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# Character formula for representations of local quaternion algebras (wildly ramified case)

By

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## Introduction

Let $F$ be a $p$-adic local field and $D$ be a quaternion division algebra over $F$. The character of an irreducible admissible representation of the multiplicative group $D^{\times}$ of $D$ was studied in [GG] and [HSY]. Especially in [HSY] the character formula is explicit and simple. But it has been dealt only the case $p \neq 2$, what we call, tamely ramified case. By Jacquet-Langlands correspondence ([JL]) between representations of $D^{\times}$ and discrete series representations of $\mathrm{GL}_2(F)$, the character foumula for $D^{\times}$ gives the character formula for $\mathrm{GL}_2(F)$ on the set of elliptic regular elements. The computation of character of the representation of $\mathrm{GL}_2$ and related gourps has been the object of much study ([SS], [Sh], [Sal], [T], [Sai]). Except [Sai], it has been also assumed $p \neq 2$. Tunnel and Saito shows ([T], [Sai]) the character of the representation is expressed by $\varepsilon$-factor of the base change lift of the representation of $\mathrm{GL}_2(F)$ to quadratic extensions (including the case $p=2$ in [Sai]). But it is not easy to compute the $\varepsilon$-factor of the base change lift when $p=2$. Here we do not treat the base change lift. Our tactics is the same as [HSY], but the wild ramification brings us many difficulties. We proceed as follows. In section 1, we treat the construction of the representation of $D^{\times}$. The set of the representations with even conductor is parameterized by the set of the regular characters of unramified quadratic extension of $F$ and their characters and completely calculated ([HSY] Corollary 1.7). Therefore we treat only the representation with odd conductor. The construction of these representation is well-known, but we need a slight modification to compute the character completely. We define a parameter for the representation, which is called 'generic data'. It is a triple $(K, \theta, \gamma)$ consisting of a ramified quadratic extension $K$ of $F$, a quasi-character of $\theta$ of $K^{\times}$ and an element $\gamma$ of $K$ which satisfy some conditions in Definition 1.1. We note if the Swan conductor $t_K$ of $K$ is 0, i.e. $p \neq 2$, the parameter $\gamma$ is dispensable since $\theta$ determines $\gamma$. We associate an irre-

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ducible representation $\pi_\Lambda$ of $D^\times$ with the generic data $\Lambda=(K, \theta, \gamma)$. Unfortunately the isomorphism class of $K$ is not an invariant of the equivalent class of $\pi_\Lambda$, but the Swan conductor $t_K$ is still an invariant of the representation. In any way, $\pi_\Lambda$ is induced from a one-dimensional representation of a subgroup H.

Section 2 is devoted to give the decomposition of $\pi_\Lambda$ as $K^\times$ module. It follows from Theorem A in [H] that each quasi-character of $K^\times$ appears at most once in the restriction of $\pi_\Lambda$ to $K^\times$. We use this repeatedly. In addition we use Mackey's theorem on induced representation and some knowledge on the local quaternion algebra. Proposition 2.8 is the main result of this section. In section 3 and 4, we assume $F/\mathbf{Q}_2$ is unramified. In section 3 we compute the character of $\pi_\Lambda$ on $K^\times$. The result of section 2 (Corollary 2.9 and (2.14)) reduces our work very much. Since we treat the wildly ramified case, we must fulfill the case by case analysis according to the relation of the conductor of the representation $\pi_\Lambda$ and the Swan conductor of $K$. Theorem 3.7 and Theorem 3.14 are character formulas for $\pi_\Lambda$ on $K^\times$. We note we can remove the assumption $F/\mathbf{Q}_2$ unramified, but the calculation becomes much more complicated and it takes much space only to state the character formula. We sketch the calculation for the general case in Appendix A. The character of $\pi_\Lambda$ outside $K^\times$ is treated in section 4. Since there exist more ramified quadratic extensions of $F$ than tamely ramified case, it becomes more complicated. The fact that the support of the character is included in a neighborhood of the conjugacy class of $K^\times$ plays an essential role. Theorem 4.6 is a character formula for $\pi_\Lambda$ outside the conjugacy class of $K^\times$.

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## Notation

Let $F$ be a finite extension of $\mathbf{Q}_2$. We denote by $\mathcal{O}_F$, $P_F$, $\tilde{\omega}_F$, $k_F$ and $v_F$ the maximal order of $F$, the maximal ideal of $\mathcal{O}_F$, a prime element of $P_F$, the residue field of $F$ and the valuation of $F$ normalized by $v_F(\tilde{\omega}_F)=1$. For a quasi-character $\theta$ of $F^\times$, we denote the exponent of its conductor by $f(\theta)$. For convenience, we regard $1+P_F^0$ as $\mathcal{O}_F^\times$. We set $q$ be the number of elements in $k_F$. Let $D$ be a quaternion division algebra over $F$, and $\mathcal{O}_D$, $P_D$, $\tilde{\omega}_D$, $k_D$ and $v_D$ the maximal order of $D$, the maximal ideal of $\mathcal{O}_D$, a prime element of $P_D$, the residue field of $D$ and the valuation of $D$ normalized by $v_D(\tilde{\omega}_D)=1$. We denote by Nr, Tr the reduced norm, and the reduced trace respectively. For $x\in D$, we denote by $\bar{x}$ the element obtained by canonical involution. For $x\in\mathbf{R}$, let $[x]$ denote the greatest integer $\leq x$.

We fix an additive character $\psi$ of $F$ whose conductor is $P_F$ i.e. $\psi(P_F)=\{1\}$ and $\psi(\mathcal{O}_F)\neq\{1\}$. Moreover we assume $\psi(x+x^2)=1$ for $x\in\mathcal{O}_F$. For an extension $K$ of $F$, let $n_K$, $tr_K$ be the norm and trace from $K$ to $F$. We denote by

$\psi_K$, $\psi_D$, the character $\psi\circ tr_K$ of $K$, and the character $\psi\circ\mathrm{Tr}$ of $D$ respectively. For an irreducible admissible representation $\pi$ of $D^\times$, the conductor $f(\pi)$, more exactly, the exponent of the conductor of $\pi$ is defined to be the minimal integer $\nu$ such that $\pi(1+P_D^{\nu-1})=\{1\}$ and $\pi(1+P_D^{\nu-2})\neq\{1\}$. Here we understand that $1+P_D^0=\mathcal{O}_D^\times$ and $f(\pi)=1$ if $\pi(\mathcal{O}_D^\times)=\{1\}$. We call $\pi$ *minimal* if $f(\pi)$ equals to the minimum of $f(\pi\otimes(\eta\circ\mathrm{Nr}))$ where $\eta$ runs through the quasi-characters of $F^\times$. Let $G$ be a totally disconnected, locally compact group. We denote by $\hat{G}$ the set of (equivalence classes of) irreducible admissible representations of $G$. For closed subgroup $H$ of $G$ and a representation $\rho$ of $H$, we denote by $\mathrm{Ind}_H^G\,\rho$ the induced representation of $\rho$ to $G$. For a representation $\pi$ of $G$, we denote by $\pi|_H$ the restriction of $\pi$ to $H$.

## 1. Construction of the representation

At first we remark that it suffices to calculate the character for the representation of $D^\times$ with minimal conductor. The character of the representation with an even minimal conductor is completely calculated by [HSY, Corollary 1.7] when the residual characteristic of $F$ is an odd prime. In fact the character formula holds for the even residual characteristic case. Therefore we shall only treat the representation with an odd conductor, which becomes automatically minimal.

**Definition 1.1.** A triple $(K,\theta,\gamma)$ is called a generic data of level $2m$ if the following conditions hold:

(1) $K$ is a ramified quadratic extension of $F$ in $D$. Let $t=t_K$ be the Swan conductor of $K/F$ i.e. $t=d_{K/F}-1$ where $d_{K/F}$ is the exponent of the different. Then $m\geq t$.

(2) $\gamma\in P_K^{1-2m}-P_K^{2-2m}$.

(3) If $m>t$, $\theta$ is a quasi-character of $K^\times$ such that the exponent of its conductor is $2m-t$ i.e. $\theta(1+P_K^{2m-t})=\{1\}$ and $\theta(1+P_K^{2m-t-1})\neq\{1\}$. And $\theta(1+x)=\psi_K(\gamma x)$ for $x\in P_K^{[(2m-t+1)/2]}$. If $m=t$, $\theta$ is a quasi-character of $K^\times$ which is trivial on $1+P_K^m$.

**Remark.** For a quadratic extension $K$ of $F$, the Swan conductor $t_K\leq 2v_F(2)$ and $t_K$ is even if and only if $t_K=2v_F(2)$. If $t_K$ is odd, $t_K=2v_F(tr_K\tilde{\omega}_K)-1$.

Let $\Lambda=(K,\theta,\gamma)$ be a generic data of level $2m$. We define a quasi-character $\psi_\gamma$ of $1+P_D^m$ by $\psi_\gamma(1+x)=\psi_D(\gamma x)$ for $x\in P_D^m$. We set $H=K^\times(1+P_D^m)$ and $\rho_{\theta,\gamma}(k(1+x))=\theta(k)\psi_\gamma(x)$ for $k\in K^\times$ and $x\in P_D^m$. Then $\rho_{\theta,\gamma}$ is an extension of $\psi_\gamma$ to $H$. We set $\pi_\Lambda=\mathrm{Ind}_H^{D^\times}\rho_{\theta,\gamma}$.

**Proposition 1.2.** *For any generic data $\Lambda$ of level $2m$, $\pi_\Lambda$ is an irreducible representation of $D^\times$ with $f(\pi_\Lambda)=2m+1$. Conversely for a positive integer $m$, every irreducible representation $\pi$ of $D^\times$ with $f(\pi)=2m+1$ can be written in the* *form $\pi_\Lambda$ for some generic data $\Lambda$ of level 2m.*

*Proof.* Let $\pi$ be an irreducible representation of $D^\times$ with $f(\pi)=2m+1$ for a positive integer $m$. Since $1+P_D^m/1+P_D^{2m}$ is abelian, $\pi|_{1+P_D^m}$ decomposes into one-dimensional representations. Therefore thare exists an element $\gamma_1\in P_D^{1-m}-P_D^{2-m}$ such that $\pi|_{1+P_D^m}$ contains $\psi_{\gamma_1}$ where $\psi_{\gamma_1}(1+x)=\psi_D(\gamma_1 x)$ for $x\in P_D^m$. (Recall that the conductor of $\psi$ is $P_F$.) It follows from [KZ, 5.2] that the normalizer $H$ of $\psi_{\gamma_1}$ in $D^\times$ is $F(\gamma_1)^\times(1+P_D^m)$. Let $K_1=F(\gamma_1)$ and $t_1$ be the Swan conductor of $K_1/F$. Any extension of $\psi_{\gamma_1}$ to $H$ is written in the form $\rho_{\theta_1,\gamma_1}$ where $\theta_1$ is a quasi-character of $K_1^\times$ with the property that $\theta_1(1+x)=\psi_{\gamma_1}(1+x)$ for $x\in P_{K_1}^m$ and $\rho_{\theta_1,\gamma_1}$ is defined on $H$ by $\rho_{\theta_1,\gamma_1}(k(1+x))=\theta_1(k)\psi_{\gamma_1}(1+x)$ for $k\in K_1^\times$ and $x\in P_D^m$. First we assume $m>t_1$. Then $f(\theta)=2m-t_1$. We need the following lemma to find a generic data $\Lambda$ satisfying $\pi_\Lambda=\mathrm{Ind}_H^{D^\times}\rho_{\theta,\gamma}$.

**Lemma 1.3.** *Let $K$ be a quadratic extension of $F$ in $D$ and $t$ be the Swan conductor of $K/F$. Then there exists $\xi\in D$ which satisfies the following conditions:*

(1) $\xi^{-1}x\xi=\bar{x}$ *for* $x\in K$.

(2) $\xi\in 1+P_D^t-(1+P_K^t+P_D^{t+1})$ *and* $\xi^2\in F^\times$.

(3) $D=K\oplus\xi K$.

(4) $\xi K=\{x\in D|\mathrm{Tr}(xy)=0$ *for all* $y\in K\}$.

*Proof.* By Skolem-Noether theorem, there exists $\xi$ satisfying (1). Since the $t$-th ramification group of $K/F$ is non-trivial and the $(t-1)$-th is trivial, $\xi$ satisfies (2), if necessary, by multiplying an appropriate element of $K^\times$. Then (3) is obvious. The last part follows from $(\xi x)^2=\xi^2 n_K(x)\in F$ for $x\in K$.

We continue the proof of Proposition 1.1. Let $\eta$ be an extension of $\psi_{\gamma_1}$ to the group $(1+P_{K_1}^{[(2m-t_1-1)/2]})(1+P_D^m)$ defined by

$$\eta((1+x)(1+y))=\psi_{\gamma_1}(1+x)\psi_{\gamma_1}(1+y)\psi_{\gamma_1}(1-xy)$$

for $x\in P_{K_1}^{[(2m-t_1-1)/2]}$ and $y\in P_D^m$. Then there exists a character $\kappa$ of $1+P_{K_1}^{[(2m-t_1-1)/2]}+P_D^m/1+P_D^m$ such that $\theta_1\cdot\psi_{\gamma_1}=\eta\otimes\kappa$ on $1+P_{K_1}^{[(2m-t_1-1)/2]}+P_D^m$. Let $\xi$ be the element which satisfies the conditions (1) - (4) in Lemma 1.3 for $K_1$. Then there exists an element $\gamma_2\in P_{K_1}^{1-m-t_1}$ such that

$$\kappa(1+x)=\psi(\mathrm{Tr}(\gamma_2(1+\xi)x))\quad \text{for } x\in P_{K_1}^{[(2m-t_1-1)/2]}+P_D^m$$

since $\gamma_2(1+\xi)\in P_D^{1-m}$ and $\psi(\mathrm{Tr}(\gamma_2(1+\xi)x))=\psi(Tr(\gamma_2 x))$ for $x\in P_{K_1}^{[(2m-t_1-1)/2]}$. Put $\gamma=\gamma_1+\gamma_2(1+\xi)$, $K=F(\gamma)$ and $t=t_K$. Then $\psi_{\gamma_1}=\psi_\gamma$ as a character of $1+P_D^m$ and $H=K_1^\times(1+P_D^m)=K^\times(1+P_D^m)$ since $\gamma_1\equiv\gamma \bmod P_D^{1-m}$. We need to show $t_{K_1}=t_K$. Take $\tilde{\omega}_{K_1}$ be a prime element of $K_1$. Then there exists a prime element $\tilde{\omega}_K$ of $K$ such that $\tilde{\omega}_{K_1} \bmod P_D^{m+1}$. Since $\mathrm{Tr}(P_D^{m+1})=P_F^{[(m+2)/2]}$ and $m\geq t_1+1$, we have $tr_{K_1}(\tilde{\omega}_{K_1})\equiv tr_K(\tilde{\omega}_K) \bmod P_F^{[(t_1+3)/2]}$. It implies $t_{K_1}=t_K$ from the remark below the Definition 1.1. It is obvious that we can take $\theta\in K^\times$ satisfying $\rho_{\theta,\gamma}=$

$\rho_{\theta_1,\gamma_1}$ on $H$. Then $\theta(1+x)=\psi_K(\gamma x)$ for $x\in 1+P_K^{[(2m-t-1)/2]}$ since $1+P_{K_1}^{[(2m-t_1-1)/2]}+P_D^m=1+P_K^{[(2m-t-1)/2]}+P_D^m$. Therefore $(K,\theta,\gamma)$ is a generic data of level $2m$ and $\pi|_H$ contains $\rho_{\theta,\gamma}$. By Clifford theory, $\mathrm{Ind}_H^{D^\times}\ \rho_{\theta,\gamma}$ is irreducible. Therefore $\pi=\mathrm{Ind}_H^{D^\times}\ \rho_{\theta,\gamma}$. Now we assume $m\le t_1$. As in the above case, $\pi=\mathrm{Ind}_H^{D^\times}\ \rho_{\theta,\gamma_1}$ for some quasi-character $\theta$ of $K_1^\times$. If $m=t_1$, $(F(\gamma_1),\theta,\gamma_1)$ is a generic data of level $2m$. Therefore we can assume $m<t_1$. If $\gamma\in P_D^{1-2m}$ satisfies $\gamma\equiv\gamma_1 \bmod P_D^{1-m}$, then $\psi_\gamma=\psi_{\gamma_1}$ on $1+P_D^m$ and $K_1^\times(1+P_D^m)=F(\gamma)^\times(1+P_D^m)$. Therefore we have only to show there exists an element $\gamma\in\gamma_1+P_D^{1-m}$ such that the Swan conductor of $F(\gamma)/F$ is $m$. Since $\mathrm{Tr}(P_D^{1-m})=P_F^{[(2-m)/2]}$ and $v_F(\mathrm{Tr}(\gamma_1))=v_F(tr_{K_1}(\gamma_1))=[(1-2m+t_1)/2]$, we can take an element $\delta\in P_D^{1-m}$ such that $v_F(\mathrm{Tr}(\gamma_1+\delta))=[(2-m)/2]$. Put $\gamma=\gamma_1+\delta$. Then the Swan conductor of $F(\gamma)/F$ is $m$. Hence our proposition.

**Remark.** If $K/F$ is tamely ramified, $\pi_\Lambda$ is determined by $\theta$ alone. But in our case $\theta$ does not determine $\psi_\gamma$. Therefore we need to use a parameter $\gamma$.

**Corollary 1.4.** *Let $\pi=\pi_\Lambda$ for a generic data $\Lambda=(K,\theta,\gamma)$ of level $2m$. Then the Swan conductor $t_K$ of $K$ is an invariant of the equivalent class of the representation $\pi$, that is, if $\pi_\Lambda\sim\pi_{\Lambda'}$ for a generic data $\Lambda'=(K',\theta',\gamma')$, then $t_K=t_{K'}$.*

*Proof.* At first assume $m>t_K$. In order to $\pi_\Lambda\sim\pi_{\Lambda'}$, it is necessary that there exists an element $g$ in $D^\times$ such that $g(K^\times(1+P_D^m))g^{-1}=K'^\times(1+P_D^m)$. Since $g(K^\times(1+P_D^m))g^{-1}=gK^\times g^{-1}(1+P_D^m))$, we have $t_K=t_{gKg^{-1}}=t_{K'}$ by the same argument to show $t_{K'}=t_K$ in the proof of Proposition 1.2. Now assume $m=t_K$. If $m>t_{K'}$, we get $t_K=t_{K'}<m$ by the above argument. Therefore $t_{K'}=m$.

The next lemma is useful to compute the character of $\pi$ when $t_K=m$.

**Lemma 1.5.** *Let $\pi=\pi_\Lambda$ for $\Lambda=(K,\theta,\gamma)$, $f(\pi)=2m+1$ and $t_K=m$. Take a quasi-character $\theta_0$ of $K^\times$ such that $\theta_0(1+x)=\psi_K(\gamma x)$ for $x\in P_K^{[(t+1)/2]}$. Then there exists a quasi-character $\eta$ of $F^\times$ such that $\pi_\Lambda=(\eta\circ\mathrm{Nr})\otimes\pi_{\Lambda'}$ where $\Lambda'=(K,\theta_0,\gamma)$.*

*Proof.* Since $\theta(1+x)=\theta_0(1+x)=\psi_K(\gamma x)$ for $x\in P_K^m=P_K^t$, $\theta$ and $\theta_0$ are trivial on $1+P_K^t$. It is easy to see the kernel of the norm map from $K^\times$ to $F^\times$ is contained in $1+P_K^t$. Thus $\theta$ and $\theta_0$ factor through the norm map i.e. $\theta=\eta'\circ n_K$, $\theta_0=\eta_0\circ n_K$ for some $\eta',\eta_0\in\widehat{F^\times}$. Then $\rho_{\theta,\gamma}=((\eta'\eta_0^{-1})\circ\mathrm{Nr})\otimes\rho_{\theta_0,\gamma}$ as a character of $K^\times(1+P_D^m)$. By virtue of the fact

$$\mathrm{Ind}_H^G(\sigma\otimes\tau|_H)=(\mathrm{Ind}_H^G\ \sigma)\otimes\tau\quad\text{for } \tau\in\hat{G}\text{ and }\sigma\in\hat{H},$$

we get our lemma.

## 2. Decomposition of $\pi_\Lambda$ as $K^\times$-module

We fix a generic data $\Lambda=(K,\theta,\gamma)$ of level $2m$ and abbreviate $t=t_K$, $\rho=\rho_{\theta,\gamma}$ and $\pi=\pi_\Lambda$. When $m=t$, we may assume $\theta(1+x)=\psi_K(\gamma x)$ for $x\in P_K^{[(t+1)/2]}$ from Lemma 1.5. Let $\xi$ be as in Lemma 1.2. In this section we determine the decomposition of $\pi$ as $K^\times$-module.

By Mackey decomposition,

$$\pi|_{K^\times}=\bigoplus_{a\in K^\times\backslash D^\times/H}\mathrm{Ind}_{aHa^{-1}\cap K^\times}^{K^\times}\rho^a. \tag{2.1}$$

where $\rho^a(x)=\rho(a^{-1}xa)$ for $x\in aHa^{-1}\cap K^\times$ and $H=K^\times(1+P_D^m)$.

First we shall give a complete system of representatives of the double coset $K^\times\backslash D^\times/H$.

**Lemma 2.1.** *$1+\xi\beta\in H$ for $\beta\in K$ is equivalent to $v_K(\beta)\geq m-t$ if $m>t$ and equivalent to $\beta\in K-(1+P_K)$ if $m=t$.*

*Proof.* Let $\tilde{\omega}_K$ be a prime element of $\mathcal{O}_K$ and $\xi=1+\xi'\tilde{\omega}_K^t$. By Lemma 1.2, we have $\mathcal{O}_D=\mathcal{O}_K\oplus\xi'\mathcal{O}_K$ and $P_D^n=P_K^n\oplus\xi'P_K^n$. It follows that $1+\xi'\beta\in H$ is equivalent to $\beta\in P_K^m$. Since $1+\xi\beta=(1+\xi'\tilde{\omega}_K^t\beta(1+\beta)^{-1})(1+\beta)$, $1+\xi\beta$ belongs to $H$ if and only if $v_K(\beta(1+\beta)^{-1})\geq m$. Hence our lemma follows.

We prepare some notations to describe the double coset $K^\times\backslash D^\times/H$. Set

$$I_\sigma=\{1+\xi\tilde{\omega}_K^\sigma\beta\mid\beta\in\mathcal{O}_K^1\backslash\mathcal{O}_K^\times/(1+P_K^{m-\sigma-t})\} \tag{2.2}$$

for $0<\sigma\leq m-t$,

$$J_\mu=\{1+\xi\beta\mid\beta\in\mathcal{O}_K^1\backslash 1+(P_K^\mu-P_K^{\mu+1})/(1+P_K^{m+2\mu-t})\} \tag{2.3}$$

for $0\leq\mu<t$ and

$$J_t=\{1+\xi\beta\mid\beta\in\mathcal{O}_K^1\backslash(1+P_K^t)/(1+P_K^{m+t})\} \tag{2.4}$$

where $\mathcal{O}_K^1=\mathrm{Ker}\ n_k$.

**Lemma 2.2.** *A complete system of representatives of the double coset $K^\times\backslash D^\times/H$ is given by the set*

$$\left(\bigcup_{\mu=0}^{t}J_\mu\right)\cup\left(\bigcup_{\sigma=1}^{m-t}I_\sigma\right)\cup\left(\bigcup_{\sigma=1}^{m-t}\xi I_\sigma\right).$$

*Proof.* First assume $m>t$, then $\xi\notin H$. It is obvious that we can take representatives of the form $1+\xi\beta$, $\beta\in\mathcal{O}_K$ or $\xi(1+\xi\beta)$, $\beta\in P_K$. For $a_1=1+\xi\beta_1$, $\beta_1\in\mathcal{O}_K$, $a_2=\xi(1+\xi\beta_2)$, $\beta_2\in P_K$ and $\alpha\in K$,

$$a_1^{-1}\alpha a_2=\mathrm{Nr}(a_1)^{-1}(\xi^2(\beta_2-\bar{\beta}_1\bar{\alpha}\alpha^{-1})+\xi(\bar{\alpha}\alpha^{-1}-\xi^2\beta_1\beta_2))\alpha.$$

Then $v_K(\bar{\alpha}\alpha^{-1}-\xi^2\beta_1\beta_2)=0$ and $v_K(\xi^2(\beta_2-\bar{\beta}_1)\bar{\alpha}\alpha^{-1})\geq 0$. By Lemma 2.1, $a_1^{-1}\alpha a_2\notin H$. Hence we have

$$D^\times/H=((1+\xi\mathcal{O}_K)\times K^\times)/H\cup\xi((1+\xi P_K)\times K^\times)/H\ (disjoint).$$

Moreover $\xi$ normalizes $K^\times$. Hence it is enough to show $\left(\bigcup_{u=0}^{t}J_\mu\right)\cup\left(\bigcup_{\sigma=1}^{m-t}I_\sigma\right)$ is a complete system of representatives of the double coset $K^\times\backslash((1+\xi\mathcal{O}_K)\times K^\times)/H$. For $a_1=1+\xi\beta_1$, $a_2=1+\xi\beta_2$, $\beta_i\in\mathcal{O}_K$ and $\alpha\in K^\times$, we have

$$a_1^{-1}\alpha a_2=\mathrm{Nr}(a_1)^{-1}(1-\xi^2\bar{\beta}_1\beta_2\bar{\alpha}\alpha^{-1}+\xi(\beta_2\bar{\alpha}\alpha^{-1}-\beta_1))\alpha. \tag{2.5}$$

If $v_K(\beta_1)>0$, it follows from Lemma 2.1 that $a_1^{-1}\alpha a_2$ is contained in $H$ for some $\alpha\in K^\times$ if and only if

$$\beta_1\equiv\alpha^1\beta_2 \qquad (\mathrm{mod}\ P_K^{m-t}),$$

for $\alpha^1\in\mathcal{O}_K^1$, because $\mathcal{O}_K^1=\{\bar{\alpha}\alpha^{-1}|\alpha\in K^\times\}$. Let $v_K(\beta_1)=0$ and $v_K(\beta_1-1)=\mu$. Then

$$v_K(\mathrm{Nr}(1+\xi\beta_1))=\begin{cases}2\mu & 0\leq\mu<t\\ t & \mu\geq t.\end{cases} \tag{2.6}$$

Since $1-\xi^2\bar{\beta}_1\beta_2\bar{\alpha}\alpha^{-1}=\mathrm{Nr}(1+\xi\beta_1)+\xi^2\bar{\beta}_1(\beta_1-\beta_2\bar{\alpha}\alpha^{-1})$, we get by Lemma 2.1 that $a_1^{-1}\alpha a_2$ is contained in $H$ for some $\alpha\in K^\times$ is equivalent to

$$\beta_1\equiv\alpha^1\beta_2 \qquad (\mathrm{mod}\ P_K^{m+2\mu-t}),$$

for $\alpha^1\in\mathcal{O}_K^1$ if $\mu<t$ and equivalent to

$$\beta_1\equiv\alpha^1\beta_2 \qquad (\mathrm{mod}\ P_K^{m+t}),$$

for $\alpha^1\in\mathcal{O}_K^1$ if $\mu=t$. Hence we get our lemma when $m>t$. For the case $m=t$, we can take representatives of the form $1+\xi\beta$, $\beta\in\mathcal{O}_K$ since $\xi\in H$. For the rest of the proof, it follows by the same argument for the case $m>t$.

Next we determine $aHa^{-1}\cap K^\times$ for the representatives of $K^\times\backslash D^\times/H$ in Lemma 2.2.

**Lemma 2.3.** *For $a\in I_\sigma$ or $\xi I_\sigma$, we have*

$$aHa^{-1}\cap K^\times=\begin{cases}F^\times(1+P_K^{m-\sigma-2t}) & \text{if}\ \ 0<\sigma<m-2t\\ K^\times & \text{if}\ \ m-2t\leq\sigma<m-t\end{cases} \tag{2.7}$$

*and for $a\in J_\mu$, we have*

$$aHa^{-1}\cap K^\times=\begin{cases}F^\times(1+P_K^{m+2\mu-2t}) & \text{if}\ \ 2\mu>2t-m\\ K^\times & \text{if}\ \ 2\mu\leq 2t-m.\end{cases} \tag{2.8}$$

*Proof.* Let $a=1+\xi\beta\in I_\sigma$. Assume $\alpha\in K^\times$ belongs to $aHa^{-1}\cap K^\times$. It is obvious that $F^\times\subset aHa^{-1}\cap K^\times$. Therefore we may assume $v_K(\alpha)=0$ or 1. Since Nr $(a)\in H$, $a^{-1}\alpha a\in H$ if and only if $\alpha-\xi^2\bar{\alpha}n_K(\beta)+\xi\beta(\bar{\alpha}-\alpha)\in H$. If $v_K(\alpha)=1$, $v_K(\alpha-\xi^2\bar{\alpha}n_K(\beta))=1$ and $v_K(\bar{\alpha}-\alpha)=t+1$. Therefore by Lemma 2.1, $a^{-1}\alpha a\in H$ if and only if $\sigma\geq m-2t$. If $v_K(\alpha)=0$, $v_K(\alpha-\xi^2\bar{\alpha}n_K(\beta))=0$. By Lemma 2.1, $a^{-1}\alpha a\in H$ if and only if $\bar{\alpha}-\alpha\in P_K^{m-\sigma-t}$. This is equivalent to $\alpha\in\mathcal{O}_F^\times(1+P_K^{m-\sigma-2t})$. Therefore we get our assertion for the case $a\in I_\sigma$. For $a\in\xi I_\sigma$, it is easy to see $aHa^{-1}\cap K^\times=(\xi a)H(\xi a)^{-1}\cap K^\times$ for $a\in I_\sigma$. For $a\in J_u$, it follows from the proof of the case $a\in I_\sigma$ and (2.6).

Let $a\in I_\sigma$ and $a'=\xi a$. Then $\rho^{a'}(x)=\rho^a(\bar{x})$ for $x\in aHa^{-1}\cap K^\times$. Therefore it suffices to consider $\rho^a$ for $a\in 1+\xi\mathcal{O}_K$.

**Lemma 2.4.** *For $a=1+\xi\beta$, $\beta\in\mathcal{O}_K$ and $\alpha\in aHa^{-1}\cap K^\times$,*

$$\rho^a\rho^{-1}(\alpha)=\rho\left(1+\frac{-\xi\beta+\xi^2n_K(\beta)}{1-\xi^2n_K(\beta)}(1-\bar{\alpha}\alpha^{-1})\right). \tag{2.9}$$

*If $a\in I_\sigma$ and $\alpha\in F^\times(1+P_K^{m-\sigma+[(1-3t)/2]})$ or $a\in J_\mu$ and $\alpha\in F^\times(1+P_K^{m+2\mu+[(1-3t)/2]})$, then we have*

$$\rho^a\rho^{-1}(\alpha)=\psi_\gamma\left(\frac{\xi^2n_K(\beta)}{1-\xi^2n_K(\beta)}(1-\bar{\alpha}\alpha^{-1})\right). \tag{2.10}$$

*Proof.* By direct calculation, we can show

$$\begin{aligned} a^{-1}\alpha a\alpha^{-1} &= (1+a-1)^{-1}\alpha(1+a-1)\alpha^{-1}\\ &= (1+a-1)^{-1}(1+\alpha(a-1)\alpha^{-1})\\ &= 1+a^{-1}(\alpha(a-1)\alpha^{-1}-(a-1))\\ &= 1+\frac{1-\xi\beta}{1-\xi^2n_K(\beta)}\xi\beta(1-\bar{\alpha}\alpha^{-1}).\end{aligned}$$

Therefore we have the first statement of our lemma. It follows from the definition of the generic data that $\rho(1+x)=\psi_\gamma(x)$ for $x\in P_K^{[(2m-t+1)/2]}+P_D^{[(2m+t+1)/2]}$. Since $\xi\in 1+P_D^t$, we see $\frac{1-\xi\beta}{1-\xi^2n_K(\beta)}\xi\beta\in P_K^\sigma+P_D^{\sigma+t}$ for $a\in I_\sigma$ and $\frac{1-\xi\beta}{1-\xi^2n_K(\beta)}\xi\beta\in P_K^{-\mu}+P_D^{-2\mu+t}$ for $a\in J_\mu$. Thus we heve

$$\rho^a\rho^{-1}(\alpha)=\psi_\gamma\left(\frac{\xi^2n_K(\beta)}{1-\xi^2n_K(\beta)}(1-\bar{\alpha}\alpha^{-1})\right)$$

since $\mathrm{Tr}(\xi\beta(1-\bar{\alpha}\alpha^{-1}))=0$.

**Corollary 2.5.** *Let the notation be as in Lemma 2.4. Then for* $a\in I_\sigma$, $\rho^a\rho^{-1}$ *is trivial on* $F^\times(1+P_K^{2m-2\sigma-2t})$ *and non-trivial on* $F^\times(1+P_K^{2m-2\sigma-2t-1})$ *if* $\sigma<m-t$*; for* $a\in I_\mu$, $\rho^a\rho^{-1}$ *is trivial on* $F^\times(1+P_K^{2m+2\mu-2t})$ *and non-trivial on* $F^\times(1+P_K^{2m+2\mu-2t-1})$.

*Proof.* This follows from Lemma 2.4. and the facts

$$v_K\left(\frac{\gamma\xi^2 n_K(\beta)}{1-\xi^2 n_K(\beta)}\right)=\begin{cases}1+2\sigma-2m & \text{for } 1+\xi\beta\in I_\sigma\\ 1-2\mu-2m & \text{for } 1+\xi\beta\in J_\mu\end{cases}$$

and $v_K(1-\bar{\alpha}\alpha^{-1})=2i+1+t$ for $a\in F^\times(1+P_K^{2i+1})-F^\times(1+P_K^{2i+2})$.

Since we use the next fact repeatedly, we state it as a lemma.

**Lemma 2.6.** *(1) The norm map* $n_K$ *from* $K^\times$ *to* $F^\times$ *induces a bijection from* $\mathcal{O}_K^\times/\mathcal{O}_K^1(1+P_K^i)$ *to* $\mathcal{O}_F^\times/1+P_F^i$ *if* $0<i\leq t$*. When* $i>t$*, the image of the induced map equals to* $n_K(\mathcal{O}_F^\times)/1+P_F^{[(i+t+1)/2]}$ *and it is index 2 in* $\mathcal{O}_F^\times/1+P_F^{[(i+t+1)/2]}$

*(2) The map* $\beta\mapsto\dfrac{\xi^2 n_K(\beta)}{1-\xi^2 n_K(\beta)}$ *induces a bijection from* $\mathcal{O}_K^\times/\mathcal{O}_K^1(1+P_K^i)$ *to* $\mathcal{O}_F^\times/1+P_F^i$ *for* $0<i\leq t$*. When* $i>t$*, it induces a bijection from* $\mathcal{O}_K^\times/\mathcal{O}_K^1(1+P_K^i)$ *to* $n_K(\mathcal{O}_K^\times)/1+P_F^{[(i+t+1)/2]}$.

*Proof.* The first part of this lemma is well-known (cf. [Se, Chap. V]). The rest of the lemma follows from the first part and the bijectivity of the map

$$x\mapsto\frac{\xi^2 x}{1-\xi^2 x}$$

from $\mathcal{O}_F^\times/1+P_F^i$ to itself.

Here we introduce some notation. Set $U_{-1}=K^\times$, $U_i=F^\times(1+P_K^{2i})$ for $i\geq 0$, and $U_i^*=U_i-U_{i+1}$. We note $F^\times(1+P_K^{2i})=F^\times(1+P_K^{2i+1})$. For $i\leq j$. let $X(i,j)$ be the set of all characters of $U_i$ that are trivial on $U_j$. Put $X^*(i,j)=X(i,j)-X(i,j-1)$. For $i=-1$, we set $X(j)=X(-1,j)$, $X^*(j)=X^*(-1,j)$. We define submodules $M_\sigma$ and $N_\mu$ of $\pi|_K$ by

$$M_\sigma=\bigoplus_{a\in I_\sigma}\operatorname{Ind}_{aHa^{-1}\cap K^\times}^{K^\times}\rho^a\rho^{-1} \tag{2.11}$$

and

$$M_\mu=\bigoplus_{a\in J_\mu}\operatorname{Ind}_{aHa^{-1}\cap K^\times}^{K^\times}\rho^a\rho^{-1}\,. \tag{2.12}$$

It follows from Corollary 2.5 that

$$M_\sigma\subset\bigoplus_{\chi\in X^*(m-\sigma-t)}\chi,\quad N_\mu\subset\bigoplus_{\chi\in X^*(m+\mu-t)}\chi \tag{2.13}$$

and we see from (2.1) and Lemma 2.2

$$\pi|_{K^\times}=\left(\theta\oplus\bar{\theta}\right)\otimes\left(\bigoplus_{\sigma=1}^{m-t}M_\sigma\right)\oplus\theta\otimes\left(\bigoplus_{\mu=0}^{t}N_\mu\right) \tag{2.14}$$

where $\bar{\theta}(x)=\theta(\bar{x})$ and $M_{m-t}$ is a trivial character of $K^\times$. By virtue of Lemma 2.6, it is easy to see that

$$\dim M_\sigma=\frac{1}{2}\left|X^*(m-\sigma-t)\right|=q^{m-\sigma-t}(q-1) \tag{2.15}$$

and

$$\dim N_\mu=\begin{cases}\frac{1}{2}\left|X^*(m+\mu-t)\right|=q^{m+\mu-t}(q-1) & \mu\neq 0,\\ q^{m-t}(q-2) & \mu=0.\end{cases} \tag{2.16}$$

From [H, Th. A], each quasi-character of $K^\times$ appears at most once in $\pi|_{K^\times}$. Thus we see that half number of characters in $X^*(m-\sigma-t)$ (resp. $X^*(m+\mu-t)$ for $\mu>0$) appear in $M_\sigma$ (resp. $N_\mu$ for $\mu>0$).

To determine which characters in $X^*(m-\sigma-t)$ (resp. $X^*(m+\mu-t)$) appear in $M_\sigma$ (resp. $N_\mu$), we start with the next lemma.

**Lemma 2.7.** *Let $a_1$, $a_2\in I_\sigma$ (resp. $a_1$, $a_2\in J_\mu$) and put $a_1=1+\xi\beta_1$, $a_2=1+\xi\beta_2$. For $0\leq i\leq\min(m-\sigma-t, t+1)$ (resp. $\mu\leq i\leq\min(m+2\mu-t, t+1)$), $\rho^{a_1}\rho^{-1}=\rho^{a_2}\rho^{-1}$ on $U_{m-\sigma-t-i}$ (resp. $U_{m+2\mu-t-i}$) if and only if $n_K(\beta_1)\equiv n_K(\beta_2)$ mod $1+P_F^i$ (multiplicative equivalence).*

*Proof.* We give the proof only for the case $a_1$, $a_2\in I_\sigma$. The other case is proved in the same way. Put $c_i=\dfrac{-\xi\beta_i+\xi^2n_K(\beta_i)}{1-\xi^2n_K(\beta_i)}(\bar{\alpha}\alpha^{-1})$ for $i=1$, 2. It is easy to see that for $\alpha\in a_iHa_i^{-1}\cap K^\times$

$$(\rho^{a_1}\rho^{-1})(\rho^{a_2}\rho^{-1})^{-1}(\alpha)=\rho(1+(c_1-c_2)(1+c_2)^{-1}).$$

Moreover if $v_D(c_1-c_2)\geq m$, we can see

$$\rho(1+(c_1-c_2)(1+c_2)^{-1})=\psi_\tau\left(\left(\frac{\xi^2n_K(\beta_1)}{1-\xi^2n_K(\beta_1)}-\frac{\xi^2n_K(\beta_2)}{1-\xi^2n_K(\beta_2)}\right)(\alpha-\bar{\alpha})\right).$$

We get from Lemma 2.6 that $\dfrac{\xi^2n_K(\beta_1)}{1-\xi^2n_K(\beta_1)}\equiv\dfrac{\xi^2n_K(\beta_2)}{1-\xi^2n_K(\beta_2)}$ mod $P_F^i$ is equivalent to $n_K(\beta_1)\equiv n_K(\beta_2)$ mod $P_F^i$. Thus we can get our lemma by induction on $i$.

**Proposition 2.8.** *Let the notation be as above.*

*(1) For $\sigma<m-2t$,*

$$M_\sigma=\bigoplus_{\chi\in M_\sigma|_{U_{m-\sigma-2t-1}}}\mathrm{Ind}_{U_{m-\sigma-2t-1}}^{K^\times}\chi,$$

and

$$M_{\sigma}|_{U_{m-\sigma-2t}} = q^{m-\sigma-2t} \bigoplus_{\chi \in X^*(m-\sigma-2t,\, m-\sigma-t)} \chi.$$

(2) For $m-2t \leq \sigma < m-t$,

$$M_{\sigma}|_{U_0} = \bigoplus_{\chi \in X^*(0,\, m-\sigma-t)} \chi.$$

(3) For $\mu > t - \frac{m}{2}$ and $\mu \neq 0$, t,

$$N_{\mu} = \bigoplus_{\chi \in N_{\mu}|_{U_{m+2\mu-2t-1}}} \mathrm{Ind}_{U_{m+2\mu-2t-1}}^{K^{\times}} \chi,$$

and

$$N_{\mu}|_{U_{m+2\mu-2t}} = q^{m+2\mu-2t} \bigoplus_{\chi \in X^*(m+2\mu-2t,\, m+\mu-t)} \chi.$$

(4) For $\mu \leq t - \frac{m}{2}$ and $\mu \neq 0$, $t$,

$$N_{\mu}|_{U_0} = \bigoplus_{\chi \in X^*(0,\, m+\mu-t)} \chi.$$

(5) For $\mu = 0 > t - \frac{m}{2}$,

$$N_0 = \bigoplus_{\chi \in N_0|_{U_{m-2t-1}}} \mathrm{Ind}_{U_{m-2t-1}}^{K^{\times}} \chi,$$

and

$$N_0|_{U_{m-2t}} = q^{m-2t} \bigoplus_{\substack{\chi \in X^*(m-2t,\, m-t) \\ \chi|_{U_{m-1}} \neq \lambda}} \chi$$

where $\lambda$ is a character of $U_{m-1}$ defined by $\lambda(\alpha) = \psi_{\tau}(1-\bar{\alpha}\alpha^{-1})$.

(6) For $\mu = 0 \leq t - \frac{m}{2}$,

$$N_0|_{U_0} = \bigoplus_{\substack{\chi \in X^*(0,\, m-t) \\ \chi|_{U_{m-1}} \neq \lambda}} \chi$$

where $\lambda$ is as in (5).

(7) For $\mu = t$,

$$N_t|_{U_{m-1}} = \bigoplus_{\chi \in N_t|_{U_{m-1}}} \mathrm{Ind}_{U_{m-1}}^{K^{\times}} \chi.$$

*Proof.* (1) Let $a_0 = 1 + \xi\beta_0$ be any element of $I_{\sigma}$ for $\sigma < m-2t$. It follows from Lemma 2.6 and Lemma 2.7 that

$$\begin{aligned}\left|\{a \in I_{\sigma} | \rho^{a}\rho^{-1} = \rho^{a_0}\rho^{-1} \text{ on } U_{m-\sigma-2t}\}\right| &= \left|\{a \in I_{\sigma} | \beta \equiv \beta_0 \bmod 1+P_K^t\}\right| \\ &= \left|n_K(1+P_K^t)/n_K(1+P_K^{m-\sigma-t})\right|\end{aligned}$$

$$=\frac{1}{2}\left|(1+P_F^t)/(1+P_F^{[(m-\sigma+t)/2]})\right|.$$

From the definition of $M_\sigma$ and [H, Th. A], each character of $U_{m-\sigma-2t}$ appears $|K^\times/F^\times(1+P_K^{m-\sigma-2t})|\times|(1+P_F^t)/(1+P_F^{[(m-\sigma+t)/2]})|=q^{m-\sigma-2t}$ times or does not appear. Therefore we have

$$M_\sigma|_{U_{m-\sigma-2t}}\subset q^{m-\sigma-2t}\bigoplus_{\chi\in X^*(m-\sigma-2t,\,m-\sigma-t)}\chi.$$

But it follows from (2.15) that the dimensions of both sides equal to $q^{m-\sigma-t}(q-1)$. Hence the second statement of (1) follows. By the same argument as above, we have

$$M_\sigma|_{U_{m-\sigma-2t-1}}\subset 2q^{m-\sigma-2t-1}\bigoplus_{\chi\in M_\sigma|_{U_{m-\sigma-2t-1}}}\chi.$$

In this case, the multiplicity $2q^{m-\sigma-2t-1}$ equals to $|K^\times/(U_{m-\sigma-2t-1})|$. By using [H, Th. A] again, the first statement follows. For (2), (3), (4) and (7), they are proved in the same way. As for (5) and (6), from the same argument for the proof of (1), it suffices to say that $N_0|_{U_{m-1}}$ does not contain $\lambda$. For $a=1+\xi\beta\in J_0$ and $\alpha\in U_{m-1}$, we have

$$\rho^a\rho^{-1}(\alpha)=\phi_r\left(\frac{\xi^2n_K(\beta)}{1-\xi^2n_K(\beta)}(1-\bar{\alpha}\alpha^{-1})\right).$$

From Lemma 2.6, the correspondence

$$\beta\mapsto\frac{\xi^2n_K(\beta)}{1-\xi^2n_K(\beta)}$$

induces the bijection from $\mathcal{O}_K^1\backslash(\mathcal{O}_K^\times-(1+P_F))/1+P_K$ to $(\mathcal{O}_F^\times-(1+P_F))/1+P_F$. Therefore $\lambda$ is not contained in $N_0|_{U_{m-1}}$.

We recall

$$K^\times=\left(\bigcup_{i=-1}^{m-1}U_i^*\right)\cup U_m\,(\text{disjoint}).$$

As a corollary of the above proposition, we can compute the trace of $M_\sigma$ and $N_\mu$ on all $U_i^*$ but one $i$.

**Corollary 2.9.** *Let the notation be as in Proposition 2.8. In addition we put*

(2.17) $$\Phi(x,\alpha)=1+\frac{-\xi x+\xi^2n_K(x)}{1-\xi^2n_K(x)}(1-\bar{\alpha}\alpha^{-1}).$$

*(1) If $\sigma<m-2t$,*

$$\operatorname{tr} M_\sigma(\alpha) = \begin{cases} 2q^{m-\sigma-2t-1} P_\sigma(\alpha) & \text{for } \alpha \in U^*_{m-\sigma-2t-1} \\ -q^{m-\sigma-t-1} & \text{for } \alpha \in U^*_{m-\sigma-t-1} \\ q^{m-\sigma-t-1}(q-1) & \text{for } \alpha \in U^*_{m-\sigma-t} \\ 0 & \text{otherwise} \end{cases}$$

*where*

$$P_\sigma(\alpha) = \sum_{x \in \tilde{\omega}_K^\sigma \mathcal{O}_K^\times / \mathcal{O}_K^1 (1+P_K^{t+1})} \rho(\Phi(x, \alpha)).$$

*(2) If* $m-2t \leq \sigma < m-t$,

$$\operatorname{tr} M_\sigma(\alpha) = \begin{cases} P_\sigma(\alpha) & \text{for } \alpha \in U^*_{-1} \\ -q^{m-\sigma-t-1} & \text{for } \alpha \in U^*_{m-\sigma-t-1} \\ q^{m-\sigma-t-1}(q-1) & \text{for } \alpha \in U^*_{m-\sigma-t} \\ 0 & \text{otherwise} \end{cases}$$

*where*

$$P_\sigma(\alpha) = \sum_{x \in \tilde{\omega}_K^\sigma \mathcal{O}_K^\times / 1+P_K^{m-\sigma-t}} \rho(\Phi(x, \alpha)).$$

*(3) If* $t - \dfrac{m}{2} < 0$,

$$\operatorname{tr} N_0(\alpha) = \begin{cases} 2q^{m-2t-1} Q_0(\alpha) & \text{for } \alpha \in U^*_{m-2t-1} \\ -q^{m-t-1}(1+\psi_T(1-\bar{\alpha}\alpha^{-1})) & \text{for } \alpha \in U^*_{m-t-1} \\ q^{m-t-1}(q-2) & \text{for } \alpha \in U^*_{m-t} \\ 0 & \text{otherwise} \end{cases}$$

*where*

$$Q_0(\alpha) = \sum_{x \in (\mathcal{O}_K^\times - (1+P_K)) / \mathcal{O}_K^1 (1+P_K^{t+1})} \rho(\Phi(x, \alpha)).$$

*(4) If* $t - \dfrac{m}{2} \geq 0$,

$$\operatorname{tr} N_0(\alpha) = \begin{cases} Q_0(\alpha) & \text{for } \alpha \in U^*_{-1} \\ -q^{m-t-1}(1+\psi_T(1-\bar{\alpha}\alpha^{-1})) & \text{for } \alpha \in U^*_{m-t-1} \\ q^{m-t-1}(q-2) & \text{for } \alpha \in U^*_{m-t} \\ 0 & \text{otherwise} \end{cases}$$

*where*

$$Q_0(\alpha) = \sum_{x \in (\mathcal{O}_K^\times - (1+P_K)) / 1+P_K^{m-1}} \rho(\Phi(x, \alpha)).$$

(5) *If* $\mu > t - \frac{m}{2}$ *and* $0 < \mu < t$,

$$\operatorname{tr} N_\mu(\alpha) = \begin{cases} 2q^{m+2\mu-2t-1} Q_\mu(\alpha) & \text{for } \alpha \in U^*_{m+2u-2t-1} \\ -q^{m+\mu-t-1} & \text{for } \alpha \in U^*_{m+\mu-t-1} \\ q^{m+\mu-t-1}(q-1) & \text{for } \alpha \in U^*_{m+\mu-t} \\ 0 & \text{otherwise} \end{cases}$$

*where*

$$Q_\mu(\alpha) = \sum_{x \in ((1+P_K^\mu) - (1+P_K^{\mu+1}))/\mathcal{O}_K^1 (1+P_K^{t+1})} \rho(\Phi(x,\alpha)).$$

(6) *If* $\mu \leq t - \frac{m}{2}$ *and* $0 < \mu < t$,

$$\operatorname{tr} N_u(\alpha) = \begin{cases} Q_\mu(\alpha) & \text{for } \alpha \in U^*_{-1} \\ -q^{m+\mu-t-1} & \text{for } \alpha \in U^*_{m+\mu-t-1} \\ q^{m+\mu-t-1}(q-1) & \text{for } \alpha \in U^*_{m+\mu-t} \\ 0 & \text{otherwise} \end{cases}$$

*where*

$$Q_\mu(\alpha) = \sum_{x \in ((1+P_K^\mu) - (1+P_K^{\mu+1}))/1+P_K^{m+2\mu-t}} \rho(\Phi(x,\alpha)).$$

(7) *For* $\mu = t$,

$$\operatorname{tr} N_t(\alpha) = \begin{cases} q^{m-1} Q_t(\alpha) & \text{for } \alpha \in U^*_{m-1} \\ q^m & \text{for } \alpha \in U^*_m \\ 0 & \text{otherwise} \end{cases}$$

*where*

$$Q_t(\alpha) = \sum_{x \in ((1+P_K^t) - (1+P_K^{t+1}))/\mathcal{O}_K^1 (1+P_K^{t+1})} \rho(\Phi(x,\alpha)).$$

*Proof.* This follows easily from Proposition 2.8, Lemma 2.4 and the fact

$$\operatorname{Ind}_{U_i}^{K^\times} \eta = \tilde{\eta} \otimes \Big( \bigoplus_{\chi \in X(i)} \chi \Big)$$

where $\eta$ is a character of $U_i$ and $\tilde{\eta}$ is any character of $K^\times$ whose restriction to $U_i$ coincides $\eta$.

## 3. Character formula of $\pi_\Lambda$ on $K^\times$ when $F$ is unramified over $\mathbf{Q}_2$

In this section we assume $F$ is unramified over $\mathbf{Q}_2$. By this assumption, the Swan conductor $t = t_K \leq 2$. Therefore the calculation of the character of $\pi$ becomes much easier. First we treat the case $t = 1$. In this case we can choose a prime element $\tilde{\omega}_K$ of $\mathcal{O}_K$ such that $tr_K(\tilde{\omega}_K) \equiv n_K(\tilde{\omega}_K) \bmod P_F^2$. Set $\tilde{\omega}_F = n_K(\tilde{\omega}_K)$. From (2.14) and Corollary 2.9, we have

**Corollary 3.1.** *Let the notation be as in Corollary 2.9.*

*(1) When $t=1$ and $m>2$,*

$$\operatorname{tr} \pi(\alpha) = \begin{cases} \theta(\alpha)(1+P_{m-2}(\alpha)) + \theta(\bar{\alpha})(1+P_{m-2}(\bar{\alpha})) & \text{if } \alpha \in U_{-1}^* \\ 2q^i(\theta(\alpha)P_{m-3-i}(\alpha) + \theta(\bar{\alpha})P_{m-3-i}(\bar{\alpha})) & \text{if } \alpha \in U_i^* \\ & \quad \text{for } 0 \le i < m-3 \\ 2q^{m-3}\theta(\alpha)Q_0(\alpha) & \text{if } \alpha \in U_{m-3}^* \\ 0 & \text{if } \alpha \in U_{m-2}^* \\ q^{m-1}\theta(\alpha)(1+Q_1(\alpha)) & \text{if } \alpha \in U_{m-1}^* \\ q^{m-1}(q+1) & \text{if } \alpha \in U_m. \end{cases}$$

*(2) When $t=1$ and $m=2$,*

$$\operatorname{tr} \pi(\alpha) = \begin{cases} \theta(\alpha)(1+Q_0(\alpha)) + \theta(\bar{\alpha}) & \text{if } \alpha \in U_{-1}^* \\ 0 & \text{if } \alpha \in U_0^* \\ q\theta(\alpha)(1+Q_1(\alpha)) & \text{if } \alpha \in U_1^* \\ q(q+1) & \text{if } \alpha \in U_2. \end{cases}$$

*(3) When $t=1$ and $m=1$,*

$$\operatorname{tr} \pi(\alpha) = \begin{cases} \theta(\alpha) & \text{if } \alpha \in U_{-1}^* \\ \theta(\alpha)(1+Q_1(\alpha)) & \text{if } \alpha \in U_0^* \\ q+1 & \text{if } \alpha \in U_1. \end{cases}$$

*Proof.* It follows from direct caluculation. We only remark we use

$$\theta(\bar{\alpha}) = \theta(\alpha)\psi_K(\gamma(\alpha-\bar{\alpha})) \qquad \text{for} \quad \alpha \in U_{m-2}^*$$

and $\theta(\bar{\alpha}) = \theta(\alpha)$ for $\alpha \in U_{m-1}$.

Thus our remaining task is to compute $P_\sigma(\alpha)$ and $Q_\mu(\alpha)$ in Corollary 2.9. For convenience, we set

$$B_\alpha = \tilde{\omega}_F^{\sigma+1} tr_K(\gamma(1-\bar{\alpha}\alpha^{-1})) \quad \text{for} \quad \alpha \in U_{m-\sigma-3}^* \tag{3.1}$$

First we calculate $P_\sigma(\alpha)$ for $0<\sigma<m-2$ and $\alpha \in U_{m-\sigma-3}^*$.

**Lemma 3.2.** *For $\alpha \in U_{m-\sigma-3}^*$,*

$$P_\sigma(\alpha) = -\frac{q}{2} h(\tilde{\omega}_K^\sigma a_\alpha, \alpha)$$

*where $a_\alpha \in \mathcal{O}_K$ is determined uniquely modulo $P_K$ by $a_\alpha^2 \equiv B_\alpha^{-1} \bmod P_F$ and*

(3.2)
$$h(x,\alpha)=\theta\left(1+\frac{x}{1-n_K(x)}(1-\bar{\alpha}\alpha^{-1})\right)\theta\left(1+\frac{n_K(x)}{1-n_K(x)}(1-\bar{\alpha}\alpha^{-1})\right)$$
$$\psi_\gamma\left(\frac{x}{1-n_K(x)}(1-\bar{\alpha}\alpha^{-1})+\left(\frac{x}{1-n_K(x)}(1-\bar{\alpha}\alpha^{-1})\right)^2\right).$$

*If* $\sigma\neq m-4$, *we have*

$$h(\bar{\omega}_K^\sigma a_\alpha,\alpha)=\psi_\gamma\left(\frac{\bar{\omega}_F^\sigma n_K(a_\alpha)}{1-\bar{\omega}_F^\sigma n_K(a_\alpha)}(1-\bar{\alpha}\alpha^{-1})\right).$$

*Proof.* We first remark that $v_D(\Phi(x,\alpha))=2m-\sigma-4$ for $\alpha\in U^*_{m-\sigma-3}$ and $x\in P_K^\sigma-P_K^{\sigma+1}$. From the definition of $P_\sigma(\alpha)$, we have

$$\begin{aligned}P_\sigma(\alpha)&=\sum_{x\in\bar{\omega}_K^\sigma\mathcal{O}_K^\times/\mathcal{O}_K^1(1+P_K^2)}\rho(\Phi(x,\alpha))\\&=\sum_{x\in\bar{\omega}_K^\sigma\mathcal{O}_K^\times/1+P_K}\ \sum_{y\in1+P_K/\mathcal{O}_K^1(1+P_K^2)}\rho(\Phi(xy,\alpha))\\&=\sum_{x\in\bar{\omega}_K^\sigma\mathcal{O}_K^\times/1+P_K}\rho(\Phi(x,\alpha))\\&=\sum_{y\in1+P_K/\mathcal{O}_K^1(1+P_K^2)}\rho\left(1+\frac{-\xi x(y-1)+\xi^2n_K(x)(n_K(y)-1)}{1-\xi^2n_K(x)}(1-\bar{\alpha}\alpha^{-1})\right).\end{aligned}$$

The last equality holds from the fact that

$$\begin{aligned}\Phi(xy,\alpha)\equiv\Phi(x,\alpha)&+\frac{-\xi x+\xi^2n_K(x)}{1-\xi^2n_K(x)n_K(y)}(1-\bar{\alpha}\alpha^{-1})\\&+\frac{-\xi x(y-1)+\xi^2n_K(x)(n_K(y)-1)}{1-\xi^2n_K(x)n_K(y)}(1-\bar{\alpha}\alpha^{-1})\quad\text{mod Ker}\rho.\end{aligned}$$

By Lemma 2.4 and the fact

$$\begin{aligned}v_K\left(\frac{-\xi x}{1-\xi^2n_K(x)}(1-\bar{\alpha}\alpha^{-1})\frac{-\xi x(y-1)}{1-\xi^2n_K(x)}(1-\bar{\alpha}\alpha^{-1})\right)&=2m-1+2(m-2-\sigma)-1\\&\geq2m-1,\end{aligned}$$

we get

$$P_\sigma(\alpha)=\sum_{x\in\bar{\omega}_K^\sigma\mathcal{O}_K^\times/1+P_K}\rho(\Phi(x,\alpha))\sum_{y\in1+P_K/\mathcal{O}_K^1(1+P_K^2)}\psi_\gamma(n_K(x)(n_K(y)-1)(1-\bar{\alpha}\alpha^{-1})).$$

By Lemma 2.6, $y\mapsto n_K(y)-1$ induces an isomorphism from $1+P_K/\mathcal{O}_K^1(1+P_K^2)$ to $(n_K(1+P_K)-1)/P_F^2$ and the latter group is index 2 in $P_F/P_F^2$. Thus there exists a unique $a_\alpha \in \mathcal{O}_K$ mod $P_K$ such that the map

$$y \mapsto \psi_\gamma\left(\tilde{\omega}_F^g n_K(a_\alpha)\,(n_K(y)-1)\,(1-\bar{\alpha}\alpha^{-1})\right)$$

is a trivial character of $1+P_K^1/\mathcal{O}_K^1(1+P_K^2)$.
Therefore

$$P_\sigma(\alpha)=\frac{q}{2}\rho\left(\Phi(\tilde{\omega}_K^g a_\alpha,\alpha)\right).$$

In fact $n_K(1+\tilde{\omega}_K y)-1=\tilde{\omega}_F(y^2+y)$ for $y\in\mathcal{O}_K$. By the assumption $\psi(x^2+x)=1$ for $x\in\mathcal{O}_F$, we have $n_K(a_\alpha)=(tr_K(\tilde{\omega}_F^{g+1}\gamma(1-\bar{\alpha}\alpha^{-1})))^{-1}$. From the definition of $\rho(\Phi(\tilde{\omega}_K^g a_\alpha,\alpha))$, we have

$$\begin{aligned}\rho(\Phi(\tilde{\omega}_K^g a_\alpha,\alpha))&=\rho\left(1+\frac{-\xi\tilde{\omega}_K^g a_\alpha+\xi^2\tilde{\omega}_F^g n_K(a_\alpha)}{1-\xi^2\tilde{\omega}_F^g n_K(a_\alpha)}\,(1-\bar{\alpha}\alpha^{-1})\right)\\&=\theta\left(1+\frac{\tilde{\omega}_K^g a_\alpha+\xi^2\tilde{\omega}_F^g n_K(a_\alpha)}{1-\xi^2 n_K(a_\alpha)}\,(1-\bar{\alpha}\alpha^{-1})\right)\\&\quad\times\psi_\gamma\left(\frac{-(\xi-1)\,\tilde{\omega}_K^g a_\alpha}{1-\xi^2\tilde{\omega}_F^g n_K(a_\alpha)}\,(1-\bar{\alpha}\alpha^{-1})\right)\\&\quad\times\psi_\gamma\left(\frac{-(\tilde{\omega}_K^g a_\alpha+\xi^2\tilde{\omega}_F^g n_K(a_\alpha))\,(1-\bar{\alpha}\alpha^{-1})\,(\xi-1)\,\tilde{\omega}_K^g a_\alpha(1-\bar{\alpha}\alpha^{-1})}{(1-\xi^2\tilde{\omega}_F^g n_K(a_\alpha))^2}\right)\\&=-h(\tilde{\omega}_K^g a_\alpha,\alpha)\end{aligned}$$

since $\psi_\gamma((\xi^2-1)\,\tilde{\omega}_F^g n_K(a_\alpha)\,(1-\bar{\alpha}\alpha^{-1}))=-1$ by virtue of $\xi^2\in(1+P_F)-n_K(1+P_K)$ and $n_K(a_\alpha)\equiv a_\alpha^2$ mod $P_F$. When $\sigma\leq m-4$,

$$h(\tilde{\omega}_K^g a_\alpha,\alpha)=\psi_\gamma\left(\frac{\tilde{\omega}_F^g n_K(a_\alpha)}{1-\tilde{\omega}_F^g n_K(a_\alpha)}\,(1-\bar{\alpha}\alpha^{-1})\right)$$

since $v_D(\Phi(\tilde{\omega}_K^g a_\alpha,\alpha))\geq m$. Hence our lemma.

Next we treat $P_{m-2}(\alpha)$ for $\alpha\in U_{-1}^*$.

**Lemma 3.3.** *(1) For $\alpha\in U_{-1}^*$,*

$$P_{m-2}(\alpha)=G_{m-2}(1-\bar{\alpha}\alpha^{-1})-1$$

*where*

(3.3) $$G_{m-2}(z)=\sum_{x\in k_F}\theta(1-\tilde{\omega}_K^{m-2}zx)\,\psi_\gamma(\tilde{\omega}_K^{m-2}zx+\tilde{\omega}_F^{m-2}(z+n_K(z))\,x^2).$$

*(2) For $z=1-\bar{\alpha}\alpha^{-1}$, $1-\alpha\bar{\alpha}^{-1}$, $G_{m-2}(z)\in\mathbf{Z}[\sqrt{-1}]$ and $|G_{m-2}(z)|=\sqrt{q}$.*

*Proof.* (1) Form the definition of $P_{m-2}$ and the fact $v_D(\Phi(x,\alpha))=m-1$ for $x\in P_K^{m-2}$, we have

$$P_{m-2}(\alpha)=\sum_{x\in\bar{\omega}_K^{m-2}\mathcal{O}_K^\times/1+P_K}\rho\left(1+\frac{-\xi x+\xi^2 n_K(x)}{1-\xi^2 n_K(x)}(1-\bar{\alpha}\alpha^{-1})\right)$$
$$=\sum_{x\in\bar{\omega}_K^{m-2}\mathcal{O}_K^\times/1+P_K}\rho\left(1+(-\xi x+\xi^2 n_K(x))(1-\bar{\alpha}\alpha^{-1})\right)$$
$$=\left(\sum_{x\in\mathcal{O}K/P_K}\rho(1+(-\xi\omega_K^{m-2}x+\xi^2\bar{\omega}_F^{m-2}n_K(x))(1-\bar{\alpha}\alpha^{-1}))\right)-1.$$

Since $\rho(1+\xi x)=\theta(1+x)\psi_T(-x+x^2)$ for $x\in P_K^{m-1}$, we get the first half of the lemma.

(2) Since $\rho\left(1+(-\xi\bar{\omega}_K^{m-2}x+\xi^2\bar{\omega}_F^{m-2}n_K(x))(1-\bar{\alpha}\alpha^{-1})\right)^4=1$, $G_{m-2}(1-\bar{\alpha}\alpha^{-1})\in\mathbf{Z}[\sqrt{-1}]$. As for the absolute value of $G_{m-2}(1-\bar{\alpha}\alpha^{-1})$, it follows from the following standard calculation. For $z\in P_K^1-P_K^2$,

$$\begin{aligned}G_{m-2}(z)\overline{G_{m-2}(z)}&=\sum_{x,y\in k_F}\theta(1-\bar{\omega}_K^{m-2}zx)\psi_T(\bar{\omega}_K^{m-2}zx+\bar{\omega}_F^{m-2}(z+n_K(z))x^2)\\&\quad\theta(1+\bar{\omega}_K^{m-2}zx+\bar{\omega}_K^{2m-4}z^2x^2)\psi_T(-\bar{\omega}_K^{m-2}zy-\bar{\omega}_F^{m-2}(z+n_K(z))y^2)\\&=\sum_{x,y\in k_F}\theta(1-\bar{\omega}_K^{m-2}z(x-y))\psi_T(\omega_K^{m-2}z(x-y))\\&\quad\psi_T(\bar{\omega}_F^{m-2}(z+n_K(z))(x-y)^2)\psi_T(\bar{\omega}_K^{2m-4}z^2x^2)\\&=\sum_{u\in k_F}\theta(1-\bar{\omega}_K^{m-2}zu)\psi_T(\bar{\omega}_K^{m-2}zu+\bar{\omega}_F^{m-2}(z+n_K(z))u^2)\\&\quad\sum_{x\in k_F}\psi_T(\bar{\omega}_K^{2m-2}z^2x^2)\\&=q.\end{aligned}$$

Next we caluculate $Q_0(\alpha)$. First we treat the case $m>2$. We define a subgroup $k_F^0$ of $k_F$ defined by

$$k_F^0=\{x+x^2|x\in k_F\}. \tag{3.4}$$

**Lemma 3.4.** *For* $\alpha\in U_{m-3}^*$,

$$Q_0(\alpha)=\begin{cases}-\frac{q}{2}(h(a'_\alpha,\alpha)+h(a''_\alpha,\alpha)) & \text{if } B_\alpha \bmod P_F\in k_F^0\\ 0 & \text{otherwise}\end{cases}$$

*where* $B_\alpha$ *as in* (3.1) *and* $a'_\alpha, a''_\alpha\in\mathcal{O}_K$ *are defined by the condition* $\frac{n_K(a'_a)}{1-n_K(a'_\alpha)}$ mod $P_F$ *and* $\frac{n_K(a''_\alpha)}{1-n_K(a''_\alpha)}$ mod $P_F$ *are solutions of* $X^2+X-(B_\alpha \bmod P_F)=0$.

*Proof.* If $|k_F|=2$, then $X^2+X-(B_\alpha \bmod P_F)$ has no solution over $k_F$ and $Q_0(\alpha)=0$ since $J_0=\phi$. Therefore we may assume $|k_F|>2$. As in the calculation for $P_\sigma(\alpha)$, we get

$$Q_0(\alpha)=\sum_{x\in\mathcal{O}_K^\times-(1+P_K)/1+P_K}\rho(\Phi(x,\alpha))\sum_{y\in 1+P_K/\mathcal{O}_K^1(1+P_K^2)}\Psi_{(\varphi(x),\alpha)}(y)$$
$$=\sum_{x\in\mathcal{O}_K^\times-(1+P_K)/1+P_K}\rho(\Phi(x,\alpha))$$
$$\sum_{y\in 1+P_K/\mathcal{O}_K^1(1+P_K^2)}\psi_T(\varphi(x)(n_K(y)-1)(1-\bar{\alpha}\alpha^{-1}))$$
$$=\sum_{x\in\mathcal{O}_K^\times-(1+P_K)/1+P_K}\rho(\Phi(x,\alpha))$$
$$\frac{1}{2}\sum_{y\in k_F}\psi_T(\varphi(x)\tilde{\omega}_F(y^2+y)(1-\bar{\alpha}\alpha^{-1})).$$

Here $\varphi(x)=\dfrac{x^2}{1-x^2}+\left(\dfrac{x^2}{1-x^2}\right)^2$ since $n_K$ induces the map $x\mapsto x^2$ on $k_F$ by the identification of $k_K$ with $k_F$. By the fact that the map $x\mapsto\dfrac{x^2}{1-x^2}$ induces a bijection from $k_F-\{0,1\}$ to itself,

$$\sum_{y\in k_F}\psi_T(\varphi(x)\omega_F(y^2+y)(1-\bar{\alpha}\alpha^{-1}))=\begin{cases} q & \text{if } B_\alpha \bmod P_F\in k_F^0\\ 0 & \textit{otherwise.}\end{cases}$$

Thus we get our lemma.

Next we treat $Q_0(\alpha)$ when $m=2$.

**Lemma 3.5.** *(1) For* $\alpha\in U_{-1}^*$,

$$Q_0(\alpha)=G_0(1-\bar{\alpha}\alpha^{-1})-1-\theta(1+1-\bar{\alpha}\alpha^{-1})$$

*where*

(3.5) $$G_0(z)=\sum_{x\in k_F}\theta(1+(x+x^2)z)\,\theta(1+x^2z)\,\psi_T((x+x^2)z+((x+x^2)z)^2).$$

*(2)* $G_0(1-\bar{\alpha}\alpha^{-1})\in\boldsymbol{Z}[\sqrt{-1}]$ *and* $|G_0(1-\bar{\alpha}\alpha^{-1})|=\sqrt{q}$.

*Proof.* If $|k_F|=2$, then $Q_0(\alpha)=0$ and $G_0(1-\bar{\alpha}\alpha^{-1})=1+\theta(1+1-\bar{\alpha}\alpha^{-1})$. Since $(\theta(1+1-\bar{\alpha}\alpha^{-1}))^4=1$, $|G_0(1-\bar{\alpha}\alpha^{-1})|=\sqrt{2}$. Thus our lemma holds. We assume $|k_F|>2$. From the definition of $Q_0$ and $\Phi$, we have

$$Q_0(\alpha)=\sum_{x\in(\mathcal{O}_K^\times-(1+P_K))/1+P_K}\rho(\Phi(x,\alpha))$$

and for $x\in\mathcal{O}_K^\times-(1+P_K)$,

$$\rho(\Phi(x,\alpha))=\theta\left(1+\frac{-x+\xi^2 n_K(x)}{1-\xi^2 n_K(x)}(1-\bar{\alpha}\alpha^{-1})\right)\psi_T\left(\frac{(-x+n_K(x))x}{(1-\xi^2 n_K(x))^2}(1-\bar{\alpha}\alpha^{-1})^2\right)$$

$$=\theta\left(1+\frac{-x}{1-n_K(x)}(1-\bar{\alpha}\alpha^{-1})\right)\theta\left(1+\frac{n_K(x)}{1-n_K(x)}(1-\bar{\alpha}\alpha^{-1})\right)$$

$$\psi_T\left(\frac{x}{1-n_K(x)}(1-\bar{\alpha}\alpha^{-1})+\left(\frac{x}{1-n_K(x)}(1-\bar{\alpha}\alpha^{-1})\right)^2\right).$$

Since $1-n_K(x)\equiv 1-x^2 \bmod P_K^2$, $\frac{x}{1-x}+\frac{x^2}{1-x^2}\equiv\frac{x}{1-x^2} \bmod P_K^2$ and $x\mapsto\frac{x}{1-x}$ induces a bijection from $k_F-\{0,1\}$ to itself, we get

$$Q_0(\alpha)=\sum_{x\in k_F-\{0,1\}}\theta(1+(x+x^2)(1-\bar{\alpha}\alpha^{-1}))\,\theta(1+x^2(1-\bar{\alpha}\alpha^{-1}))$$

$$\psi_T((x+x^2)(1-\bar{\alpha}\alpha^{-1})+((x+x^2)(1-\bar{\alpha}\alpha^{-1}))^2).$$

Hence we get the first half of our lemma. $G_0(1-\bar{\alpha}\alpha^{-1})\in\mathbf{Z}[\sqrt{-1}]$ follows from $\theta(1-\bar{\alpha}\alpha^{-1})^4=1$. The absolute value can be calculated in the same way for $G_{m-2}(1-\bar{\alpha}\alpha^{-1})$ when $m>2$.

The last term we must calculate is $Q_1(\alpha)$ for $\alpha\in U_{m-1}^*$. The next lemma holds for all $m\geq 1$.

**Lemma 3.6.** *For* $\alpha\in U_{m-1}^*$,

$$Q_1(\alpha)=\sum_{x\in k_F}\psi_T\left(\frac{1-\bar{\alpha}\alpha^{-1}}{\tilde{\omega}_F(x^2+x+b)}\right)$$

*where* $b\in k_F-k_F^0$.

*Proof.* This follows from the following direct calculation:

$$Q_1(\alpha)=\sum_{x\in 1+P_K/\mathcal{O}_K^1(1+P_K^2)}\rho(\Phi(x,\alpha))$$

$$=\sum_{n_K(x)\in n_K(1+P_K)/(1+P_K^2)}\psi_T\left(\frac{\xi^2 n_K(x)}{1-\xi^2 n_K(x)}(1-\bar{\alpha}\alpha^{-1})\right)$$

$$=\frac{1}{2}\sum_{x\in k_F}\psi_T\left(\frac{\xi^2(1+\tilde{\omega}_F(x^2+x))}{1-\xi^2(1+\tilde{\omega}_F(x^2+x))}(1-\bar{\alpha}\alpha^{-1})\right)$$

$$=\frac{1}{2}\sum_{x\in k_F}\psi_T\left(\frac{1}{1-\xi^2-\tilde{\omega}_F(x^2+x)}(1-\bar{\alpha}\alpha^{-1})\right).$$

From $(\xi^2-1)/\tilde{\omega}_F \bmod P_F\notin k_F^0$, we get our lemma.

Now we can state the character formula for $t=1$.

**Theorem 3.7.** *Let $\Lambda=(K, \theta, \gamma)$ be a generic data of level $2m$ and $\pi=\pi_\Lambda$. (See section 1 for the definition of generic data and $\pi_\Lambda$.) Assume $t=t_K=1$. Take a prime element $\tilde{\omega}_K$ of $\mathcal{O}_K$ and a prime element $\tilde{\omega}_F$ satisfying $tr_K(\tilde{\omega}_K)=n_K(\tilde{\omega}_K)$ and $\tilde{\omega}_F=n_K(\tilde{\omega}_K)$. Let $k_F^0$ be an index 2 subgroup of $k_F$ defined by $k_F^0=\{x^2+x \mid x\in k_F\}$ and take $b\in\mathcal{O}_F$ such that $(b \bmod P_F)\in k_F-k_F^0$.*

(1) *If $m>2$, then*

$$\operatorname{tr}\pi(\alpha)=\begin{cases}\theta(\alpha)G_{m-2}(1-\bar{\alpha}\alpha^{-1})+\theta(\bar{\alpha})G_{m-2}(1-\alpha\bar{\alpha}^{-1}) \\ \qquad\qquad \text{if } \alpha\in U_{-1}^* \\ -q^{i+1}(\theta(\alpha)h(\tilde{\omega}_K^{m-i-3}a_\alpha, \alpha)+\theta(\bar{\alpha})h(\tilde{\omega}_K^{m-i-3}a_{\bar{\alpha}}, \bar{\alpha})) \\ \qquad\qquad \text{if } \alpha\in U_i^* \text{ for } 0\leq i\leq m-3 \\ -q^{m-2}\theta(\alpha)(h(a_\alpha', \alpha)+h(a_\alpha'', \alpha)) \\ \qquad\qquad \text{if } \alpha\in U_{m-3}^* \text{ and } B_\alpha \bmod P_F\in k_F^0 \\ 0 \qquad\qquad \text{if } \alpha\in U_{m-3}^* \text{ and } B_\alpha \bmod P_F\notin k_F^0 \\ 0 \qquad\qquad \text{if } \alpha\in U_{m-2}^* \\ q^{m-1}\left(1+\theta(\alpha)\sum_{x\in k_F}\psi_\gamma\left(\dfrac{1-\bar{\alpha}\alpha^{-1}}{\tilde{\omega}_F(x^2+x+b)}\right)\right) \\ \qquad\qquad \text{if } \alpha\in U_{m-1}^* \\ q^{m-1}(q+1) \qquad \text{if } \alpha\in U_m\end{cases}$$

*where $B_\alpha=\tilde{\omega}_F^{\sigma+1}\,tr_K(\gamma(1-\bar{\alpha}\alpha^{-1}))$ for $\alpha\in U_{m-\sigma-3}^*$, $a_\alpha\in\mathcal{O}_K$ is determined uniquely modulo $P_K$ by $a_\alpha^2\equiv B_\alpha^{-1} \bmod P_K$, $h(x, a)$ as in (3.2), $a_\alpha', a_\alpha''\in\mathcal{O}_K$ are defined by the condition $\dfrac{n_K(a_\alpha')}{1-n_K(a_\alpha')} \bmod P_F$, $\dfrac{n_K(a_\alpha'')}{1-n_K(a_\alpha'')} \bmod P_F$ are solutions of $X^2+X-(B_\alpha \bmod P_K)=0$, $\psi_\gamma(1+x)=\psi(tr_K(\gamma x))$ for $x\in P_K^m$* and *$G_{m-2}$ as in (3.3). $G_{m-2}(1-\bar{\alpha}\alpha^{-1})$, $G_{m-2}(1-\alpha\bar{\alpha}^{-1})$ belong to $\boldsymbol{Z}[\sqrt{-1}]$ and their absolute value is $\sqrt{q}$.*

(2) *If $m=2$,*

$$\operatorname{tr}\pi(\alpha)=\begin{cases}\theta(\bar{\alpha})-\theta(\alpha-\bar{\alpha})+\theta(\alpha)G_0(1-\bar{\alpha}\alpha^{-1}) & \text{if } \alpha\in U_{-1}^* \\ 0 & \text{if } \alpha\in U_0^* \\ q\theta(\alpha)\left(1+\sum_{x\in k_F}\psi_\gamma\left(\dfrac{1-\bar{\alpha}\alpha^{-1}}{\tilde{\omega}_F(x^2+x+b)}\right)\right) & \text{if } \alpha\in U_{m-1}^* \\ q(q+1) & \text{if } \alpha\in U_m\end{cases}$$

*where $G_0$ as in (3.5) and $G_0(1-\bar{\alpha}\alpha^{-1})$ satisfies $G_0(1-\bar{\alpha}\alpha^{-1})\in\boldsymbol{Z}[\sqrt{-1}]$ and $|G_0(1-\bar{\alpha}\alpha^{-1})|=\sqrt{q}$.*

(3) *If $m=1$,*

$$\operatorname{tr} \pi(\alpha)=\begin{cases} \theta(\alpha) & \text{if } \alpha\in U_{-1}^{*} \\ \theta(\alpha)\left(1+\sum_{x\in k_F}\psi_{\tau}\left(\dfrac{1-\bar{\alpha}\alpha^{-1}}{\tilde{\omega}_F(x^2+x+b)}\right)\right) & \text{if } \alpha\in U_0^{*} \\ q+1 & \text{if } \alpha\in U_1. \end{cases}$$

Now we assume $t=t_K=2$. In this case we can choose a prime element $\tilde{\omega}_K$ of $\mathcal{O}_K$ such that $\tilde{\omega}_K^2\in F$ and $\tilde{\omega}_K^2\equiv 2 \bmod P_F^2$. Set $\tilde{\omega}_F=n(\tilde{\omega}_K)$. As in the case $t=1$, we have from (2.14) and Corollary 2.9

**Corollary 3.8.** *Let the notation be as in Corollary* 2.9.
*(1) When* $t=2$ *and* $m>4$,

$$\operatorname{tr} \pi(\alpha)=\begin{cases} \theta(\alpha)(1+P_{m-3}(\alpha)+P_{m-4}(\alpha)) & \\ \qquad +\theta(\bar{\alpha})(1+P_{m-3}(\bar{\alpha})+P_{m-4}(\bar{\alpha})) & \text{if } \alpha\in U_{-1}^{*} \\ 2q^{i}(\theta(\alpha)P_{m-4-i}(\alpha)+\theta(\bar{\alpha})P_{m-4-i}(\bar{\alpha})) & \text{if } \alpha\in U_i^{*} \\ & \text{for } 0\leq i<m-5 \\ 2q^{m-5}\theta(\alpha)Q_0(\alpha) & \text{if } \alpha\in U_{m-5}^{*} \\ 0 & \text{if } \alpha\in U_{m-4}^{*} \\ 2q^{m-3}\theta(\alpha)Q_1(\alpha) & \text{if } \alpha\in U_{m-3}^{*} \\ 0 & \text{if } \alpha\in U_{m-2}^{*} \\ q^{m-1}\theta(\alpha)(1+Q_2(\alpha)) & \text{if } \alpha\in U_{m-1}^{*} \\ q^{m-1}(q+1) & \text{if } \alpha\in U_m. \end{cases}$$

*(2) When* $t=2$ *and* $m=4$,

$$\operatorname{tr} \pi(\alpha)=\begin{cases} \theta(\alpha)(1+P_1(\alpha)+Q_0(\alpha)) & \\ \qquad +\theta(\bar{\alpha})(1+P_1(\bar{\alpha})) & \text{if } \alpha\in U_{-1}^{*} \\ 0 & \text{if } \alpha\in U_0^{*} \\ 2q\theta(\alpha)Q_1(\alpha) & \text{if } \alpha\in U_1^{*} \\ 0 & \text{if } \alpha\in U_2^{*} \\ q^3\theta(\alpha)(1+Q_2(\alpha)) & \text{if } \alpha\in U_3^{*} \\ q^3(q+1) & \text{if } \alpha\in U_4. \end{cases}$$

*(3) When* $t=2$ *and* $m=3$,

$$\operatorname{tr} \pi(\alpha)=\begin{cases} \theta(\alpha)(1+Q_0(\alpha))+\theta(\bar{\alpha}) & \text{if } \alpha\in U_{-1}^{*} \\ 2\theta(\alpha)Q_1(\alpha) & \text{if } \alpha\in U_0^{*} \\ 0 & \text{if } \alpha\in U_1^{*} \\ q^2\theta(\alpha)(1+Q_2(\alpha)) & \text{if } \alpha\in U_2^{*} \\ q^2(q+1) & \text{if } \alpha\in U_3. \end{cases}$$

*(4) When* $t=2$ *and* $m=2$,

$$\operatorname{tr}\pi(\alpha)=\begin{cases}\theta(\alpha)(1+Q_1(\alpha)) & \text{if } \alpha\in U_{-1}^*\\ 0 & \text{if } \alpha\in U_0^*\\ q\theta(\alpha)(1+Q_2(\alpha)) & \text{if } \alpha\in U_1^*\\ q(q+1) & \text{if } \alpha\in U_2.\end{cases}$$

As in the case $t=1$, we set

$$B_\alpha=\omega_F^{\sigma+2}\, tr_K(\gamma(1-\bar{\alpha}\alpha^{-1})) \quad \text{for} \quad \alpha\in U_{m-\sigma-5}^* \tag{3.6}$$

We first calculate $P_\sigma(\alpha)$ for $\sigma<m-4$ and $\alpha\in U_{m-\sigma-5}^*$.

**Lemma 3.9.** *(1) For $\sigma<m-4$ and $\alpha\in U_{m-\sigma-5}^*$,*

$$P_\sigma(\alpha)=-\frac{q}{2}h(\tilde{\omega}_K^\sigma a_\alpha,\alpha)G_0(\alpha)$$

*where $a_\alpha\in\mathcal{O}_K$ is determined uniquely modulo $P_K$ by $a_\alpha^2\equiv B_\alpha^{-1}$ mod $P_K$, $h(x,\alpha)$ as in (3.2). The Gauss sum part $G_0$ is defined by*

$$G_0(\alpha)=\sum_{x\in k_F}\Psi_{(\varphi(\tilde{\omega}_K^\sigma a_\alpha),\alpha)}(1+\tilde{\omega}_K x) \tag{3.7}$$

*where*

$$\varphi(x)=\frac{n_K(x)}{1-n_K(x)}+\left(\frac{n_K(x)}{1-n_K(x)}\right)^2 \tag{3.8}$$

*and*

$$\Psi_{(x,\alpha)}(y)=\phi_\gamma\Big(x((n_K(y)-1)(1-\bar{\alpha}\alpha^{-1})+(y-1)n_K(1-\bar{\alpha}\alpha^{-1}))\Big). \tag{3.9}$$

*If $\sigma\neq m-6$, we have*

$$h(\tilde{\omega}_K^\sigma a_\alpha,\alpha)=\phi_\gamma\left(\frac{\tilde{\omega}_F^\sigma n_K(a_\alpha)}{1-\tilde{\omega}_F^\sigma n_K(a_\alpha)}(1-\bar{\alpha}\alpha^{-1})\right).$$

*(2) The Gauss sum $G_0(\alpha)$ belongs to $\mathbf{Z}[\sqrt{-1}]$ and its absolute value is $\sqrt{q}$.*

*Proof.* (1) By the argument as in Lemma 3.2, we can show

$$P_\sigma(\alpha)=\frac{q}{2}\sum_{x\in 1+P_K/1+P_K^2}\rho(\Phi(\tilde{\omega}_K^\sigma a_\alpha x,\alpha))$$

where $a_\alpha\in\mathcal{O}_K$ is defined uniquely modulo $P_K$ by $a_\alpha^2\equiv(tr_K(\tilde{\omega}_F^{\sigma+2}\gamma(1-\bar{\alpha}\alpha^{-1})))^{-1}$ mod $P_K$. For $x\in P_K^\sigma-P_K^{\sigma+1}$ and $y\in 1+P_K$, we have

$$\begin{aligned}\Phi(xy,\alpha)\equiv{}&\Phi(x,\alpha)\\ &+\frac{-\xi x(y-1)+\xi^2 n_K(x)(n_K(y)-1)}{1-\xi^2 n_K(x)}(1-\bar{\alpha}\alpha^{-1})\\ &+\frac{\xi^2 n_K(x)(-\xi x-\xi x(y-1)+\xi^2 n_K(x))(n_K(y)-1)}{(1-\xi^2 n_K(x))^2}(1-\bar{\alpha}\alpha^{-1})\end{aligned}$$

$$\text{mod } 1+P_K^{2m-2}+P_D^{2m} \quad \text{(multiplicative equivalence)}.$$

Therefore we get from Lemma 2.4 that

$$\begin{aligned}\rho(\Phi(\tilde{\omega}_K^{\sigma}a_\alpha x,\alpha)) &= \rho(\Phi(\tilde{\omega}_K^{\sigma}a_\alpha,\alpha)) \\ &\quad\times\psi_\gamma(\varphi(\tilde{\omega}_K^{\sigma}a_\alpha)(n_K(x)-1)(1-\bar{\alpha}\alpha^{-1})) \\ &\quad\times\psi_\gamma\left(\frac{\xi^2 n_K(\tilde{\omega}_K^{\sigma}a_\alpha)}{(1-\xi^2 n_K(\tilde{\omega}_K^{\sigma}a_\alpha))^2}(x-1)n_K(1-\bar{\alpha}\alpha^{-1})\right) \\ &= \rho(\Phi(\tilde{\omega}_K^{\sigma}a_\alpha,\alpha))\Psi_{(\varphi(\tilde{\omega}_K^{\sigma}a_\alpha),\alpha)}(x)\end{aligned}$$

for $x\in 1+P_K$. As in the proof of Lemma 3.2, we can show $\rho(\Phi(\tilde{\omega}_K^{\sigma}a_\alpha,\alpha))=-h(\tilde{\omega}_K^{\sigma}a_\alpha,\alpha)$. If $\sigma\neq m-6$, then $v_D(\Phi(a_\alpha,\alpha))\geq m-1$. Thus

$$h(a_\alpha,\alpha)=\psi_\gamma\left(\frac{n_K(a_\alpha)}{1-n_K(a_\alpha)}(1-\bar{\alpha}\alpha^{-1})\right).$$

(2) Since $v_K(2)=2$, we have $v_K(2\Psi_{(\varphi(\tilde{\omega}_K^{\sigma}a_\alpha),\alpha)}(x))=2m-3$ for $x\in 1+P_K$ and $\alpha\in U^*_{m-\sigma-3}$. Thus $\Psi_{(\varphi(\tilde{\omega}_K^{\sigma}a_\alpha),\alpha)}(x)^2\neq 1$ for some $x\in 1+P_K$ and $\Psi_{(\varphi(\tilde{\omega}_K^{\sigma}a_\alpha),\alpha)}(x)^4=1$ for any $x\in 1+P_K$. Hence $G_\sigma(\alpha)\in\mathbf{Z}[\sqrt{-1}]$. As for the absolute value of $G_\sigma(\alpha)$, it follows from the following standard calculation:

$$\begin{aligned}G_0(\alpha)\overline{G_0(\alpha)} &= \sum_{x,y\in 1+P_K/1+P_K^2}\Psi_{(\varphi(\tilde{\omega}_K^{\sigma}a_\alpha),\alpha)}(x)\overline{\Psi_{(\varphi(\tilde{\omega}_K^{\sigma}a_\alpha),\alpha)}(y)} \\ &= \sum_{x,y\in 1+P_K/1+P_K^2}\psi_\gamma(\varphi(\tilde{\omega}_K^{\sigma}a_\alpha)(n_K(x)-n_K(y))(1-\bar{\alpha}\alpha^{-1})) \\ &\qquad \psi_\gamma((x-y)n_K(1-\bar{\alpha}\alpha^{-1})).\end{aligned}$$

Put $x=1+\tilde{\omega}_K a$, $y=1+\tilde{\omega}_K b$ for $a,b\in k_F$, then

$$\begin{aligned}n_K(x)-n_K(y) &= tr_K(\tilde{\omega}_K)(a-b)+n_K(\tilde{\omega}_K)(a^2-b^2) \\ &= tr_K(\tilde{\omega}_K)(a-b)+n_K(\tilde{\omega}_K)(a-b)^2+2n_K(\tilde{\omega}_K)b(a-b).\end{aligned}$$

Hence we get

$$\begin{aligned}G_\sigma(\alpha)\overline{G_\sigma(\alpha)} &= \sum_{c\in k_F}\Psi_{(\varphi(\tilde{\omega}_K^{\sigma}a_\alpha),\alpha)}(1+\tilde{\omega}_K c)\sum_{b\in k_F}\psi_\gamma(2n_K(\tilde{\omega}_K cb)) \\ &= q.\end{aligned}$$

Next we calculate the term $P_{m-4}(\alpha)$.

**Lemma 3.10.** *For $\alpha\in U^*_{-1}$,*

$$P_{m-4}(\alpha)=\begin{cases}-qh(\tilde{\omega}_K^{m-4}a_\alpha,\alpha) & \text{if } \alpha\in\gamma U_0^* \\ 0 & \text{otherwise}\end{cases}$$

*where $a_\alpha\in\mathcal{O}_F$ is defined by the condition*

$$a_\alpha^2\equiv\tilde{\omega}_F^{5-m}tr_K(\gamma(1-\bar{\alpha}\alpha^{-1}))(n_K(1-\bar{\alpha}\alpha^{-1})tr_K(\gamma\omega_K))^{-2} \mod P_F.$$

*Proof.* From Corollary 2.9 (2), we have

$$P_\sigma(\alpha) = \sum_{x \in \tilde{\omega}_K^{m-4}\mathcal{O}_K^\times/1+P_K^2} \rho(\Phi(x, \alpha))$$

$$= \sum_{x \in \tilde{\omega}_K^{m-4}\mathcal{O}_K^\times/1+P_K} \rho(\Phi(x, \alpha)) \sum_{y \in 1+P_K/1+P_K^2} \Psi_{(\varphi(x),\alpha)}(y)$$

*where*

(3.10) $\Psi_{(z,\alpha)}(y) = \psi_\gamma(z((n_K(y)-1)(1-\bar{\alpha}\alpha^{-1}) + n_K(1-\bar{\alpha}\alpha^{-1})(y-1)))$.

If $\alpha \in \gamma U_1$, then $v_F(tr_K(\gamma(1-\bar{\alpha}\alpha^{-1}))) \geq 5-2m$ and

$$\Psi_{(\varphi(x),\alpha)}(y) = \psi_\gamma(\varphi(x)(n_K(1-\bar{\alpha}\alpha^{-1})(y-1))$$

for all $y \in 1+P_K$. Therefore the map $y \mapsto \Psi_{(\varphi(x),\alpha)}(y)$ is a non-trivial character of $1+P_K/1+P_K^2$. Hence $P_{m-4}(\alpha) = 0$ if $\alpha \in \gamma U_1$. Now we assume $\alpha \in \gamma U_0^*$. Then

$$\Psi_{(\varphi(x),\alpha)}(1+\tilde{\omega}_K y) = \psi(n_K(x)(\tilde{\omega}_F tr_K(\gamma(1-\bar{\alpha}\alpha^{-1}))y^2 + n_K(1-\bar{\alpha}\alpha^{-1})tr_K(\gamma\tilde{\omega}_K)y))$$

for $y \in k_F$. Since $\varphi(x)(\tilde{\omega}_F tr_K(\gamma(1-\bar{\alpha}\alpha^{-1}))) \not\equiv 0 \mod P_F$, $y \mapsto \Psi_{(\varphi(x),\alpha)}(y)$ is a non-trivial character of $1+P_K/1+P_K^2$ if and only if

$$n_K(x) \equiv \tilde{\omega}_F tr_K(\gamma(1-\bar{\alpha}\alpha^{-1}))(n_K(1-\bar{\alpha}\alpha^{-1})tr_K(\gamma\tilde{\omega}_K))^{-2} \mod P_F^{m-3}.$$

This implies our lemma.

Now we shall calculate $P_{m-3}(\alpha)$ for $\alpha \in U_{-1}^*$.

**Lemma 3.11.**

$$P_{m-3}(\alpha) = \begin{cases} -1 & \text{if } \alpha \in \gamma U_0^* \\ q-1 & \text{if } \alpha \in \gamma U_1 \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Since $v_D(\Phi(x, \alpha)) = 2m-4$, we have

$$\Phi(x, \alpha) = \psi_\gamma\left(\frac{n_K(x)}{1-n_K(x)}(1-\bar{\alpha}\alpha^{-1})\right).$$

Thus it follows from Lemma 2.6 that

$$P_\sigma(\alpha) = \sum_{x \in \tilde{\omega}_K^{m-3}\mathcal{O}_K^\times/1+P_K} \psi_\gamma\left(\frac{n_K(x)}{1-n_K(x)}(1-\bar{\alpha}\alpha^{-1})\right)$$

$$= \sum_{x \in \tilde{\omega}_F^{m-3}\mathcal{O}_K^\times/1+P_F} \psi(tr_K(\gamma(1-\bar{\alpha}\alpha^{-1}))x)$$

$$= \sum_{x \in \mathcal{O}_F/P_F} \psi(tr_K(\gamma(1-\bar{\alpha}\alpha^{-1})\tilde{\omega}_F^{m-3})x) - 1$$

$$= \begin{cases} -1 & \text{if } v_F(tr_K(\gamma(1-\bar{\alpha}\alpha^{-1}))) = -m+3 \\ q-1 & \text{if } v_F(tr_K(\gamma(1-\bar{\alpha}\alpha^{-1}))) \geq -m+4 \\ 0 & \text{otherwise.} \end{cases}$$

Since $v_F(tr_K(\gamma(1-\bar{\alpha}\alpha^{-1}))) = -m+3$ is equivalent to $\alpha \in \gamma U_0^*$, we get our lemma.

Next we treat the terms $Q_\mu(\alpha)$. Most of them can be calculated by the same way as in the case $t=1$.

**Lemma 3.12.** *(1) For $m>4$ and $\alpha \in U_{m-5}^*$,*

$$Q_0(\alpha) = \begin{cases} \frac{q}{2}(h(a'_\alpha, \alpha)G_0(a'_\alpha, \alpha) + h(a''_\alpha, \alpha)G_0(a''_\alpha, \alpha)) & \text{if } B_\alpha \bmod P_F \in k_F^0 \\ 0 & \text{otherwise.} \end{cases}$$

*where $a'_\alpha, a''_\alpha \in \mathcal{O}_K$ are defined by the condition $\frac{n_K(a'_\alpha)}{1-n_K(a'_\alpha)} \bmod P_F$, $\frac{n_K(a''_\alpha)}{1-n_K(a''_\alpha)}$ $\bmod P_F$ are solutions of $X^2+X-(B_\alpha \bmod P_F)=0$, $h(x, \alpha)$ as in (3.2) and*

$$G_0(z, \alpha) = \sum_{x \in 1+P_K/1+P_K^2} \theta\left(1+\frac{-z(x-1)}{1-n_K(z)}(1-\bar{\alpha}\alpha^{-1})\right) \tag{3.11}$$
$$\psi_\gamma\left(\frac{z(x-1)}{1-n_K(z)}(1-\bar{\alpha}\alpha^{-1})\right)\Psi_{(\varphi(z),\alpha)}(x).$$

*For $z=a'_\alpha, a''_\alpha$, $G_0(z, \alpha) \in \mathbf{Z}[\sqrt{-1}]$ and $|G_0(z, \alpha)| = \sqrt{q}$.*

*(2) For $m>2$ and $\alpha \in U_{m-3}^*$,*

$$Q_1(\alpha) = -\frac{q}{2}h(a_\alpha, \alpha).$$

*(3) For $\alpha \in U_{m-1}^*$,*

$$Q_2(\alpha) = \sum_{x \in k_F} \psi_\gamma\left(\frac{1-\bar{\alpha}\alpha^{-1}}{\tilde{\omega}_F(x^2+x+b)}\right)$$

*where $b \in k_F - k_F^0$.*

*Proof.* (1) Except the assertion about Gauss sum $G_0(z, \alpha)$, it follows from the same argument in the proof of Lemma 3.4. With respect to the Gauss sum $G_0(z, \alpha)$, we can show the statement by the usual calculation as above.

(2) It follows easily from the definition of $Q_1(\alpha)$ and our routine calculation. We remark this holds including the case $m \geq 3$.

(3) We can show this by the same way as the proof of Lemma 3.6.

Thus we have only to calculate $Q_0(\alpha)$ for $m=3, 4$ and $Q_1(\alpha)$ for $m=2$.

**Lemma 3.13.** *(1) For* $m=4$ *and* $\alpha\in U^*_{-1}$,

$$Q_0(\alpha)=\begin{cases}-\frac{q}{2}(h(a'_\alpha,\alpha)+h(a''_\alpha,\alpha)) & \text{if } \alpha\in\gamma U^*_0 \text{ and } \tilde{\omega}_F^{-1}B_\alpha \bmod P_F\in k_F^0\\ 0 & \text{otherwise}\end{cases}$$

*where* $\frac{n_K(a'_\alpha)}{1-n_K(a'_\alpha)}, \frac{n_K(a''_\alpha)}{1-n_K(a''_\alpha)}$ *are solutions of* $X^2+X-(\tilde{\omega}_F^{-1}B_\alpha \bmod P_F)=0$.

*(2) For* $m=3$ *and* $\alpha\in U^*_1$,

$$Q_0(\alpha)=\begin{cases}-1-\psi(tr_K(\gamma(1-\bar{\alpha}\alpha^{-1}))) & \text{if } \alpha\in\gamma U_0^*\\ q-1-\psi(tr_K(\gamma(1-\bar{\alpha}\alpha^{-1}))) & \text{if } \alpha\in\gamma U_1\\ 0 & \text{otherwise.}\end{cases}$$

*(3) For* $m=2$ *and* $\alpha\in U^*_3$,

$$Q_1(\alpha)=\begin{cases}-1 & \text{if } \alpha\in\gamma U_0^*\\ q-1 & \text{if } \alpha\in\gamma U_1.\end{cases}$$

*Proof.* By combining the arguments in the calculation of $P_\sigma(\alpha)$ when $\sigma\geq m-4$ and in the calculation of $Q_0(\alpha)$ when $m>4$, we get the first part of this lemma.

For $m=3$ and $\alpha\in U^*_{-1}$, we have as in the calculation of $P_{m-3}(\alpha)$ that

$$\begin{aligned}Q_0(\alpha)&=\sum_{x\in(\mathcal{O}_K^\times-(1+P_K))/1+P_K}\psi_\gamma\left(\frac{n_K(x)}{1-n_K(x)}(1-\bar{\alpha}\alpha^{-1})\right)\\&=\sum_{x\in(\mathcal{O}_F^\times-(1+P_F))/1+P_F}\psi_\gamma\left(\frac{x}{1-x}(1-\bar{\alpha}\alpha^{-1})\right)\\&=\left(\sum_{x\in\mathcal{O}_F/P_F}\psi(tr_K(\gamma(1-\bar{\alpha}\alpha^{-1}))x)\right)-(1+\psi(tr_K(\gamma(1-\bar{\alpha}\alpha^{-1}))))\end{aligned}$$

Therefore we get the second part.

For $m=2$ and $\alpha\in U^*_{-1}$,

$$Q_1(\alpha)=\sum_{x\in((1+P_K)-(1+P_K^2))/1+P_K^2}\psi_\gamma\left(\frac{\xi^2 n_K(x)}{1-\xi^2 n_K(x)}(1-\bar{\alpha}\alpha^{-1})\right)$$

since $v_K(\Phi(x,\alpha))=1=m-\frac{t}{2}$. It follows from $\xi^2\in 1+P_F^2$ and $n_K(x)\in 1+P_F$ that

$$Q_1(\alpha)=\sum_{x\in((1+P_K)-(1+P_K^2))/1+P_K^2}\psi_\gamma\left(\frac{1}{1-n_K(x)}(1-\bar{\alpha}\alpha^{-1})\right).$$

Since $n_K$ induces a bijection from $((1+P_K)-(1+P_K^2))/1+P_K^2$ to $P_F^{-1}-\mathcal{O}_F/\mathcal{O}_F$, we have

$$Q_1(\alpha)=\sum_{y\in k_F-\{0\}}\psi_\tau\Big(\frac{y}{2}(1-\bar{\alpha}\alpha^{-1})\Big)$$
$$=\begin{cases}-1 & \text{if } \alpha\in\gamma U_0^* \\ q-1 & \text{if } \alpha\in\gamma U_1.\end{cases}$$

Now we can state the character formula of $\pi$ when $t=2$.

**Theorem 3.14.** *Let $\Lambda=(K,\theta,\gamma)$ be a generic data of level $2m$ and $\pi=\pi_\Lambda$. (See section 1 for the definition of generic data and $\pi_\Lambda$.) Assume $t=t_K=2$. Take a prime element $\tilde{\omega}_K$ of $\mathcal{O}_K$ such that $\tilde{\omega}_K^2\in F$ and $\tilde{\omega}_K^2\equiv 2 \bmod P_K^2$. Set $\tilde{\omega}_F=n_K(\tilde{\omega}_K)$. Let $k_F^0$ be an index 2 subgroup of $k_F$ defined by $k_F^0=\{x^2+x|x\in k_F\}$ and take $b\in\mathcal{O}_F$ such that $b \bmod P_F\in k_F-k_F^0$.*

(1) *If $m>4$, then*

$$\operatorname{tr}\pi(\alpha)=\begin{cases}q(\theta(\alpha)+\theta(\bar{\alpha})) & \text{if } \alpha\in\gamma U_1\\ -q(h(\tilde{\omega}_K^{m-4}a_\alpha,\alpha)+h(\tilde{\omega}_K^{m-4}a_{\bar{\alpha}},\bar{\alpha})) & \\ & \text{if } \alpha\in\gamma U_0^*\\ -q^{i+1}(\theta(\alpha)h(\tilde{\omega}_K^{m-i-5}a_\alpha,\alpha)G_{m-i-5}(\alpha) & \\ \quad +\theta(\bar{\alpha})h(\tilde{\omega}_K^{m-i-5}a_{\bar{\alpha}},\bar{\alpha})G_{m-i-5}(\bar{\alpha})) & \\ & \text{if } \alpha\in U_i^* \text{ for } 0\le i\le m-5\\ -\frac{q}{2}(h(a_\alpha',\alpha)G_0(a_\alpha',\alpha)+h(a_\alpha'',\alpha)G_0(a_\alpha'',\alpha)) & \\ & \text{if } \alpha\in U_{m-5}^* \text{ and } B_\alpha \bmod P_F\in k_F^0\\ 0 & \text{if } \alpha\in U_{m-5}^* \text{ and } B_\alpha \bmod P_F\notin k_F^0\\ 0 & \text{if } \alpha\in U_{m-4}^*\\ -q^{m-2}\theta(\alpha)h(a_\alpha,\alpha) & \\ & \text{if } \alpha\in U_{m-3}^*\\ 0 & \text{if } \alpha\in U_{m-2}^*\\ q^{m-1}\theta(\alpha)\Big(1+\sum_{x\in k_F}\psi_\tau\Big(\dfrac{1-\bar{\alpha}\alpha^{-1}}{\tilde{\omega}_F(x^2+x+b)}\Big)\Big) & \\ & \text{if } \alpha\in U_{m-1}^*\\ q^{m-1}(q+1) & \text{if } \alpha\in U_m\end{cases}$$

*where $B_\alpha=\tilde{\omega}_F^{\sigma+2}tr_K(\gamma(1-\bar{\alpha}\alpha^{-1}))$ for $\alpha\in U_{m-\sigma-5}^*$, $a_\alpha\in\mathcal{O}_K$ is determined uniquely modulo $P_K$ by*

$$a_\alpha^2 \equiv \begin{cases} \tilde{\omega}_F^{5-m} tr_K(\gamma(1-\bar{\alpha}\alpha^{-1}))\,(n_K(1-\bar{\alpha}\alpha^{-1})\, tr_K(\gamma\tilde{\omega}_K))^{-2} \bmod P_K \\ \qquad \text{if } \alpha\in\gamma U_0^* \\ B_\alpha^{-1} \bmod P_K \\ \qquad \text{if } \alpha\in U_{m-\sigma-5}, \end{cases}$$

*$h(x, \alpha)$ as in* (3.2), *$a_\alpha'$, $a_\alpha'' \in \mathcal{O}_K$ are defined by the condition* $\dfrac{n_K(a_\alpha')}{1-n_K(a_\alpha')} \bmod P_F$, $\dfrac{n_K(a_\alpha'')}{1-n_K(a_\alpha'')} \bmod P_F$ *are solutions of* $X^2+X-(B_\alpha \bmod P_F)=0$ *and $G_i(z)$ as in* (3.7), (3.11).

(2) *If* $m=4$,

$$\operatorname{tr}\pi(\alpha) = \begin{cases} q(\theta(\alpha)+\theta(\bar{\alpha})) & \text{if } \alpha\in\gamma U_1 \\ -\frac{q}{2}(h(a_\alpha', \alpha)+h(a_\alpha'', \alpha)) & \\ & \text{if } \alpha\in\gamma U_0^* \text{ and } \tilde{\omega}_F^{-1}B_\alpha \bmod P_F\in k_F^0 \\ 0 & \text{if } \alpha\in\gamma U_0^* \text{ and } \tilde{\omega}_F^{-1}B_\alpha \bmod P_F\notin k_F^0 \\ 0 & \text{if } \alpha\in U_0^* \\ -q\theta(\alpha)h(a_\alpha, \alpha) & \text{if } \alpha\in U_1^* \\ 0 & \text{if } \alpha\in U_2^* \\ q^3\theta(\alpha)\left(1+\sum_{x\in k_F}\psi_T\left(\dfrac{1-\bar{\alpha}\alpha^{-1}}{\tilde{\omega}_F(x^2+x+b)}\right)\right) & \\ & \text{if } \alpha\in U_3^* \\ q^3(q+1) & \text{if } \alpha\in U_4 \end{cases}$$

*where $a_\alpha'$, $a_\alpha''\in\mathcal{O}_K$ are defined by the condition* $\dfrac{n_K(a_\alpha')}{1-n_K(a_\alpha')} \bmod P_F$ *and* $\dfrac{n_K(a_\alpha'')}{1-n_K(a_\alpha'')} \bmod P_F$ *are solutions of* $X^2+X-(\tilde{\omega}_F^{-1}B_\alpha \bmod P_F)=0$ *and other notations are as in* (1).

(3) *If* $m=3$,

$$\operatorname{tr}\pi(\alpha) = \begin{cases} q\theta(\alpha) & \text{if } \alpha\in\gamma U_1 \\ 0 & \text{if } \alpha\in\gamma U_0^* \\ -q\theta(\alpha)h(a_\alpha, \alpha) & \text{if } \alpha\in U_0^* \\ 0 & \text{if } \alpha\in U_1^* \\ q^2\theta(\alpha)\left(1+\sum_{x\in k_F}\psi_T\left(\dfrac{1-\bar{\alpha}\alpha^{-1}}{\tilde{\omega}_F(x^2+x+b)}\right)\right) & \text{if } \alpha\in U_2^* \\ q^2(q+1) & \text{if } \alpha\in U_3. \end{cases}$$

(4) *If* $m=2$,

$$\operatorname{tr}\pi(\alpha)=\begin{cases} q\theta(\alpha) & \text{if } \alpha\in\gamma U_1 \\ 0 & \text{if } \alpha\in\gamma U_0^*\cup U_0^* \\ q\theta(\alpha)\left(1+\sum_{x\in k_F}\psi_\gamma\left(\dfrac{1-\bar{\alpha}\alpha^{-1}}{\tilde{\omega}_F(x^2+x+\mathrm{b})}\right)\right) & \text{if } \alpha\in U_1^* \\ q(q+1) & \text{if } \alpha\in U_2. \end{cases}$$

## 4. Character formula outside the conjugacy class of $K^\times$

We use the same notations as above. We note we fix a generic data $\Lambda(K, \theta, \gamma)$ and denote simply $\pi_\Lambda$ by $\pi$. As in section 3, we assume $F$ is unramified over $\mathbf{Q}_2$. First we define a kind of distance between $K^\times$ and other elliptic tori. We denote by $O(X)$ the conjugacy class of $X$ in $D^\times$.

**Definition 4.1.** For $x, y\in D$ and $X, Y\subset D$, we set

$$d(x, y)=v_D(x-y)-\min(v_D(x), v_D(y))$$

and $d(X, Y)=\min\{d(x, y)\,|\,x\in X, y\in Y\}$. Let $E$ be a quadratic extension of $F$. We define

$$d(O(E))=d(O(E^\times-F^\times(1+P_E)), K^\times-F^\times(1+P_E))$$

and

$$d(E)=d(E^\times, K^\times-F^\times(1+P_E)).$$

It is easy to see that if $E/F$ is ramified,

(4.1) $$d(O(E))=\min\{d(x^g, y)\,|\,v_E(x)=1, v_K(y)=1, g\in D^\times\}$$

and if $E/F$ is unramified,

(4.2) $$d(O(E))=\min\{d(x^g, y)\,|\,v_E(x)=0, v_K(y)=0, g\in D^\times\}.$$

**Lemma 4.2.** *Let $a, b\in K$ and $E$ is a quadratic extension of $F$ in $D$.*

(1) $d(a+\xi b, K)=v_K(b)+t-v_D(a+\xi b)$. $(t=t_K)$

(2) *If $E$ is unramified,* $d(E)=0$.

(3) *If* $O(E^\times)\neq O(K^\times)$, $d(E)\leq 2t$.

*Proof.* (1) Since $a+\xi b=a+b+(\xi-1)b$, $a+b$ is one of the closest elements of $K^\times$ to $a+\xi b$. Thus $d(a+\xi b, K)=v_D(\xi-1)+v_K(b)-v_D(a+\xi b)$.

(2) When $E$ is unramified, $\mathcal{O}_E\neq F^\times(1+P_E)$. For $x\in\mathcal{O}_E-F^\times(1+P_E)$, $d(x, K)=0$.

(3) It suffices to show that if $v_D(a+\xi b)=1$ and $d(a+\xi b, K)>2t$, there exists $x\in K$ such that $(1+\xi x)^{-1}(a+\xi b)(1+\xi x)\in K$. By the direct calculation, we have

$$(1+\xi x)^{-1}(a+\xi b)(1+\xi x)=\frac{(a-\xi^2 n_K(b)\bar{a}+\xi^2(x\bar{b}-\bar{x}b))}{1-\xi^2 n_K(\beta)}$$

$$+\frac{\xi(-\xi^2\bar{b}x^2+(\bar{a}-a)x+\bar{b})}{1-\xi^2 n_K(\beta)}.$$

It can belong to $K$ if and only if $(\bar{a}-a)^2-4\xi^2 n_K(b)\in K^{\times 2}$. The assumption $d(a+\xi b, K)>2t$ and $v_K(a+\xi b)=1$ implies $v_K(b)-v_K(a)>t$ and $v_K(a)=1$. Then $v_K(\bar{a}-a)=t+1$ and $v_K(n_K(b))>2(t+1)$. Therefore $(\bar{a}-a)^2-4\xi^2 n_K(b)\in K^{\times 2}(1+P_K^{t+1})$. Since $1+P_K^{t+1}\subset K^{\times 2}$, we get our lemma.

The support of $\chi_\pi$ is relatively small on $E^\times$. We may assume $d(O(E))=d(E)$, if necessary, replacing it with its conjugate.

**Lemma 4.3.** *Let $E$ be a quadratic extension of $F$ satisfying $d(E)=d(O(E))$. Set $d=d(E)$.*

(1) *If $E/F$ is unramified, $\chi_\pi(x)=0$ for $x\notin F^\times(1+P_E^m)$.*

(2) *If $E/F$ is ramified and $d\neq 0$, $\chi_\pi(x)=0$ for $x\notin F^\times(1+P_E^{2m-2d})$.*

(3) *If $E/F$ is ramified and $d=0$, $\chi_\pi(x)=0$ for $x\notin F^\times(1+P_E^{2m-1})$.*

*Proof.* By the definition of $\pi$,

$$\chi_\pi(x)=\sum_{g\in D^\times/K^\times(1+P_D^m)}\rho(g^{-1}xg).$$

It follows from the definition of $d(E)$ that $O(x)$ does not intersect $K^\times(1+P_D^m)$ if $x\notin F^\times(1+(P_D^{m-d}\cap E))$. Thus we may assume $m-d\le v_E(x-1)$. Set $r=v_D(x-1)$. Then we have

$$\chi_\pi(x)=\frac{1}{q^{[(2m-r)/2]}}\sum_{g\in D^\times/K^\times(1+P_D^m)}\ \sum_{k\in P_D^{[(2m+1-r)/2]}/P_D^{2m-r}}\rho((1+k)^{-1}g^{-1}xg(1+k)).$$

Set $g^{-1}xg=1+h$. Since $(1+k)^{-1}g^{-1}xg\ (1+k)\equiv 1+hk-kh \bmod P_D^{2m}$, $\rho((1+k)^{-1}g^{-1}xg(1+k))=\psi(\mathrm{Tr}(\gamma h-h\gamma)k)$. Moreover $h\in P_D^r$ and $h\notin P_K^r+P_D^{r+d+1}$. Thus the map $k\mapsto\psi(\mathrm{Tr}(\gamma h-h\gamma)k)$ is a non-trivial character of $P_D^{[(2m+1-r)/2]}/P_D^{2m-r}$ if $r<2m-1$. Therefore $\chi_\pi(x)=0$.

**Corollary 4.4.** *If $E$ is unramified quadratic extension,*

$$\chi_\pi(x)=\begin{cases}0 & x\notin F^\times(1+P_E^m)\\ \theta(c) & x=c(1+k)\in F^\times(1+P_E^m).\end{cases}\tag{4.3}$$

When $E$ is ramified, we have only to calculate $\chi_\pi$ on $F^\times(1+P_E^{2m-2d})-F^\times(1+P_E^{2m-d})$ and $F^\times(1+P_E^{2m-1})-F^\times(1+P_E^{2m})$ when $d=0$.

**Lemma 4.5.** *Let $E$ be a ramified quadratic extension of $F$ and $x\in F^\times(1+P_E^r)-F^\times(1+P_E^{r+1})$. Then $x$ can be written in the form $x=c(1+a)(1+b)$ where $c\in F^\times$, $a\in P_K^r-P_K^{r+1}$, $b\in P_D^{r+d}$. Here we set $r=0$ if $x\in E^\times-F^\times(1+P_E)$.*

(1) *If $r\ge 2m-d$, then $\chi_\pi(x)=\theta(c)\chi_\pi(1+a)$.*

(2) *If $2m-2d\le r<2m-d$, then $\chi_\pi(x)=\theta(c)\psi_\gamma(b)\chi_\pi(1+a)$.*

(3) *If* $d=0$ *and* $r=2m-1$, *then* $x$ *can be written in the form* $x=c(1+a+\xi a(1+b))$ *where* $c\in F^{\times}$, $a\in P_K^{2m-t-1}-P_K^{2m-t}$, $b\in P_K^t$ *and*

$$\chi_\pi(x)=\theta(c)\psi_r(a)\left(1+\sum_{y\in k_F}\psi_r\left(\frac{\bar{a}\bar{b}-ab+(\overline{\tilde{\omega}_K^t a}-\tilde{\omega}_K^t a)y}{\tilde{\omega}_K^t(y^2+y+\delta)}\right)\right)$$

*where* $\delta=\dfrac{\xi^2-1}{\tilde{\omega}_F^t}$.

*Proof.* When $r\geq 2m-d$, $x=c(1+a)$ mod $\mathrm{Ker}\pi$. Thus $\chi_\pi(x)=\theta(c)\chi_\pi(1+a)$. Next we treat the case $2m-2d\leq r<m-d$. As in the proof of Lemma 4.3, we can show

$$\chi_\pi(x)=\theta(c)\sum_{g\in D^\times/K^\times(1+P_D^{2m-r-d})}\rho(1+g^{-1}bg)$$
$$\sum_{h\in K^\times(1+P_D^{2m-r-d})/K^\times(1+P_D^m)}\rho(1+h^{-1}g^{-1}agh)$$

and the last sum is proportional to

$$\sum_{h\in K^\times(1+P_D^{2m-r-d})/K^\times(1+P_D^m)}\ \sum_{k\in P_D^d/P_D^{2d}}\rho((1+k)^{-1}g^{-1}xg(1+k)).$$

Put $a'=(gh)^{-1}agh$. If $r\geq m$, we have

$$\sum_{k\in P_D^d/P_D^{2d}}\rho((1+k)^{-1}g^{-1}xg(1+k))=\sum_{k\in P_D^d/P_D^{2d}}\psi(\mathrm{Tr}(\gamma a'-a'\gamma)k).$$

It follows from the same argument as in the proof of Lemma 4.3 that this sum is 0 if $gh\notin K^\times(1+P_D^{2m-r-d})$. It implies

$$\chi_\pi(x)=\theta(c)\psi(\mathrm{Tr}\gamma b)\sum_{h\in K^\times(1+P_D^{2m-r-d})/K^\times(1+P_D^m)}\rho(h^{-1}(1+a)h).$$

On the other hand,

$$\chi_\pi(1+a)=\sum_{g\in D^\times/K^\times(1+P_D^{2m-r-d})}\ \sum_{h\in K^\times(1+P_D^{2m-r-d})/K^\times(1+P_D^m)}\rho(1+h^{-1}g^{-1}agh)$$
$$=\sum_{h\in K^\times(1+P_D^{2m-r-d})/K^\times(1+P_D^m)}\rho(1+h^{-1}ah)$$

Therefore we get $\chi_\pi(x)=\theta(c)\psi(\mathrm{Tr}(\gamma b))\chi_\pi(1+a)$. Now we assume $r<m$. Since $(1+k)^{-1}(1+a')(1+k)=(1+a')(1+(1+a')^{-1}(a'k-ka'))$ and $1+(1+a')^{-1}(a'k-ka')\in 1+P_D^m$, $\rho((1+k)^{-1}(1+a')(1+k))=0$ unless $a'\equiv a$ mod $P_D^m$. When $a'-a\in P_D^m$,

$$\sum_{k\in P_D^m/P_D^{2m}}\rho((1+k)^{-1}(1+a')(1+k))$$
$$=\sum_{k\in P_D^m/P_D^{2m}}\psi_r((1+a')^{-1}a'k-(1+a')^{-1}ka')$$

$$= \sum_{k \in P_D^m / P_D^{2m}} \psi(\mathrm{Tr}(\gamma a' - a'\gamma) k)$$

$$= 0$$

if $gh \notin K^\times (1+P_D^{2m-r-d})$. Therefore we can show $\chi_\pi(x) = \theta(c)\psi(\mathrm{Tr}(\gamma b))\chi_\pi(1+a)$ by the same way for the case $r \geq m$. Finally we assume $d=0$ and $r=2m-1$. It follows from Lemma 2.2 that the set $\{1\} \cup \{1+\xi\beta | \beta \in 1+P_K^t/1+P_K^{t+1}\}$ gives a complete system of representatives of $D^\times/K^\times(1+P_D)$. It implies $x \in F^\times(1+P_E^{2m-1})$ can be written in the form $x = c(1+a+\xi a(1+b))$ where $c \in F^\times$, $a \in P_K^{2m-t-1} - P_K^{2m-t}$, $b \in P_K^t$ and for this $x$

$$\chi_\pi(x) = q^{m-1}\theta(c)\left(1 + \sum_{\beta \in 1+P_K^t/1+P_K^{t+1}} \rho((1+\xi\beta)^{-1}(1+a+\xi a(1+b))(1+\xi\beta))\right).$$

Since

$$\begin{aligned}(1+\xi\beta)^{-1}(1+a+\xi a(1+b))(1+\xi\beta) = 1 &+ a + \frac{\xi^2 n_K(\beta)}{1-\xi^2 n_K(\beta)}(a-\bar{a}) \\ &+ \frac{\xi^2(\overline{ab}\beta - ab\bar{\beta})}{1-\xi^2 n_K(\beta)} \\ &+ (\xi K \text{ part})\end{aligned}$$

and

$$\frac{\xi^2 n_K(\beta)}{1-\xi^2 n_K(\beta)}(a-\bar{a}) + \frac{\xi^2(\overline{ab}\beta - ab\bar{\beta})}{1-\xi^2 n_K(\beta)} \equiv \frac{\overline{ab} - ab + \bar{a}(\beta-1) - a(\bar{\beta}-1)}{1-\xi^2 n_K(\beta)} \mod P_K^{2m-t},$$

we have

$$\rho((1+\xi\beta)^{-1}(1+a+\xi ab)(1+\xi\beta)) = \psi_r\left(a + \frac{\overline{ab} - ab + \bar{a}(\beta-1) - a(\bar{\beta}-1)}{1-\xi^2 n_K(\beta)}\right).$$

Therefore by replacing $\beta = 1 + \tilde{\omega}_K^t y$, we get

$$\chi_\pi(x) = q^{m-1}\theta(c)\left(1 + \psi_r(a) \sum_{y \in k_F} \psi_r\left(\frac{\overline{ab} - ab + (\bar{a}-a)\tilde{\omega}_K^t y}{\tilde{\omega}_F^t (y^2+y+\delta)}\right)\right).$$

Now we can state the main result of this section.

**Theorem 4.6.** *Let $\Lambda = (K, \theta, \gamma)$ be a generic data of level $2m$ (cf. Definition 1.1), $\pi = \pi_\Lambda$ the irreducible representation of $D^\times$ associated with $\Lambda$ (cf. Proposition 1.2). Set $t = t_K$ take prime elements $\tilde{\omega}_F$ and $\tilde{\omega}_K$ such that $tr_K(\tilde{\omega}_K) \equiv n_K(\tilde{\omega}_K) = \tilde{\omega}_F \bmod P_F^2$ when $t=1$ and $\tilde{\omega}_F^2 = -\tilde{\omega}_F \equiv 2 \bmod P_F^2$* when t=2. *Let $E$ be a quadratic extension of $F$ in $D$ satisfying $d(O(E)) = d(E)$ (cf. Definition 4.1) and set $d = d(E)$.*

(1) *If $E/F$ is unramified,*

$$\chi_\pi(x) = \begin{cases} 0 & x \notin F^\times(1+P_E^m) \\ q^m\theta(c) & x=c(1+a),\, c\in F^\times,\, a\in P_E^m. \end{cases}$$

(2) *If E/F is ramified and* $d>0$,

$$\chi_\pi(x) = \begin{cases} 0 & x \notin F^\times(1+P_E^{2m-2d}) \\ \theta(c)\psi_\gamma(b)\chi_\pi(1+a) & x=c(1+a)(1+b),\, c\in F^\times, \\ & \quad a\in P_E^r - P_E^{r+1},\, b\in P_D^{r+d} \\ & \quad \text{for } 2m-2d\leq r<2m-d \\ \theta(c)\chi_\pi(1+a) & x=c(1+a)(1+b),\, c\in F^\times, \\ & \quad a\in P_E^r,\, b\in P_D^{r+d} \\ & \quad \text{for } 2m-d\leq r \end{cases}$$

*where* $\chi_\pi(x)$ *for* $x\in K^\times$ *as in Theorem* 3.7 *and Theorem* 3.14.

(3) *If E is ramified and* $d=0$,

$$\chi_\pi(x) = \begin{cases} 0 & x \notin F^\times(1+P_E^{2m-1}) \\ \theta(c)\left(1+\psi_\gamma(a)\sum_{y\in k_F}\psi_\gamma\left(\dfrac{\overline{ab}-ab+(\overline{\tilde{\omega}_K^t a}-\tilde{\omega}_K^t a)y}{\tilde{\omega}_F^t(y^2+y+\delta)}\right)\right) & \\ x=c(1+a+\xi a(1+b)),\, c\in F^\times,\, a\in P_K^{2m-t-1}-P_K^{2m-t},\, b\in P_K^t & \\ q^m\theta(c) & x=c(1+a),\, c\in F^\times,\, a\in P_E^{2m} \end{cases}$$

*where* $\delta=\dfrac{\xi^2-1}{\tilde{\omega}_F^t}$.

***Remark***. The above theorem holds without the assumption $F$ is unramified over $\mathbf{Q}_2$. But we give the character formula of $\pi$ on $K^\times$ only when $F$ is unramified over $\mathbf{Q}_2$. Therefore we state it under the assumption $F/\mathbf{Q}_2$ is unramified.

## Appendix A. Calculation for general case

Here we show how to compute $P_\sigma(\alpha)$ and $Q_u(\alpha)$ in Corollary 2.9 without the assumption $F/\mathbf{Q}_2$ unramified. This amounts to the character formula for $\pi=\pi_\Lambda$. We use the same notation as in Section 3. Since we have already calculated the character when $t=t_K=1$, we may and do assume $t=t_K>1$. We devide the calculation into 7 parts according to Corollary 2.9.

We start with the calculation of $P_\sigma(\alpha)$.

**Proposition A. 1.** *Let the notation be as above and assume* $0<\sigma<m-2t$ *and* $\alpha\in U^*_{m-\sigma-2t-1}$.

(1) *When t is odd,*

$$P_\sigma(\alpha) = -\frac{q^{(t+1)/2}}{2}h(a_\alpha,\alpha)$$

*where* $a_\alpha \in P_K^q$ *is defined uniquely modulo* $P_K^{q+(t+1)/2}$ *by the condition that* $\Psi_{(\varphi(a_\alpha),\alpha)}$ *is trivial on* $1+P_K^{(t+1)/2}$ *and* $h$, $\varphi$, $\Psi$ *are as in* (3.2), (3.8), (3.9) *respectively.*

(2) *When t is even,*

$$P_\sigma(\alpha) = -\frac{q^{t/2}}{2} h(a_\alpha, \alpha) G_\sigma(\alpha)$$

*where* $a_\alpha \in P_K^q$ *is defined uniquely modulo* $P_K^{q+t/2}$ *by the condition that* $\Psi_{(\varphi(a_\alpha),\alpha)}$ *is trivial on* $1+P_K^{(t+2)/2}$ *and*

(A.1) $$G_\sigma(\alpha) = \sum_{x \in 1+P_K^{t/2}/1+P_K^{(t+2)/2}} \Psi_{(\varphi(a_\alpha),\alpha)}(x).$$

*The absolute value of* $G_\sigma(\alpha)$ is $q^{1/2}$ *and* $G_\sigma(\alpha)$ *belongs to* $\mathbf{Z}[\sqrt{-1}]$.

*Proof.* From the same argument in the proof of Lemma 3.2 and Lemma 3.9, we have

$$P_\sigma(\alpha) = \frac{q}{2} \sum_{x \in a_0(1+P_K)/1+P_K^{[(t+2)/2]}} \rho(\Phi(x,\alpha)) \sum_{y \in 1+P_K^{[(t+2)/2]}/1+P_K^t} \Psi_{(\varphi(x),\alpha)}(y)$$

where $a_0 \in P_K^q$ is determined uniquely mod $P_K^{q+1}$ such that the map

$$y \mapsto \psi_T(n_K(a_0)(n_K(y)-1)(1-\bar{\alpha}\alpha^{-1}))$$

is a trivial character of $1+P_K^t/\mathcal{O}_K^1(1+P_K^{t+1})$ and $\Psi$ is as in (3.9).

For $x \in n_K(a_0)(1+P_F)$, the map $y \mapsto \Psi_{(\varphi(x),\alpha)}(y)$ is a character of $1+P_K^{(t+1)/2}/1+P_K^t$.
Therefore

$$\sum_{y \in 1+P_K^{(t+1)/2}/1+P_K^t} \Psi_{(\varphi(x),\alpha)}(y) = 0$$

unless $\Psi_{(\varphi(x),\alpha}$ is a trivial character of $1+P_K^{(t+1)/2}$.

**Lemma A.2.** *There exists a unique element* $x \in n_K(a_0)(1+P_F)/1+P_F^{[(t+1)/2]}$ *such that* $\Psi_{(x,\alpha)}$ *is a trivial character of* $1+P_K^{[(t+2)/2]}$.

*Proof.* For $y \in 1+P_K^{t-1}$ and $x = n_K(a_0)+x_1$, $x_1 \in P_F^{q+1}$,

$$\begin{aligned}\Psi_{(x,\alpha)}(y) &= \psi_T\Big((n_K(a_0)+x_1)((n_K(y)-1)(1-\bar{\alpha}\alpha^{-1}) + (y-1)n_K(1-\bar{\alpha}\alpha^{-1}))\Big) \\ &= \psi_T\Big(n_K(a_0)((n_K(y)-1)(1-\bar{\alpha}\alpha^{-1}) + (y-1)n_K(1-\bar{\alpha}\alpha^{-1}))\Big) \\ &\quad \times \psi_T(x_1(n_K(y)-1)(1-\bar{\alpha}\alpha^{-1})).\end{aligned}$$

Since $x_1 \mapsto (y \mapsto \psi_T(x_1(n_K(y)-1)(1-\bar{\alpha}\alpha^{-1})))$ induces a bijection from $P_K^{q+1}/P_F^{q+2}$ to $(1+P_K^{t-1}/1+P_K^t)^\wedge$, there exists a unique element $x_1 \in P_F^{q+1}/P_F^{q+2}$ such that $\Psi_{(n_K(a_0)+x_1,\alpha)}$ is trivial on $1+P_K^{t-1}$. By repeating this process for $y \in 1+P_K^i$, $i = t-2, \cdot \cdot \cdot, [(t+2)/2]$, we can show that there exists a unique element $x \in$

$P_F^{\sigma+1}/P_F^{\sigma+[(t+1)/2]}$ such that $\Psi_{(n_K(a_0)+x,\,\alpha)}$ is trivial on $1+P_K^{[(t+2)/2]}$.

Since $\varphi$ induces a bijection from $a_0(1+P_K)/1+P_K^{[(t+1)/2]}$ to $n_K(a_0)(1+P_F)/1+P_F^{[(t+1)/2]}$, it follows from the above lemma that

$$(A.2)\qquad P_\sigma(\alpha)=\begin{cases}\dfrac{q^{(t+1)/2}}{2}\rho(\Phi(a_\alpha,\alpha)) & \text{if } t \text{ odd}\\ \dfrac{q^{t/2}}{2}\displaystyle\sum_{x\in(1+P_K^{t/2})/1+P_K^{t/2+1}}\rho(\Phi(a_\alpha x,\alpha)) & \text{if } t \text{ even}\end{cases}$$

where $a_\alpha\in P_K^\sigma$ is defined uniquely modulo $P_K^{\sigma+[(t+1)/2]}$ by the condition that $\Psi_{(\varphi(a_\alpha),\,\alpha)}$ is trivial on $1+P_K^{[(t+2)/2]}$.

The rest of the lemma is also proved by the same way as in Lemma 3.2 and Lemma 3.9.

**Proposition A.3.** *Let the notation be as above and assume $m-2t\le\sigma<m-t$ and $\alpha\in U_{-1}^*$.*

(1) *If $\sigma\ge m-\frac{3}{2}t$,*

$$P_\sigma(\alpha)=\begin{cases}-q^{m-\sigma-t-1} & \textit{if } \alpha\in\gamma U_{m-\sigma-t-1}^*\\ q^{m-\sigma-t-1}(q-1) & \textit{if } \alpha\in\gamma U_{m-\sigma-t}\\ 0 & \textit{otherwise}.\end{cases}$$

(2) *If $t$ odd and $\sigma<m-(3t+1)/2$,*

$$P_\sigma(\alpha)=\begin{cases}-q^{(t-1)/2}h(a_\alpha,\alpha)G_\sigma(\alpha) & \textit{if } \alpha\in\gamma U_{\sigma-m+2t}^*\\ 0 & \textit{otherwise}\end{cases}$$

*where $a_\alpha\in P_K^\sigma$ is defined uniquely modulo $P_K^{\sigma+(m-\sigma-(3t-1)/2)}$ by the condition $\Psi_{(\varphi(a_\alpha),\,\alpha)}$ is trivial on $1+P_K^{m-\sigma-(3t-1)/2}$ and*

$$(A.3)\qquad G_\sigma(\alpha)=\sum_{x\in 1+P_K^{m-\sigma-(3t+1)/2}/1+P_K^{m-\sigma-(3t-1)/2}}\theta\Big(1+\frac{-a_\alpha(x-1)}{1-n_K(a_\alpha)}(1-\bar\alpha\alpha^{-1})\Big)\,\psi_\tau\Big(\frac{a_\alpha(x-1)}{1-n_K(a_\alpha)}(1-\bar\alpha\alpha^{-1})\Big)\Psi_{(\varphi(a_\alpha),\,\alpha)}(x).$$

*$G_\sigma(\alpha)$ satisfies $G_\sigma(\alpha)\in\mathbf{Z}[\sqrt{-1}]$ and $|G_\sigma(\alpha)|=\sqrt{q}$.*

(3) *If $t$ odd and $\sigma<m-(3t+1)/2$,*

$$P_\sigma(\alpha)=\begin{cases}q^{(t-1)/2}(G_\sigma(\alpha)-1) & \textit{if } \alpha\in\gamma U_{\sigma-m+2t}^*\\ 0 & \textit{otherwise}.\end{cases}$$

*where*

$$(A.4)\qquad G_\sigma(\alpha)=\sum_{x\in k_F}\theta\big(1-(1-\bar\alpha\alpha^{-1})\tilde\omega_K^{m-(3t+1)/2}x\big)\,\psi_\tau\Big((1-\bar\alpha\alpha^{-1})\tilde\omega_K^{m-(3t+1)/2}x\Big)$$

$$\psi_\gamma\Big(((1-\bar{\alpha}\alpha^{-1})+n_K(1-\bar{\alpha}\alpha^{-1}))n_K(\bar{\omega}_K^{(m-(3t+1)/2)})x^2)\Big)$$

*and* $G_\sigma(\alpha)\in\mathbf{Z}[\sqrt{-1}]$, $|G_\sigma(\alpha)|=\sqrt{q}$.

(4) *If* $t$ *even and* $\sigma<m-\frac{3}{2}t$,

$$P_\sigma(\alpha)=\begin{cases} -q^{t/2}h(a_\alpha,\alpha) & \text{if } \alpha\in\gamma U^*_{\sigma-m+2t} \\ 0 & \text{otherwise} \end{cases}$$

*where* $a_\alpha\in P_K^\sigma$ *is defined uniquely modulo* $P_K^{\sigma+(m-\sigma-3t/2)}$ *by the condition that* $\Psi_{(\varphi(a_\alpha),\alpha)}$ *is trivial on* $1+P_K^{m-\sigma-3t/2}$.

*Proof.* In this case, $v_D(\Phi(x,\alpha))=2\sigma+t$ for $\alpha\in U^*_{-1}$ and $x\in P_K^\sigma-P_K^{\sigma+1}$. First we assume $\sigma\geq m-\frac{3}{2}t$. Then

$$\Phi(x,\alpha)=\psi_\gamma\Big(\frac{n_K(x)}{1-n_K(x)}(1-\bar{\alpha}\alpha^{-1})\Big)$$

for $\alpha\in U^*_{-1}$ and $x\in P_K^\sigma-P_K^{\sigma+1}$. Thus it follows from Lemma 2.6 and the argument in the proof of Lemma 3.11 that

$$\begin{aligned} P_\sigma(\alpha) &= \sum_{x\in\bar{\omega}_K^\sigma\mathcal{O}_K^\times/1+P_K^{m-\sigma-t}} \psi_\gamma\Big(\frac{n_K(x)}{1-n_K(x)}(1-\bar{\alpha}\alpha^{-1})\Big) \\ &= \sum_{x\in\mathcal{O}_F/P_F^{m-\sigma-t}} \psi_\gamma(tr_K(\gamma(1-\bar{\alpha}\alpha^{-1})\bar{\omega}_F^\sigma)x) \\ &\quad - \sum_{x\in P_F/P_F^{m-\sigma-t}} \psi_\gamma(tr_K(\gamma(1-\bar{\alpha}\alpha^{-1})\bar{\omega}_F^{\sigma+1})x) \end{aligned}$$

$$=\begin{cases} -q^{m-\sigma-t-1} & \text{if } v_F(tr_K(\gamma(1-\bar{\alpha}\alpha^{-1})))=-\sigma \\ q^{m-\sigma-t-1}(q-1) & \text{if } v_F(tr_K(\gamma(1-\bar{\alpha}\alpha^{-1})))\geq 1-\sigma \\ 0 & \text{otherwise.} \end{cases}$$

Since $v_F(tr_K(\gamma(1-\bar{\alpha}\alpha^{-1})))=-\sigma$ is equivalent to $\alpha\in\gamma U^*_{m-\sigma-t-1}$, we get the first part of the lemma.

Next we treat the case $t$ odd and $\sigma<m-3t/2$. By the same argument in the case $0<\sigma<m-2t$, we get

$$P_\sigma(\alpha)=\sum_{x\in\bar{\omega}_K^\sigma\mathcal{O}_K^\times/1+P_K^{2m-2\sigma-3t}} \rho(\Phi(x,\alpha))$$

$$\sum_{y\in P_F^{2m-2\sigma-3t}/P_F^{m-\sigma-t}} \psi_K(\gamma\varphi(x)(1-\bar{\alpha}\alpha^{-1})y).$$

Hence

$$P_\sigma(\alpha)=\begin{cases} q^{\sigma-m+2t}\displaystyle\sum_{x\in\bar{\omega}_K^\sigma\mathcal{O}_K^\times/1+P_K^{2m-2\sigma-3t}} \rho(\Phi(x,\alpha)) & \text{if } \alpha\in\gamma U_{\sigma-m+2t} \\ 0 & \text{otherwise} \end{cases}$$

since $v_F(tr_K(\gamma(1-\bar{\alpha}\alpha^{-1})))\geq\sigma+3t-2m+1$ is equivalent to $\alpha\in\gamma U_{\sigma-m+2t}$. Now assume $v_F(tr_K(\gamma(1-\bar{\alpha}\alpha^{-1})))\geq\sigma+3t-2m+1$, then

$$P_\sigma(\alpha)=q^{\sigma-m+2t}\sum_{x\in\tilde{\omega}_K^\sigma\mathcal{O}_K^\times/1+P_K^{m-\sigma-(3t-1)/2}}\rho(\Phi(x,\alpha))$$

$$\sum_{y\in 1+P_K^{m-\sigma-(3t-1)/2}/1+P_K^{2m-2\sigma-3t/2}}\Psi_{(\varphi(x),\alpha)}(y).$$

Let us assume $\sigma\neq m-(3t+1)/2$. Since $\Psi_{(\varphi(x),\alpha)}(y)=\psi_\gamma(\varphi(x)n_K(1-\bar{\alpha}\alpha^{-1})(y-1))$ for $y\in 1+P_K^{2m-2\sigma-3t-1}$ and $\alpha\in\gamma U_{\sigma-m+2t+1}$, the map $y\mapsto\Psi_{(\varphi(x),\alpha)}(y)$ is a non-trivial character of $1+P_K^{m-\sigma-(3t-1)/2}/1+P_K^{2m-2\sigma-3t/2}$ when $\alpha\in\gamma U_{\sigma-m+2t+1}$. It follows $P_\sigma(\alpha)=0$ unless $\alpha\in\gamma U^*_{\sigma-m+2t}$. Therefore we may assume $\alpha\in\gamma U^*_{\sigma-m+2t}$. Then

$$\Psi_{(\varphi(x),\alpha)}(1+\tilde{\omega}_K^{2m-2\sigma-3t-1}y)=\psi(\varphi(x)(n_K(\tilde{\omega}_K^{2m-2\sigma-3t-1})tr_K(\gamma(1-\bar{\alpha}\alpha^{-1}))y^2)$$
$$\times\psi(\varphi(x)(n_K(1-\bar{\alpha}\alpha^{-1})tr_K(\gamma\tilde{\omega}_K^{2m-2\sigma-3t-1})y))$$

for $y\in k_F$. Since $\varphi(x)n_K(\tilde{\omega}_K^{2m-2\sigma-3t-1})tr_K(\gamma(1-\bar{\alpha}\alpha^{-1}))\not\equiv 0 \bmod P_F$, there exists a unique $a_0\in P_K^\sigma-P_K^{\sigma+1} \bmod P_K^{\sigma+1}$ satisfying $\Psi_{(\varphi(a_0),\alpha)}(y)=1$ for all $y\in 1+P_K^{2m-2\sigma-3t-1}$. By applying the argument in the proof of Lemma A.2 to this case, we have

$$P_\sigma(\alpha)=q^{\sigma-m+2t}\,q^{m-\sigma-(3t-1)/2}\sum_{x\in 1+P_K^{m-\sigma-(3t+1)/2}/1+P_K^{m-\sigma-(3t-1)/2}}\rho(\Phi(a_\alpha x,\alpha))$$

where $a_\alpha\in P_K^\sigma$ is defined uniquely modulo $P_K^{\sigma+(m-\sigma-(3t-1)/2)}$ by the condition that $\Psi_{(\varphi(a_\alpha),\alpha)}$ is trivial on $1+P_K^{m-\sigma-(3t-1)/2}$. Thus we get

$$P_\sigma(\alpha)=\begin{cases} q^{(t-1)/2}\rho(\Phi(a_\alpha,\alpha))G_0(\alpha) & \text{if } \alpha\in\gamma U^*_{\sigma-m+2t}\\ 0 & \text{otherwise.}\end{cases}$$

In this expression, we can prove $G_0(\alpha)\in\mathbf{Z}[\sqrt{-1}]$ and $|G_0(\alpha)|=\sqrt{q}$ by the same way as above and we can show $\rho(\Phi(a_\alpha,\alpha))=-h(a_\alpha,\alpha)$. (See (3.2) for the definition of $h(a_\alpha,\alpha)$.)

When $\sigma=m-(3t+1)/2$, it is proved by the same way as Lemma 3.3.

When $t$ is even and $\sigma<m-\frac{3}{2}t$, the calculation of $P_\sigma(\alpha)$ for $m-2t\leq\sigma<m-t$ is easier since Gauss sum $G_0(\alpha)$ does not appear. We omit the proof.

Next we treat the term $Q_0(\alpha)$.

**Proposition A.4.** *Let the notation be as above and assume $t-m/2<0$ and $\alpha\in U^*_{-1}$.*

*If t odd,*

$$Q_0(\alpha)=\begin{cases} -\dfrac{q^{(t+1)/2}}{2}(h(a_\alpha,\alpha)+h(a'_\alpha,\alpha)) & \text{if } A_\alpha \bmod P_K\in k_F^0\\ 0 & \text{otherwise}\end{cases}$$

*and if t even,*

$$Q_0(\alpha)=\begin{cases} -\frac{q^{t/2}}{2}(h(a_\alpha,\alpha)G_0(\alpha,a_\alpha)+h(a'_\alpha,\alpha)G_0(\alpha,a'_\alpha)) & \textit{if } A_\alpha \bmod P_K\in k_F^0 \\ 0 & \textit{otherwise} \end{cases}$$

*where* $\frac{n_K(a_\alpha)}{1-n_K(a_\alpha)}$, $\frac{n_K(a'_\alpha)}{1-n_K(a'_\alpha)}$ *are solutions of* $X^2+X-A_\alpha=0$ and $A_\alpha\in\mathcal{O}_K$ *is determined uniquely modulo* $P_K^{[(t+1)/2]}$ *by the condition* $\Psi_{(A_\alpha,\alpha)}(A_\alpha((n_K(x)-1)(1-\bar{\alpha}\alpha^{-1})+(y-1)n_K(1-\bar{\alpha}\alpha^{-1})))=1$ *for all* $y\in P_K^{[(t+2)/2]}$ *and*

$$G_0(z,\alpha)=\sum_{x\in 1+P_K^{t/2}/1+P_K^{(t+2)/2}}\theta\Big(1+\frac{-z(x-1)}{1-n_K(z)}(1-\bar{\alpha}\alpha^{-1})\Big)$$

(A.5)

$$\psi_T\Big(\frac{z(x-1)}{1-n_K(z)}(1-\bar{\alpha}\alpha^{-1})\Big)\Psi_{(\varphi(z),\alpha)}(x).$$

*The absolute value of* $G_0(a_\alpha,\alpha)$ *and* $G_0(a'_\alpha,\alpha)$ *is* $q^{1/2}$ *and they belong to* $\mathbf{Z}[\sqrt{-1}]$.

*Proof.* First we assume $|k_F|>2$. As in the calculation for $P_\sigma(\alpha)$, we get

$$Q_0(\alpha)=\sum_{x\in\mathcal{O}_K^\times-(1+P_K)/1+P_K^t}\rho(\Phi(x,\alpha))\sum_{y\in 1+P_K^t/\mathcal{O}_k^\times(1+P_K^{t+1})}\Psi_{(\varphi(x),\alpha)}(y)$$

$$=\sum_{x\in\mathcal{O}_K^\times-(1+P_K)/1+P_K^t}\rho(\Phi(x,\alpha))$$

$$\sum_{y\in 1+P_K^t/\mathcal{O}_k^\times(1+P_K^{t+1})}\psi_T(\varphi(x)(n_K(y)-1)(1-\bar{\alpha}\alpha^{-1}))$$

$$=\sum_{x\in\mathcal{O}_K^\times-(1+P_K)/1+P_K^t}\rho(\Phi(x,\alpha))$$

$$\frac{1}{2}\sum_{y\in k_F}\psi_T(\varphi(x)n_K(C)(y^2+y)(1-\bar{\alpha}\alpha^{-1}))$$

where $C$ be an element of $P_K^t$ satisfying $tr_K(C)\equiv n_K(C) \bmod P_K^{t+1}$. As in the proof of Lemma 3.4, we have

$$\sum_{y\in k_F}\psi_T(\varphi(x)n_K(C)(y^2+y)(1-\bar{\alpha}\alpha^{-1}))=\begin{cases} q & \text{if } a_0 \bmod P_K\in k_F^0 \\ 0 & \text{otherwise}\end{cases}$$

where $a_\alpha\in\mathcal{O}_K$ is determined uniquely modulo $P_K$ by the condition $\psi_T(n_K(C)(1-\bar{\alpha}\alpha^{-1})a_0(y^2+y))=1$ for all $y\in\mathcal{O}_F$ and $k_F^0$ as in (3.4). Assume $X^2+X-a_0$ is reducible and let $X^2+X+a_0=(X+a'_0)(X+a''_0)$. Then

$$Q_0(\alpha)=\frac{q}{2}\Big(\sum_{x\in a'_0(1+P_K)/1+P_K^t}\rho(\Phi(x,\alpha))+\sum_{x\in a''_0(1+P_K)/1+P_K^t}\rho(\Phi(x,\alpha))\Big).$$

Hence we have the lemma by the same way as in the calculation $P_\sigma(\alpha)$.

If $|k_F|=2$, then $Q_0(\alpha)=0$ since $J_0=\phi$. On the other hand, $X^2+X+A_\alpha$ has no solution over $k_F$. Therefore the formula holds including the case $|k_F|=2$.

**Proposition A.5.** *Let the notation be as above and assume* $t-m/2<0$ *and* $\alpha\in U_{-1}^*$.

(1) *If* $m-\frac{3}{2}t\geq 0$,

$$Q_0(\alpha)=\begin{cases} -q^{m-t-1}(1+\psi(tr_K(\gamma(1-\bar{\alpha}\alpha^{-1})))) & \text{if } \alpha\in\gamma U_{m-t-1}^* \\ q^{m-t-1}(q-(1+\psi(tr_K(\gamma(1-\bar{\alpha}\alpha^{-1}))))) & \text{if } \alpha\in\gamma U_{m-t} \\ 0 & \text{otherwise.}\end{cases}$$

(2) *If t odd and* $m-(3t-1)/2>0$,

$$Q_0(\alpha)=\begin{cases} -\frac{q^{(t-1)/2}}{2}(h(a_\alpha,\alpha)G_0(a_\alpha,\alpha)+h(a'_\alpha,\alpha)G_0(a'_\alpha,\alpha)) & \\ \qquad \text{if } \alpha\in\gamma U_{2t-m}^* \text{ and } A_\alpha \bmod P_K\in k_F^0 & \\ 0 \qquad \text{otherwise} & \end{cases}$$

*where* $\frac{n_K(a_\alpha)}{1-n_K(a_\alpha)}$, $\frac{n_K(a'_\alpha)}{1-n_K(a'_\alpha)}$ *are solutions of* $X^2+X-A_\alpha=0$, $A_\alpha\in\mathcal{O}_K$ *is determined uniquely modulo* $P_K^{m-(3t+1)/2}$ *by the condition* $\Psi_{(A_\alpha,\alpha)}(y)=1$ *for all* $y\in 1+P_K^{m-(3t-1)/2}$ *and*

$$\text{(A.6)}\quad \begin{aligned} G_0(z,\alpha)=&\sum_{x\in 1+P_K^{m-(3t+1)/2}/1+P_K^{m-(3t+1)/2}}\theta\left(1+\frac{-z(x-1)}{1-n_K(z)}(1-\bar{\alpha}\alpha^{-1})\right)\\ &\psi_\gamma\left(\frac{z(x-1)}{1-n_K(z)}(1-\bar{\alpha}\alpha^{-1})\right)\Psi_{(\varphi(z),\alpha)}(x).\end{aligned}$$

*The absolute value of* $G_0(z,\alpha)$ *is* $q^{1/2}$ *and* $G_0(z,\alpha)$ *belongs to* $\mathbf{Z}[\sqrt{-1}]$ *when* $z=a_\alpha, a'_\alpha$.

(3) *If t odd and* $m=(3t+1)/2$,

$$Q_0(\alpha)=\begin{cases} q^{(t-1)/2}(G_0(\alpha)-1-\theta(1+1-\bar{\alpha}\alpha^{-1})) & \text{if } \alpha\in\gamma U_{(t-1)/2}^* \\ 0 & \text{otherwise}\end{cases}$$

*where*

$$\text{(A.7)}\quad \begin{aligned} G_0(\alpha)=&\sum_{x\in k_F}\theta(1+(x+x^2)(1-\bar{\alpha}\alpha^{-1}))\theta(1+x^2(1-\bar{\alpha}\alpha^{-1}))\\ &\psi_\gamma((x+x^2)(1-\bar{\alpha}\alpha^{-1})+((x+x^2)(1-\bar{\alpha}\alpha^{-1}))^2)\end{aligned}$$

*and* $G_0(\alpha)\in\mathbf{Z}[\sqrt{-1}]$, $|G_0(\alpha)|=\sqrt{q}$.

(4) *If t even and* $m-\frac{3}{2}t>0$,

$$Q_0(\alpha)=\begin{cases}-\frac{q^{t/2}}{2}(h(a_\alpha,\alpha)+h(a'_\alpha,\alpha)) \\ \quad \text{if } \alpha\in\gamma U^*_{2t-m} \text{ and } A_\alpha \bmod P_K\in k_F^0 \\ 0 \qquad \text{otherwise}\end{cases}$$

*where* $\frac{n_K(a_\alpha)}{1-n_K(a_\alpha)}$, $\frac{n_K(a'_\alpha)}{1-n_K(a'_\alpha)}$ *are solutions of* $X^2+X-\mathrm{A}_\alpha=0$, $A_a\in\mathcal{O}_K$ *is determined uniquely modulo* $P_K^{m-\frac{3}{2}t}$ *by the condition* $\Psi_{(\mathrm{A}_\alpha,\alpha)}(y)=1$ *for all* $y\in 1+P_K^{m-\frac{3}{2}t}$.

*Proof.* This is proved by combining the arguments in the Proposition A.3, Proposition A.4, Lemma 3.5 and Lemma 3.13. We omit the detail.

Now we start the calculation of $Q_\mu(\alpha)$. This is much more complicated than the case $t\leq 2$. We set

$$(\mathrm{A}.8)\quad \Xi_{(x,\alpha)}(y)=\Psi_{(\varphi(x),\alpha)}(y)\,\psi_\gamma\Big(\Big(\frac{\xi^2 n_K(x)}{1-\xi^2 n_K(x)}\Big)^3(n_K(y)-1)^2(1-\bar{\alpha}\alpha^{-1})\Big)$$

and define a subset $S_\mu$ of $U^*_{m+2\mu-2t-1}$ by

$$(\mathrm{A}.9)\quad S_\mu=\{\alpha\in U^*_{m+2\mu-2t-1}\mid \Xi_{(x,\alpha)}|_{1+P_K^{[(t+2)/2]}}=1 \text{ for some } x\in a_0(1+P_K^{\mu+1})\}$$

when $4\mu\leq t$ and

$$(\mathrm{A}.10)\quad S_\mu=\{\alpha\in U^*_{m+2\mu-2t-1}\mid \Xi_{(x,\alpha)}|_{1+P_K^{[(t+2\mu+3)/3]}}=1 \text{ for some } x\in a_0(1+P_K^{\mu+1})\}$$

when $4\mu\geq t$.

**Proposition A.6.** *Let the notation be as above and assume* $\mu>t-\frac{m}{2}$, $0<\mu<t$ *and* $\alpha\in U^*_{m+2\mu-2t-1}$.

(1) *If t odd and* $4\mu\leq t$,

$$Q_\mu(\alpha)=\begin{cases}-q^{(t+1)/2}(h(a_\alpha,\alpha)+h(a'_\alpha,\alpha)) & \text{if } \alpha\in S_\mu \\ 0 & \text{otherwise}\end{cases}$$

*where* $a_\alpha$, $a'_\alpha\in 1+P_K^\mu$ *are determined by the condition that* $\Xi_{(a_\alpha,\alpha)}(y)=1$ *for all* $y\in 1+P_K^{(t+1)/2}$.

(2) *If t even and* $4\mu<t$,

$$Q_\mu(\alpha)=\begin{cases}-q^{t/2}(h(a_\alpha,\alpha)G_\mu(a_\alpha,\alpha)+h(a'_\alpha,\alpha)G_\mu(a'_\alpha,\alpha)) & \text{if } \alpha\in S_\mu \\ 0 & \text{otherwise}\end{cases}$$

*where* $a_\alpha$, $a'_\alpha\in 1+P_K^\mu$ *are determined by the condition that* $\Xi_{(a_\alpha,\alpha)}(y)=1$ *for all* $y\in 1+P_K^{(t+2)/2}$ *and*

$$G_\mu(z,\alpha)=\sum_{x\in 1+P_K^{t/2}/1+P_K^{(t+2)/2}}\theta\Big(1+\frac{-z(x-1)}{1-n_K(z)}(1-\bar{\alpha}\alpha^{-1})\Big)$$

(A.11)
$$\psi_T\Big(\frac{z(x-1)}{1-n_K(z)}(1-\bar{\alpha}\alpha^{-1})\Big)\Xi_{(\varphi(z),\alpha)}(x).$$

*For* $z=a_\alpha, a'_\alpha$, *the absolute value of* $G_\mu(z,\alpha)$ *is* $q^{1/2}$ *and* $G_\mu(z,\alpha)$ *belongs to* $\mathbf{Z}[\sqrt{-1}]$.

(3) *If* $4\mu>t$ *and* $2\mu+t\equiv 2 \bmod 3$,

$$Q_\mu(\alpha)=\begin{cases} -q^{(2t-2\mu+2)/3}h(a_\alpha,\alpha) & \text{if } \alpha\in S_\mu \\ 0 & \text{otherwise}\end{cases}$$

*where* $a_\alpha\in 1+P_K^\mu$ *are determined by the condition that* $\Xi_{(a_\alpha,\alpha)}(y)=1$ *for all* $y\in 1+P_K^{(t+2\mu+1)/3}$.

(4) *If* $4\mu>t$ *and* $2\mu+t\not\equiv 2 \bmod 3$, *we have*

$$Q_\mu(\alpha)=\begin{cases} -q^{(2t-2\mu+2)/3}h(a_\alpha,\alpha)H_\mu(\alpha) & \text{if } \alpha\in S_\mu \\ 0 & \text{otherwise}\end{cases}$$

*where* $a_\alpha\in 1+P_K^\mu$ *are determined by the condition that* $\Xi_{(a_\alpha,\alpha)}(y)=1$ *for all* $y\in 1+P_K^{[(t+2\mu+3)/3]}$ *and*

(A.12)
$$H_\mu(\alpha)=\sum_{x\in 1+P_K^{[(t+2\mu)/3]}/1+P_K^{[(t+2\mu+3)/3]}}\theta\Big(1+\frac{-a_\alpha(x-1)}{1-n_K(a_\alpha)}(1-\bar{\alpha}\alpha^{-1})\Big)$$

$$\psi_T\Big(\frac{a_\alpha(x-1)}{1-n_K(a_\alpha)}(1-\bar{\alpha}\alpha^{-1})\Big)\Xi_{(\varphi(a_\alpha),\alpha)}(x)$$

$$\psi_T\Big(\Big(\frac{\xi^2 n_K(a_\alpha)}{1-\xi^2 n_K(a_\alpha)}\Big)^4(n_K(x)-1)^3(1-\bar{\alpha}\alpha^{-1})\Big).$$

Unfortunately we cannot call $H_\mu(\alpha)$ Gauss sum since the absolute value of $H_\mu(\alpha)$ is not $q^{1/2}$.

*Proof.* By repeating the routine calculation, we have

$$Q_\mu(\alpha)=q\sum_{x\in a_0(1+P_K^{\mu+1})/1+P_K^t}\rho(\Phi(x,\alpha))$$

*where* $a_0\in 1+P_K^\mu-1+P_K^{\mu+1}$ is determined by the condition $\psi_T(\varphi(a_0)((n_K(y)-1)(1-\bar{\alpha}\alpha^{-1})))=1$ for all $y\in 1+P_K^t$. Let $\Xi_{(x,\alpha)}$ be as in (A.8). When $i\geq \max[(t+2)/2],[(t+2\mu+3)/3)]$,

$$\rho(\Phi(xy,\alpha))=\rho(\Phi(x,\alpha))\Xi_{(x,\alpha)}(y)$$

for $y \in 1+P_K^i$ and the map $1+y \mapsto \Xi_{(x,\alpha)}(y)$ is a character of $1+P_K^i$ for $x \in a_0(1+P_K^{\mu+1})$.Thus if $4\mu \geq t$ (resp. $4\mu > t$), the map $1+y \mapsto \Xi_{(x,\alpha)}(y)$ is a character of $1+P_K^{[(t+2)/2]}$ (resp. $1+P_K^{[(t+2\mu+3)/3]}$) for $x \in a_0(1+P_K^{\mu+1})$.

The next lemma is an analogue of Lemma A.2.

**Lemma A.7.** (1) *Where $4\mu \leq t$ and $\alpha \in S_\mu$, there exist two elements $x \in a_0(1+P_K^{\mu+1})/1+P_K^{[(t+1)/2]}$ such that $\Xi_{(x,\alpha)}$ is a trivial character of $1+P_K^{[(t+2)/2]}$. (See (A.9) for $S_\mu$.)*

(2) *When $4\mu > t$ and $\alpha \in S_\mu$, there exist a unique element $x \in a_0(1+P_K^{\mu+1})/1+P_K^{[(t+2\mu+1)/3]}$ such that $\Xi_{(x,\alpha)}$ is a trivial character of $1+P_K^{[(t+2\mu+3)/3]}$. (See (A.10) for $S_\mu$.)*

*Proof.* For $y \in 1+P_K^{t-1}$ and $x=a_0x_1$, $x_1 \in 1+P_K^{\mu+1}$,

$$\begin{aligned}\Xi_{(x,\alpha)}(y) &= \psi_r(\varphi(a_0 x_1)((n_K(y)-1)(1-\bar{\alpha}\alpha^{-1})+(y-1)n_K(1-\bar{\alpha}\alpha^{-1})))\\ &\quad \times \psi_r\Big(\Big(\frac{\xi^2 n_K(a_0x_1)}{1-\xi^2 n_K(a_0x_1)}\Big)^3(n_K(y)-1)^2(1-\bar{\alpha}\alpha^{-1})\Big)\\ &= \psi_r(\varphi(a_0)((n_K(y)-1)(1-\bar{\alpha}\alpha^{-1})+(y-1)n_K(1-\bar{\alpha}\alpha^{-1})))\\ &\quad \times \psi_r\Big(\Big(\frac{\xi^2 n_K(a_0)}{1-\xi^2 n_K(a_0)}\Big)^3(n_K(y)-1)^2(1-\bar{\alpha}\alpha^{-1})\Big)\end{aligned}$$

because $\Big(\frac{\xi^2 n_K(a_0x_1)}{1-\xi^2 n_K(a_0x_1)}\Big)^2 \equiv \Big(\frac{\xi^2 n_K(a_0)}{1-\xi^2 n_K(a_0)}\Big)^2 \bmod P_K^{-2\mu+2}$. It implies $\Xi_{(x,\alpha)}(y) = 1$ for all $y \in 1+P_K^{t-1}$ by the assumption $\alpha \in S_\mu$. For $y \in 1+P_K^{t-2}$ and $x=a_0x_1$, $x_1 \in 1+P_K^{\mu+1}$,

$$\begin{aligned}\Xi_{(x,\alpha)}(y) &= \psi_r(\varphi(a_0)((n_K(y)-1)(1-\bar{\alpha}\alpha^{-1})+(y-1)n_K(1-\bar{\alpha}\alpha^{-1})))\\ &\quad \times \psi_r\Big(\Big(\frac{\xi^2 n_K(a_0)}{1-\xi^2 n_K(a_0)}\Big)^3(n_K(y)-1)^2(1-\bar{\alpha}\alpha^{-1})\Big)\\ &\quad \times \psi_r\Big(\Big(\frac{\xi^2 n_K(a_0)}{1-\xi^2 n_K(a_0)}\Big)^4(n_K(x_1)-1)^2(n_K(y)-1)(1-\bar{\alpha}\alpha^{-1})\Big)\end{aligned}$$

because

$$\begin{aligned}\Big(\frac{\xi^2 n_K(a_0x_1)}{1-\xi^2 n_K(a_0x_1)}\Big)^2 &\equiv \Big(\frac{\xi^2 n_K(a_0)}{1-\xi^2 n_K(a_0)}\Big)^2\\ &\quad + \Big(\frac{\xi^2 n_K(a_0)}{1-\xi^2 n_K(a_0)}\Big)^4(n_K(x_1)-1)^2 \bmod P_F^{-2\mu+4}.\end{aligned}$$

Since the map

$$x_1 \mapsto \Big(y \mapsto \psi_r\Big(\Big(\frac{\xi^2 n_K(a_0)}{1-\xi^2 n_K(a_0)}\Big)^4(n_K(x_1)-1)^2(n_K(y)-1)(1-\bar{\alpha}\alpha^{-1})\Big)$$

induces a bijection from $1+P_K^{\mu+1}/1+P_K^{\mu+2}$ to $(1+P_K^{t-2}/1+P_K^{t-1})^{\wedge}$, there exists a unique element $x_1 \in 1+P_K^{\mu+1}/1+P_K^{\mu+2}$ such that $\Xi_{(a_0x,\alpha)}$ is trivial on $1+P_K^{t-2}$. By repeating this procedure, we get our lemma for the case $4\mu>t$. For the case $4\mu\leq t$, we have there exists a unique element $x^{(\mu-1)} \in 1+P_K^{\mu+1}/1+P_K^{2\mu}$ such that $\Xi_{(a_0x^{(\mu-1)},\alpha)}$ is trivial on $1+P_K^{t-2\mu-1}$. For $y \in 1+P_K^{t-2}$ and $x=a_0x^{(\mu-1)}x_\mu$, $x_\mu \in 1+P_K^{2\mu}$,

$$\Xi_{(a_0x,\alpha)}(y)=\Xi_{(a_0x^{(\mu-1)},\alpha)}$$

$$\times\psi_\tau\Big(\Big(\frac{\xi^2n_K(a_0x^{(\mu-1)})}{1-\xi^2n_K(a_0x^{(\mu-1)})}\Big)^2(n_K(x_\mu)-1)(n_K(y)-1)(1-\bar{\alpha}\alpha^{-1})\Big)$$

$$\times\psi_\tau\Big(\Big(\frac{\xi^2n_K(a_0x^{(\mu-1)})}{1-\xi^2n_K(a_0x^{(\mu-1)})}\Big)^4(n_K(x_\mu)-1)^2(n_K(y)-1)(1-\bar{\alpha}\alpha^{-1})\Big)$$

because

$$\varphi(a_0x^{(\mu-1)}x_\mu)\equiv\Big(\frac{\xi^2n_K(a_0x^{(\mu-1)})}{1-\xi^2n_K(a_0x^{(\mu-1)})}\Big)^2$$

$$+\Big(\frac{\xi^2n_K(a_0x^{(\mu-1)})}{1-\xi^2n_K(a_0x^{(\mu-1)})}\Big)^2(n_K(x_\mu)-1)$$

$$+\Big(\frac{\xi^2n_K(a_0x^{(\mu-1)})}{1-\xi^2n_K(a_0x^{(\mu-1)})}\Big)^4(n_K(x_\mu)-1)^2 \bmod P_F.$$

Since the map $x\mapsto(y\mapsto\psi_\tau(x(n_K(y)-1)(1-\bar{\alpha}\alpha^{-1})))$ induces a bijection from $\mathcal{O}_F/P_F$ to $(1+P_K^{t-2\mu}/1+P_K^{t-2\mu+1})^{\wedge}$, there exist two $x_\mu$'s satisfying $\Xi_{(a_0x^{(\mu-1)}x_\mu,\alpha)}(y)=1$ for all $y\in 1+P_K^{t-2\mu}$ by the assumption $\alpha\in S_\mu$. For $[(t+2)/2]+1\leq i<t-2\mu$, we can show by the same way as in the proof of Lemma A.2 that if $x^{(t-i-\mu-1)}\in 1+P_K^{\mu+1}/1+P_K^{t-i}$ satisfies $\Xi_{(a_0x^{(t-i-\mu-1)},\alpha)}(y)=1$ for all $y \in 1+P_K^{i}$, there exists a unique element $x^{t-i-\mu}\in 1+P_K^{t-i}/1+P_K^{t-i-\mu+1}$ such that $\Xi_{(a_0x^{(t-i-\mu-1)}x_{t-i-\mu},\alpha)}(y)=1$ for all $y\in 1+P_K^{i-1}$. Hence our lemma.

By the above lemma and our routine calculation, we get our proposition.

The next term is $Q_\mu(\alpha)$ for $\alpha\in U_{-1}^*$ when $\mu\leq t-\frac{m}{2}$ and $0<\mu<t$. We set

(A.13)

$$S_{(-1,\mu)}=\{\alpha\in\gamma U^*_{2t-m-2\mu}\mid \Xi_{(x,\alpha)}|_{1+P_K^{[(2m+4\mu-3t+1)/2]}}=1 \quad \text{for some } x\in a_0(1+P_K^{\mu+1})\}$$

when $2m-3t>0$ and

(A.14)

$$S_{(-1,\mu)}=\{\alpha\in\gamma U^*_{2t-m-2\mu}\mid \Xi_{(x,\alpha)}|_{1+P_K^{[(2m+6\mu-3t+2)/3]}}=1 \quad \text{for some } x\in a_0(1+P_K^{\mu+1})\}$$

when $2m-3t\leq 0$.

**Proposition A.8.** *Let the notation as above and assume* $\mu \geq t - \frac{m}{2}$, $0 < \mu < t$ *and* $\alpha \in U_{-1}^{*}$.

(1) *If* $\mu \leq t - \frac{2}{3}m + \frac{1}{3}$,

$$Q_\mu(\alpha) = \begin{cases} -q^{m+\mu-t-1} & \text{if } \alpha \in \gamma U_{m+\mu-t-1}^{*} \\ q^{(m+\mu-t-1)}(q-1) & \text{if } \alpha \in \gamma U_{m+\mu-t} \\ 0 & \text{otherwise.} \end{cases}$$

(2) *If* $\mu > t - \frac{2}{3}m + \frac{1}{3}$, $2m - 3t > 0$ *and* $t$ *even,*

$$Q_\mu(\alpha) = \begin{cases} -q^{t/2}(h(a_\alpha, \alpha) + h(a'_\alpha, \alpha)) & \text{if } \alpha \in S_{(-1,\mu)} \\ 0 & \text{otherwise} \end{cases}$$

*where* $a_\alpha$, $a'_\alpha \in 1 + P_K^{\mu}$ *are determined by the condition that* $\Xi_{(a_\alpha, \alpha)}(y) = 1$ *for all* $y \in 1 + P_K^{(2m+4\mu-3t)/2}$.

(3) *If* $\mu > t - \frac{2}{3}m + \frac{1}{3}$, $2m - 3t > 0$ *and* $t$ *odd,*

$$Q_\mu(\alpha) = \begin{cases} -q^{(t-1)/2}(h(a_\alpha, \alpha) G_\mu(a_\alpha, \alpha) + h(a'_\alpha, \alpha) G_\mu(a'_\alpha, \alpha)) & \text{if } \alpha \in S_{(-1,\mu)} \\ 0 & \text{otherwise} \end{cases}$$

*where* $a_\alpha$, $a'_\alpha \in 1 + P_K^{\mu}$ are determined by the condition that $\Xi_{(a_\alpha, \alpha)}(y) = 1$ *for all* $y \in 1 + P_K^{(2m+4\mu-3t+1)/2}$ *and*

(A.15)

$$G_\mu(z, \alpha) = \sum_{x \in 1+P_K^{(2m+4\mu-3t-1)/2}/1+P_K^{(2m+4\mu-3t+1)/2}} \theta\left(1 + \frac{-z(x-1)}{1-n_K(z)}(1-\bar{\alpha}\alpha^{-1})\right)$$

$$\psi_\gamma\left(\frac{z(x-1)}{1-n_K(z)}(1-\bar{\alpha}\alpha^{-1})\right) \Xi_{(\varphi(z),\alpha)}(x).$$

*For* $z = a_\alpha$, $a'_\alpha$, *the absolute value of* $G_\mu(z, \alpha)$ *is* $q^{1/2}$ *and* $G_\mu(z, \alpha)$ *belongs to* $\mathbf{Z}[\sqrt{-1}]$.

(4) *If* $\mu > t - \frac{2}{3}m + \frac{1}{3}$, $2m - 3t \leq 0$ *and* $m \equiv 0 \bmod 3$,

$$Q_\mu(\alpha) = \begin{cases} -q^{m/3} h(a_\alpha, \alpha) & \text{if } \alpha \in S_{(-1,\mu)} \\ 0 & \text{otherwise} \end{cases}$$

*where* $a_\alpha \in 1 + P_K^{\mu}$ are determined by the condition that $\Xi_{(a_\alpha,\alpha)}(y) = 1$ *for all* $y \in 1 + P_K^{(2m+6\mu-3t)/3}$.

(5) *If* $\mu > t - \frac{2}{3}m + \frac{1}{3}$, $2m - 3t \geq 0$ *and* $m \not\equiv 0 \bmod 3$,

$$Q_\mu(\alpha)=\begin{cases}-q^{[m/3]}h(a_\alpha,\alpha)H_\mu(a_\alpha,\alpha) & \text{if } \alpha\in S_{(-1,\mu)}\\ 0 & \text{otherwise}\end{cases}$$

*where* $a_\alpha\in 1+P_K^\mu$ *are determined by the condition that* $\Xi_{(a_\alpha,\alpha)}(y)=1$ *for all* $y\in 1+P_K^{[(2m+6\mu-3t+2)/3]}$ *and*

(A.16)

$$H_\mu(\alpha)=\sum_{x\in 1+P_K^{[(2m+6\mu-3t)/3]}/1+P_K^{[(2m+6\mu-3t+2)/3]}}\theta\Big(1+\frac{-a_\alpha(x-1)}{1-n_K(a_\alpha)}(1-\bar\alpha\alpha^{-1})\Big)$$
$$\psi_\gamma\Big(\frac{a_\alpha(x-1)}{1-n_K(a_\alpha)}(1-\bar\alpha\alpha^{-1})\Big)\Xi_{(\varphi(a_\alpha),\alpha)}(x)$$
$$\psi_\gamma\Big(\Big(\frac{\xi^2n_K(a_\alpha)}{1-\xi^2n_K(a_\alpha)}\Big)^4(n_K(x)-1)^3(1-\bar\alpha\alpha^{-1})\Big).$$

*Proof.* First we assume $\mu\le t-\frac{2}{3}m+\frac{1}{3}$. Since

$$Q_\mu(\alpha)=\sum_{x\in((1+P_K^\mu)-(1+P_K^{\mu+1}))/1+P_K^{m+2\mu-t-1}}\rho(\Phi(x,\alpha))$$
$$\sum_{y\in 1+P_K^{m+2\mu-t-1}/1+P_K^{m+2\mu-t}}\psi(tr_K(\gamma(1-\bar\alpha\alpha^{-1}))\varphi(x)(n_K(y)-1)),$$

$\alpha\in\gamma U_1$ is necessary for $Q_\mu(\alpha)\ne 0$. Next we consider $y\in 1+P_K^{m+2\mu-t-2}$, then we get $\alpha\in\gamma U_2$ is necessary for $Q_\mu(\alpha)\ne 0$. Repeating this procedure, we get

$$Q_\mu(\alpha)=\begin{cases}\displaystyle\sum_{x\in((1+P_K^\mu)-(1+P_K^{\mu+1}))/1+P_K^{m+2\mu-t}}\psi_\gamma\Big(\frac{n_K(x)}{1-n_K(x)}(1-\bar\alpha\alpha^{-1}))\Big) & \text{if } \alpha\in\gamma U_{m+\mu-t-1}\\ 0 & \text{otherwise.}\end{cases}$$

Since the map $x\mapsto\frac{n_K(x)}{1-n_K(x)}$ induces a bijection from $((1+P_K^\mu)-(1+P_K^{\mu+1}))/1+P_K^{m+2\mu-t}$ to $\tilde\omega_F^{-\mu}\mathcal{O}_F^\times/1+P_F^{m+\mu-t}$, we have

$$Q_\mu(\alpha)=\begin{cases}-q^{m+\mu-t-1} & \text{if } \alpha\in\gamma U^*_{m+\mu-t-1}\\ q^{m+\mu-t-1}(q-1) & \text{if } \alpha\in\gamma U_{m+\mu-t}\\ 0 & \text{otherwise.}\end{cases}$$

Now we assume $\mu>t-\frac{2}{3}m+\frac{1}{3}$. As above we get

$$Q_\mu(\alpha)=\sum_{x\in((1+P_K^\mu)-(1+P_K^{\mu+1}))/1+P_K^{2m+4\mu-3t}}\rho(\Phi(x,\alpha))$$
$$\sum_{y\in P_F^{2m+4\mu-3t}/P_F^{m+2\mu-t}}\psi(tr_k(\gamma(1-\bar\alpha\alpha^{-1}))\varphi(x)y).$$

Hence we have

$$Q_\mu(\alpha)=\begin{cases} q^{2t-m-2\mu}\displaystyle\sum_{x\in((1+P_K^\mu)-(1+P_K^{\mu+1}))/1+P_K^{2m+4\mu-3t}}\rho(\Phi(x,\alpha)) & \text{if } \alpha\in\gamma U_{2t-m-2\mu} \\ 0 & \text{otherwise.}\end{cases}$$

Applying the calculation for $P_\sigma(\alpha)$ when $\sigma<m-\frac{3}{2}t$ and for $Q_\mu(\alpha)$ when $\mu>t-\frac{m}{2}$, we get our proposition.

The last part is $Q_t(\alpha)$. It is easily calculated by the same way as the case $t\geq 2$. (See Lemma 3.6.)

**Proposition A.9.** *Let the notation as avove. For $\alpha\in U_{m-1}^*$,*

$$Q_t(\alpha)=\sum_{x\in k_F}\phi_\gamma\left(\frac{1-\bar{\alpha}\alpha^{-1}}{C(x^2+x+b)}\right)$$

*where $b\in k_F-k_F^0$ and $C$ be an element of $P_K^t$ such that $tr_K(C)\equiv n_K(C)$ mod $P_F^{t+1}$.*

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