J. Math. Kyoto Univ. (JMKYAZ)
36-1 (1996) 45-60

Lie algebra of the infinitesimal automorphisms on S^3 and its central extension

By

Tosiaki KORI

0. Introduction

In this paper we shall deal with a central extension of the Lie algebra of infinitesimal automorphisms on S^3. Such a central extension on the circle is famous in the name of Virasoro algebra. The Lie algebra *Vect* (S^1) of infinitesimal automorphisms on the circle is generated by (the restriction on S^1) of

$$L_m = z^m\left(z\frac{d}{dz}\right), \quad m = 0, \pm 1, \cdots,$$

where we look $S^1 = \{z \in C; |z| = 1\}$, with the commutation relation

$$[L_m, L_n] = (n-m) L_{m+n}.$$

A two cocycle on *Vect* (S^1) is given by the formula

$$c(L_m, L_n) = -\frac{1}{12} n(n^2-1)\delta_{n+m,0}. \tag{0-1}$$

Virasoro algebra is the central extension associated with this two cocycle. A highest weight representation of the Virasoro algebra is generated by a highest weight representation of the affine Lie algebra S^1g (Sugawara construction) [K]. Though we have not a satisfactory theory on the highest weight representation of the (abelian) extension of S^3g [M-R] and do not know about the action of *Vect* (S^3) on the representation space of current algebra the author thinks it is worth trying to have a central extension of *Vect* (S^3).

In [K-K] it was shown that the two cocycle (0-1) is derived from the non-commutative residue on the cotangent bundle of S^1, that is,

$$c(X, Y) = \int_{|z|=1} res\ [\ln|\zeta|^2, symb\ X] \cdot symb\ Y.$$

Here *symb* X is the pseudodifferential symbol and ζ denotes fiber coordinate. (Actually their derivation of (0-1) should be corrected a little. See the

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Communicated by Prof. T. Hirai, January 6, 1995

discussion in section 5.) We shall extend this method to have our central extension of $Vect(S^3)$.

In the above explanation z^m's are spherical functions for a heighest weight representation of Lie group $U(1)$ acting on S^1; $z\frac{d}{dz}=mz^m$. These weight functions enjoy the property that they are closed under products. In sections 1 to 3 we shall give a class of spherical functions for a heighest weight representation of $SU(2)$ acting on S^3 such that the product is expressed by their linear combination. Such a property has been investigated in [Ru, V] earlier and we present a new (dual pair of) basis of the space of spherical functions. (The author thinks this is the only new point through sections 1 to 3.) These spherical functions are very commode to describe the Lie algebra $Vect(S^3)$ and to construct a two-cocycle on it. The Lie algebra $Vect(S^3)$ is introduced in 4.2 and the commutation relations are given in Proposition 4.2, The draft of this work was distributed in 1992 as volume No.92-12 of Report of Science and Engineering Research Laboratory of Waseda University.

1. Harmonic polynomials on C^2

1. 1. We introduce first the following vector fields that form a frame on $C^2-\{0\}$:

$$\nu=z_1\frac{\partial}{\partial z_1}+z_2\frac{\partial}{\partial z_2},\quad \bar{\nu}=\bar{z}_1\frac{\partial}{\partial \bar{z}_1}+\bar{z}_2\frac{\partial}{\partial \bar{z}_2},$$

(1-1-1)

$$\varepsilon=-\bar{z}_2\frac{\partial}{\partial z_1}+\bar{z}_1\frac{\partial}{\partial z_2},\quad \bar{\varepsilon}=-z_2\frac{\partial}{\partial \bar{z}_1}+z_1\frac{\partial}{\partial \bar{z}_2}.$$

Put

$$\mathbf{n}=\frac{1}{2}(\nu+\bar{\nu}),\ \theta_0=\frac{1}{2\sqrt{-1}}(\nu-\bar{\nu}),\ \theta_1=\frac{1}{2}(\varepsilon+\bar{\varepsilon}),\ \theta_2=\frac{1}{2\sqrt{-1}}(\varepsilon-\bar{\varepsilon}).$$

$\mathbf{n}$ is the normal to the sphere $\{|z|=const\}$ and $\{\theta_0, \varepsilon, \bar{\varepsilon}\}$ form a basis for the induced tangential Cauchy-Riemann structure on the sphere.

There is another quartet of frame on $C^2-\{0\}$,

$$\mu=z_2\frac{\partial}{\partial z_2}+\bar{z}_1\frac{\partial}{\partial \bar{z}_1},\ \bar{\mu}=\bar{z}_2\frac{\partial}{\partial \bar{z}_2}+z_1\frac{\partial}{\partial z_1},$$

(1-1-2)

$$\delta=\bar{z}_2\frac{\partial}{\partial \bar{z}_1}-z_1\frac{\partial}{\partial z_2},\ \bar{\delta}=z_2\frac{\partial}{\partial z_1}-\bar{z}_1\frac{\partial}{\partial \bar{z}_2}.$$

These vector fields give also a frame on $C^2-\{0\}$. We have $\mathbf{n}=\frac{1}{2}(\mu+\bar{\mu})$. Put

$$\tau_0=\frac{1}{2\sqrt{-1}}(\mu-\bar{\mu}),\ \tau_1=\frac{1}{2}(\delta+\bar{\delta}),\ \tau_2=\frac{1}{2\sqrt{-1}}(\delta-\bar{\delta}).$$

On the unit sphere $S^3=\{|z|=1\}$ we have the following commutation relations:

$$(1\text{-}1\text{-}3)\qquad \begin{aligned}&[\theta_0, \varepsilon]=\sqrt{-1}\varepsilon, \quad [\theta_0, \bar{\varepsilon}]=-\sqrt{-1}\bar{\varepsilon}, \quad [\varepsilon, \bar{\varepsilon}]=2\sqrt{-1}\theta_0.\\ &[\tau_0, \delta]=\sqrt{-1}\delta, \quad [\tau_0, \bar{\delta}]=-\sqrt{-1}\bar{\delta}, \quad [\delta, \bar{\delta}]=2\sqrt{-1}\tau_0.\end{aligned}$$

$$(1\text{-}1\text{-}4)\qquad [\varepsilon, \delta]=[\bar{\varepsilon}, \delta]=[\varepsilon, \bar{\delta}]=[\bar{\varepsilon}, \bar{\delta}]=[\theta_0, \tau_0]=0.$$

1. 2 On C^2 we consider the natural metric $dz_1 \otimes d\bar{z}_1 + dz_2 \otimes d\bar{z}_2$, and on the sphere $S^3=\{|z|=1\}$ we consider the induced metric. With respect to this metric $\{\sqrt{2}\,\theta_0, \sqrt{2}\,\theta_1, \sqrt{2}\,\theta_2\}$ form an orthonormal frame on S^3. Similarly $\{\sqrt{2}\,\tau_0, \sqrt{2}\,\tau_1, \sqrt{2}\,\tau_2\}$ also give an orthonormal frame for the same metric. $\sqrt{2}\,\mathbf{n}$ is the unit normal to the sphere. Laplacian on C^2 is given by $\Delta=\frac{\partial^2}{\partial z_1 \partial \bar{z}_1}+\frac{\partial}{\partial z^2 \partial \bar{z}_2}$. The Laplace-Beltrami operator on $C^2-\{0\}$ is given by $\Delta_1=(\theta_0^2+\theta_1^2+\theta_2^2)=(\tau_0^2+\tau_1^2+\tau_2^2)$. We have the decomposoition;

$$\Delta=\frac{1}{|z|^2}(\mathbf{n}^2+\mathbf{n}+\Delta_1).$$

The separation of variable method to obtain the spherical expansion of harmonic functions by the eigenvectors of the Laplace-Beltrami operator on the boundary is well known. We note that we have two candidates depending on which frame of vector fields θ_i or τ_i we use.

Let Δ_1 be Laplace-Beltrami operator on the unit sphere $S^3=\{|z|=1\}$. $-\Delta_1$ being a second order elliptic differential operator, the eigenvalues of $-\Delta_1$ are nonnegative with only accumulation point at infinity and the eigenfunctions form a complete system in $L^2(S^3, d\sigma)$, where σ is the normalized surface measure. Let $\{\phi_\lambda\}_{\lambda\geq 0}$ be the set of eigenfunctions of Δ_1 on the unit sphere; $\Delta_1\phi_\lambda=\lambda\phi_\lambda$. Then every harmonic function h in a unit ball $D=\{|z|<1\}$ with L^2- boundary value on S^3 has the expansion of the form;

$$(1\text{-}2\text{-}1)\qquad h(z)=\sum_\lambda c_\lambda a_\lambda(|z|)\,\phi_\lambda\left(\frac{z}{|z|}\right),$$

where $a_\lambda(t)=t^{\sqrt{4\lambda+1}-1}$.

1. 3

a. A polynomial P on C^2 is said to be *of type* (p, q) if

$$(1\text{-}3\text{-}1)\qquad P(az_1, az_2, b\bar{z}_1, b\bar{z}_2)=a^p b^q P(z_1, z_2, \bar{z}_1, \bar{z}_2).$$

Let $\widehat{S}^{p,q}$ be the set of polynomials of type (p, q).
Similarly a polynomial that satisfies

$$(1\text{-}3\text{-}2)\qquad P(az_1, bz_2, b\bar{z}_1, a\bar{z}_2)=a^k b^l P(z_1, z_2, \bar{z}_1, \bar{z}_2)$$

is called *of class* (k, l). The set of polynomials of class (k, l) is denoted by $S_{k,l}$.

Let H be the set of harmonic polynomials on C^2 and put

$$\widehat{H}^{p,q}=H\cap\widehat{S}^{p,q},\quad H_{k,l}=H\cap S_{k,l}.$$

The following facts are proved routinely [T].

Proposition 1. 1.

(1)

$$\widehat{S}^{p,q}=\widehat{H}^{p,q}\oplus|z|^2\widehat{S}^{p-1,q-1},\quad S_{p,q}=H_{p,q}\oplus|z|^2S_{p-1,q-1}.$$

(2)

$$\dim\widehat{H}^{p,q}=\dim H_{p,q}=p+q+1.$$

We have the following decomposition of H to direct sums:

$$H=\sum_{p,q}\widehat{H}^{p,q},\quad H=\sum_{k,l}H_{k,l}. \tag{1-3-3}$$

We shall see in the next section that these are decompositions of H as irreducible representation spaces of $SU(2)$.

b. In the sequel we shall use the multiindices $\alpha=(\alpha_1,\alpha_2)$, α_i's being non-negative integers, and the notation $z^\alpha=z_1^{\alpha_1}z_2^{\alpha_2}$ for $z=(z_1, z_2)\in C^2$. The meaning of the notations S_α, H_α or $\widehat{H}^\alpha$ will be obvious from **a**. We shall also write $|\alpha|=\alpha_1+\alpha_2$.

Put

$$h_\alpha^q(z)=\varepsilon^q(z^\alpha),\quad \text{for } 0\leq q\leq|\alpha|. \tag{1-3-4}$$

Proposition 1. 2. *For each* α, h_α^q; $q=0,1,\cdots,|\alpha|$, *give a basis of* H_α

There is on the other hand a series of polynomials generated by the operation of δ that constitute a basis of $\widehat{H}^\alpha$.

Put

$$\widehat{h}_q^\alpha(z)=\delta^q(\bar{z}_1^{\alpha_1}z_2^{\alpha_2}). \tag{1-3-5}$$

We see that $\widehat{h}_q^\alpha(z)$ is a harmonic polynomial.

Proposition 1. 3. *For every* α, $\widehat{h}_q^\alpha$; $q=0,1,\cdots,|\alpha|$, *give a basis of* $\widehat{H}^\alpha$.

We have the following relations;

Lemma 1. 4.

$$(-1)^{b+q}(a+b-q)!\,\overline{h_{(a,b)}^q}=q!\,h_{(b,a)}^{a+b-q},\quad (-1)^a b!\,\widehat{h}_a^{(a+b-q,q)}=q!\,h_{(a,b)}^{a+b-q}.$$

Proposition 1. 5.

(1)

$$\sum_{k=0}^r H_{k,r-k}=\sum_{k=0}^r\widehat{H}^{r-k,k}.$$

(2)

$$H_{k,r-k}\cap\widehat{H}^{s-q,q}=\begin{cases}0 \text{ if } s\neq r\\ Ch^{q}_{(k,r-k)} \text{ if } s=r\end{cases}$$

The proposition follows from Proposition 1.1 and Lemma 1.4.

1. 4.

a. We shall describe the operations of θ_0, ε, etc. on the space of harmonic polynomials H. These will give an infinitesimal representation of $su(2)$ as we shall see in the next section.

Lemma 1. 6.

(1)

$$\theta_0\varepsilon^q=\varepsilon^q\theta_0+\sqrt{-1}q\varepsilon^q,\quad \bar{\varepsilon}\varepsilon^q=\varepsilon^q\bar{\varepsilon}-2q\sqrt{-1}\varepsilon^{q-1}\theta_0+q(q-1)\varepsilon^{q-1}.$$

(2)

$$\tau_0\delta^q=\delta^q\tau_0+\sqrt{-1}q\delta^q,\quad \bar{\delta}\delta^q=\delta^q\bar{\delta}-2q\sqrt{-1}\delta^{q-1}\tau_0+q(q-1)\delta^{q-1}.$$

The lemma follows from the commutation relations (1-1-4). This lemma implies the following calculation.

Proposition 1. 7.

(1) $\sqrt{-1}\,\theta_0 h^q_\alpha=\left(\frac{|\alpha|}{2}-q\right)h^q_\alpha$ *for* $q=0,1,\cdots,|\alpha|$.

(2) $\varepsilon h^q_\alpha=h^{q+1}_\alpha$.

(3) $\bar{\varepsilon}h^q_\alpha=-q(|\alpha|-q+1)h^{q-1}_\alpha$.

Similarly we have;

Proposition 1. 8.

(1) $\sqrt{-1}\,\tau_0\widehat{h}^\alpha_q=\left(\frac{|\alpha|}{2}-q\right)\widehat{h}^\alpha_q$ *for* $q=0,1,\cdots,|\alpha|$.

(2) $\delta\widehat{h}^\alpha_q=\widehat{h}^\alpha_{q+1}$.

(3) $\bar{\delta}\widehat{h}^\alpha_q=-q(|\alpha|-q+1)\widehat{h}^{|\alpha|-k,k}_{q-1}$.

Proposition 1. 9.

$$\Delta_1\widehat{h}^\alpha_q=-\frac{|\alpha|(|\alpha|+2)}{4}\widehat{h}^\alpha_q,$$
$$\Delta_1 h^q_\alpha=-\frac{|\alpha|(|\alpha|+2)}{4}h^q_\alpha.$$

These follow from 1.2 and the above lemmas.

(1-2-1) and Proposition 1.9 yield that every harmonic function h with L^2- boundary values on $\{|z|<1\}$ has the expansion

(1-4-1) $$h(z)=\sum_{\alpha,p}c^p_\alpha h^p_\alpha(z),$$

which converges compact uniformly.

b. As is shown in the following the decomposition (1-3-3) is orthogonal with respect to the spherical measure on S^3. The 3-form which gives the spherical measure $\sigma(dz)$ is defined by $i_n(dz \wedge d\bar{z}) = -\frac{1}{4}\theta_0^* \wedge \theta_1^* \wedge \theta_2^* = \frac{\sqrt{-1}}{2}$ $\theta_0^* \wedge \varepsilon^* \wedge \bar{\varepsilon}^*$, where i indicates the inner derivation and θ_0^* etc. are dual 1-forms of θ_0 etc.. The inner product of two functions on S^3 is

$$(f, g) = \int_{\{|z|=1\}} f(z)\overline{g(z)}\,\sigma(dz).$$

We see that the adjoint operator of ε is $-\bar{\varepsilon}$ and θ_0 is selfadjoint.

(1) **Proposition 1. 10.**

$$(h_\alpha^p, h_\beta^q) = \delta_{p,q}\delta_{\alpha,\beta}\frac{\alpha\,!}{(|\alpha|+1)}\frac{p\,!}{(|\alpha|-p)\,!}$$

(2)

$$(\widehat{h}_p^\alpha, \widehat{h}_q^\beta) = \delta_{p,q}\delta_{\alpha,\beta}\frac{\alpha\,!}{(|\alpha|+1)}\frac{p\,!}{(|\alpha|-p)\,!}$$

where $\alpha\,! = \alpha_1\,!\,\alpha_2\,!$.

We have used the formula

$$\int_B |z_1^a z_2^b|^2 d\sigma = \frac{a\,!\,b\,!}{(a+b+1)\,!}.$$

2. Infinitesimal representation of SU(2)

2. 1. Let $SU(2)$ be the special unitary group and $su(2)$ be its Lie algebra. We regard often $z \in S^3$ as the element of $SU(2)$ given by

(2-1-1) $$\dot{z} = \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}.$$

The left action of $SU(2)$ on S^3 is defined for $g = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}$ and $z = (z_1, z_2)$ by

(2-1-2) $$g \cdot z = (az_1 - \bar{b}z_2,\ bz_1 + \bar{a}z_2).$$

Similarly the right action is defiened by

(2-1-3) $$z \cdot g = (\bar{a}z_1 + b\bar{z}_2,\ \bar{a}z_2 - b\bar{z}_1).$$

Both actions are free and transitive.

For a continuous function on S^3 we put

(2-1-4) $$L_g f(z) = f(g^{-1} \cdot z), \quad R_g f(z) = f(z \cdot g).$$

L_g (resp. R_g) is extended to a unitary operator on $L^2(S^3, d\sigma)$ and give a unit-

ary representation of $SU(2)$.
We take a basis of the Lie algebra $su(2)$ given as follows;
(2-1-5)

$$e_0=\frac{1}{2}\begin{pmatrix}\sqrt{-1} & 0\\ 0 & \sqrt{-1}\end{pmatrix},\quad e_1=\frac{1}{2}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\quad e_2=\frac{1}{2}\begin{pmatrix}0 & \sqrt{-1}\\ \sqrt{-1} & 0\end{pmatrix}.$$

Proposition 2. 1.

$$dR(e_0)=\theta_0,\quad dR(e_1)=\theta_1,\quad dR(e_2)=\theta_2.$$

Proposition 2. 2.

$$dL(e_0)=-\tau_0,\quad dL(e_1)=-\tau_1,\quad dL(e_2)=-\tau_2.$$

Propsitions 1.7 and 2.1 yield, for each r and α with $|\alpha|=r$, the following $(r+1)$ − dimensional representation (dR, H_α) of Lie algebra $sl(2, C)$ with highest weight $\frac{r}{2}$:

(2-1-6) $$dR(e_0)h_\alpha^q=-\sqrt{-1}\left(\frac{r}{2}-q\right)h_\alpha^q \text{ for } q=0,1,\cdots,r,$$

(2-1-7) $$dR(e_-)h_\alpha^q=-h_\alpha^{q+1},\quad dR(e_-)h_\alpha^r=0,$$

(2-1-8) $$dR(e_+)h_\alpha^q=q(r-q+1)h_\alpha^{q-1},\quad dR(e_+)h_\alpha^0=0,$$

where

$$e_-=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\quad e_+=\begin{pmatrix}0 & -1\\ 0 & 0\end{pmatrix}.$$

and dR is extended to $sl(2, C)$. All weights are half odd integers.
Similar formula for the representation $(dL, \widehat{H}^\alpha)$, $|\alpha|=r$, holds.

Theorem 2. 3. (1) *The space H of harmonic polynomials on C^2 is decomposed by the action R of $SU(2)$ into*

$$H=\sum_r\sum_{|\alpha|=r}H_\alpha.$$

Each induced representation $R_\alpha=(R, H_\alpha)$, with $|\alpha|=r$, is an irreducible representation with highest weight $\frac{r}{2}$.

(2) *The decomposition of H by the action L of $SU(2)$ is given by*

$$H=\sum_r\sum_{|\alpha|=r}\widehat{H}^\alpha.$$

Each induced representation $L^\alpha=(L, \widehat{H}^\alpha)$, with $|\alpha|=r$, is an irreducible representation with highest weight $\frac{r}{2}$.

Let C be the Casimir operator of $su(2)$;

(2-1-9) $$C=\frac{1}{2}e_0^2+\frac{1}{4}\{e_+e_-+e_-e_+\}.$$

Then we have the following;

Proposition 2. 4.

$$dR_\alpha(C)=dL^\alpha(C)=\frac{|\alpha|(|\alpha|+2)}{8}I.$$

3. Representation of $SO(4)$

3. .1. Let A and B be two elements of $SU(2)$ and consider the application

(3-1-1) $$\ddot{z}\longrightarrow A^{-1}\ddot{z}B,$$

where

$$\ddot{z}=\begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}\in SU(2)$$

which we regard as a point z on S^3, (2-1-1). This establishes a homomorphism $A^\#$ from $SU(2)\times SU(2)$ into $O(4)$. The kernel of the homomorphism consists only of the pairs (I, I) and $(-I, -I)$. It can be observed that the diagonal subgroup K (subgroup for which $A=B$) leaves the point $(1,0)\in \mathbf{C}^2$ invariant and generates the subgroup of rotations in the 3-dimensional space perpendicular to $(1,0)$. From this we can show that $A^\#$ is a homomorphism onto the connected component of the identity in $O(4)$. Thus we have established the isomorphism

(3-1-2) $$A^\#\colon G=\frac{SU(2)\times SU(2)}{\pm(I, I)}\longrightarrow SO(4).$$

As was remarked in the above the isotropy subgroup of $(1,0)$ by the action (3-1-2) is isomorphic to

$$K\simeq\frac{SU(2)}{\pm I}\simeq SO(3)$$

and we have

(3-1-3) $$G/K\simeq SO(4)/SO(3)\simeq S^3.$$

Every finite dimensional representation σ of $SO(4)$ is realized by the finite dimensional representation ρ of $SU(2)\times SU(2)$ whose kernel contains $(\pm(I, I))$;

$$\rho(g)=\sigma(A^\#(g)).$$

Let $R_\alpha=(R, H_\alpha)$ and $L^\beta=(L, \widehat{H^\beta})$ be the representation of $SU(2)$ described in

Theorem 2.3. The ternsor product $L^{\beta} \otimes R_{\alpha}$ is a finite dimensional representation of $SU(2) \times SU(2)$ on the space

$$F_{\alpha}^{\beta}=\widehat{H}^{\beta} \otimes H_{\alpha}$$

given by

$$(L^{\beta} \otimes R_{\alpha})_{(g,g')}f(z, z')=f(g \cdot z, \quad z' \cdot g'), \quad \text{for } f \in F_{\alpha}^{\beta}.$$

We have

$$\dim F_{\alpha}^{\beta}= (|\alpha|+1)\,(|\beta|+1),$$

and

$$\widehat{h}_{p}^{\beta} \otimes h_{\alpha}^{q}; \quad p=0,1,\cdots,|\beta|, q=0,1,\cdots,|\alpha|$$

form a basi of F_{α}^{β}. The weights of representation are integers or half odd integers according to either $\frac{|\alpha|+|\beta|}{2}$ is integer or half odd integer.

Let $g=\exp(\nu e_0)=\begin{pmatrix} n & 0 \\ 0 & \bar{n} \end{pmatrix}$, $n=e^{\frac{i}{2}\nu}$, and $g'=\exp(\mu e_0)=\begin{pmatrix} m & 0 \\ 0 & \bar{m} \end{pmatrix}$, $m=e^{\frac{i}{2}\mu}$.
We have

$$(L^{\beta} \otimes R_{\alpha})_{(g,g')}(\widehat{h}_{p}^{\beta} \otimes h_{\alpha}^{q})=n^{2p-|\beta|}m^{2q-|\alpha|}\widehat{h}_{p}^{\beta} \otimes h_{\alpha}^{q}.$$

In particular, if $\nu=\mu=2\pi$ we have $(L^{\beta} \otimes R_{\alpha})(-I, -I)(\widehat{h}_{p}^{\beta} \otimes h_{\alpha}^{q})=(-1)^{|\alpha|+|\beta|}$ $\widehat{h}_{p}^{\beta} \otimes h_{\alpha}^{q}$. Hence, for $(L^{\beta} \otimes R_{\alpha})$ to be a representation of $O(4)$ it is necessary that $|\alpha|+|\beta|$ is an even number. In this case all weights are integers. The converse is true and, for each pair (α, β) such that $|\alpha|+|\beta|$ is an even number, we have a representation σ_{α}^{β} of $SO(4)$ such that

$$(L^{\beta} \otimes R_{\alpha})=\sigma_{\alpha}^{\beta} \circ A^{\#}.$$

The characteristic function of the representation $(L^{\beta} \otimes R_{\alpha})$ being

$$\chi_{\beta,\alpha}((e^{i\nu}, e^{i\mu}))=\frac{\sin(|\beta|+1)\nu}{\sin\nu} \cdot \frac{\sin(|\alpha|+1)\mu}{\sin\mu}, \tag{3-1-4}$$

the representation $(L^{\beta} \otimes R_{\alpha})$ is irreducible and for $|\alpha|+|\beta|$ even the representation σ_{α}^{β} is irreducible.
Thus we have;

Theorem 3.1. (1) *For every r, s such that $r+s$ is an even number and for every indices α, β with $|\alpha|=r$, $|\beta|=s$,*

$$(F_{\alpha}^{\beta}, \sigma_{\alpha}^{\beta})$$

gives the irreducible representation of $SO(4)$ of highest weight $\frac{r+s}{2}$.

(2) *The polynomials* $\widehat{h}^{\beta}_{p} \otimes h^{q}_{\alpha}$, $0 \leq p \leq s$, $0 \leq q \leq r$, *form a basis of weight vectors for* σ^{β}_{α}

4. Algebra of infinitesimal automorphisms on S^3

4. 1. For indices $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ we shall put $\alpha \pm \beta = (\alpha_1 \pm \beta_1, \alpha_2 \pm \beta_2)$.
1 denotes the index (1,1).

Lemma 4. 1.

$$(4\text{-}1\text{-}1) \qquad h^{p}_{\alpha} \cdot h^{q}_{\beta} = \sum_{k=0}^{p+q} C_k |z|^{2k} h^{p+q-k}_{\alpha+\beta-k\mathbf{1}}$$

for some rational numbers $C_k = C_k(\alpha, p; \beta, q)$; $k=0, \cdots, p+q$, *where, for terms with a negative index*, $C_k = 0$.

In fact, $h^{p}_{\alpha} \cdot h^{q}_{\beta} \in S_{\alpha+\beta} \cap \widehat{S}^{(p+q, |\alpha|+|\beta|-p-q)}$. Repeated applications of Proipositions 1.1 and 1.4 yield the assertion.
To have the constants $C_k(\alpha, p; \beta, q)$ is very cumbersome. We must solve linear equations:

$$(1\text{-}4\text{-}2) \qquad \sum_{k} L(n, k) C_k = R(n) \quad n=1, \cdots, p+q,$$

with the coefficients

$$R(n) = \sum_{i=0}^{n} p\,!\,q\,! \binom{\alpha_1}{i} \binom{\alpha_2}{p-i} \binom{\beta_1}{n-i} \binom{\beta_2}{q-n+i}$$

(4-1-3)

$$L(n\ k) = \sum_{i=0}^{k} (-1)^i (p+q-k)\,! \binom{k}{i} \binom{\alpha_1+\beta_1-k}{n-i} \binom{\alpha_2+\beta_2-k}{\alpha_2+\beta_2-p-q+n-i}.$$

Evidently $C_0(\alpha, 0; \beta, 0) = 1$. Integrating both sides of (4-1-1) we have from Lemma 1.4 and Proposition 1.10

$$(4\text{-}1\text{-}4) \qquad C_{|\alpha|}(\alpha, q; \hat{\alpha}, |\alpha|-p) = (-1)^{\alpha_2+p} \frac{\alpha\,!}{|\alpha|+1}.$$

Example.

$$h^{0}_{1,0} \cdot h^{1}_{1,1} = \frac{2}{3} h^{1}_{2,1} + \frac{1}{3} |z|^2 h^{0}_{1,0}$$

$$h^{2}_{2,0} \cdot h^{2}_{2,1} = \frac{1}{10} h^{4}_{4,1} - \frac{4}{15} |z|^2 h^{3}_{3,0}$$

$$h^{1}_{2,0} \cdot h^{1}_{0,2} = \frac{1}{3} h^{2}_{2,2} - \frac{2}{3} |z|^4 h^{0}_{0,0}.$$

The equations to obtain the coefficients in the last example are

$$2C_0+C_1+C_2=0,\ 8C_0-2C_2=4,\ 2C_0-C_1+C_2=0.$$

There are some recurrent formulas among the numbers $C_k(\alpha, p; \beta, q)$ but here we do not write down them.

The multiplication of two harmonic polynomials on C^2 is not harmonic but its restriction on $B=\{|z|=1\}$ is again the restriction of some harmonic polynomial. We have given in (4-1-1) the formula of this multiplication;

$$h_\alpha^p \cdot h_\beta^q=\sum_{k=0}^{p+q} C_k h_{\alpha+\beta-k1}^{p+q-k} \text{ on } B. \tag{4-1-5}$$

The same investigations on C^n for $n\geq 2$ have already appeared in [Ru].

On $B=\{|z|=1\}$ we consider the following graded algebra of (the restrictions on B of) harmonic polynomials;

$$H(n)=\sum_{r=0}^{n} gr_r H$$

$$gr_r H=\sum_{|\alpha|=r} H_\alpha=\sum_{|\alpha|=r} \widehat{H}^\alpha.$$

Then we have

$$H(r)\cdot H(s)\subset H(r+s). \tag{4-1-6}$$

4. 2. Let $\mathcal{V}(S^3)$ denote the Lie algebra of smooth vector fields on $B=\{|z|=1\}$. Every $X\in\mathcal{V}(S^3)$ is written in the form

$$X(z)=f_0(z)\theta_0(z)+f_1(z)\theta_1(z)+f_2(z)\theta_2(z),\quad z\in B,$$

or

$$X(z)=f_0(z)\theta_0(z)+f_+(z)\varepsilon(z)+f_-(z)\bar{\varepsilon}(z), \tag{4-2-1}$$

with smooth functions as coefficients. The topology of $\mathcal{V}(S^3)$ is given by the uniform convergence of the coefficients. Since the polynomials $\{h_\alpha^q\}$ form a dense set, by a theorem of Weierstrass, every vector field is expanded in

$$X=\sum_{\alpha,p} h_\alpha^p\{a_0(\alpha,p)\theta_0+a_+(\alpha,p)\varepsilon+a_-(\alpha,p)\bar{\varepsilon}\}. \tag{4-2-2}$$

Put

$$L_\alpha^p=h_\alpha^p\theta_0 \quad E_\alpha^p=h_\alpha^p\varepsilon \quad F_\alpha^p=h_\alpha^p\bar{\varepsilon}. \tag{4-2-3}$$

Let $Vect(S^3)\subset\mathcal{V}(S^3)$ be the Lie subalgebra generated by L_α^p, E_α^p and F_α^p. Here are the commutation relations between the generators L_α^p, E_α^p, F_α^p, that give the structure constants of Lie algebra $Vect(S^3)$.

Proposition 4. 2.

$$[L_\alpha^p, L_\beta^q]=\sqrt{-1}\left(q-p+\frac{1}{2}\left(|\alpha|-|\beta|\right)\right)\sum_{\mu=0}^{p+q} C_\mu(p, \alpha; q, \beta)L_{\alpha+\beta-\mu\cdot 1}^{p+q-\mu}$$

$$[E_\alpha^p, E_\beta^q]=\sum_{\mu=0}^{p+q+1}(C_\mu(p, \alpha; q+1, \beta)-C_\mu(p+1, \alpha; q, \beta))E_{\alpha+\beta-\mu\cdot 1}^{p+q+1-\mu}$$

$$[F_\alpha^p, F_\beta^q]=\sum_{\mu=0}^{p+q-1}(p(|\alpha|-p+1)C_\mu(p-1, \alpha; q, \beta)$$
$$-q(|\beta|-q+1)C_\mu(p, \alpha; q-1, \beta))F_{\alpha+\beta-\mu\cdot 1}^{p+q-1-\mu}$$

$$[L_\alpha^p, E_\beta^q]=\sqrt{-1}\left(q-\frac{1}{2}|\beta|+1\right)\sum_{\mu=0}^{p+q} C_\mu(p, \alpha; q, \beta)E_{\alpha+\beta-\mu\cdot 1}^{p+q-\mu}$$
$$-\sum_{\mu=0}^{p+q+1} C_\mu(p+1, \alpha; q, \beta)L_{\alpha+\beta-\mu\cdot 1}^{p+q+1-\mu}$$

$$[L_\alpha^p, F_\beta^q]=\sqrt{-1}\left(q-\frac{1}{2}|\beta|-1\right)\sum_{\mu=0}^{p+q} C_\mu(p, \alpha; q, \beta)F_{\alpha+\beta-\mu\cdot 1}^{p+q-\mu}$$
$$+p(|\alpha|-p+1)\sum_{\mu=0}^{p+q-1} C_\mu(p-1, \alpha; q, \beta)L_{\alpha+\beta-\mu\cdot 1}^{p+q-1-\mu}$$

$$[E_\alpha^p, F_\beta^q]=\sum_{\mu=0}^{p+q+1} C_\mu(p, \alpha; q+1, \beta)F_{\alpha+\beta-\mu\cdot 1}^{p+q+1-\mu}$$
$$+p(|\alpha|-p+1)\sum_{\mu=0}^{p+q-1} C_\mu(p-1, \alpha; q, \beta)E_{\alpha+\beta-\mu\cdot 1}^{p+q-1-\mu}$$
$$-2\sum_{\mu=0}^{p+q} C_\mu(p, \alpha; q, \beta)L_{\alpha+\beta-\mu\cdot 1}^{p+q-\mu},$$

where

$$\alpha+\beta-\mu\cdot 1=(\alpha_1+\beta_1-\mu, \alpha_2+\beta_2-\mu).$$

Let

$$V(r)=\{X\in Vect(S^3);\ \text{the coefficients of } X \text{ are in } H(r)\}.$$

Proposition 4. 3. $[V(r), V(s)]\subset V(r+s+1)$.

We have form Proposition 1.9 (1)

$$\overline{L_{\alpha}^{p}}=(-1)^{\alpha_2+p}\frac{p\,!}{(|\alpha|-p)\,!}L_{\hat{\alpha}}^{|\alpha|-p}$$

(4-2-4) $$\overline{E_{\alpha}^{p}}=(-1)^{\alpha_2+p}\frac{p\,!}{(|\alpha|-p)\,!}F_{\hat{\alpha}}^{|\alpha|-p}$$

$$\overline{F_{\alpha}^{p}}=(-1)^{\alpha_2+p}\frac{p\,!}{(|\alpha|-p)\,!}E_{\hat{\alpha}}^{|\alpha|-p}.$$

Thus $V\ (r)$ is closed under complex conjugation.

5. Radul-Kravchenko-Khesin cocycle on $Vect(S^3)$

5. 1. A. O. Radul [R] introduced after Kravchenko-Khesin the following formula for the cocycle on the ring of classical pseudodifferential operators on a manifold.

Let

$$CL(M^n)=\{a=\sum_{-\infty<k\leq d} a_k(x,\xi)\}$$

be the ring of formal pseudodifferential symbols on a riemannian manifold M^n. Here $x=(x_1,\cdots,x_n)$ are local coordinates, $\xi=(\xi_1,\cdots,\xi_n)$ is a non-zero covector, $a_k(x,\xi)$ are functions on the cotangent bundle T^*M with zero section removed that satisfiy the homogeneity condition $a_k(x,t\xi)=t^k a_k(x,\xi)$, $t>0$. The multiplication in $CL(M)$ is defined by

(5-1-1) $$a\cdot b=\sum_{\alpha}\frac{1}{\alpha\,!}\partial_{\xi}^{\alpha}a\ \partial_{x}^{\alpha}b,$$

where α denotes a multiindex. Let α be the canonical 1-form on T^*M; $\alpha=\Sigma\xi_i dx_i$, and let $\omega=d\alpha$. The noncommutative residue of M. Wodzicki [W] of a symbol $a\in CL(M)$ is defined by the formula

(5-1-2) $$Res\ a=\int_{S_z^*M}a_{-n}(z,\xi)\,\alpha\wedge\omega^{n-1},$$

which is a differential n-form on M and where S_z^*M is the fiber over z of unit cosphere bundle S^*M. Integrating $Res\ a$ on M we obtain the trace formula on $CL(M)$:

(5-1-3) $$Tr\ a=\int_M Res\ a.$$

We have $Tr[a,b]=0$. Let S be an elliptic differential operator of order m on M with the leading symbol $s_m(x,,\xi)>0$ for $\xi\neq 0$. Then the formula

(5-1-4) $$c(a,b)=Tr([\ln s_m,a]\cdot b)\quad a,b\in CL(M)$$

gives a 2-cocycle [R].

Here we note that, though $\ln s_m(x,\xi)\notin CL(M)$, we have $[\ln s_m(x,\xi), CL(M)]\subset CL(M)$. The cocycle properties are proved by the following fact:

$$Tr\,[\ln s_m, a]=0.$$

Now we shall change the definition of Wodzicki's residue to have a concordant result with Kravchenko-Khesin's explanation of Virasoro term, that is, the cocycle for the central extension of $Vect(S^1)$. The Lie algebra $Vect(S^1)$ is generated by

$$L_m=z^{m+1}\frac{d}{dz},\quad m=0,\ \pm 1,\ \cdots,$$

where we look $S^1=\{z\in C;\ |z|=1\}$. The symbol of L_m is $z^{m+1}\zeta$ while the symbol of the square root of Laplacian is $|\zeta|$. Thus

$$L_m[\ln|\zeta|, L_n]=\sum_{k\geq 1}\frac{(-1)^{k+1}(n+1)\cdots(n-k+2)}{2k}z^{n+m+2-k}\zeta^{2-k}$$
$$+\sum_{k\geq 1}\frac{(-1)^{k+1}(n+1)\cdots(n-k+1)}{2k}z^{n+m+1-k}\zeta^{1-k},$$

The homogeneity order (-1) term is $-\frac{1}{12}n(n^2-1)z^{n+m-1}$. If we use here Wodzicki's formula we must integrate it on $S_z^*S^1=\{\pm 1\}$, which leads to 0. So we change the definition (5-1-2) to have a correct result. Let $P(T^*M)$ be the projective cotangent bundle whose fiber over a point $z\in M$ is the projective space $P(T_z^*M)$. We revise our definition of $Res\ \alpha$ by

$$\text{(5-1-5)}\qquad Res\ a=\int_{P(T_z^*M)}a_{-n}(z,\xi)\,\alpha\wedge\omega^{n-1},$$

We note especially that $P(T_z^*S^1)$ is one point. The 2-cocycle becomes

$$c(L_n, L_m)=\int_{|z|=1}Res_\zeta(L_m[\ln|\zeta|, L_n])\,dz=-\frac{1}{12}n(n^2-1)\,\delta_{n+m,0}.$$

Thus we get the Kravchenko-Khesin's formula.

Now we shall investigate the cocycle on $Vect(S^3)$. We shall continue to denote $B=\{z\in C^2;\ |z|=1\}\cong S^3$.

Any covector is written by $\xi_0\theta_0^*+\xi_1\theta_1^*+\xi_2\theta_2^*$ or equivalently by $\eta\theta_0^*+\zeta\varepsilon^*+\bar{\zeta}\bar{\varepsilon}^*$, where $\bar{\varepsilon}^*$ is the dual 1-form of $\bar{\varepsilon}$. We take η, ζ, $\bar{\zeta}$ as the coordinates on T_z^*B. Then the canonical 1-form α becomes $\alpha=\eta\theta_1^*+\zeta\varepsilon^*+\bar{\zeta}\bar{\varepsilon}^*$. Let $\omega=d\alpha$. The 5-form $\alpha\wedge\omega^2$ restricted on the local coordinate $U_0=\{(z, [\eta. \zeta, \bar{\zeta}]);\ \eta\neq 0\}\subset P(T^*B)$ is given by

$$\alpha\wedge\omega^2=\frac{1}{\eta}d\zeta\wedge d\bar{\zeta}\wedge dV,\quad dV=\theta_0^*\wedge\theta_1^*\wedge\theta_2^*,$$

where $\eta^2+|\zeta|^2=1$. By the polar coordinates $\eta=\cos\phi$, $\zeta=\sin\phi\, e^{i\theta}$, $\bar{\zeta}=\sin\phi\, e^{-i\theta}$, we have $\alpha\wedge\omega^2|_B=\sin\phi\ d\phi\ d\theta\wedge dV$, $0\leq\phi<\frac{\pi}{2}$, $0\leq\theta<2\pi$. The symbol of the first order differential operators L^p_α, E^p_α, F^p_α are respectively $h^p_\alpha\eta$, $h^p_\alpha\zeta$, $h^p_\alpha\bar{\zeta}$, and the symbol of Laplace-Beltrami operator is $\eta^2+|\zeta|^2$.

For X, $Y\in$ $Vect(B)$ we put

$$R(X,Y)dV=Res\{(symbY)\cdot[\ln(\eta^2+|\zeta|^2),(symbX)]\}. \tag{5-1-6}$$

Then the formula

$$c(X,Y)=\int_{S^3}R(X,Y)dV \tag{5-1-7}$$

defines a 2-cocycle on $Vect(S^3)$ and we have the central extension of $Vect(S^3)$ associated with this 2-cocycle.

Proposition 5.1. *$R(X, Y)$ for every X, Y in $Vect(S^3)$ is written by a linear combination of Beta functions*

$$B(u,v)=\int_0^{\frac{\pi}{2}}(\sin\phi)^{2u-1}(\cos\phi)^{2v-1}d\phi,$$

with its coefficients polynomials in z, $\bar{z}$.

Proof. Since $Vect(S^3)$ is the linear hull of $\{L^p_\alpha, E^p_\alpha, F^p_\alpha\}$ it is enough to give calculation of Res for these vector fields. We shall look $R(L^q_\beta, L^p_\alpha)$. The others are obtained by the same calculation. Put $r=(\eta^2+|\zeta|^2)^{\frac{1}{2}}$. We have

$$[\ln r, f(z)\eta]$$
$$=\sum_{p+p'+q\geq 1}\sum_{k}^{\min(p,p')}C_{p,p',k}\bar{\zeta}^{p-k}\zeta^{p'-k}\sum_{q,j}^{j\leq[q/2]}D_{q,j}\eta^{\delta_{q,j}+1}P\begin{pmatrix}p & p' & q\\ \varepsilon & \bar{\varepsilon} & \theta_0\end{pmatrix}f(z)r^{-2(p+p'-k-q-\delta_{q,j})}$$

where $C_{p,p',k}$ and $D_{q,j}$ are some constants and $\delta_{q,j}=2j$ or $2j+1$ according to q is even or odd. $P\begin{pmatrix}p & p' & q\\ \varepsilon & \bar{\varepsilon} & \theta_0\end{pmatrix}$ denotes the sum of all differentiations that are p (resp. p', q) times with respect to ε (resp. $\bar{\varepsilon}$, θ_0). The term of homogeneity order -3 of $g(z)\eta\cdot[\ln r, f(z)\eta]\in CL(B)$, where $\eta=symb\theta_0$, is given on B by

$$\sum_{p+p'+q=5}\sum_{k}C_{p,p',k}\zeta^{p'-k}\bar{\zeta}^{p-k}\sum_{j=0,1,2}D_{q,j}\eta^{2j+3}g(z)P\begin{pmatrix}p & p' & q\\ \varepsilon & \bar{\varepsilon} & \theta_0\end{pmatrix}f(z)$$
$$+\sum_{p+p'+q=4}\sum_{k}C_{p,p',k}\zeta^{p'-k}\bar{\zeta}^{p-k}\sum_{j=0,1,2}D_{q,j}\eta^{2j+1}g(z)\theta_0P\begin{pmatrix}p & p' & q\\ \varepsilon & \bar{\varepsilon} & \theta_0\end{pmatrix}f(z).$$

It is enough to consider only those terms with $p=p'$, for the other terms vanish after the integration by $d\zeta d\bar{\zeta}$. Then q becomes necessarily odd and the integration on the fiber $P(T_z^*B)$ becomes a linear combination of the following type of integrals with polynomial coefficients;

$$\int_0^\pi (\sin\phi)^{2p-2k+1}(\cos\phi)^{2j+1}d\phi$$

Remark. *If we took in the definition of Res the integration on S_z^*B instead of the projective cotangent bundle $P(T^*B)$ we would have $R(X, Y)=0$.*

WASEDA UNIVERSITY

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