\begin{cases} -C_2\log|x|, & n=2 \\ C_n|x|^{2-n}, & n>2. \end{cases}$$

Throughout this paper, for every point $x\in\mathbf{R}^n$ we write $x=(x',x'')$, $x'\in\mathbf{R}^{n'}$, $x''\in\mathbf{R}^{n''}$, $n=n'+n''$ where n' and n'' are natural numbers. Moreover, by $\mathbf{N}_0$ we denote the set of nonnegative integers.

In this paper we characterize the harmonic functions near their singularities. In fact, it is well known that if u is harmonic and positive in the deleted unit ball $U\setminus\{0\}$ then u can be written as

$$u(x)=v(x)+aE(x), \quad x\in U\setminus\{0\}$$

for some constant $a\geq 0$ and a harmonic function $v(x)$ in U. This is, so called, Bôcher's theorem. More generalized decomposition theorem with no positivity condition was given in [CKL] and [ABR1] . These theorems eventually describe that every harmonic function with singularities can be expressed as the sum of derivatives of $E(x)$ modulo harmonic function in the whole domain.

In this paper we give a characterization of harmonic functions u in $\Omega\setminus K$ where Ω is an open set and $K=K'\times\{0\}$, K' is a compact subset of $\mathbf{R}^{n'}$, $\mathbf{R}^n=\mathbf{R}^{n'}\times\mathbf{R}^{n''}$, as follows:

Communicated by Prof. K, Ueno, August 7, 1995
Partially supported by GARC-KOSEF and Korea Research Foundation

(2.11) $$u(x)=v(x)+\sum_{\beta\in\mathbf{N}_0^{n''}} f_\beta(x')*\partial^\beta{}_{x''}E(x',x''),\quad x\in\Omega\backslash K$$

where v is a harmonic function in Ω and f_β are analytic functionals in $\mathbf{R}^{n'}$ with support in K'. This result refines the decomposition theorem in [ABR1] . In addition to this result we give a characterization of harmonic function with some restrictive growth near K. In fact, it is proved that if a harmonic function $u(x)$ satisfies

$$|u(x)|\leq C[d(x,K)]^{-M}\quad\text{near}\quad K$$

for some $C>0$ and $M>0$ then the infinite sum in (2.11) above can be reduced to the finite sum with the distributions f_β supported by K'. Moreover, especially if $u(x)$ satisfies the more restrictive condition

$$|u(x)|\leq C[d(x,K)]^{2-n}\quad\text{for}\quad n>2,$$

and

$$|u(x)|\leq C\,|\log d(x,K)|\quad\text{for}\quad n=2$$

near K then $u(x)$ can be written in a much simpler form

$$u(x)=v(x)+g_0(x')*E(x',x''),\quad x\in\Omega\backslash K.$$

Throughout this paper every theory can be developed by virtue of the generalized function theory without appealing to the general potential theory, such as, the maximum principle, the mean value theorm, etc. Basically we depend on the Sato hyperfunction theory or sometimes on the Schwartz distribution theory. This is another point of this paper.

2. A review on hyperfunctions and main theorems

At first we give a brief introduction to hyperfunctions. See [H,S] for more details.

As usual the strong dual $\mathscr{E}'$of the space of C^∞functions in $\mathbf{R}^n$ is called the space of (Schwartz) distributions with compact support. Similarly the strong dual $\mathscr{A}'$of the space $\mathscr{A}$ of analytic functions is called the space of analytic functionals. Here it is precisely given as follows:

Definition 2.1. Let K be a compact subset of $\mathbf{R}^n$. Then we denote by $\mathscr{A}'(K)$ the space of continuous linear functionals u on the space $\mathscr{A}$ of entire functions such that for every complex neighborhood ω of K

(2.1) $$|u(\phi)|\leq C_\omega\sup_{z\in\omega}|\phi(z)|,\quad \phi\in\mathscr{A}.$$

We call the element of $\mathscr{A}'(K)$ an analytic functional with support in K. For an open set Ω we denote by $\mathscr{A}'(\Omega)$ the set of all analytic functionals whose supports are compact subsets of Ω.

The following theorem characterizes the analytic functionals with support in a hyperplane.

Theorem 2.2. *Let $u \in \mathscr{A}'(\mathbf{R}^n)$ with support in $K = K' \times \{0\}$ where K' is a compact subset of $\mathbf{R}^{n'}$. Then there exists a sequence of analytic functionals u_β in $\mathbf{R}^{n'}$ with support in K' such that*

$$u = \sum_{\beta \in \mathbf{N}_0^{n''}} u_\beta \otimes \delta^{(\beta)}(x''). \tag{2.2}$$

In other words,

$$u(\phi) = \sum_{\beta \in \mathbf{N}_0^{n''}} u_\beta(\partial^\beta{}_{x''}\phi(x',0)), \quad \phi \in \mathscr{A}. \tag{2.3}$$

Moreover, we have $\|u_\beta\|_{K'_\varepsilon} \leq C_\varepsilon \varepsilon^{|\beta|}/\beta!$ for every $\varepsilon > 0$ where $\|\cdot\|_{K'_\varepsilon}$ denotes the operator norm in (2.1) and $K'_\varepsilon = \{z' \in \mathbf{C}^{n'} | d(z', K') < \varepsilon\}$.

Proof. Let $\phi(x)$ be an entire function. Expanding ϕ in Taylor series with respect to x'' variable we may write

$$\phi(x) = \sum_{\beta \in \mathbf{N}_0^{n''}} \partial^\beta{}_{x''}\phi(x',0)x''^\beta/\beta!. \tag{2.5}$$

For each $\beta \in \mathbf{N}_0^{n''}$ and $\phi(x') \in \mathscr{A}(\mathbf{C}^{n'})$ we define

$$u_\beta(\phi) = u(\phi(x')x''^\beta/\beta!). \tag{2.6}$$

Then for every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$\begin{aligned} |u_\beta(\phi)| &= |u(\phi(x')x''^\beta/\beta!)| \\ &\leq C_\varepsilon \sup_{\substack{z' \in K'_\varepsilon \\ |z''| < \varepsilon}} |\phi(z')z''^\beta/\beta!| \\ &\leq C_\varepsilon \frac{\varepsilon^{|\beta|}}{\beta!} \sup_{z' \in K'_\varepsilon} |\phi(z')|, \end{aligned}$$

which means that

$$\|u_\beta\|_{K'_\varepsilon} \leq C_\varepsilon \varepsilon^{|\beta|}/\beta!$$

for each β and that u_β belongs to $\mathscr{A}'(\mathbf{R}^{n'})$ with support in K'. Then using (2.5) and the continuity of u we can obtain

$$u(\phi) = \sum_{\beta \in \mathbf{N}_0^{n''}} u\ (\partial^\beta_{x''}\phi(x',0)x''^\beta)/\beta!$$

$$= \sum_{\beta \in \mathbf{N}_0^{n''}} u_\beta \ (\partial^\beta{}_{x''} \phi(x', 0))$$

$$= \sum_{\beta \in \mathbf{N}_0^{n''}} [u_\beta \otimes \delta^{(\beta)}(x'')] \ \ (\phi),$$

which completes the proof.

Remark. (i) In fact, every u of the form (2.2) defines an analytic functional with support in $K = K' \times \{0\}$, which is the converse of the above theorem.

(ii) For the distributions it is well known that every distribution in $\mathbf{R}^n$ with support in $K = K' \times \{0\}$ can written as a finite sum

(2.7)
$$u = \sum_{|\beta| \leq N} u_\beta \otimes \delta^{(\beta)},$$

where u_β are distributions in $\mathbf{R}^{n'}$ with support in K'. This fact will be used also later.

Now we define a hyperfunction in a bounded open subset Ω of $\mathbf{R}^n$.

Definition 2.3. (i) The space $B(\Omega)$ of hyperfunctions in Ω is defined by

$$B(\Omega) = \mathscr{A}'(\overline{\Omega}) / \mathscr{A}'(\partial\Omega).$$

(ii) For $u \in \mathscr{A}'(\overline{\Omega})$ the support of the class $\dot{u}$ of u in $B(\Omega)$ is defined by

$$\operatorname{supp} \dot{u} = \Omega \cap \operatorname{supp} u.$$

For the notions of hyperfunctions on more general open set we refer to [H] and [K] .

The followings are the basic properties of hyperfunctions, which will be very helpful later.

Theorem 2.4. ([S] , *Lemma* 121, *Theorems* 122, 141) . *Let Ω be an open subset of $\mathbf{R}^n$ and $\Omega_1 \subset \Omega$ be open.*

(i) *For every $u \in \mathrm{B}(\Omega_1)$ there exists $\tilde{u} \in \mathrm{B}(\Omega)$ which extends u.*

(ii) *The space $\mathscr{D}'(\Omega)$ of Schwartz distributions in Ω is continuously imbedded into $B(\Omega)$*

(iii) *For an elliptic partial differential operator $p(\partial)$ with constant coefficients, if $u \in B(\Omega)$ and $p(\partial)u = 0$ in Ω then u is analytic. In other words, every elliptic partial differential operator with constant coefficents is analytic-hypoelliptic in the category of hyperfunctions.*

We are now in a position to state and prove the main theorem of this paper.

Theorem 2.5. *Let Ω be an open subset of $\mathbf{R}^n$ and $K=K'\times\{0\}\subset\Omega$ for a compact subset K' of $\mathbf{R}^{n'}$. If u is a harmonic function in $\Omega\backslash K$ then there exist a harmonic function $v(x)$ in Ω and $f_\beta\in\mathscr{A}'(\mathbf{R}^{n'})$ such that*

$$u(x)=v(x)+\sum_{\beta\in\mathbf{N}_0^{n''}} f_\beta(x)*\partial^\beta{}_{x''}E(x',x''),\quad x\in\Omega\backslash K, \tag{2.8}$$

where $*$ denotes the convolution product with respect to x'variable. The coefficients satisfy that $f_\beta\in\mathscr{A}'(K')$ and $\|f_\beta\|_{K'_\varepsilon}\leq C_\varepsilon\varepsilon^{|\beta|}/\beta!$. Moreover, the expression (2.8) is uniquely determined.

Proof. For a bounded open subset Ω_0 with $K\subset\Omega_0\subset\Omega$ we assume for a moment that (2.8) holds for $x\in\Omega_0\backslash K$. Since

$$\sum_{\beta\in\mathbf{N}_0^{n''}} f_\beta*\partial^\beta{}_{x''}E(x',x'')$$

is harmonic in $\mathbf{R}^n\backslash K$ we can extend $v(x)$ harmonically to Ω via (2.8).Therefore, we may assume that Ω is a bounded open set.

Since $u(x)$ is real analytic in $\Omega\backslash K$ we may regard it as a distribution in $\Omega\backslash K$. Then by virtue of Theorem 2.4 (i), (ii) there exists $\tilde{u}\in B(\Omega)$ which is an extension of u. Since u is harmonic we obtain

$$\Delta\tilde{u}=0\quad\text{in}\quad\Omega\backslash K,$$

which implies that $\Delta\tilde{u}$ is a hyperfunction supported by K. Then we can write

$$\Delta\tilde{u}=w_1+w_2,$$

for some $w_1\in\mathscr{A}'(\overline{\Omega})$ and $w_2\in\mathscr{A}'(\partial\Omega)$.

Since

$$(\mathrm{supp}w_1)\cap\Omega=\mathrm{supp}\Delta\tilde{u}\subset K$$

we have supp $w_1\subset K\cup\partial\Omega$. Since $K\cap\partial\Omega=\phi$ (or using the decomposition $\mathscr{A}'(K_1\cup K_2)=\mathscr{A}'(K_1)+\mathscr{A}'(K_2)$ given in [H, Theorem 9.1.8]) we may write

$$\Delta\tilde{u}=w_1+w_2$$

for some $w_1\in\mathscr{A}'(K)$ and $w_2\in\mathscr{A}'(\partial\Omega)$. Then in view of Theorem 2.2 there are $f_\beta\in\mathscr{A}'(K')$ with$\|f_\beta\|_{K'_\varepsilon}\leq C_\varepsilon\varepsilon^{|\beta|}/\beta!$ for each β such that $w_1=\sum_{\beta\in\mathbf{N}_0^{n''}}f_\beta\otimes\delta^{(\beta)}(x'')$.

If we define

$$v=\tilde{u}-\sum_{\beta\in\mathbf{N}_0^{n''}} f_\beta*\partial^\beta{}_{x''}E(x',x'')\in B(\Omega) \tag{2.9}$$

then since $f_\beta(x')$ has compact support and $E(x)$ belongs to $B(\Omega)$ it is easy to see $v \in B(\Omega)$. Also,

$$\Delta v = \Delta\tilde{u} - \sum_{\beta\in\mathbf{N}_0^{n''}} \Delta\Big(f_\beta * \partial^\beta{}_{x''}E(x',x'')\Big)$$

$$= \Delta\tilde{u} - \sum_{\beta\in\mathbf{N}_0^{n''}} \Delta\Big([f_\beta\otimes\delta^{(\beta)}(x'')] * E(x',x'')\Big)$$

$$= \Delta\tilde{u} - \sum_{\beta\in\mathbf{N}_0^{n''}} f_\beta\otimes\delta^{(\beta)}(x'')$$

$$= \Delta\tilde{u} - w_1 = w_2 \in A'(\partial\Omega).$$

Thus we have $\Delta v \equiv 0$ as a hyperfunction in Ω. Applying Theorem 2.4 (iii) we can see that v is a harmonic function. Moreover, since each term in (2.9) is exactly C^∞ functions in $\Omega\backslash K$ we can obtain that in $\Omega\backslash K$

$$u(x) = \tilde{u}(x) = v(x) + \sum_{\beta\in\mathbf{N}_0^{n''}} f_\beta(x') * \partial^\beta{}_{x''}E(x)$$

Finally, to prove the uniqueness of the expression (2.8) we suppose that

$$u(x) = v(x) + \sum_{f\in\mathbf{N}_0^{n''}} f_\beta(x') * \partial^\beta{}_{x''}E(x)$$

$$u(x) = w(x) + \sum_{\beta\in\mathbf{N}_0^{n''}} g_\beta(x') * \partial^\beta{}_{x''}E(x)$$

for some harmonic functions $v(x)$, $w(x)$ in Ω and f_β, $g_\beta \in A'(K')$ with

$$\|f_\beta\|_{K'_\varepsilon} \le C_\varepsilon \varepsilon^{|\beta|}/\beta!, \quad \|g_\beta\|_{K'_\varepsilon} \le C_\varepsilon \varepsilon^{|\beta|}/\beta!.$$

Then applying the Laplace operator Δ we get

$$\sum_{\beta\in\mathbf{N}_0^{n''}} (f_\beta - g_\beta)\otimes\delta^{(\beta)}(x'') = 0.$$

Applying an entire function $\phi(x) = \psi(x')x''^\beta$ as a test function of both sides we obtain $f_\beta = g_\beta$ for every β, so that $v(x) = w(x)$. This completes the proof.

In the above theorem if K is noncompact subset of Ω then we obtain a little different conclusion when Ω is a bounded open subset. But for arbitrary open subset Ω it may not be true.

Theorem 2.6. *Let Ω be a bounded open subset of $\mathbf{R}^n$ and $\Omega' = \Omega \cap \{x' = 0\}$. If u is a harmonic function in $\Omega\backslash\Omega'$ then exist a harmonic function $v(x)$ in Ω*

and $f_\beta \in \mathscr{A}'(\mathbf{R}^{n'})$ *such that*

$$u(x) = v(x) + \sum_{\beta \in \mathbf{N}_0^{n''}} f_\beta(x) * \partial^\beta_{x''} E(x', x''), \quad x \in \Omega \backslash \Omega'$$

and

$$f_\beta \in \mathscr{A}'(\overline{\Omega'}) \quad \text{and} \quad \|f_\beta\|_{\overline{\Omega'}\varepsilon} \leq C_\varepsilon \varepsilon^{|\beta|} / \beta!$$

where $\overline{\Omega'}$ *denotes the closure of* Ω' *in* $\mathbf{R}^n$.

Proof. Let $\tilde{u}$ be a hyperfunctional extension of u to Ω. Then as in the proof of Theorem 2.5 we may write

$$\Delta \tilde{u} = w_1 + w_2$$

for some $w_1 \in \mathscr{A}'(\overline{\Omega'})$ and $w_2 \in \mathscr{A}'(\partial\Omega)$. Then similar argument as before gives the conclusion.

But, in general, if Ω is not bounded then we cannot expect to get expression (2.8). For example, consider $\Omega = \mathbf{R}^2$ and

$$u(x, y) = \begin{cases} y & y \geq 0, \\ -y & y > 0. \end{cases}$$

Then $u(x, y)$ is harmonic in $\mathbf{R}^2 \backslash \Omega'$ where $\Omega' = \{(x, 0) \mid x \in \mathbf{R}\} = \mathbf{R} \times \{0\}$. Then a calculation gives

$$\Delta u(x, y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2\delta(y) \quad \text{in} \quad \mathbf{R}^2.$$

On the other hand, if $u(x, y)$ can be written as in (2.8) then we have

$$\Delta u(x, y) = \sum_{j=0}^{\infty} f_j(x) \otimes \delta^{(j)}(y).$$

Comparing these relations we obtain

$$f_0(x) = 2, \quad f_j(x) = 0 \quad \text{for} \quad j = 1, 2, \cdots.$$

Then it follows that

$$u(x, y) = f_0(x) * E(x, y)$$

$$= -2C_2 \int \log\sqrt{x^2 + y^2}\, dx.$$

But the last integral does not converge, which leads a contradiction.

In fact, for the Laplace operator Δ we use only the ellipticity in the proof of Theorem 2.5. Therefore, we can state a similar result for an elliptic operator with constant coefficients as follows:

Corollary 2.7. *Let $P(D)$ be an elliptic partial differential operator with constant coefficients. If Ω is an open subset of $\mathbf{R}^n$ and $K=K'\times\{0\}$ for a compact subset K' of $\mathbf{R}^{n'}$ then every solution $u(x)$ of $P(D)u=0$ in $\Omega\backslash K$ can be written as*

$$u(x)=v(x)+\sum_{\beta\in\mathbf{N}_0^{n''}} f_\beta(x')*\partial^\beta{}_{x''}F(x',x''),\quad x\in\Omega\backslash K$$

for some solution $v(x)$ of $P(D)v(x)=0$ in Ω and $f_\beta(x')\in A'(\mathbf{R}^{n'})$ with $\|f_\beta\|_{K_\varepsilon}\leq C_\varepsilon\varepsilon^{|\beta|}/\beta!$ where $F(x)$ is a fundamental solution of the partial differential operator $P(D)$.

In particular, if $K=\{0\}\subset\Omega$ we can easily obtain the following corllary which has already been proved in [CKL] .

Corollary 2.8. *Let u be harmonic in $\Omega\backslash\{0\}$. Then there exist a harmonic function v in Ω and constants a_α such that*

$$u(x)=v(x)+\sum_{\alpha\in\mathbf{N}_0^{n''}} a_\alpha\partial^\alpha E(x),\quad x\in\Omega\backslash\{0\} \tag{2.10}$$

and

$$|a_\alpha|\leq C_\varepsilon\varepsilon^{|\alpha|}/\alpha! \quad \text{for every} \quad \varepsilon>0.$$

For a harmonic function with some restrictive growth near the singular set K we can give a much simpler characterization. In fact, for those harmonic functions in $\Omega\backslash K$ the infinite sum in (2.8) or (2.10) can be reduced to a finite sum.

Theorem 2.9. *Let u be a harmonic function in $\Omega\backslash K$, $K=K'\times\{0\}$ where K' is a compact subset of $\mathbf{R}^{n'}$. If there exist some constants $M>0$ and $C>0$ such that*

$$|u(x)|\leq C[d(x,K)]^{-M} \tag{2.11}$$

near K then there exist a harmonic function $v(x)$ in $\Omega\backslash K$ and a finite number of distributions g_β supported by K' such that

$$u(x)=v(x)+\sum_{|\beta|\leq N} g_\beta*\partial^\beta{}_{x''}E(x',x''),\quad x\in\Omega\backslash K \tag{2.12}$$

for some $N>0$.

Proof. In view of the growth (2.11) near K we can extend $u(x)$ to be a distribution $\tilde{u}$ defined in the whole of Ω. Using the structure theorem (2.7) given in Remark (ii) we can proceed the proof similarly as in the proof of

Theorem 2.5.

In fact, the constant N in the sum of (2.12) inform us how badly the harmonic function $u(x)$ behaves near K. Hence, if a harmonic function $u(x)$ has more restrictive growth than (2.11) it can be written in a much simpler from than (2.12).

Theorem 2.10. *Let Ω be an open subset of $\mathbf{R}^n$ and $K=K'\times\{0\}\subset\Omega$ where $K'$is a compact subset of $\mathbf{R}^{n'}$. Then every harmonic function in $\Omega\backslash K$ with*

$$|u(x)|\leq C\ [d(x,K)]^{2-n}\quad for\quad n>2 \tag{2.13}$$

or

$$|u(x)|\leq C|\log d(x,K)|\quad for\quad n=2$$

near K, can be represented as

$$u(x)=v(x)+g_0*E(x',x''),\quad x\in\Omega\backslash K \tag{2.14}$$

for some harmonic function $v(x)$ and distributions g_0 supported by K'.

Proof. The rate of growth near K allows us to consider $u(x)$ as a locally integrable function in Ω. By virtue of Theorem 2.7 we may write

$$u(x)=v(x)+\sum_{|\beta|\leq N}g_\beta(x')*\partial^\beta{}_{x''}E(x',x''),\quad x\in\Omega\backslash K \tag{2.15}$$

where g_β are distributions supported by K'. If $N\leq 2$ then we have done it. So we assume that $N>2$. If we take

$$J(x)=\sum_{|\beta|\leq N}g_\beta(x')*\partial^\beta{}_{x''}E(x),\quad x\in\Omega\backslash K$$

then $J(x)=u(x)-v(x)$ can be considered as a locally integrable function in Ω. Thus we may assume that $J(x)$ is a distribution in Ω.

Define

$$J_1(\phi)=\int_\Omega J(x)\Delta\phi(x)dx,\quad \phi\in C_0^\infty(\Omega).$$

Then it follows that for some constant C_J depending only on J,

$$|J_1(\phi)|\leq C_J\sup_{x\in\Omega}|\Delta\phi(x)| \tag{2.16}$$

and in the sense of distribution

$$\begin{aligned} J_1(\phi)&=(\Delta J(x),\phi(x))\\ &=\left(\Delta\sum_{|\beta|\leq N}g_\beta(x')*\partial^\beta{}_{x''}E(x),\phi(x)\right)\end{aligned} \tag{2.17}$$

$$= \sum_{|\beta|\leq N} (-1)^{|\beta|} (g_\beta(x'), \partial^\beta{}_{x''}\phi(x',0))$$

for every $\phi \in C_0^\infty(\Omega)$.

Especially, choose a point $x_0 = (x'_0, 0) \in \Omega$ and a function $\phi \in C^\infty{}_0$ with its support in a ball in centered at x_0 and $\phi(x_0) = 1$. Applying $\phi(x', kx'')$, for k large enough, to (2.16) and (2.17) we obtain

$$\left| \sum_{|\beta|\leq N} (-k)^{|\beta|} \left(g_\beta(x'), \partial^\beta{}_{x}{}''\phi(x',0) \right) \right| \leq C_J k^2 \sup_{x\in\Omega} \left| \sum_{j=1}^{n} \frac{\partial^2}{\partial x^2{}_j} \phi(x) \right|.$$

Letting $k \to \infty$ we have $g_\beta(x') = 0$ for every β with $|\beta| > 2$. Thus we may write

$$J(x) = g_0(x) * E(x', x'') + \sum_{j=1}^{n} g_j(x') * \frac{\partial E(x', x'')}{\partial x''_j} + \sum_{i,j=1}^{n} g_{ij}(x') * \frac{\partial^2 E(x', x'')}{\partial x''_i \partial x''_j}$$

where g_0, g_j and g_{ij} are distributions with support in K'. But since each derivative of $E(x', x'')$ goes to infinity more rapidly than $E(x', x'')$ as $x'' \to 0$ and $x' \in K'$ it follows that for $x' \in K'$

$$\text{(2.19)} \qquad \left\{ g_j(x') * \frac{\partial E(x', x'')}{\partial x''_j} \right\} / E(x', x'') = \left(g_j, \frac{\partial E(x' - \cdot, x'')}{\partial x''_j} / E(x', x'') \right)$$

goes to infinity as $x'' \to 0$ and $x' \in K'$. Similarly, we can see also

$$\left\{ g_{ij}(x') * \frac{\partial^2 E(x', x'')}{\partial x''_i \partial x''_j} \right\} / E(x', x'') = \left(f_{ij}, \frac{\partial^2 E(x' - \cdot, x'')}{\partial x''_j} / E(x', x'') \right)$$

goes to infinity more rapidly than (2.19) as $x'' \to 0$ and $x' \in K$. Therefore it must be true that $g_j = 0$ for $j = 1, 2, \cdots, n$ and $g_{ij} = 0$ for $i, j = 1, 2, \cdots, n$.

This completes the proof.

Finally we add a well known decomposition theorem which is very useful in the harmonic function theory. We prove it in a much easier way based on the hyperfunction theory.

Theorem 2.11. *Let Ω be an open subset of $\mathbf{R}^n$ and K be a compact subset of Ω. Then if $u(x)$ is harmonic in $\Omega \backslash K$ then u has a unique decompostition of the form*

$$\text{(2.20)} \qquad u = v + w,$$

where v is harmonic in Ω and w is harmonic in $\mathbf{R}^n \backslash K$. Here w is such that

$$w(x) = w_0 * E(x)$$

for some $w_0 \in \mathscr{A}'(K)$ *and that* $\lim_{x \to \infty} w(x) = 0$ *for* $n > 2$, $\lim_{x \to \infty} [w(x) - c \log |x|] = 0$ *for some constant c, for* $n = 2$.

Proof. Here we may also assume that Ω is bounded. Let $\tilde{u}$ be a hyperfunctional extension of u on Ω. Then $\Delta\tilde{u}$ is a hyperfunction with support in K. Here we can express $\Delta\tilde{u} = w_0 + w_1$ for some $w_0 \in \mathscr{A}'(K)$ and $w_1 \in \mathscr{A}'(\partial\Omega)$. Taking $w = w_0 * E(x)$ and $v = \tilde{u} - w$ we can easily obtain the results. In fact, since w_0 is an analytic functional with support in K it follows taht for $n > 2$,

$$\begin{aligned} |w(x)| &= |w_0(E(x - \zeta))| \\ &\le C_\varepsilon \sup_{\zeta \in K_\varepsilon} \frac{1}{|x - \zeta|^{n-2}} \to 0 \end{aligned}$$

as $x \to \infty$.

Acknowledgements. The author gratefully acknowledges the refree for suggesting valuable comments. Moreover, the author would like to express his thanks to Dr. Dohan Kim for helpful discussion on this paper.

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