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A probabilistic scheme for collapse of metrics

Dedicated to Professor Masatoshi Fukushima on his 60th birthday

By

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1. Introduction

After the development of the theory of collapse of Riemannian manifolds [1, 3], Ikeda and the first author spelled out the correspondence between the collapse of Riemannian metrics on a manifold and the convergence of the Brownian motions associated with them in [5, 8]. In [8], the first author employed the monotone convergence theorem for Dirichlet forms to investigate the convergence of resolvents, semigroups, and eigenvalues corresponding to the Laplace-Beltrami operators associated with the converging sequence of Riemannian metrics on a manifold. However, the advantage of the monotone convergence theorem bears much more than what was investigated in the paper. Indeed, we can establish a probabilistic scheme to treat the collapse of "metrics" on an infinite dimensional space such as a path group space over a Lie group, which is the main motivation of this paper.

From a point of view of the theory of Dirichlet forms, the based state space need not to be a manifold, and we can develop an analytic argument for generalized "Riemannian metrics" on a more general space. Namely, consider a separable metric space X as a "manifold" and a family of separable real Hilbert spaces H_x, $x \in X$ as a family of its tangent spaces at x. Then the space $\mathbf{S}$ of families A of non-negative definite symmetric operators $A(x): H_x \to H_x^*$ is regarded as a space of generalized "Riemannian metrics", where the symmetry and non-negativity are defined in a usual manner identifying H_x^* with H_x. Roughly speaking, our first aim is to see the convergence of associated bilinear forms, resolvents and semigroups when $A_n \in \mathbf{S}$ converges to A, and the second is to specify the limit bilinear form. For details, see Section 2.

A typical example covered by the above scheme is a path group

$$X \equiv \{\mathbf{x}: [0, 1] \to G : \mathbf{x} \text{ is continuous and } \mathbf{x}(0) = e\}$$

over a Lie group G with an AdG-invariant inner product $< \cdot , \cdot >_{\mathcal{G}}$ on the Lie algebra $\mathcal{G}$. In this case, due to the group structure on X, all H_x coincide with a Hilbert space of functions $\mathbf{h}: [0, 1] \to \mathcal{G}$ with $\mathbf{h}(0) = 0$ which are abso-

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lutely continuous and possess square integrable derivatives. To an $A \in \mathbf{S}$ vanishing on $H_0 \equiv \{\mathbf{h} \in h: \mathbf{h}(1) = 0\}$, we apply the above general scheme and specify the Dirichlet form corresponding to A. Further, with the help of the ergodic theorem by Gross [4], we shall show the coincidence of the Dirichlet form with a certain Dirichlet form on G. These observations will be given in Section 3.

In Section 4, turning to the case where the based space is a manifold, we revisit the results in [5, 8] with our general scheme. As another application of this general scheme, we shall make clear when to impose the differential geometrical bypotheses on foliations assumed in [5, 8]. In fact, we shall see that no geometric assumption is needed before the identification of the limit Dirichlet form with one on a submanifold.

2. A general scheme

Let X be a separable metric space and m be an everywhere positive probability measure on $(X, \mathscr{B}(X))$, $\mathscr{B}(X)$ being the topological Borel field of X. Throughout this section, assume that

(A. 1) there exists a family $\mathscr{H} = \{H_x\}_{x \in X}$ of separable real Hibert spaces H_x with inner product $\langle \cdot, \cdot \rangle_x$ and norm $\| \cdot \|_x$.

Thinking of H_x as a tangent space of X at x, we regard the disjoint sum $\bigcup_{x \in X} H_x^*$, H_x^* being the dual space of H_x, as a cotangent bundle of X. Then, a mapping $\omega: X \to \bigcup_{x \in X} H_x^*$ is said to be a measurable section if $\omega(x) \in H_x^*$, $x \in X$, and the mapping $x \mapsto \| \iota^*(x)[\omega(x)] \|_x$ is measurable, where $\iota^*(x): H_x^* \to H_x$ is the natural imbedding. We denote by $\Gamma(X)$ the space of measurable sections. For $a, b \in \mathbf{R}$ and $\omega_1, \omega_2 \in \Gamma(X)$, a linear combination $a\omega_1 + b\omega_2$ is defined by point-wise sum; $(a\omega_1 + b\omega_2)(x) = a\omega_1(x) + b\omega_2(x)$, $x \in X$. Throughout this section, in addition to (A.1), we assume that

(A. 2) there exist a subspace $\mathscr{C} \subset L^2(X; m)$ and a mapping $D: \mathscr{C} \to \Gamma(X)$ such that

(i) if $a, b \in \mathbf{R}$ and $u, v \in \mathscr{C}$, then $aDu + bDv \in \Gamma(X)$ and $D(au + bv) = aDu + bDv$,

(ii) the symmetric bilinear form

(2.1) $\mathscr{E}(u,v) = \int_X (\iota^*(x)[Du(x)], Dv(x))_x m(dx)$, $u, v \in \mathscr{C}$ is closable on $L^2(X; m)$, where $(\cdot, \cdot)_x$ is the natural pairing of H_x and H_x^*.

We shall make two remarks on the assumption. First, the measurability of the mapping $x \to (\iota^*(x)[Du(x)], Dv(x))_x$, which has been indispensable to define the bilinear form (2.1), follows from the assumption (A.2) (i). The second is that only the linearity of D is required in this section, while the D's enjoying also the derivation property will be dealt with in the latter sections.

The closure of the bilinear form given in (2.1) will be denoted by the same letter $\mathscr{E}$ again and its domain will be done by $\mathscr{F}$. The space $\mathscr{F}$ is a real

Hilbert space with inner product $\mathcal{E}_1(u,v) = \int_X uvdm + \mathcal{E}(u,v)$. For each $u \in \mathcal{F}$, there exists a family $\{Du(x)\}_{x\in X}$ so that $\int_X \|Du_n(x) - Du(x)\|_x^2 m(dx) \to 0$ whenever $u_n \to u$ in $\mathcal{F}$. The family $\{Du(x)\}$is unique up to m-a.e. equivalence.

A linear operators $T: H_x \to H_x^*$ with $\mathrm{Dom}(T) = H_x$ is called symmetric if $(h, T[k])_x = (k, T[h])_x$ for any $h,k \in H_x$, and is said to be non-negative definite, $T \geq 0$ in notation, if $(h, T[h])_x \geq 0$ for every $h \in H_x$. Denote by $\mathbf{S}$ the set of families $A = \{A(x)\}_{x\in X}$ such that

(1) for every $x \in X$,

  (a) $A(x)$ is a linear operator from H_x to H_x^* with $\mathrm{Dom}(A(x)) = H_x$,

  (b) $A(x)$ is symmetric and non-negative definite,

(2) there is an $M < \infty$ so that $M\iota(x) - A(x) \geqq 0$, $x \in X$,

(3) the mapping $x \to (A(x)^n[Du(x)], Du(x))_x$ is measurable for every $n \in \mathbf{N}$ and $u \in \mathcal{F}$.

In the above, $\iota(x)$ is the adjoint operator of $\iota^*(x)$, and the operator $A(x)^n: H_x \to H_x^*$ is defined after identifying H_x^* with H_x in the standard manner. The third condition is fulfilled if the measurability is verified for all $u \in \mathcal{C}$. For $A, A' \in \mathbf{S}$, write $A \geq A'$ to indicate that $A(x) - A'(x) \geq 0$ m-a.e. $x \in X$, and do $A \gg A'$to mean that $A - A' \geq \varepsilon\iota$ for some $\varepsilon > 0$, where $\varepsilon\iota = \{\varepsilon\iota(x)\}_{x\in X}$. Put

$$\mathbf{S}_+ = \{A \in \mathbf{S}: A \gg 0\}.$$

Obviously $A + \frac{1}{n}\iota \in \mathbf{S}_+$ if $A \in \mathbf{S}$. Moreover, if $A \in \mathbf{S}_+$, then the mapping $x \mapsto (A(x)^{-1}[Du(x)], Du(x))_x$ is measurable for any $u \in \mathcal{F}$. In fact, identify H_x^* with H_x, and hence think of $\iota(x)$ as the identity mapping on H_x. Then the desired measurability follows from the Neumann series expansion of inverse operators;

$$A(x)^{-1} = \frac{1}{M+1} \sum_{n=0}^{\infty} \left(\iota(x) - \frac{1}{M+1} A(x)\right)^n,$$

where $M < \infty$ is the constant in the condition (2) above.

A subclass $\mathcal{P}_+$ of $L^1(X; m)$ are defined by

$$\mathcal{P}_+ = \left\{ \phi \in L^1(X; m) : \begin{array}{l} \int_X \phi dm = 1, \ \operatorname*{ess\,inf}_{x\in X} \phi(x) > 0, \\ \text{and } \operatorname*{ess\,sup}_{x\in X} \phi(x) < \infty \end{array} \right\}.$$

If $A \in \mathbf{S}_+$ and $\phi \in \mathcal{P}_+$, then the symmetric bilinear form

$$\begin{cases} \mathrm{Dom}(\mathcal{E}^{A,\phi}) = \mathcal{F}, \\ \mathcal{E}^{A,\phi}(u,v) = \int_X (A(x)^{-1}[Du(x)], Dv(x))_x \phi(x) m(dx), \quad u,v \in \mathrm{Dom}(\mathcal{E}^{A,\phi}), \end{cases}$$

is well-defined and closed on $L^2(X; m^\phi)$, where $dm^\phi = \phi dm$. In fact, the well-definedness and the measurability of the mapping $x \to (A(x)^{-1}[Du(x)], Dv(x))_x$ have been seen above, and one can easily conclude, from the boundedness and positivity of A and ϕ, the existence of $\delta > 0$ so that

$$\delta\mathscr{E}(u, u) \leq \mathscr{E}^{A,\phi}(u, u) \leq \delta^{-1}\mathscr{E}(u,u), \quad u \in \mathscr{F}.$$

We shall describe a monotone convergence theorem for symmetric bilinear forms, due to Schwartz [10], Kato [6], Robinson [9], and Simon [11], in the following proposition. In contrast with theirs, we do not assume that the domains of bilinear forms are dense in the Hilbert space. But we can see that the assertion is still valid by the similar arguments to those in [6, 11]. For a general closed non-negative symmetric bilinear form $\mathscr{A}$ on a Hilbert space G with inner product $(\cdot,\cdot)_G$, we define the resolvent $\{R_\alpha\}_{\alpha>0}$ through the relation $\mathscr{A}(R_\alpha u,v) + \alpha(R_\alpha u,v)_G = (u,v)_G$, $u \in G$, $v \in \mathrm{Dom}(\mathscr{A})$. Our monotone convergence theorem reads

Proposition 2.2. *Let G be a real separable Hilbert space with inner product $(\cdot,\cdot)_G$, and $\{\mathscr{A}_n\}$ be a sequence of closed non-negative symmetric bilinear forms on G such that*

$$\mathrm{Dom}(\mathscr{A}_{n+1}) \subset \mathrm{Dom}(\mathscr{A}_n), \quad \textit{and} \quad \mathscr{A}_n(u,u) \leq \mathscr{A}_{n+1}(u,u), \quad u \in \mathrm{Dom}(\mathscr{A}_{n+1}).$$

Define

$$\begin{cases} \mathrm{Dom}(\mathscr{A}_\infty) = \{u \in \cap_{n=1}^\infty \mathrm{Dom}(\mathscr{A}_n) : \sup_n \mathscr{A}_n(u,u) < \infty\}, \\ \mathscr{A}_\infty(u,u) = \lim_{n\to\infty} \mathscr{A}_n(u,u), \quad u,v \in \mathrm{Dom}(\mathscr{A}_\infty). \end{cases}$$

Then

(*i*) *$\mathscr{A}_\infty$ is a closed non-negative symmetric bilinear form on G,*

(*ii*) $R_\alpha^{(n)} \to R_\alpha^{(\infty)}$ strongly in G for any u ∈ G, where *$\{R_\alpha^{(\cdot)}\}_{\alpha>0}$ is the resolvent corresponding to $\mathscr{A}_\cdot$.*

On account of the proposition, for $A \in \mathbf{S}$ and $\phi \in \mathscr{P}_+$, one can then define a closed bilinear form $\mathscr{E}^{A,\phi}$ on $L^2(X;m^\phi)$ by

$$\begin{cases} \mathrm{Dom}(\mathscr{E}^{A,\phi}) = \{u \in \mathscr{F} : \sup_{n\in\mathbf{N}} \mathscr{E}^{A+\frac{1}{n}\iota,\phi}(u,u) < \infty\}, \\ \mathscr{E}^{A,\phi}(u,v) = \lim_{n\to\infty} \mathscr{E}^{A+\frac{1}{n}\iota,\phi}(u,v), \quad u,v \in \mathrm{Dom}(\mathscr{E}^{A,\phi}). \end{cases}$$

Then, on has that

$$\lim_{n\to\infty} G_\alpha^{A+\frac{1}{n}\iota,\phi} = G_\alpha^{A,\phi} \quad \text{strongly in } L^2(X;m^\phi),$$

where $\{G^{A,\phi}{}_{\alpha}\}_{\alpha>0}$ is the resolvent associated with the form $\mathscr{E}^{A,\phi}$. Notice that, in case of $A \in \mathbf{S}_{+}$, the symmetric form $(\mathscr{E}^{A,\phi}, \mathrm{Dom}(\mathscr{E}^{A,\phi}))$ defined just above coincides with the previous one defined for $A \in \mathbf{S}_{+}$.

For $A, A_n \in \mathbf{S}$, we say $A_n \to A$ if there is a sequence $\{\varepsilon_n\} \subset \mathbf{R}$ such that $\varepsilon_n \to 0$ and $-\varepsilon_n \iota \leq A_n - A \leq \varepsilon_n \iota$.

Lemma 2.3. *For any $A_n \in \mathbf{S}_{+}$, $A \in \mathbf{S}$ with $A_n \gg A$, $A_n \geq A_{n+1}$, and $A_n \to A$, and for any $\phi \in \mathscr{P}_{+}$, it holds that*

$$\mathrm{Dom}(\mathscr{E}^{A,\phi}) = \{u \in \mathscr{F}: \sup_{n\in\mathbf{N}} \mathscr{E}^{A_n,\phi}(u,u) < \infty\},$$

$$\mathscr{E}^{A,\phi}(u,v) = \lim_{n\to\infty} \mathscr{E}^{A_n,\phi}(u,v), \quad u,v \in \mathrm{Dom}(\mathscr{E}^{A,\phi}),$$

$$\lim_{n\to\infty} G_{\alpha}^{A_n,\phi} = G_{\alpha}^{A,\phi} \text{ strongly in } L^2(X;m^{\phi}).$$

Moreover, for $A,B \in \mathbf{S}$ with $A \leq B$ and $\phi \in \mathscr{P}_{+}$, it holds that

$$\mathrm{Dom}(\mathscr{E}^{A,\phi}) \subset \mathrm{Dom}(\mathscr{E}^{B,\phi}),$$

$$\mathscr{E}^{A,\phi}(u,u) \geq \mathscr{E}^{B,\phi}(u,u), \; u \in \mathrm{Dom}(\mathscr{E}^{A,\phi}),$$

$$\int_X u\, G_{\alpha}^{A,\phi} u\, dm^{\phi} \leq \int_X u G_{\alpha}^{B,\phi} u\, dm^{\phi}, \; u \in L^2(X; m^{\phi}).$$

Proof. Due to Proposition 2.2, one has a closed symmetric bilinear form $\mathscr{G}$ on $L^2(X;m^{\phi})$ given by

$$\begin{cases} \mathrm{Dom}(\mathscr{G}) = \{u \in \mathscr{F}: \sup_{n\in\mathbf{N}} \mathscr{E}^{A_n,\phi}(u,u) < \infty,\} \\ \mathscr{G}(u,v) = \lim_{n\to\infty} \mathscr{E}^{A_n,\phi}(u,v), \quad u, v \in \mathrm{Dom}(\mathscr{G}). \end{cases}$$

Notice that for every $n \in \mathbf{N}$, there is an $m_n \in \mathbf{N}$ such that $A_m \leq A + \frac{1}{n}\iota$ and $A + \frac{1}{m}\iota \leq A_n$ for any $m \geq m_n$. It then follows that

$$\mathrm{Dom}(\mathscr{E}^{A,\phi}) = \mathrm{Dom}(\mathscr{G}) \quad \text{and} \quad \mathscr{E}^{A,\phi}(u,u), u \in \mathrm{Dom}(\mathscr{G}).$$

In conjunction with Proposition 2.2, this implies the first half of the assertion.

The first two parts of the second assertion follow from the very definition of $\mathscr{E}^{A,\phi}$. Finally, the ordering that $\mathscr{E}^{A,\phi}(u,u) \geq \mathscr{E}^{B,\phi}(u,u)$ yields the last inequality. Namely, if one sets $\mathscr{E}_{\alpha}^{A,\phi}(\cdot,\cdot) = \mathscr{E}^{A,\phi}(\cdot,\cdot) + \alpha < \cdot,\cdot >_{L^2(X;m^{\phi})}$, then one has that

$$\begin{aligned} < G_{\alpha}^{A,\phi} u, u >_{L^2(X;m^{\phi})} &= \mathscr{E}_{\alpha}^{B,\phi}(G_{\alpha}^{A,\phi} u, G_{\alpha}^{B,\phi} u) \\ &\leq \mathscr{E}_{\alpha}^{B,\phi}(G_{\alpha}^{A,\phi} u, G_{\alpha}^{A,\phi} u)^{1/2} \mathscr{E}_{\alpha}^{B,\phi}(G_{\alpha}^{B,\phi} u, G_{\alpha}^{B,\phi} u)^{1/2} \\ &\leq \mathscr{E}_{\alpha}^{A,\phi}(G_{\alpha}^{A,\phi} u, G_{\alpha}^{A,\phi} u)^{1/2} \mathscr{E}_{\alpha}^{B,\phi}(G_{\alpha}^{B,\phi} u, G_{\alpha}^{B,\phi} u)^{1/2} \end{aligned}$$

$$= \langle G_\alpha^{A,\phi}\ u,u\rangle^{1/2}_{L^2(X;m^\phi)} \langle G_\alpha^{B,\phi}\ u,u\rangle^{1/2}_{L^2(X;m^\phi)},$$

which means that the last inequality holds.

Proposition 2.4. *Let $A, A_n \in \mathbf{S}$ and $\phi \in \mathcal{P}_+$, and suppose that $A_n \geq A$ and $A_n \to A$. Then it holds that*

$$\mathcal{E}^{A,\phi}(u,v) = \lim_{n\to\infty} \mathcal{E}^{A_n,\phi}(u,v), \quad u,v \in \mathrm{Dom}(\mathcal{E}^{A,\phi}),$$

$$G_\alpha^{A,\phi} = \lim_{n\to\infty} G_\alpha^{A_n,\phi} \text{ strongly in } L^2(X;m^\phi).$$

Proof. For each $n \in \mathbf{N}$, one can find $m_n \in \mathbf{N}$ so that $A \leq A_m \leq A + \frac{1}{n}\iota$ for any $m \geq m_n$. Now the first assertion follows from Lemma 2.3.

We shall now see the strong convergence of $G_\alpha^{A_n,\phi}$ to $G_\alpha^{A,\phi}$. Due to Lemma 2.3, one has that

$$\langle G_\alpha^{A,\phi}u,u\rangle_{L^2(X;m^\phi)} \leq \langle G_\alpha^{A_n,\phi}u,u\rangle_{L^2(X;m^\phi)} \leq \langle G_\alpha^{A+\frac{1}{m}\iota,\phi}u,u\rangle_{L^2(X;m^\phi)}, \quad m \geq m_n,$$

where m_n is the same number as above. Hence $G_\alpha^{A_n,\phi}$ converges to $G_\alpha^{A,\phi}$ weakly in $L^2(X;m^\phi)$.

There is a resolution of the identity $\{E_\lambda^{A_n,\phi}\}$ on $L^2(X;m^\phi)$ so that

$$(2.5)\qquad \langle G_\alpha^{A_n,\phi}u,v\rangle_{L^2(X;m^\phi)} = \int_0^\infty \frac{1}{\lambda+\alpha} d\langle E_\lambda^{A_n,\phi}u,v\rangle_{L^2(X;m^\phi)}, \quad u,v \in L^2(X;m^\phi).$$

Namely, for an arbitrarily fixed α, one can find a resolution of the identiy so that the identity holds, because $G_\alpha^{A_n,\phi}$ is symmetric. Then, applying the resolvent equation $G_\alpha^{A_n,\phi} - G_\beta^{A_n,\phi} + (\alpha-\beta)\, G_\alpha^{A_n,\phi} G_\beta^{A_n,\phi} = 0$, the identity extends to general α's. For the proof of resolvent equation, see[2, Theorem 1.3.2].

Notice that the total variation of $\langle E_\lambda^{A_n,\phi}u,v\rangle_{L^2(X;m^\phi)}$ is dominated by the product $\| u \|_{L^2(X;m^\phi)}$. Hence, for any subsequence $\{A_n'\}$ of $\{A_n\}$, one can find a subsequence $\{A_{n_k}'\}$ of $\{A_n'\}$ and a system of linear operators $\{E_\lambda'\}$ in $L^2(X;m^\phi)$ such that

$$\lim_{k\to\infty} \langle E_\lambda^{A'_{n_k},\phi}u, v\rangle_{L^2(X;m^\phi)} = \langle E_\lambda' u, v\rangle_{L^2(X;m^\phi)}$$

at any continuity point λ of the right hand side for any $u,v \in L^2(X;m^\phi)$. One further finds a resolution of the identity $\{E_\lambda\}$ in $L^2(X;m^\phi)$ such that

$$\langle G_\alpha^{A,\phi}u,v\rangle_{L^2(X;m^\phi)} = \int_0^\infty \frac{1}{\lambda+\alpha} d < E_\lambda u,v >_{L^2(X;m^\phi)}, \quad u,v \in L^2(X;m^\phi).$$

Remember now that $G_\alpha^{A_{n_k},\phi} \to G_\alpha^{A,\phi}$ weakly in $L^2(X;m^\phi)$ to observe that the above E_λ' coincides with E_λ. Thereby, one concludes that

$$\lim_{n\to\infty} \langle E_\lambda^{A_n,\phi}u,v\rangle_{L^2(X;m^\phi)} = < E_\lambda u,v\rangle_{L^2(X;m^\phi)}$$

at any continuity point λ of the right hand side for any $u,v \in L^2(X;m^\phi)$.

Let $\{T_t^{A_n,\phi}\}$ and $\{T_t^{A,\phi}\}$ be the semigroup associated with $\mathscr{E}^{A_n,\phi}$ and $\mathscr{E}^{A,\phi}$, respectively. On account of the spectral representation

$$T_t^{A_n,\phi}=\int_0^\infty e^{-\lambda t}\,dE_\lambda^{A_n,\phi} \text{ and } T_t^{A,\phi}=\int_0^\infty e^{-\lambda t}\,dE_\lambda,$$

one obtains that $T_t^{A_n,\phi}\longrightarrow T_t^{A,\phi}$ weakly in $L^2(X;m^\phi)$. By the semigroup property and the symmetry of $T_t^{A_n,\phi}$ and $T_t^{A,\phi}$, then one has that

$$\lim_{n\to\infty}\|T_t^{A_n,\phi}u\|^2_{L^2(X;m^\phi)}=\|T_t^{A,\phi}u\|^2_{L^2(X;m^\phi)},$$

and hence that

$$\lim_{n\to\infty}\|T_t^{A_n,\phi}u-T_t^{A,\phi}u\|^2_{L^2(X;m^\phi)}$$
$$=\lim_{n\to\infty}\{\|T_t^{A_n,\phi}u\|^2_{L^2(X;m^\phi)}+\|T_t^{A,\phi}u\|^2_{L^2(X;m^\phi)},-2\langle T_t^{A_n,\phi}u,\,T_t^{A,\phi}u\rangle_{L^2(X;m^\phi)}\}$$
$$=0.$$

Thus $T_t^{A^n,\phi}\longrightarrow T_t^{A,\phi}$ strongly in $L^2\,(X;m^\phi)$, and hence so does $G_\alpha^{A_n,\phi}$ to $G_\alpha^{A,\phi}$.

Let $\|\cdot\|_\infty$ be the norm of $L^\infty(X;m)$. We now have the following continuity theorem.

Theorem 2.6 *Let $A,A_n\in\mathbf{S}$ and $\phi_n,\phi\in\mathscr{P}_+$. Suppose that $A_n\geq A$, $A_n\to A$, and $\|\phi_n-\phi\|_\infty\longrightarrow 0$. Then it holds that*

(2.7) $$\mathscr{E}^{A,\phi}(u,v)=\lim_{n\to\infty}\mathscr{E}_n^{A_n,\phi}(u,v),\quad u,v\in \mathrm{Dom}\,(\mathscr{E}^{A,\phi}),$$

(2.8) $$G_\alpha^{A,\phi}=\lim_{n\to\infty}G_\alpha^{A_n,\phi_n}\qquad \textit{strongly in } L^2(X;m^\phi),$$

(2.9) $$T_t^{A,\phi}=\lim_{n\to\infty}T_t^{A_n,\phi_n}\qquad \text{strongly in } \mathrm{L}^2(X;m^\phi),$$

where $\{T_t^{A,\phi}\}_{t\geq0}$ denotes the semigroup associated with $\mathscr{E}^{A,\phi}$.

Proof. It is elementary to see that

$$|\mathscr{E}^{A,\phi}(u,u)-\mathscr{E}^{A,\psi}(u,u)|\leq\|\phi-\psi\|_\infty\,\mathscr{E}^{A,1}(u,u),\quad u\in\mathscr{F},$$

for every $A\in\mathbf{S}_+$ and $\phi,\psi\in\mathscr{P}_+$. Notice that Dom $(\mathscr{E}^{B,\phi})=$ Dom $(\mathscr{E}^{B,\psi})$ for any $B\in\mathbf{S}$. Thus the inequality remains valid for $A\in\mathbf{S}$ and $u\in$ Dom $(\mathscr{E}^{A,\phi})$, which, combined with Proposition 2.4, implies that (2.7) holds.

Let $A\in\mathbf{S}_+$ and $\phi,\psi\in\mathscr{P}_+$. Setting $\mathscr{E}_\alpha^{A,\phi}(u,v)=\mathscr{E}^{A,\phi}(u,v)+\alpha\langle u,v\rangle_{L^2(X;m^\phi)}$ and then recalling that

$$\mathscr{E}_\alpha^{A,\phi}(G_\alpha^{A,\phi}u,v)=\langle u,v\rangle_{L^2(X;m^\phi)},\quad u\in L^2(X;m^\phi),\quad v\in\mathscr{F},$$

one sees that

$$\mathscr{E}_\alpha^{A,\phi}(G_\alpha^{A,\psi}u,v)-\mathscr{E}_\alpha^{A,\phi}(G_\alpha^{A,\phi}u,v)$$
$$=\int_X (A(x)^{-1}[D(G_\alpha^{A,\psi}u)(x)],Dv(x))_x(\phi-\psi)m(dx)$$
$$+\alpha\int_X((G_\alpha^{A,\psi}u)v)(x)(\phi-\psi)(x)m(dx)+\int_X(uv)(x)(\psi-\phi)(x)m(dx),$$

whence one concludes that

$$(2.10)\qquad |\mathscr{E}_\alpha^{A,\phi}(G_\alpha^{A,\psi}u-G_\alpha^{A,\phi}u,v)|$$
$$\le 2\,||\psi-\phi||_\infty\mathscr{E}_\alpha^{A,1}(G_\alpha^{A,\psi}u,\ G_\alpha^{A,\psi}u)^{\frac{1}{2}}\mathscr{E}_\alpha^{A,1}(v,v)^{\frac{1}{2}}$$
$$+||\psi-\phi||_\infty||u||_{L^2(X;m)}||v||_{L^2(X;m)},\quad u\in L^2(X;m),\ v\in\mathscr{F}.$$

Recall that

$$\mathscr{E}_\alpha^{A,\phi}(G_\alpha^{A,\phi}u,\ G_\alpha^{A,\phi}u)=\langle G_\alpha^{A,\phi}u,v\rangle_{L^2(X;m^\phi)}\le\frac{1}{\alpha}||u||^2_{L^2(X;m^\phi)},$$
$$\mathscr{E}_\alpha^{A,\phi}(u,u)\le\left\|\frac{\phi}{\psi}\right\|_\infty\le\mathscr{E}_\alpha^{A,\psi}(u,u),\quad u\in\mathscr{F},$$

and then observe that

$$\mathscr{E}_\alpha^{A,1}(G_\alpha^{A,\psi}u-G_\alpha^{A,\phi}u,\ G_\alpha^{A,\psi}u-G_\alpha^{A,\phi}u)$$
$$\le\frac{2}{\alpha}\left(\left\|\frac{1}{\phi}\right\|_\infty||\phi||_\infty+\left\|\frac{1}{\psi}\right\|_\infty||\phi||_\infty\right)||u||^2_{L^2(X;m)}.$$

Plugging this into (2.10), one comes to

$$||G_\alpha^{A,\psi}u-G_\alpha^{A,\phi}u||^2_{L^2(X;m)}$$
$$\le C\left(\frac{1}{\alpha},\left\|\frac{1}{\phi}\right\|_\infty,\left\|\frac{1}{\psi}\right\|_\infty,\left\|\phi\right\|_\infty,\left\|\psi\right\|_\infty\right)||\phi-\psi||_\infty||u||^2_{L^2(X;m)},$$

where C (...) denotes a constant depending only on {...} continuously. By an approximation argument, it is easily seen that the estimation continues to hold for $A\in\mathbf{S}$. Since

$$M\equiv\sup_n C\left(\frac{1}{\alpha},\left\|\frac{1}{\phi}\right\|_\infty,\left\|\frac{1}{\phi_n}\right\|_\infty,||\phi||_\infty,||\phi_n||_\infty\right)<\infty,$$

due to Proposition 2.4, it holds that

$$||G_\alpha^{A_n,\phi_n}u-G_\alpha^{A,\phi}u||_{L^2(X;m^\phi)}$$
$$\le||\phi||_\infty M^{1/2}||\phi-\phi_n||_\infty||u||_{L^2(X;m^\phi)}+||G_\alpha^{A_n,\phi}u-G_\alpha^{A,\phi}u||_{L^2(X;m^\phi)}\to 0\quad\text{as } n\to\infty,$$

which means that (2.8) holds.

To see the identity (2.9), let $\{E_\lambda^n\}$ be a resolution of the identity in $L^2(X;m^{\phi_n})$ associated with $G_\alpha^{A_n,\phi_n}$;

$$\langle G_\alpha^{A_n,\phi_n}u,v\rangle_{L^2(X;m^{\phi_n})}=\int_0^\infty \frac{1}{\lambda+\alpha}d\langle E_\lambda^n u,v\rangle_{L^2(X;m^{\phi_n})},\quad u,v\in L^2(X;m^{\phi_n}).$$

See the remark after (2.5). Since

$$\begin{aligned}&\text{the total variation of } \lambda\mapsto \langle E_\lambda^n u,v\rangle_{L^2(X;m^{\phi_n})}\\&\leq(\sup_n\|\phi_n\|_\infty)\|u\|_{L^2(X;m)}\|v\|_{L^2(X;m)},\end{aligned}$$

in repetition of the argument employed to see the second assertion of Proposition 2.4, one can conclude (2.9).

Remark 2.11. Several parts of the arguments used in the proofs of Lemma 2.3, Proposition 2.4, and Theorem 2.6 have already appeared in [8] in the case when X is a Riemannian manifold. We have repeated some of the arguments for the completeness of the present paper.

We now proceed to a characterization of $(\mathcal{E}^{A,\phi}, \mathrm{Dom}(\mathcal{E}^{A,\phi}))$. To do this, for $A\in\mathbf{S}$, let $\pi_A(x):H_x\to H_x$ be the orthogonal projection onto $\mathrm{Ker}(A(x))^\perp$, the orthogonal complement of $\mathrm{Ker}(A(x))$ in H_x. The symmetry of $A(x):H_x\to H_x^*$ implies that $A(x)\circ\pi_A(x)=\pi_A(x)^*\circ A(x)$, and hence that, for m-a.e. $x\in X$,

$$A(x)=A(x)\circ\pi_A(x)=\pi_A(x)^*\circ A(x)=\pi_A(x)^*\circ A(x). \tag{2.12}$$

This yields that the mapping

$$A(x)+\frac{1}{n}\iota(x)|_{\mathrm{Ker}(A(x))^\perp}:\mathrm{Ker}(A(x))^\perp\to\iota(x)(\mathrm{Ker}(A(x))^\perp)$$

is bijective for m-a.e. $x\in X$. In fact, by an elementary computation, one sees that the inverse is given by

$$\begin{aligned}&\left\{(A(x)+\frac{1}{n}\iota(x))|_{\mathrm{Ker}(A(x))^\perp}\right\}^{-1}\\&\qquad=\pi_A(x)\circ(A(x)+\frac{1}{n}\iota(x))^{-1}\circ\pi_A(x)^*|_{\iota(x)(\mathrm{Ker}(A(x))^\perp)}\end{aligned}$$

where on the right hand side $(A(x)+\frac{1}{n}\iota(x))^{-1}$ denotes the inverse mapping of the bijection $A(x)+\frac{1}{n}\iota(x):H_x\to H_x^*$, m-a.e. Combining this with (2.12), one obtains that

$$\begin{aligned}&(A(x)+\frac{1}{n}\iota(x))^{-1}\\&\qquad=\pi_A(x)\circ\left\{(A(x)+\frac{1}{n}\iota(x))|_{\mathrm{Ker}(A(x))^\perp}\right\}^{-1}\circ\pi_A(x)^*\\&\qquad\quad+n(\mathbf{I}_{H_x}-\pi_A(x))\circ\iota^*(x),\quad m\text{-a.e.}\end{aligned} \tag{2.13}$$

Now, define

$$\mathbf{S}_{p+}=\left\{ A \in \mathbf{S}: \begin{array}{l} A(x) \geq \varepsilon(x)\, \iota \circ \pi_A(x) \ m\text{-a.e. for some } \varepsilon: X \to (0,\infty) \text{ and} \\ \text{the mapping } x \to (\iota^*(x)[\pi_A(x)^*[Du(x)]],\ \pi_A(x)^*[Du(x)])_x\}. \\ \text{is measurable for every } u \in \mathcal{F} \end{array} \right\}$$

Suppose that $A \in \mathbf{S}_{p+}$. Due to the Neumann series expression of inverse operators, one can conclude from (2.13) that $A(x)|_{\mathrm{Ker}(A(x))^\perp}$: Ker $(A(x))^\perp)$ $\to \iota(X)$ (Ker $(A(x))^\perp)$ is bijective for m-a.e. $x \in X$, and that its inverse operator $(A(x)|_{\mathrm{Ker}(A(x))^\perp})^{-1}$ satisfies that

$$\left\{A(x)+\frac{1}{n}\iota(x)|_{\mathrm{Ker}(A(x))^\perp}\right\}^{-1} \nearrow \{A(x)|_{\mathrm{Ker}(A(x))^\perp}\}^{-1} \text{as operators.}$$

In particular, it follows from this and (2.13) that the mapping

$$x \to (\{A(x)|_{\mathrm{Ker}(A(x))^\perp}\}^{-1}[\pi_A(x)^*[Du(x)]], \pi_A(x)^*[Du(x)])_x$$

is measurable for every $u \in \mathcal{F}$. Further, by the monotone convergence theorem and (2.13), one comes to the identity

$$\begin{aligned} &\lim_{n\to\infty} \mathcal{E}^{A+\frac{1}{n}\iota,\phi}(u,u) \\ &= \int_X (\{A(x)|_{\mathrm{Ker}(A(x))^\perp}\}^{-1}[\pi_A(x)^*[Du(x)]], \pi_A(x)^*[Du(x)])_x \phi(x) m(dx) \\ &\quad + \infty \cdot \int_X ((\mathbf{I}_{H_x} - \pi_A(x)) \ \{\iota^*(x)[Du(x)]], Du(x))_x \phi(x) m(dx), \quad u \in \mathcal{F}, \end{aligned}$$

which leads one to the following characterization.

Theorem 2.14. *If $A \in \mathbf{S}_{p+}$ and $\phi \in \mathcal{P}_+$, then it holds that*

$$\mathrm{Dom}(\mathcal{E}^{A,\phi}) = \left\{ u \in \mathcal{F}: \begin{array}{l} Du(x) = \pi_A(x)^*[Du(x)]\, m\text{-a.e. } x \in X \textit{ and} \\ \int_X ((A(x)|_{\mathrm{Ker}(A(x))^\perp})^{-1}[Du(x)],\ Du(x))_x m^\phi(dx) \\ \textit{is finite} \end{array} \right\},$$

$$\mathcal{E}^{A,\phi}(u,v) = \int_X ((A(x)|_{\mathrm{Ker}(A(x))^\perp})^{-1}[Du(x)],\ Dv(x))_x m^\phi(dx),$$
$$u, v \in \mathrm{Dom}(\mathcal{E}^{A,\phi}).$$

Remark 2.15. For $A \in \mathbf{S}_{p+}$, one can give the same characterization of $\mathcal{E}^{A,\phi}$ with general $A_n \in \mathbf{S}_{p+}$ such that $A_n \geq A_{n+1} \geq$ A and $A_n \to A$ instead of the special sequence $\left\{A+\frac{1}{n}\iota\right\}$. In fact, choose a sequence $\varepsilon_n > 0$ decreasing to 0 so that $A_n \leq A + \varepsilon_n \iota$. If $u \in$ Dom $(\varepsilon^{A,\phi})$, then as in the observation before Theorem 2.14 one has that

$$
\begin{aligned}
+\infty &> \lim_{n\to\infty} \mathscr{E}^{A_n,\phi}(u,u) \\
&\geq \lim_{n\to\infty} \mathscr{E}^{A+\varepsilon_n\iota,\phi}(u,u) \\
&= \int_X (\{A(x)|_{\mathrm{Ker}(A(x))^\perp}\}^{-1}[\pi_A(x)^*[Du(x)]], \pi_A(x)^*[Du(x)])_x \phi(x) m(dx) \\
&+\infty \cdot \int_X ((\mathbf{I}_{H_x} - \pi_A(x))[\iota^*(x)[Du(x)]], Du(x))_x \phi(x) m(dx), \quad u \in \mathscr{F}.
\end{aligned}
$$

Hence

$$
\begin{cases}
\mathrm{Dom}(\mathscr{E}^{A,\phi}) \subset \mathscr{D}^{A,\phi} \\
\mathscr{E}^{A,\phi}(u,u) \geq \int_X ((A(x)|_{\mathrm{Ker}(A(x))^\perp})^{-1}[Du(x)], Du(x))_x \phi(x) m(dx),
\end{cases}
$$

where

$$
\mathscr{D}^{A,\phi} = \left\{ u \in \mathscr{F}: \begin{array}{c} Du(x) = \pi_A(x)^*[Du(x)] \ m\text{-a.e. } x \in X \text{ and} \\ \int_X ((A(x)|_{\mathrm{Ker}(A(x))^\perp})^{-1}[Du(x)], Du(x))_x \phi(x) m(dx) < \infty \end{array} \right\}.
$$

To see the converse inclusion and inequality, note that $A_n \geq A$ and then that

$$
\pi_A(x)^* \circ \left\{A_n(x) + \frac{1}{m}\iota(x)\right\}^{-1} \circ \pi_A \leq \pi_A(x)^* \circ \left\{A(x) + \frac{1}{m}\iota(x)\right\}^{-1} \circ \pi_A, \quad m\text{-a.e.}
$$

Let $m \to \infty$ and see that

$$
\pi_A(x)^* \circ A_n(x)^{-1} \circ \pi_A \leq \pi_A(x)^* \circ \ \{A(x)|_{\mathrm{Ker}(A(x))^\perp})^{-1} \circ \pi_A, \quad m\text{-a.e.}
$$

Hence if $u \in \mathscr{D}^{A,\phi}$, then

$$
\begin{aligned}
&\mathscr{E}^{A_n,\phi}(u,u) \\
&= \int_X (A_n(x)^{-1}[\pi_A(x)[Du(x)]], [\pi_A(x)[Du(x)]])_x \phi(x) m(dx) \\
&\leq \int_X \left(\left\{A(x)|_{\mathrm{Ker}(A(x))^\perp}\right\}^{-1} [\pi_A(x)[Du(x)]], [\pi_A(x)[Du(x)]]\right)_x \phi(x) m(dx),
\end{aligned}
$$

which implies the converse inclusion and inequality. Thus the characterization with general A_n's has been given.

Another slightly complicated approach to the characterization with general A_n's was given in [8] in the case where X is a compact manifold.

3. On path groups

Let G be a simply connected Lie group with Lie algebra $\mathscr{G}$ ($\equiv$the space of right invariant vector fields). We assume throughout this section that G is of compact type; $\mathscr{G}$ admits an AdG-invariant inner product $\langle\cdot,\cdot\rangle_{\mathscr{G}}$, which we fix. As is well known (cf. [7]), G is of compact type if and only if it is a product of a compact group and $\mathbf{R}^N$. As usual, $\mathscr{G}$ is identified with T_eG ($\equiv$ the tangent space of G at the identity element e), and the inner product $\langle\cdot,\cdot\rangle_{\mathscr{G}}$ is extended to T_gG, $g \in G$, so to be right invariant Riemannian metric.

Let

$$X=\{\mathbf{x}:[0,1]\to G : \mathbf{x} \text{ is continuous and } \mathbf{x}(0)=e\}.$$

Then X is a topological group under pointwise multiplication $(\mathbf{xy})(s)=\mathbf{x}(s)\mathbf{y}(s)$ and the topology of uniform convergence. Let $d=\dim G$ and $\xi_1,\cdots,\xi_d$ be an orthonormal basis of $\mathscr{G}$, and m be a probability measure on X induced by the solution of the Stratonovich stochastic differential equation on G:

$$\begin{cases} d\mathbf{x}(s)=\sum_{i=1}^{d}\xi_i(\mathbf{x}(s))\circ dB^i(s), \\ \mathbf{x}(0)=e, \end{cases}$$

where $(B^1(t),\cdots,B^d(t))$ is the standard Brownian motion on $\mathbf{R}^d$. Note that the probability measure m is the same for any choice of orthonormal basis $\xi_1,\cdots,\xi_d$. Let

$$H=\left\{\mathbf{h}:[0,1]\to\mathscr{G}: \begin{array}{l}\mathbf{h} \text{ is absolutely continuous, } \mathbf{h}(0)=0 \text{ and} \\ \text{the derivative } \dot{\mathbf{h}} \text{ satisfies that } \int_0^1|\dot{\mathbf{h}}(s)|_{\mathscr{G}}^2ds<\infty\end{array}\right\}$$

where $|\cdot|_{\mathscr{G}}=\sqrt{\langle\cdot,\cdot\rangle_{\mathscr{G}}}$. Then H is a real Hilbert space with the inner product

$\langle\mathbf{h},\mathbf{k}\rangle_H=\int_0^1\langle\dot{\mathbf{h}}(s),\dot{\mathbf{k}}(s)\rangle_{\mathscr{G}}ds,\quad \mathbf{h},\mathbf{k}\in H$. Set

$$\mathscr{C}=\left\{u:X\to\mathbf{R}: \begin{array}{l}u(\mathbf{x})=f(\mathbf{x}(s_1),\cdots,\mathbf{x}(s_m)) \text{ for some } f\in C_0^\infty(G^m), \\ 0\le s_1<\cdots<s_m\le 1 \text{ and } m\in\mathbf{N}\end{array}\right\}.$$

For $\xi\in\mathscr{G}$, denote by $\{e^{t\xi}\}_{t\ge 0}$ the integral curve along ξ starting at e;

$$\frac{d}{dt}e^{t\xi}=\xi(e^{t\xi}),\ t\in[0,1],\quad\text{and}\quad e^{t\xi}|_{t=0}=e,$$

and, for $\mathbf{h}\in H$, define $e^{t\mathbf{h}}\in X$ by $(e^{t\mathbf{h}})(s)=e^{t\mathbf{h}(s)}$, $s\in[0,1]$. For $u\in\mathscr{C}$, $Du: X\to H^*$ is given by

$$(\mathbf{h},Du(\mathbf{x}))=\frac{d}{dt}u(e^{t\mathbf{h}}\mathbf{x})|_{t=0},\quad \mathbf{x}\in X,\ \mathbf{h}\in H,$$

where $(\cdot,\cdot)$ stands for the pairing of H and H^*. If one represents $u\in\mathscr{C}$ as $u(\mathbf{x})=f(\mathbf{x}(s_1),\cdots,\mathbf{x}(s_m))$ with $f\in C_0^\infty(G^m)$ and $0\le s_1<\cdots<s_m\le 1$, then it holds that

$$(\mathbf{h}, Du(\mathbf{x})) = \sum_{i=1}^{m} (\mathbf{h}(s_i)^{[i]}f)\,(\mathbf{x}(s_1), \cdots, \mathbf{x}(s_m)),$$

where, for $\xi \in \mathscr{G}$,

$$(\xi^{[i]}f)\,(g_1, \cdots, g_m) = \frac{d}{dt} f(g_1, \cdots, g_{i-1}, e^{t\xi}g_i, g_{i+1}, \cdots, g_m)\big|_{t=0}.$$

As an elementary application of the Girsanov formula, one sees that the operator

$$D: L^2(X; m) \supset \mathscr{C} \ni u \mapsto Du \in L^2(X; H^*, m)$$

is closable (cf. [4, §3]), where $L^2(X; H^*, m) = \{v: X \to H^*: \int_X \|v\|^2_{H^*} dm < \infty\}$. We continue to denote by the same letter D the minimal closed extension of the above D.

Set $H_x = H$, $x \in X$, and $\mathscr{H} = \{H_x\}_{x \in X}$. Then observe that $(X, m, \mathscr{H}, \mathscr{C}, D)$ satisfies Assumptions (A.1) and (A.2). As in Section 2, denote by $(\mathscr{E}, \mathscr{F})$ the closure of the bilinear form in (2.1);

$$\begin{cases} \mathscr{F} = \mathrm{Dom}(D), \\ \mathscr{E}(u, v) = \int_X (\iota^*[Du(\mathbf{x})], Dv(\mathbf{x}))\, m(d\mathbf{x}), \quad u, v \in \mathscr{F}, \end{cases}$$

where $\iota^*: H^* \to H$ is the natural imbedding. Define now $\mathbf{S}$, $\mathbf{S}_+$, $\mathbf{S}_{p+}$ and $\mathscr{P}_+$ in exactly the same way as in Section 2, only this time relative to the above $(\mathscr{E}, \mathscr{F})$. In particular, $\mathscr{E}^{A,\phi}$'s, $A \in \mathbf{S}$, $\phi \in \mathscr{P}_+$, are given as stated just before Lemma 2.3 with this Dirichlet form.

Define a subspace H_0 of H by

$$H_0 = \{\mathbf{h} \in H: \mathbf{h}(1) = 0\}.$$

It was seen by Gross [4, Theorem 2.5 and Lemma 5.2] that

(3.1) if $u \in \mathscr{F}$ satisfies that $(\mathbf{h}, Du(\mathbf{x})) = 0$ m-a.e. $\mathbf{x} \in X$ for every $\mathbf{h} \in H_0$, then there is an $f \in L^2(G; p_1(g)dg)$ such that $u(\mathbf{x}) = f(\mathbf{x}(1))$,

where dg is the Haar measure on G and $\{p_t(g)\}_{t \geq 0}$ is the heat kernel on G associated with $\frac{1}{2}\Delta = \frac{1}{2}\sum_{i=1}^{d} \xi^2_i$. Notice that the function f above is given by $f(g) = \Pi u(g)$ where $\Pi u(g) = \mathbf{E}[u | \mathbf{x}(1) = g]$.

In this section, we investigate an $A \in \mathbf{S}$ of special form. Namely, denote by π the orthogonal projection of H onto the orthogonal complement $H_0^\perp$ of H_0 in H, fix an arbitrary measurable $A_0: X \to H^* \otimes H$ so that $\{A_0(x)\}_{x \in X} \in \mathbf{S}_+$, and define $A \in \mathbf{S}$ by

$$A(\mathbf{x})[\mathbf{h}] = (\pi^* \circ A_0(\mathbf{x}) \circ \pi)[\mathbf{h}], \quad \mathbf{h} \in H.$$

Obviously $A \in \mathbf{S}_{p+}$, because

$$\varepsilon \iota \circ \pi \leq A \leq \varepsilon^{-1} \iota \circ \pi \quad \text{for some } \varepsilon > 0,$$

where $\iota: H \to H^*$ is the natural imbedding of H into H^*. Notice that

$$\mathrm{Ker}(A(\mathbf{x})) = H_0 \quad \text{for every } \mathbf{x} \in X,$$

and hence, by the observation in the previous section, $A(\mathbf{x})|_{H_0^\perp}: H_0^\perp \to \iota(H_0^\perp)$ is bijective for every $\mathbf{x} \in X$. Fix a $\phi \in \mathscr{P}_+$ arbitrarily. Due to Theorem 2.14, one then has that

$$\begin{cases} \mathrm{Dom}(\mathscr{E}^{A,\phi}) = \mathrm{Dom}(\mathscr{E}^{\iota\circ\pi,1}) \\ \qquad = \{u \in \mathscr{F}: Du(\mathbf{x})[\mathbf{h}] = 0 \ m\text{-a.e. } \mathbf{x} \in X \text{ for any } \mathbf{h} \in H_0\}, \\ \mathscr{E}^{A,\phi}(u,v) = \int_X ((A(\mathbf{x})|_{H_0^\perp})^{-1}[\pi^*[Du(\mathbf{x})]], \pi^*[Dv(\mathbf{x})])\phi(\mathbf{x}) m(d\mathbf{x}). \end{cases}$$

It follows from (3.1) that

$$(3.2) \qquad u \in \mathrm{Dom}(\mathscr{E}^{\iota\circ\pi,1}) \Rightarrow u = \Pi u \circ \psi,$$

where $\psi: X \to G$ is given by $\psi(\mathbf{x}) = \mathbf{x}(1)$. Let $\tilde{\mathscr{F}}$ be the closure of $C_0^\infty(G)$ with respect to the norm

$$\|f\|_{L^2(G;p_1(g)dg)} + \|df\|_{L^2(G,T^*G;p_1(g)dg)},$$

where df is the exterior derivative of f and its norm in T^*G is taken with respect to the Riemannian metric induced by $\langle\cdot,\cdot\rangle_{\mathscr{G}}$. For $\xi \in \mathscr{G}$, denote by ξ^* the formal adjoint of ξ acting on $C_0^\infty(G)$ with respect to $p_1(g)dg$, and by $\mathbf{h}^\xi$ the element of H given by $\mathbf{h}^\xi(s) = s\xi$. Observe then that

$$\int_G \xi f_1(g) f_2(g) p_1(g) dg = \int_X (\mathbf{h}^\xi, D(f_1\circ\psi)(\mathbf{x})) f_2(\psi(\mathbf{x})) m(d\mathbf{x}), \quad f_1, f_2 \in C^\infty{}_0(G),$$

so that

$$\int_G (\Pi u)(g)(\xi^* f)(g) p_1(g) dg = \int_X u(\mathbf{x}) D^*((f\circ\psi)\mathbf{h}_\xi) m(d\mathbf{x}),$$

$$u \in \mathrm{Dom}(\mathscr{E}^{\iota\circ\pi,1}),\ f \in C_0^\infty(G),\ \xi \in \mathscr{G},$$

where D^* is the adjoint operator of $D: L^2(X;m) \to L^2(X, H^*; m)$. Hence for $u \in \mathrm{Dom}(\mathscr{E}^{\iota\circ\pi,1})$, Πu is differentiable in the sense of Sobolev and

$$\|d(\Pi u)\|_{L^2(G,T^*G;p_1(g)dg)} < \infty.$$

Thus one concludes that

$$(3.3) \qquad u \in \mathrm{Dom}(\mathscr{E}^{\iota\circ\pi,1}) \Leftrightarrow u = \Pi u \circ \psi \text{ and } \Pi u \in \tilde{\mathscr{F}}.$$

Moreover, in this case, one has that

$$(3.4) \qquad (\mathbf{h}, Du(\mathbf{x})) = {}_{\mathscr{G}}(\mathbf{h}(1), (R_g)_e^*[d(\Pi u)(g)])_{\mathscr{G}^*}|_{g=x(1)},$$

where ${}_{\mathscr{G}}(\cdot,\cdot)_{\mathscr{G}^*}$ is the natural pairing of $\mathscr{G}$ and $\mathscr{G}^*$, $R_g g' = g'g$, $g, g' \in G$, and $(R_g)_e^*: T_g^*G \to T_e^*G$ is its induced pull-back at e.

Thinking of H as a tangent space of X, one can define a differential of the mapping $\phi: X \to G$, i.e., a linear mapping $\phi_*: H \to \mathscr{G} = T_e G$ so that $\phi_*[\mathbf{h}] = \mathbf{h}(1)$, $\mathbf{h} \in H$. Then the pull-back $\phi^*: \mathscr{G}^* \to H^*$ is given by

$$(\mathbf{h}, \phi^* \eta) = {}_{\mathscr{G}}(\phi_* \mathbf{h}, \eta)_{\mathscr{G}^*} = {}_{\mathscr{G}}(\mathbf{h}(1), \eta)_{\mathscr{G}^*}, \quad \mathbf{h} \in H, \eta \in \mathscr{G}^*.$$

Since $\phi^* \eta \in \iota(H_0^\perp), \eta \in \mathscr{G}^*$, one can now define an inner product $\alpha_g^{A,\phi}$ on $T_g^* G$ for dg-a.e. $g \in G$ by

$$\alpha_g^{A,\phi}(\eta, \zeta) = \mathbf{E}[\phi(\mathbf{x})((A(\mathbf{x})|_{H_0^\perp})^{-1}[\phi^*[(R_g)_e^* \eta]], \phi^*[(R_g)_e^* \zeta])|\mathbf{x}(1) = g],$$
$$\eta, \zeta \in T_g^* G.$$

If one sets

$$\begin{cases} \mathrm{Dom}(\tilde{\mathscr{E}}^{A,\phi}) = \tilde{\mathscr{F}}, \\ \tilde{\mathscr{E}}^{A,\phi}(f_1, f_2) = \int_G \alpha_g^{A,\phi}(df_1(g), df_2(g)) p_1(g) dg, \quad f_1, f_2 \in \mathrm{Dom}(\tilde{\mathscr{E}}^{A,\phi}), \end{cases}$$

then one can easily conclude from (3.3) and (3.4) that

$$\mathrm{Dom}(\tilde{\mathscr{E}}^{A,\phi}) = \Pi(\mathrm{Dom}(\mathscr{E}^{A,\phi})),$$
$$\tilde{\mathscr{E}}^{A,\phi}(\Pi u, \Pi v) = \mathscr{E}^{A,\phi}(u, v), \quad u, v \in \mathrm{Dom}(\mathscr{E}^{A,\phi}).$$

Recall that $\mathrm{Dom}(\tilde{\mathscr{E}}^{A,\phi}) = \mathrm{Dom}(\mathscr{E}^{\iota \circ \pi, 1})$, and notice that, for some $\delta > 0$,

$$\delta \mathscr{E}^{\iota \circ \pi, 1}(u, u) \leq \mathscr{E}^{A,\phi}(u, u) \leq \delta^{-1} \mathscr{E}^{\iota \circ \pi, 1}),$$

and that

$$\alpha_g^{\iota \circ \pi, 1}(\eta, \zeta) = {}_{\mathscr{G}}\langle (R_g)_*^{-1} \eta, (R_g)_*^{-1} \zeta \rangle_{\mathscr{G}^*}, \quad \eta, \zeta \in T_g^* G.$$

Then these identities implies that $(\tilde{\mathscr{E}}^{A,\phi}, \mathrm{Dom}(\tilde{\mathscr{E}}^{A,\phi}))$ is a regular Dirichlet form on $L^2(G; P_1(g) dg)$. Further, by the very definition of resolvent, one sees that

$$G_\alpha^{A,\phi} u = \tilde{G}_\alpha^{A,\phi}[\Pi u] \circ \phi, \quad T^{A,\phi}{}_t u = \tilde{T}_t^{A,\phi}[\Pi u] \circ \phi, \quad u \in L^2(X; m^\phi),$$

where $\{\tilde{G}_\alpha^{A,\phi}\}_{\alpha > 0}$ and $\{\tilde{T}_t^{A,\phi}\}_{t > 0}$ are the resolvent and the semigroup corresponding to $\tilde{\mathscr{E}}^{A,\phi}$. Thus one finally arrives at

Theorem 3.5. *Let a measurable $A_0: X \to H^* \otimes H$ satisfy $\{A_0(x)\}_{x \in X} \in$ $\mathbf{S}_+$, and define $A \in \mathbf{S}_{p+}$ as above. Suppose that $A_n \in \mathbf{S}$ and $\phi, \phi_n \in \mathscr{P}_+$ enjoy that $A_n \geq A$, $A_n \to A$, and $\|\phi_n - \phi\|_\infty \to 0$. Then it holds that*

$$\lim_{n \to \infty} \mathscr{E}^{A_n, \phi_n}(u, v) = \tilde{\mathscr{E}}^{A,\phi}(\Pi u, \Pi v), \quad u, v \in \mathrm{Dom}(\mathscr{E}^{A,\phi}),$$
$$\lim_{n \to \infty} G^{A_n, \phi_n}{}_\alpha = \phi^* \circ \tilde{G}_\alpha^{A,\phi} \circ \Pi \quad \text{strongly in } L^2(X; m^\phi),$$
$$\lim_{n \to \infty} T_t^{A_n, \phi_n} = \phi^* \circ \tilde{T}_t^{A,\phi} \circ \Pi \quad \text{strongly in } L^2(X; m^\phi),$$

where $\phi^*: L^2(G; p_1(g) dg) \to L^2(X; m)$ is defined by $\phi^* f = f \circ \phi$.

4. On Riemannian manifolds

Let X be an N-dimensional Riemannian manifold with Riemannian metric g_0, and Ω_0 be the Riemannian volume element. Throughout this section, suppose that $\mathrm{vol}(X, g_0) < \infty$, and set $m = |\Omega_0|/\mathrm{vol}(X, g_0)$. For each $x \in X$, denote by H_x the Hilbert space obtained by equipping the tangent space T_xX at x with the inner product $\langle \cdot, \cdot \rangle_x$ induced by g_0. Observe that $(X, m, \mathscr{H} \equiv \{H_x\}_{x \in X}, C_0^\infty(X), d)$, d being the exterior derivative, satisfies Assumptions (A.1) and (A.2). The closure of the bilinear form in (2.1) with $D = d$ will be denoted by $(\mathscr{E}, \mathscr{F})$. Every $u \in \mathscr{F}$ has a derivative $du \in L^2(X, T^*M; m)$ with $du(x) \in T_x^*M$ m-a.e., and hence, for any smoothe vector field Y on X, Yu can be defined through the measurable mapping $x \to (du(x), Y(x))_x$.

Let $T_2^0 X$ be the bundle of $(0, 2)$-tensors over X with the projection τ: $T_2^0 X \to X$. A measurable mapping $g: X \to T_2^0 X$ with $\tau(g(x)) = x$, $x \in X$, is said to be symmetric (resp. non-negative definite) if so is each $g(x) \in T_x^* X \otimes T_x^* X$, $x \in X$. Let

$$\mathbf{T} = \left\{ g: X \to T_2^0 X: \begin{array}{l} g \text{ is measurable, with } \tau(g(x)) = x,\ x \in X, \\ \text{symmetric, non-negative definite, and} \\ \sup_{x \in X} \sup_{\substack{\xi \in H_x \\ \|\xi\|_x = 1}} |g(x)(\xi, \xi)| < \infty \end{array} \right\}.$$

Identify $T_x^*X \otimes T_x^*X$ with the space of linear mappings of T_xX to T_x^*X, to get

$$\mathbf{T} \subset \mathbf{S}.$$

Due to the definition of the topology in $\mathbf{S}$, we also see that, for $g_n, g \in \mathbf{T}$,

$$g_n \to g \quad \text{in } \mathbf{S} \Leftrightarrow d_T(g_n, g) \to 0,$$

where

$$d_T(g_n, g) = \sup \{ |g_n(x)(h,h) - g(x)(h,h)| : h \in H_x, \|h\|_x \le 1, x \in X \}.$$

Moreover, one has that

$$\mathbf{T} \subset \mathbf{S}_{p+}, \tag{4.1}$$

where $\mathbf{S}_{p+}$ is that given just after (1.13). Indeed, let $g \in \mathbf{T}$, and $\pi_g(x)$ be the projection of H_x onto $\mathrm{Ker}(g(x))^\perp$. A simple computation leads to

$$\pi_g(x) = \lim_{n \to \infty} (g(x) + \frac{1}{n}\iota(x))^{-1} g(x), \quad x \in X.$$

Thus the mapping

$$x \mapsto (\iota^*(x)[\pi_g(x)^*[du(x)]], \pi_g(x)^*[du(x)])_x$$

is measurable for every $u \in C_0^\infty(X)$. Since $\dim H_x = N$,

$$g(x) \geq \varepsilon(x)\iota(x)\circ\pi_g(x),\ x \in X \quad \text{for some } \varepsilon: X\to(0,\infty),$$

and hence (4.1) has been verified.

Due to Theorems 2.6 and 2.14, one now has that

Theorem 4.2. *Let $g, g_n \in \mathbf{T}$ and $\phi_n, \phi \in \mathscr{P}_+$.*

(i) *Suppose that $g_n \geq g$, $d_T(g_n, g)\to 0$, and $\|\phi_n-\phi\|_\infty\to 0$. Then it holds that*

$$\mathscr{E}^{g,\phi}(u,v) = \lim_{n\to\infty}\mathscr{E}^{g_n,\phi_n}(u,v), \quad u, v \in \mathrm{Dom}(\mathscr{E}^{g,\phi}),$$

$$G_\alpha^{g,\phi} = \lim_{n\to\infty} G_\alpha^{g_n,\phi_n} \qquad \textit{strongly in } L^2(X; m^\phi),$$

$$T_t^{g,\phi} = \lim_{n\to\infty} G_t^{g_n,\phi_n} \qquad \textit{strongly in } L^2(X; m^\phi).$$

(ii) *It holds that*

$$\mathrm{Dom}(\mathscr{E}^{g,\phi}) = \left\{ u \in \mathscr{F}: \begin{array}{l} du(x) \perp \mathrm{Ker}(g(x))\ m\text{-a.e.}x \in X \textit{ and} \\ \int_X g^*(x)(du(x), du(x))m^\phi(dx) < \infty \end{array} \right\},$$

$$\mathscr{E}^{g,\phi}(u,v) = \int_X g^*(x)(du(x), dv(x))m^\phi(dx), \qquad u,v \in \mathrm{Dom}(\mathscr{E}^{g,\phi}),$$

where for $\eta \in H_x^*$ we mean by $\eta \perp \mathrm{Ker}(g(x))$ *that* $(\xi, \eta)_x = 0$ for any $\xi \in \mathrm{Ker}(g(x))$, *and $g^*(x)$ is given by*

$$g^*(x)(l_1,l_2) = ((g(x)|_{\mathrm{Ker}(g(x))^\perp})^{-1}[l_1], l_2)_x \qquad \textit{for } l_1,l_2 \in \iota(x)(\mathrm{Ker}(g(x))^\perp).$$

Remark 4.3. In the case where X is compact, the assertion was seen in [5, 8] under an assumption on complete integrability of distribution $x \mapsto \mathscr{D}_x \equiv \mathrm{Ker}(g(x))$.

In the remainder of this section, we shall give a characterization of $\mathscr{E}^{g,\phi}$ analogous to that made before Theorem 3.5. To do this, let $g \in \mathbf{T}$ and $\phi \in \mathscr{P}_+ \cap C^0(X)$, and T_0^1X be the tangent bundle over X. Assume first that

(F.1) the C^∞ differential system $\mathscr{D}\,(\equiv \{Y \in T_0^1X: Y(x) \in \mathscr{D}_x, x \in X\})$, is an N_0-dimensional completely integrable differential distribution.

Let F_x be the maximal connected integral manifold of $\mathscr{D}$ passing through $x \in X$, and introduce an equivalent relation $\sim$ on X so that $x\sim y$ if $F_x=F_y$. Denote by M the quotient space with respect to the equivalent relation, and assume secondly that

(F.2) M is an $N'\,(=N-N_0)$-dimensional manifold and the projection $\psi: X\to M$ is C^1.

Let $\psi_* m^\phi$ be the induced measure of m^ϕ on M through ψ. Define a contraction mapping $\Pi: L^2(X; m^\phi)\to L^2(M; \psi_* m^\phi)$ by

$$\Pi u(p) = \mathbf{E}^{m^\phi}[u|\psi=p].$$

Then one has that

$$u=\Pi u\circ\psi \quad \text{for every } u \in \mathrm{Dom}(\mathscr{E}^{g,\phi}). \tag{4.4}$$

In fact, fix an arbitrary $x \in X$, and apply Frobenius' theorem to find a cubic coordinate system $(U, x^1,\cdots,x^N)$ around x so that $\{z \in U : x^i(z)=x^i(y),\ 1\leq i \leq N'\}$ is an integral manifold of $\mathfrak{D}$ through y for every $y \in U$. For $Y \in \mathfrak{D}$ with $\mathrm{supp}[Y] \supset U$ and $u \in \mathrm{Dom}(\mathscr{E}^{g,\phi})$, it then holds that

$$0=\int_U |Yu(x^1,\cdots,x^N)|^2 k(x^1,\cdots,x^N)\,dx^1\cdots dx^N,$$

where

$$k(x)=\frac{\phi(x^1,\cdots,x^N)\,|\Omega_0|(dx^1\cdots dx^N)}{dx^1\cdots dx^N}(x) \in C^0(U) \text{ and } >0 \text{ on } U.$$

Since $Y=\sum_{i=N'+1}^{N} Y^i(x)\,(\partial/\partial x^i)$ on U, this means that

$$u(y)=u^*(x^1(y),\cdots,x^{N'}(y)),\quad dx^1\cdots dx^N\text{-a.e. } y \in U \text{ for some } u^*.$$

Due to the connectedness of F_y, one comes to (4.4).

One can now define a symmetric bilinear form $\tilde{\mathscr{E}}^{g,\phi}$ on $L^2(M;\psi_*m^\phi)$ by

$$\begin{cases} \mathrm{Dom}(\tilde{\mathscr{E}}^{g,\phi})=\{\Pi u : u \in \mathrm{Dom}(\mathscr{E}^{g,\phi})\}, \\ \tilde{\mathscr{E}}^{g,\phi}(\Pi u,\Pi v)=\mathscr{E}^{g,\phi}(u,v), \qquad u,v \in \mathscr{E}^{g,\phi}). \end{cases}$$

Noticing that the definition of Π and (4.4) imply that

$$\begin{cases} \|\Pi u\|_{L^2(M;\psi_*m^\phi)} \leq \|u\|_{L^2(X;m_\phi)}, & u \in L^2(X;m^\phi), \\ \|\Pi u\|_{L^2(M;\psi_*m^\phi)} = \|u\|_{L^2(X;m^\phi)}, & u \in \mathrm{Dom}(\mathscr{E}^{g,\phi}), \end{cases}$$

one sees that $\tilde{\mathscr{E}}^{g,\phi}$ with domain $\mathrm{Dom}(\tilde{\mathscr{E}}^{g,\phi})$ is a closed symmetric bilinear form on $L^2(M;\psi_*m^\phi)$.

Apply Frobenius' theorem again to observe that each $f \in \mathrm{Dom}(\tilde{\mathscr{E}}^{g,\phi})$ admits a locally ψ_*m^ϕ-integrable exterior derivative df, and that the pull-backed space $(\psi^*)_x(T^*_{\psi(x)}M)$ is contained in $\iota(x)(\mathrm{Ker}(g(x))^\perp)$. For $f_1,f_2 \in \mathrm{Dom}(\tilde{\mathscr{E}}^{g,\phi})$, we now define

$$g^*(\psi^*df_1,\psi^*df_2)(x)=g^*(x)\,((\psi^*)_x(df_1)(\psi(x)),(\psi^*)_x(df_2)(\psi(x))) \quad \text{for } m^\phi\text{-a.e. } x \in X,$$

and

$$h^*(df_1,df_2)(p)=\Pi\,[g^*(\psi^*df_1,\psi^*df_2)](p) \qquad \text{for } \psi_*m^\phi\text{-a.e. } p \in M.$$

By Theorem 4.2(ii) and (4.4), one then has that

$$\tilde{\mathscr{E}}^{g,\phi}(f_1,f_2)=\int_M h^*(df_1,df_2)(p)\,\psi_*m^\phi(dp), \qquad f_1,f_2 \in \mathrm{Dom}(\tilde{\mathscr{E}}^{g,\phi}). \tag{4.5}$$

By the very definition of resolvents, one further sees that

$$G_\alpha^{g,\phi}u=\widetilde{G}_\alpha^{g,\phi}[\Pi u]\circ\psi, \quad \text{and hence} \quad T_t^{g,\phi}u=\widetilde{T}_t^{g,\phi}[\Pi u]\circ\psi, \quad u \in L^2(X; m^\phi),$$

where $\{\tilde{G}_\alpha^{g,\phi}\}_{\alpha>0}$ and $\{\tilde{T}_t^{g,\phi}\}_{t>0}$ are the resolvent and the semigroup corresponding to $\tilde{\mathcal{E}}^{g,\phi}$. Thus we finally arrive at

Theorem 4.6. *Let $g \in \mathbf{T}$ and $\phi \in \mathcal{P}_+$* be as above. *Suppose that $g_n \in \mathbf{T}$ and $\phi_n \in \mathcal{P}_+$ enjoy that $g_n \geq g$, $d_T(g_n, g) \to 0$, and $\|\phi_n-\phi\|_\infty \to 0$. Then it holds that*

$$\lim_{n\to\infty}\mathcal{E}^{g_n,\phi_n}(u, v)=\tilde{\mathcal{E}}^{g,\phi}(\Pi u, \Pi v), \qquad u,v \in \mathrm{Dom}(\mathcal{E}^{g,\phi}),$$

$$\lim_{n\to\infty} G_\alpha^{g_n,\phi_n}=\psi^*\circ\widetilde{G}^{g,\phi}{}_\alpha\circ\Pi \qquad \textit{strongly in } L^2(X; m^\phi),$$

$$\lim_{n\to\infty} T_t^{g_n,\phi_n}=\psi^*\circ\widetilde{T}^{g,\phi}{}_t\circ\Pi \qquad \textit{strongly in } L^2(X; m^\phi),$$

where $\psi^*: L^2(X; m^\phi) \to L^2(M; \psi_* m^\phi)$ is defined by $\psi^* f=f\circ\psi$.

Remark 4.7. If M admits a complete Riemannian metric, then it has a nice cut-off function, whence the closed symmetric bilinear form $\tilde{\mathcal{E}}^{g,\phi}$ in (4.5) is a C_0-regular Dirichlet form. Moreover, in this case, the form has the local property.

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