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# Construction of the Green function on Riemannian manifold using harmonic coordinates

By

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## 0. Introduction

Let $(M, g)$ be a compact Riemannian manifold of dimension $n \geq 3$ without boundary. We denote the Levi-Civita Connection of $(M, g)$ by $\nabla$, and the Laplace operator by $\Delta$. In this paper, we will prove an $L^p$-estimate for the Laplace operator:

$$\|\nabla^2 u\|_p \leq C\|\Delta u\|_p.$$

Naturally, the constant $C$ depends on geometric data of $(M, g)$. The main purpose of this paper is to estimate the constant $C$ in terms of the diameter, the injectivity radius, and the lower bound of the Ricci tensor.

For the purpose of this, we construct the Green function using a parametrix. In [2, 3], Aubin used the Riemannian distance function $d(x, y)$ to construct a parametrix of the Green function. However, the second derivatives of the distance function cannot be estimated in terms of the Ricci tensor. In fact, we need a bound of Riemann curvature tensor in order to yeild an estimate of $\Delta d(x, y)$. (Here the Laplace operator $\Delta$ acts on $d(x, y)$ with respect to the argument $y$.) Therefore we construct a parametrix utilizing the harmonic coordinate of [1]. In the course of this we estimate the Green function and its first derivatives near the singularity in Section 6, and, using the estimate of the second derivative of the parametrix, we show the Calderon-Zygmund type inequalities in Section 6, from which we can easily obtain an $L^p$-estimate for the Laplace operator.

We denote the diameter by $D$, the injectivity radius by $i_0$, the volume by $V$, and the Ricci tensor by Ric. We fix a non-negative constant $\Lambda$ for which the bound $\mathrm{Ric} \geq -(n-1)\Lambda g$ is satisfied.

For $x \in M$, the Green function $G_x$ is a unique smooth functions on $M \setminus \{x\}$ that satisfies $\Delta G_x = \delta_x - V^{-1}$ as distributions and $\int_M G_x d\mu = 0$, where $\delta_x$ is the Dirac function at $x$ and $d\mu$ is the Riemannian volume form.

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## 1. Preliminaries

In this section we prepare some analytic tools. For $p \geq 1$ and $0 < \alpha \leq 1$, we consider the following norms for functions on an Euclidean ball $B_0(r) = \{\xi \in \mathbf{R}^n : |\xi| < r\}$:

$$\|f\|_{p,r} = \|f\|_{L^p(B_0(r))} = \left\{\int_{B_0(r)} |f|^p d\xi\right\}^{1/p};$$

$$\|\partial f\|_{p,r} = \left\{\sum_i \int_{B_0(r)} |\partial_i f|^p d\xi\right\}^{1/p};$$

$$\|f\|_{\infty,r} = \|f\|_{C^0(B_0(r))} = \sup_{\xi \in B_0(r)} |f(\xi)|;$$

$$[f]_{\alpha,r} = \sup_{\substack{\xi,\zeta \in B_0(r) \\ \xi \neq \zeta}} \frac{|f(\xi) - f(\zeta)|}{|\xi - \zeta|^\alpha}.$$

The Sobolev space $L_1^p(B_0(r))$ is the set of measurable functions for which the norm

$$\|f\|_{L_1^p(B_0(r))} = \|f\|_{p,r} + \|\partial f\|_{p,r}$$

is finite. The Hölder Space $C^\alpha(B_0(r))$ is the set of functions for which the norm

$$\|f\|_{C^\alpha(B_0(r))} = \|f\|_{\infty,r} + [f]_{\alpha,r}$$

is finite.

We use Sobolev's embedding theorem in the following form. For the verification, see the proof of [5, Theorem 7.17].

**Theorem 1.1.** *Assume $p > n$ and set $\alpha = 1 - n/p$. For $f \in L_1^p(B_0(2r))$, we have Sobolev's inequalities*

$$\|f\|_{\infty,r} \leq C(\|f\|_{p,2r} + r^\alpha \|f\|_{p,2r})$$

and

$$[f]_{\alpha,r} \leq C\|\partial f\|_{p,2r},$$

*where $C = C(n, p)$ is a constant that depends only on $n$ and $p$.*

We next consider the regularity for an elliptic partial differential equation

$$\sum_{i,j} a^{ij} \partial_{ij}^2 u = f. \qquad (1.1)$$

The elliptic regularity theorem [5, Theorem 6.2] can be restated as follows.

**Theorem 1.2.** *Assume that the coefficients $a^{ij}$ are smooth functions on $B_0(2r)$ and satisfy for some constant $\kappa > 0$ the conditions*

$$(1+\kappa)^{-2}\delta^{ij} \leq a^{ij}(\xi) \leq (1+\kappa)^{2}\delta^{ij} \ (\textit{as symmetric bilinear forms})$$

and

$$r^{\alpha}[a^{ij}]_{\alpha,2r} \leq \kappa.$$

*If $u$ is a bounded weak solution of* (1.1) *for $f \in C^{\alpha}(B_0(2r))$, then we have*

$$r\|\partial u\|_{\infty,r} + r^2\|\partial^2 u\|_{\infty,r} + r^{2+\alpha}[\partial^2 u]_{\alpha,r} \leq C(\|u\|_{\infty,2r} + r^2\|f\|_{\infty,2r} + r^{2+\alpha}[f]_{\alpha,2r})$$

*for some constant $C=C(n,\alpha,\kappa)$.*

For a compact Riemannian manifold $(M, g)$, we can also define the norms

$$\|f\|_p = \|f\|_{L^p(M)} = \left\{\int_M |f|^p d\mu\right\}^{1/p},$$
$$\|f\|_\infty = \|f\|_{C^0(M)} = \sup_{x\in M}|f(x)|,$$

and

$$[f]_\alpha = \sup_{\substack{x,y\in M\\ x\neq y}} \frac{|f(x)-f(y)|}{d(x,y)^\alpha}.$$

We define the Sobolev space $L_1^p(M)$ using the norm

$$\|f\|_{L_1^p(M)} = \|f\|_p + \|\nabla f\|_p,$$

where $\|\nabla f\|_p$ is the $L^p$-norm of $|\nabla f|$, the pointwise Riemannian norm of the covariant derivative $\nabla f$, and the Hölder space $C^\alpha(M)$ using the norm

$$\|f\|_{C^\alpha(M)} = \|f\|_\infty + [f]_\alpha.$$

It is well known the bound

$$\mathrm{Ric} \geq -(n-1)\Lambda g \tag{1.2}$$

yields the lower bound for the Sobolev constant (cf. [6]). We state it as follows.

**Theorem 1.3** *There is a constant $C_S$, depending only on $n$, $\Lambda D^2$, and $D^n/V$, such that*

$$\|f\|_{\frac{2n}{n-2}} \leq C_S\|\nabla f\|_2 \tag{1.3}$$

*for any $f \in L_1^2(M)$ satisfying $\int_M f d\mu = 0$.*

We denote by $B_x(r)$ the geodesic ball of $M$ centered at $x$ and of radius $r$, by $S_x$ the unit sphere of $T_xM$ with respect to $g$, and by $d\omega$ the standard volume form of the unit sphere $S_x = S^{n-1}$. Under the identification via the exponential mapping $\mathbf{R}_+ \times S_x \ni (r, v) \mapsto \exp_x(rv) \in M$, we define a positive function $a(r, v)$ on $\mathbf{R}_+ \times S_x$ by the equation $d\mu = a(r,v)^{n-1}dr d\omega$ if the geodesic $[0, r] \ni t \mapsto \exp_x(tv)$ is minimizing, and $a(r, v) = 0$ otherwise. Set $\gamma = e^{(n-1)\sqrt{\Lambda}D}$. We also restate Bishop-Gromov's volume comparison theorem in the following form.

**Theorem 1.4** *The function $a(r, v)$ satisfies $a(r, v) \leq \gamma^{1/(n-1)} r$. For $0 < r \leq R$, we have* $\mathrm{Vol}(B_x(R))/\mathrm{Vol}(B_x(r)) \leq \gamma (R/r)^{n-1}$.

## 2. Harmonic coordinates

First, we recall the result of Anderson and Cheeger [1] concerning the harmonic coordinate which is useful in considering regularity problems on a Riemannian manifold.

**Theorem 2.1.** *Suppose that $(M, g)$ is a compact Riemannian manifold without boundary satisfying the bound $\mathrm{Ric} \geq -(n-1)\Lambda g$ for some constant $\Lambda \geq 0$. Given $\kappa > 0$, $p > n$, there are constants $C_1$ and $C_2$, depending only on $n$, $\kappa$, and $p$, such that there is a coordinate $u = (u^1, \cdots, u^n)$ on any geodesic ball $B_x(r)$ for $r \leq \min\{C_1/\sqrt{\Lambda}, C_2 i_0\}$ satisfying the following conditions:*

(1) $u(x) = 0$.

(2) *Each $u^k$ $(k = 1, \cdots, n)$ is a harmonic function on $B_x(r)$ with respect to $g$.*

(3) *The functions $g_{ij} = g(\partial/\partial u^i, \partial/\partial u^j)$ satisfy*

$$g_{ij}(x) = \delta_{ij};$$

$$(1+\kappa)^{-2}\delta_{ij} \leq g_{ij} \leq (1+\kappa)^2\delta_{ij} \ (\text{as symmetric bilinear forms});$$

$$r^{1-n/p}\|\partial g_{ij}\|_{L^p(B_x(r))} \leq \kappa.$$

Let $p > n$ and set $\alpha = 1 - n/p$. Fixing $\kappa = 1$, we restate Theorem 2.1 in the following form.

**Theorem 2.2** *There is a constant $C_H$, depending only on $n$, $p$, and $\Lambda D^2$, such that there is a diffeomorphism $F: B_0(r) \to M$ for any $x \in M$ and $r \leq C_H i_0$ satisfying the following properties.*

(1) $F(0) = x$.

(2) *The local representation of $g$ by $F$, which we denote by $g_{ij}$, satisfies $4^{-1}\delta_{ij} \leq g_{ij} \leq 4\delta_{ij}$ as symmetric bilinear forms on $B_0(r)$ and $g_{ij}(0) = \delta_{ij}$.*

(3) *The functions $g_{ij}$ satisfy*

$$r^{1-n/p}\|\partial g_{ij}\|_{p,r} \leq 1 \quad \text{and} \quad r^{\alpha}\ [g_{ij}]_{\alpha,r} \leq 1.$$

(4) *The inverse mapping $F^{-1} = (f^1, \cdots, f^n)$ can be considered as a function $F^{-1}$: $B_x(4r) \mapsto \mathbf{R}^n$ and each component $f^k$ is a harmonic function with respect to $g$.*

Proof. Set $C_3 = \min\{C_1/\sqrt{\Lambda}D, C_2\}$. Clearly the properties of Theorem 2.1 hold for $r \leq C_3 i_0$. By taking $F = u^{-1}$, we easily see that the properties of Theorem 2.2 are satisfied for $r \leq C_3 i_0/4$ except for the estimate of $[g_{ij}]_{\alpha,r}$. Applying Sobolev's inequality (Theorem 1.1), we can show that there is constant $C_4$, depending only on $n$ and $p$, such that

$$r^{\alpha}\ [g_{ij}]_{\alpha,r/2} \leq C_4 r^{-n/p}\|\partial g_{ij}\|_{p,r} \leq C_4.$$

We now set $C_5 = \min\{C_4^{-1/\alpha}, 1/2\}$. The theorem is valid for $C_H = C_3C_5/4$.

**Definition.** We call the diffeomorphism $F$ in Theorem 2.2 a $p$-harmonic coordinate around $x$.

We fix $p$ such that $np/(p-n)$ is not an integer and set $r_0 = C_H i_0/2$. In a $p$-harmonic coordinate $F: B_0(2r_0) \to M$, the Laplace operator $\Delta$ is given by

$$\Delta = -\sum_{ij} g^{ij}\partial_{ij}^2.$$

If two $p$-harmonic coordinates $F, F': B_0(r_0) \to M$ overlap, i.e.,

$$F(B_0(r_0)) \cap F'(B_0(r_0)) \neq \emptyset,$$

then

$$F(B_0(2r_0)) \subset B_{F(0)}(4r_0) \subset B_{F'(0)}(8r_0).$$

Each component of the transition function $F'^{-1}\circ F$ can be considered as a function on $B_0(2r_0)$ which is harmonic with respect to $g_{ij}$, that is

$$\Delta(F'^{-1}\circ F) = -\sum_{ij} g^{ij}\partial_{ij}^2(F'^{-1}\circ F) = 0.$$

Then Theorem 1.2 implies that there is a constant $C$, which depends only on $n$ and $p$, such that

$$\begin{aligned} \|\partial_i(F'^{-1}\circ F)\|_{\infty,r_0} &\leq C; \\ r_0\|\partial_{ij}^2(F'^{-1}\circ F)\|_{\infty,r_0} &\leq C; \qquad (2.1) \\ r_0^{1+\alpha}\ [\partial_{ij}^2(F'^{-1}\circ F)]_{\alpha,r_0} &\leq C. \end{aligned}$$

Thus we obtain the estimate of $C^{2+\alpha}$-norms of the transition functions of $p$-harmonic coordinates.

Set $t_0 = r_0/12$. Let $\{B_{x_\lambda}(t_0/8)\}_{\lambda=1}^{Q}$ be a maximal family of disjoint geodesic balls of radius $t_0/8$. We can choose a $p$-harmonic coordinate $F_\lambda: B_0(r_0) \to M$ around each $x_\lambda$. It is easy to see that $\{B_{x_\lambda}(t_0/4)\}_{\lambda=1}^{Q}$ covers $M$. Hence $\{F_\lambda(B_0(t_0/2))\}_{\lambda=1}^{Q}$ also covers $M$.

Set $m(x) = \#\{\lambda : x \in F_\lambda(B_0(t_0))\}$ for $x \in M$. Bishop-Gromov's volume comparison theorem yields an estimate of $Q$ in terms of $n$, $\Lambda$, $D$, $V$ and $t_0$. Moreover,

**Proposition 2.3** *There is an upper bound $m_0$ for $m(x)$ that depends only on $n$ and $\Lambda D^2$.*

*Proof.* Let $\{\lambda_i\}_{i=1}^{m(x)}$ be the subset of the indices $\{\lambda\}_{\lambda=1}^{Q}$ such that $x \in F_{\lambda_i}(B_0(t_0))$. Since $B_{x_{\lambda_i}}(t_0/8) \subset B_x(3t_0) \subset B_{x_{\lambda_i}}(5t_0)$, we have

$$m(x) \leq \max_i \frac{\mathrm{Vol}(B_x(3t_0))}{\mathrm{Vol}(B_{x_{\lambda_i}}(t_0/8))} \leq \max_i \frac{\mathrm{Vol}(B_{x_{\lambda_i}}(5t_0))}{\mathrm{Vol}(B_{x_{\lambda_i}}(t_0/8))}.$$

Thus the result follows from Bishop-Gromov's volume comparison theorem.

Let $\chi$ be a smooth non-increasing function on $\mathbf{R}_+$ satisfying

$$\chi(s)=1 \quad \text{for } s \le t_0/2; \quad \chi(s)=0 \quad \text{for } s \ge t_0;$$

$$-4/t_0 \le \chi'(s) \le 0; \quad |\chi''(s)| \le 32\, t_0^2; \quad |\chi'''(s)| \le 512/t_0^3.$$

We set $\tilde{\chi}_\lambda(x) = \chi(|F_\lambda^{-1}(x)|)$ for $x \in F_\lambda(B_0(t_0))$ and $\tilde{\chi}_\lambda(x)=0$ otherwise. Then we see that

$$1 \le \sum_{\lambda=1}^{Q} \tilde{\chi}_\lambda(x) \le m_0.$$

Thus we can construct a partition of unity $\{\chi_\lambda\}_{\lambda=1}^{Q}$ subordinate to the covering $\{F_\lambda(B_0(t_0))\}_{\lambda=1}^{Q}$ by setting

$$\chi_\lambda(x) = \frac{\tilde{\chi}_\lambda(x)}{\sum_{\nu=1}^{Q} \tilde{\chi}_\nu(x)}.$$

The $C^{2+\alpha}$-norm of $\chi_\lambda \circ F_\mu$ can be estimated by $t_0$, $n$, $p$, and $\Lambda D^2$. In particular,

$$|\chi_\lambda(x) - \chi_\lambda(y)| \le C r_0^{-1} d(x, y) \tag{2.2}$$

for some constant $C = C(n, p, \Lambda D^2)$.

## 3. Parametrix of the Green function

In this section, we construct a parametrix of the Green function using the $p$-harmonic coordinates $\{F_\lambda\}_{\lambda=1}^{Q}$. We denote by $g_{ij}^\lambda$ and $g_\lambda^{ij}$ the metric tensor and its inverse in the coordinate $F_\lambda$. From now on, we adopt Einstein's convention.

For $\zeta \in B_0(t_0)$, we define a non-negative function $d_\zeta^\lambda$ on $\mathbf{R}^n$ by

$$\{d_\zeta^\lambda(\xi)\}^2 = g_{ij}^\lambda(\zeta)(\xi^i - \zeta^i)(\xi^j - \zeta^j).$$

Choose a smooth increasing function $\phi$: $\mathbf{R}_+ \to \mathbf{R}_+$ such that

$$\phi(s) = s \text{ for } s \le t_0/6; \quad \phi \equiv t_0/3 \text{ for } s \ge t_0/2;$$

$$0 \le \phi' \le 1; \quad -6/t_0 \le \phi'' \le 0.$$

We now define a function $h_\zeta^\lambda$ on $\mathbf{R}^n$ by

$$h_\zeta^\lambda(\xi) = \frac{\{\phi(d_\zeta^\lambda(\xi))\}^{2-n} - (t_0/3)^{2-n}}{(n-2)\,\omega},$$

where $\omega$ is the volume of the standard $(n-1)$-sphere. Notice that $h_\zeta^\lambda(\xi) = 0$ if $d_\zeta^\lambda(\xi) \ge t_0/2$. The first derivatives are given by

$$\partial_i h_\zeta^\lambda(\xi) = -\frac{1}{\omega}\{\phi(d_\zeta^\lambda(\xi))\}^{1-n} \phi'(d_\zeta^\lambda(\xi))\, \{d_\zeta^\lambda(\xi)\}^{-1} g_{ij}^\lambda(\zeta)(\xi^j - \zeta^j).$$

Since $\phi'(s) = 0$ for $s \ge t_0/2$, we see that $\partial_i h_\zeta^\lambda(\xi) = 0$ if $d_\zeta^\lambda(\xi) \ge t_0/2$. If $d_\zeta^\lambda(\xi) \le t_0/2$, using the estimates $2s/3 \le \phi(s) \le s$ for $s \le t_0/2$ and $|\xi - \zeta|/2 \le d_\zeta^\lambda(\xi) \le 2|\xi - \zeta|$, we obtain

$$|\partial_i h_\zeta^\lambda(\xi)| \le C|\xi - \zeta|^{1-n}$$

for some constant $C = C(n)$.

Similarly we can estimate the second derivatives of $h_\zeta^\lambda$, which are given by

$$\partial_{ij}^2 h_\zeta^\lambda(\xi) = \frac{1}{\omega}\{\psi(d_\zeta^\lambda(\xi))\}^{-n}\Psi_1(d_\zeta^\lambda(\xi))\{d_\zeta^\lambda(\xi)\}^{-2}g_{ik}^\lambda(\zeta)g_{jl}^\lambda(\zeta)(\xi^k-\zeta^k)(\xi^l-\zeta^l)$$
$$-\frac{1}{\omega}\{\psi(d_\zeta^\lambda(\xi))\}^{1-n}\psi'(d_\zeta^\lambda(\xi))\{d_\zeta^\lambda(\xi)\}^{-1}g_{ij}^\lambda(\zeta),$$

where we set $\Psi_1(s)=(n-1)\{\psi'(s)\}^2-\psi(s)\psi''(s)+\psi(s)\psi'(s)/s$. Since $\Psi_1(s)=n$ for $s\le t_0/6$ and $\Psi_1(s)=0$ for $s\ge t_0/2$, $\partial_{ij}^2 h_\zeta^\lambda$ vanishes for $d_\zeta^\lambda(\xi)\ge t_0/2$ and we have

$$|\partial_{ij}^2 h_\zeta^\lambda(\xi)|\le C|\xi-\zeta|^{-n}$$

for some constant $C=C(n)$.

The following will be needed in the next section.

**Lemma 3.1** *There is a constant C depending only on n such that if* $|\xi-\zeta|\ge 2|\xi-\xi'|$, then

$$|\partial_{ij}^2 h_\zeta^\lambda(\xi)-\partial_{ij}^2 h_\zeta^\lambda(\xi')|\le C|\xi-\zeta|^{-n-1}|\xi-\xi'|,$$

*and if* $|\xi-\zeta|\ge 2|\zeta-\zeta'|$, *then*

$$|\partial_{ij}^2 h_\zeta^\lambda(\xi)-\partial_{ij}^2 h_{\zeta'}^\lambda(\xi)|\le C\{r_0^{-\alpha}|\xi-\zeta|^{-n}|\zeta-\zeta'|^\alpha+|\xi-\zeta|^{-n-1}|\zeta-\zeta'|\}.$$

*Proof.* We apply the mean value theorem with attention to the fact that

$$|\xi'-\zeta|\ge\frac{1}{2}|\xi-\zeta| \quad \text{for} \quad |\xi-\zeta|\ge 2|\xi-\xi'|$$

and

$$|\xi-\zeta'|\ge\frac{1}{2}|\xi-\zeta| \quad \text{for} \quad |\xi-\zeta|\ge 2|\zeta-\zeta'|.$$

We also notice that either $\xi$ or $\xi'$ does not appear in the left-hand sides of the inequalities as the argument of $g_{ij}^\lambda$

Next, we will estimate $\Delta h_\zeta^\lambda$, which are given by

$$\Delta h_\zeta^\lambda(\xi)=-g_\lambda^{ij}(\xi)\partial_{ij}^2 h_\zeta^\lambda(\xi)$$
$$=-\frac{1}{\omega}\{\psi(d_\zeta^\lambda(\xi))\}^{-n}\Psi_1(d_\zeta^\lambda(\xi))\{d_\zeta^\lambda(\xi)\}^{-2}$$
$$\times g_\lambda^{ij}(\xi)g_{ik}^\lambda(\zeta)g_{jl}^\lambda(\zeta)(\xi^k-\zeta^k)(\xi^l-\zeta^l)$$
$$+\frac{1}{\omega}\{\psi(d_\zeta^\lambda(\xi))\}^{1-n}\psi'(d_\zeta^\lambda(\xi))\{d_\zeta^\lambda(\xi)\}^{-1}g_\lambda^{ij}(\xi)g_{ij}^\lambda(\zeta).$$
$$=-\frac{1}{\omega}\{\psi(d_\zeta^\lambda(\xi))\}^{-n}\Psi_2(d_\zeta^\lambda(\xi))$$
$$-\frac{1}{\omega}\{\psi(d_\zeta^\lambda(\xi))\}^{1-n}\psi'(d_\zeta^\lambda(\xi))\{d_\zeta^\lambda(\xi)\}^{-1}g_\lambda^{ij}(\xi)\{g_{ij}^\lambda(\xi)-g_{ij}^\lambda(\zeta)\}$$
$$+\frac{1}{\omega}\{\psi(d_\zeta^\lambda(\xi))\}^{-n}\Psi_1(d_\zeta^\lambda(\xi))\{d_\zeta^\lambda(\xi)\}^{-2}$$

$$\times\, g_\lambda^{ij}(\xi) g_{ik}^\lambda(\zeta)\, \{g_{jl}^\lambda(\xi) - g_{jl}^\lambda(\zeta)\}\, (\xi^k - \zeta^k)\, (\xi^l - \zeta^l) \tag{3.1}$$

where $\Psi_2(s) = \Psi_1(s) - n\phi(s)\phi'(s)/s$. Notice that $\Psi_2$ satisfy

$$\Psi_2(s) = 0 \quad \text{for } s \leq t_0/6 \text{ or } s \geq t_0/2.$$

Since $\partial_{ij}^2 h_\zeta^\lambda$ vanishes for $d_\zeta^\lambda(\xi) \geq t_0/2$, $\Delta h_\zeta^\lambda$ vanishes for $d_\zeta^\lambda(\xi) \geq t_0/2$. For $d_\zeta^\lambda(\xi) \leq t_0/6$,

$$\Delta h_\zeta^\lambda(\xi) = -\frac{1}{\omega}\{d_\zeta^\lambda(\xi)\}^{-n} g_\lambda^{ij}(\xi)\, \{g_{ij}^\lambda(\xi) - g_{ij}^\lambda(\zeta)\}$$

$$+\frac{n}{\omega}\{d_\zeta^\lambda(\xi)\}^{-n-2} g_\lambda^{kl}(\xi) g_{lj}^\lambda(\zeta)\, \{g_{ik}^\lambda(\xi) - g_{ik}^\lambda(\zeta)\}\, (\xi^i - \zeta^i)\, (\xi^j - \zeta^j).$$

Then (3) of Theorem 2.2 implies that, if $d_\zeta^\lambda(\xi) \leq t_0/6$,

$$|\, \Delta h_\zeta^\lambda(\xi)\, | \leq C r_0^{-\alpha} |\, \xi - \zeta\, |^{\alpha - n}$$

for some constat $C = C(n)$. For $t_0/6 \leq d_\zeta^\lambda(\xi) \leq t_0/2$, the estimate of $|\, \partial_{ij}^2 h_\zeta^\lambda(\xi)\, |$ implies that, if $t_0/6 \leq d_\zeta^\lambda(\xi) \leq t_0/2$,

$$|\, \Delta h_\zeta^\lambda(\xi)\, | \leq C r_0^{-n}$$

for some constant $C = C(n)$.

Combining these results, we obtain

$$|\, \Delta h_\zeta^\lambda(\xi)\, | \leq C r_0^{-\alpha} |\, \xi - \zeta\, |^{\alpha - n}$$

where $C$ is a constant that depends only on $n$ and $p$.

Fix $x \in M$ and take $\lambda$ for which $x \in F_\lambda(B_0(t_0))$. For $y \in F_\lambda(B_0(3t_0))$, set

$$H_x^\lambda(y) = h_{F_\lambda^{-1}(x)}^\lambda(F_\lambda^{-1}(y)).$$

Notice that $H_x^\lambda \equiv 0$ outside $F_\lambda(B_0(2t_0))$. Therefore we can smoothly extend $H_x^\lambda$ over $M$ to be zero outside $F_\lambda(B_0(2t_0))$. Using the partition of unity $\{\chi_\lambda\}_{\lambda=1}^Q$ constructed in Section 2, we define

$$H_x(y) = \sum_{\lambda=1}^{Q} \chi_\lambda(x) H_x^\lambda(y).$$

It is clear that $H_x(y)$ is a smooth function on $M \times M$ minus the diagonal that satisfies

$$C_1 d(x, y)^{2-n} - C_2 r_0^{2-n} \leq H_x(y) \leq C_3 d(x, y)^{2-n} \tag{3.2}$$

for some positive constants $C_1$, $C_2$, and $C_3$, which depend only on $n$ and $p$. The function $H_x(y)$ vanishes when $d(x, y) \geq 2t_0$. Notice that we have

$$\nabla H_x(y) = \sum_{\lambda=1}^{Q} \chi_\lambda(x)\ \nabla H_x^\lambda(y)$$

and

$$\Delta H_x(y) = \sum_{\lambda=1}^{Q} \chi_\lambda(x) \Delta H_x^\lambda(y).$$

From the estimate on $h_\zeta^\lambda$, we obtain the estimates on $H_x^\lambda$ in the harmonic coordinate $F_\lambda$. Moreover, in view of (2.1), $H_x^\lambda$ can be estimated in any other

$p$-harmonic coordinate.

Hence the above argument shows:

**Proposition 3.2** *There is a constant $C$, depending only on $n$ and $p$, such that*

$$|\nabla H_x(y)| \le Cd(x, y)^{1-n}$$

*and*

$$|\Delta H_x(y)| \le Cr_0^{-\alpha} d(x, y)^{\alpha-n}.$$

We can now prove Green's formula.

**Lemma 3.3** *For any $\varphi \in C^2(M)$*

$$\varphi(x) = \int_M H_x(y)\Delta\varphi(y)d\mu(y) - \int_M \Delta H_x(y)\varphi(y)d\mu(y).$$

*Proof.* Take a $p$-harmonic coordinate $F$ around $x$. Using integration by parts, we obtain

$$\int_{M\setminus F(B_0(\epsilon))} H_x(y)\Delta\varphi(y)d\mu(y) - \int_{M\setminus F(B_0(\epsilon))} \Delta H_x(y)\varphi(y)d\mu(y)$$

$$= \int_{F(\partial B_0(\epsilon))} H_x(y)\nabla_\nu\varphi(y)d\sigma(y) - \int_{F(\partial B_0(\epsilon))} \nabla_\nu H_x(y)\varphi(y)d\sigma(y), \qquad (3.3)$$

where $\nu$ is the outward normal vector field of $\partial F(B_0(\epsilon)) = F(\partial B_0(\epsilon))$ and $d\sigma$ is the volume element of $F(\partial B_0(\epsilon))$. Let $g_{ij}$ and $g^{ij}$ be the metric tensor and its inverse in the harmonic coordinate $F$. Then $\nu$ and $d\sigma$ are given by

$$\nu(\xi) = \{g^{kl}(\xi)\xi_k\xi_l\}^{-1/2} g^{ij}(\xi)\xi_i\partial_j$$

and

$$d\sigma(\xi) = |\xi|^{-1}\{g^{kl}(\xi)\xi_k\xi_l\}^{1/2}\sqrt{\det g_{ij}(\xi)}\, d\omega_\epsilon(\xi),$$

where $d\omega_\epsilon$ is the volume element of the $(n-1)$-sphere of radius $\epsilon$ in the Euclidean space.

The estimate (3.2) implies that the first integral of the right-hand side of (3.3) tends to 0 as $\epsilon \to 0$. If $x \in F_\lambda(B_0(t_0))$, by putting $F_\lambda^{-1} = (f_\lambda^1, \cdots, f_\lambda^n)$ and changing the variable, we have

$$-\int_{F(\partial B_0(\epsilon))} \nabla_\nu H_x^\lambda(y)\varphi(y)d\sigma(y)$$

$$= -\int_{\partial B_0(\epsilon)} |\xi|^{-1} g^{ij}(\xi)\xi_i\partial_j(H_x^\lambda \circ F)(\xi)\varphi(F(\xi))\sqrt{\det g_{ij}(\xi)}\, d\omega_\epsilon(\xi)$$

$$= -\int_{\partial B_0(\epsilon)} |\xi|^{-1} g^{ij}(\xi)\xi_i\partial_k h^\lambda_{F_\lambda^{-1}(x)}(F_\lambda^{-1}\circ F(\xi))\partial_j(f_\lambda^k\circ F)(\xi)$$
$$\times\ \varphi(F(\xi))\sqrt{\det g_{ij}(\xi)}\, d\omega_\epsilon(\xi)$$

$$= \frac{1}{\omega}\int_{\partial B_0(\epsilon)} \{d^\lambda_{F_\lambda^{-1}\circ F(0)}(F_\lambda^{-1}\circ F(\xi))\}^{-n} |\xi|^{-1} g^{ij}(\xi) g^\lambda_{kl}(F_\lambda^{-1}\circ F(0))$$
$$\times\ \xi_i\partial_j(f_\lambda^k\circ F)(\xi)(f_\lambda^l\circ F(\xi) - f_\lambda^l\circ F(0))$$
$$\times\ \varphi(F(\xi))\sqrt{\det g_{ij}(\xi)}\, d\omega_\epsilon(\xi). \qquad (3.4)$$

Using Taylor's formula and the transformation law

$$g_{ij}(0)=g^{\lambda}_{kl}(F_{\lambda}^{-1}\circ F(0))\,\partial_i(f^k_{\lambda}\circ F)(0)\,\partial_j(f^l_{\lambda}\circ F)(0),$$

we obtain

$$\begin{aligned} &d^{\lambda}_{F_{\lambda}^{-1}\circ F(0)}(F_{\lambda}^{-1}\circ F(\xi)) \\ &=\{g^{\lambda}_{ij}(F_{\lambda}^{-1}\circ F(0))\,(f^i_{\lambda}\circ F(\xi)-f^i_{\lambda}\circ F(0))\,(f^j_{\lambda}\circ F(\xi)-f^j_{\lambda}\circ F(0))\}^{1/2} \\ &=\{g_{ij}(0)\xi^i\xi^j+O(|\xi|^3)\}^{1/2} \\ &=|\xi|(1+O(|\xi|)) \end{aligned}$$

and

$$\begin{aligned} &g^{ij}(\xi)g^{\lambda}_{kl}(F_{\lambda}^{-1}\circ F(0))\,\xi_i\partial_j(f^k_{\lambda}\circ F)(\xi)\,(f^l_{\lambda}\circ F(\xi)-f^l_{\lambda}\circ F(0)) \\ &=g^{ij}(0)g_{jk}(0)\xi_i\xi^k+O(|\xi|^3) \\ &=|\xi|^2(1+O(|\xi|)). \end{aligned}$$

Hence the integrand of the last integral of (3.4) is

$$\epsilon^{1-n}\varphi(F(0))\,(1+O(\epsilon))$$

and the integral tends to $\varphi(F(0))=\varphi(x)$ as $\epsilon\to 0$. Multiplying (3.4) by $\chi_{\lambda}(x)$, summing it up over $\lambda$, and passing to the limit, we obtain the lemma.

## 4. Estimate for singular integrals

We set $\Gamma^1_x(y)=-\Delta H_x(y)$ and define functions $\Gamma^k_x$ inductively by

$$\Gamma^{k+1}_x(y)=\int_M \Gamma^k_x(z)\Gamma^1_z(y)\,d\mu(z).$$

**Proposition 4.1** *Suppose $k<n/\alpha$. Then $\Gamma^k_x(y)=0$ for $d(x,y)\geq 2kt_0$ and*

$$|\Gamma^k_x(y)|\leq Cr_0^{-k\alpha}d(x,y)^{k\alpha-n}$$

*for some constant $C=C(n,p,\Lambda D^2)$.*

*Proof.* Set $\rho=d(x,y)$. We denote by $\hat{z}$ the middle point of a minimizing geodesic joining $x$ and $y$. The first assertion is obvious from the fact that $\Gamma^1_x(y)=0$ for $d(x,y)\geq 2t_0$. The second assertion follows from the estimate of the integral

$$\int_{B_{\hat{z}}\left(\frac{\rho}{2}+2t_0\right)} d(x,z)^{k\alpha-n}d(z,y)^{\alpha-n}\,d\mu(z)$$

for $d(x,y)\leq 2(1+k)t_0$. We split the domain of the integral into

$$B_x\left(\frac{\rho}{2}\right),\quad B_y\left(\frac{\rho}{2}\right),\quad B_{\hat{z}}(\rho)\backslash\left(B_x\left(\frac{\rho}{2}\right)\cup B_y\left(\frac{\rho}{2}\right)\right),\quad \text{and}\quad B_{\hat{z}}\left(\frac{\rho}{2}+2t_0\right)\backslash B_{\hat{z}}(\rho).$$

By Bishop's theorem, we can estimate the integrals as follows:

$$\int_{B_x\left(\frac{\rho}{2}\right)} d(x,z)^{k\alpha-n} d(z,y)^{\alpha-n} d\mu(z) \le \gamma\omega\left(\frac{\rho}{2}\right)^{\alpha-n}\int_0^{\frac{\rho}{2}} r^{k\alpha-1} dr = \frac{\gamma\omega}{k\alpha}\left(\frac{\rho}{2}\right)^{(k+1)\alpha-n},$$

$$\int_{B_y\left(\frac{\rho}{2}\right)} d(x,z)^{k\alpha-n} d(z,y)^{\alpha-n} d\mu(z) \le \gamma\omega\left(\frac{\rho}{2}\right)^{k\alpha-n}\int_0^{\frac{\rho}{2}} r^{\alpha-1} dr = \frac{\gamma\omega}{\alpha}\left(\frac{\rho}{2}\right)^{(k+1)\alpha-n},$$

$$\int_{B_{\bar{z}}(\rho)\setminus\left(B_x\left(\frac{\rho}{2}\right)\cup B_y\left(\frac{\rho}{2}\right)\right)} d(x,z)^{k\alpha-n} d(z,y)^{\alpha-n} d\mu(z) \le \gamma\omega\left(\frac{\rho}{2}\right)^{(k+1)\alpha-2n}\int_0^{\rho} r^{n-1} dr$$

$$= \frac{2^n\gamma\omega}{n}\left(\frac{\rho}{2}\right)^{(k+1)\alpha-n},$$

$$\int_{B_{\bar{z}}\left(\frac{\rho}{2}+2t_0\right)\setminus B_{\bar{z}}(\rho)} d(x,z)^{k\alpha-n} d(z,y)^{\alpha-n} d\mu(z)$$

$$\le \gamma\omega\int_{\rho}^{\frac{\rho}{2}+2t_0}\left(r-\frac{\rho}{2}\right)^{(k+1)\alpha-2n} r^{n-1} dr$$

$$\le 2^{n-1}\gamma\omega\int_{\frac{\rho}{2}}^{2t_0} r^{(k+1)\alpha-n-1} dr$$

$$= \begin{cases} \dfrac{2^{n-1}\gamma\omega}{n-(k+1)\alpha}\left\{\left(\dfrac{\rho}{2}\right)^{(k+1)\alpha-n} - (2t_0)^{(k+1)\alpha-n}\right\} & \text{if } (k+1)\alpha < n, \\ 2^{n-1}\gamma\omega\log\dfrac{4t_0}{\rho} & \text{if } (k+1)\alpha = n, \\ \dfrac{2^{n-1}\gamma\omega}{(k+1)\alpha-n}\left\{(2t_0)^{(k+1)\alpha-n} - \left(\dfrac{\rho}{2}\right)^{(k+1)\alpha-n}\right\} & \text{if } (k+1)\alpha > n. \end{cases}$$

Notice that we have put $\gamma = e^{(n-1)\sqrt{\Lambda}D}$. The last integral vanishes when $\rho \ge 4t_0$. The claim now follows by induction.

Recall that $n/\alpha = np/(p-n)$ is not an integer. The proof of Proposition 4.1 also yields the following estimate.

**Proposition 4.2.** *Set* $N = [n/\alpha] + 1$. *Then*

$$\Gamma_x^N(y) = 0 \quad \text{for } d(x,y) \ge 2Nt_0,$$

*and*

$$|\Gamma_x^N(y)| \le C r_0^{-n}$$

*for some constant* $C = C(n, p, \Lambda D^2)$.

The following estimate will be used later.

**Corollary 4.3** *Let* $1 \le k \le N$ *and* $f$ *be a function on* $M$. *Set*

$$u(x) = \int_M \Gamma_y^k(x) f(y)\, d\mu(y).$$

*Then there is a constant* $C$, *depending only on* $n$, $p$, *and* $\Lambda D^2$, *such that*

$$\|u\|_q \leq C\|f\|_q$$

*for* $1 \leq q \leq \infty$. *The similar estimate holds for*

$$u(x) = \int_M \Gamma_x^k(y) f(y) d\mu(y).$$

*Proof.* From the previous propositions, we have

$$\int_M |\Gamma_y^k(x)| d\mu(y) \leq C$$

and

$$\int_M |\Gamma_y^k(x)| d\mu(x) \leq C$$

for some constant $C = C(n, p, \Lambda D^2)$. For $1 \leq q < \infty$, we have by Hölder's inequality,

$$\begin{aligned} |u(x)|^q &\leq \left\{\int_M |\Gamma_y^k(x)| d\mu(y)\right\}^{q-1} \left\{\int_M |\Gamma_y^k(x)||f(y)|^q d\mu(y)\right\} \\ &\leq C^{q-1} \int_M |\Gamma_y^k(x)||f(y)|^q d\mu(y), \end{aligned}$$

from which we obtain (by integration in $x$)

$$\begin{aligned} \int_M |u(x)|^q d\mu(x) &\leq C^{q-1} \int_M \left\{\int_M |\Gamma_y^k(x)| d\mu(x)\right\} |f(y)|^q d\mu(y) \\ &\leq C^q \int_M |f(y)|^q d\mu(y). \end{aligned}$$

This completes the proof for $1 \leq q < \infty$. For $q = \infty$, the corollary follows from

$$|u(x)| \leq \int_M |\Gamma_y^k(x)| d\mu(y) \cdot \|f\|_\infty.$$

We next estimate $\Gamma_x^{N+1}(y)$.

**Proposition 4.4** (1) *There is a constant* $C = C(n, p, \Lambda D^2)$ *such that*

$$\Gamma_x^{N+1}(y) = 0 \quad \textit{for } d(x, y) \geq 2(N+1) t_0$$

*and*

$$|\Gamma_x^{N+1}(t)| \leq C r_0^{-n}.$$

(2) *The function* $\Gamma_x^{N+1}$ *is of* $C^\beta$*-class for any* $0 < \beta < \alpha$. *More precisely, in any* $p$*-harmonic coordinate* $F: B_0(r_0) \to M$, *we have*

$$r_0^\beta [\Gamma_x^{N+1} \circ \mathrm{F}]_{\beta, r_0} \leq C r_0^{-n}$$

*for some constant* $C = C(n, p, \beta, \Lambda D^2)$.

*Proof.* The claim (1) can be proved easily by straightforward calculation as in the proof of Proposition 4.1. To prove (2), we need the following lemma.

**Lemma 4.5.** *Suppose that* $k_1(\xi, \zeta)$ *and* $k_2(\xi, \zeta)$ *are smooth functions on* $B_0(R) \times B_0(R)$ *minus the diagonal satisfying*

$$|k_1(\xi, \zeta)| \leq C_1 R^{-\alpha} |\xi - \zeta|^\alpha, \quad |k_1(\xi, \zeta) - k_1(\xi', \zeta)| \leq C_2 R^{-\alpha} |\xi - \xi'|^\alpha,$$

$$|k_2(\xi,\zeta)| \leq C_3|\xi-\zeta|^{-n}, \quad |\frac{\partial k_2}{\partial \xi}(\xi,\zeta)| \leq C_4|\xi-\zeta|^{-n-1}.$$

*Set* $k(\xi,\zeta)=k_1(\xi,\zeta)k_2(\xi,\zeta)$ *and*

$$u(\xi)=\int_{B_0(R)} k(\xi,\zeta)f(\zeta)d\zeta$$

*for* $f\in C^0(B_0(R))$. *Then* $u\in C^\beta(B_0(R))$ *for any* $0<\beta<\alpha$. *More precisely, there exists a constant* $C$, *depending only on* $n$, $\alpha,\beta$, $C_1$, $C_2$, $C_3$, *and* $C_4$, *such that*

$$[u]_{\beta,R} \leq CR^{-\beta}\|f\|_{\infty,R}.$$

*Proof of Lemma.* Set $\rho=|\xi-\xi'|$ and $\bar{\xi}=(\xi+\xi')/2$. We have

$$\begin{aligned}|u(\xi)-u(\xi')| \leq &\Big\{\int_{B_\xi(\frac{3\rho}{2})}|k(\xi,\zeta)|d\zeta+\int_{B_{\xi'}(\frac{3\rho}{2})}|k(\xi',\zeta)|d\zeta\\ &+\int_{B_0(R)\setminus B_{\bar{\xi}}(\rho)}|k_1(\xi',\zeta)|\cdot|k_2(\xi,\zeta)-k_2(\xi',\zeta)|d\zeta\\ &+\int_{B_0(R)\setminus B_{\bar{\xi}}(\rho)}|k_1(\xi,\zeta)-k_1(\xi',\zeta)|\cdot|k_2(\xi,\zeta)|d\zeta\Big\}\cdot\|f\|_{\infty,R}.\end{aligned}$$

The first and the second integrals in the braces are estimated by

$$C_1C_3\omega R^{-\alpha}\int_0^{\frac{3\rho}{2}} r^{\alpha-1}dr=\frac{3^\alpha C_1C_3\omega R^{-\alpha}}{2^\alpha\alpha}\rho^\alpha \leq \frac{3^\alpha C_1C_3\omega R^{-\beta}}{2^\beta\alpha}\rho^\beta.$$

When $|\bar{\xi}-\zeta|\geq\rho$,

$$|k_2(\xi,\zeta)-k_2(\xi',\zeta)|=\rho|\frac{\partial k_2}{\partial\xi}(\tilde{\xi},\zeta)|$$

for some $\tilde{\xi}$ which lies in the segment connecting $\xi$ and $\xi'$. Since $|\tilde{\xi}-\bar{\xi}|\leq\rho/2 \leq|\bar{\xi}-\zeta|/2$,

$$|\tilde{\xi}-\zeta|\geq|\bar{\xi}-\zeta|-|\tilde{\xi}-\bar{\xi}|\geq|\bar{\xi}-\zeta|/2.$$

Then the third integral is estimated by

$$\begin{aligned}&C_1C_4R^{-\alpha}\rho\int_{B_0(R)\setminus B_{\bar{\xi}}(\rho)}|\xi-\zeta|^\alpha|\tilde{\xi}-\zeta|^{-n-1}d\zeta\\ &\leq 2^{n+1-\alpha}C_1C_4R^{-\alpha}\rho\int_{B_0(R)\setminus B_{\bar{\xi}}(\rho)}|\bar{\xi}-\zeta|^{\alpha-n-1}d\zeta\\ &\leq 2^{n+1-\alpha}C_1C_4\omega R^{-\alpha}\rho\int_\rho^{2R} r^{\alpha-2}dr\\ &\leq \frac{2^{n+1-\alpha}C_1C_4\omega R^{-\alpha}}{1-\alpha}\rho^\alpha\\ &\leq \frac{2^{n+1-\beta}C_1C_4\omega R^{-\beta}}{1-\alpha}\rho^\beta.\end{aligned}$$

Similarly, the last integral is estimated by

$$C_2C_3R^{-\alpha}\rho^\alpha\int_{B_0(R)\setminus B_{\bar{\xi}}(\rho)}|\xi-\zeta|^{-n}d\zeta \leq 2^nC_2C_3R^{-\alpha}\rho^\alpha\int_{B_0(R)\setminus B_{\bar{\xi}}(\rho)}|\bar{\xi}-\zeta|^{-n}d\zeta$$

$$\le 2^n C_2 C_3 \omega R^{-\alpha}\rho^{\alpha}\int_{\rho}^{2R} r^{-1}dr$$
$$= 2^n C_2 C_3 \omega R^{-\alpha}\rho^{\alpha}\log\frac{2R}{\rho}$$
$$\le \frac{2^{n+\alpha-\beta}C_2C_3\omega R^{-\beta}}{(\alpha-\beta)e}\rho^{\beta}$$

because the function $\rho\longrightarrow\rho^{\alpha-\beta}\log(2R/\rho)$ takes its maximum at $\rho=2Re^{-1/(\alpha-\beta)}$. The lemma has been proved.

We now return to the proof of Proposition 4.4. By definition,

$$\Gamma_x^{N+1}(y) = -\sum_{\lambda=1}^{Q}\int_M \Gamma_x^N(z)\chi_\lambda(z)\Delta H_z^\lambda(y)\,d\mu(z).$$

We rewrite each term of the sum in the harmonic coordinate $F_\lambda$:

$$u_\lambda(\xi) \equiv -\int_{B_0(t_0)}\Gamma_x^N(F_\lambda(\zeta))\chi_\lambda(F_\lambda(\zeta))\Delta h_\zeta^\lambda(\xi)\sqrt{\det g_{ij}^\lambda(\zeta)}\,d\zeta.$$

In view of (2.1), it suffices to estimate the $C^\beta$-norm of $u_\lambda$. It is a consequence of straightforward calculation that $\Delta h_\zeta^\lambda(\xi)$ expressed in (3.1) is a sum of the functions which satisfy the condition of Lemma 4.5: for the first term, with $k_1(\xi,\zeta)=\Psi_2(d_\zeta^\lambda(\xi))$; for the second term, with $k_1(\xi,\zeta)=g_\lambda^{ij}(\xi)\{g_{ij}^\lambda(\xi)-g_{ij}^\lambda(\zeta)\}$; and with $k_1(\xi,\zeta)=g_\lambda^{ij}(\xi)g_{ik}^\lambda(\zeta)\{g_{jl}^\lambda(\xi)-g_{jl}^\lambda(\zeta)\}$ for the last term. Then the claim follows by applying Lemma 4.5. with

$$f(\zeta)=\Gamma_x^N(F_\lambda(\zeta))\chi_\lambda(F_\lambda(\zeta))\sqrt{\det g_{ij}^\lambda(\zeta)}.$$

## 5. Construction of the Green function

We are now ready to construct the Green function by using $H_x(y)$ and $\Gamma_x^k(y)$. Recall Green's formula,

$$\varphi(x)=\int_M H_x(y)\Delta\varphi(y)\,d\mu(y)+\int_M \Gamma_x^1(y)\varphi(y)\,d\mu(y).$$

By putting $\varphi(x)\equiv 1$ in Green's formula, we obtain

$$\int_M \Gamma_x^1(y)\,d\mu(y)=1.$$

Iterating Green's formula, we also obtain

$$\varphi(x)=\int_M K_x(y)\Delta\varphi(y)\,d\mu(y)+\int_M \Gamma_x^{N+1}(y)\varphi(y)\,d\mu(y) \tag{5.1}$$

where

$$K_x(y)=H_x(y)+\int_M\sum_{k=1}^{N}\Gamma_x^k(z)H_z(y)\,d\mu(z).$$

From the results of the previous section, It is easy to see that

$$|K_x(y)|\le Cd(x,y)^{2-n}$$

for some constant $C=C(n,p,\Lambda D^2)$ and that $K_x(y)=0$ for $d(x,y)\ge 2(N+1)t_0$. Hence we have

$$\int_M |K_x(y)|\,d\mu(y) \leq Cr_0^2 \tag{5.2}$$

for some $C=C(n, p, \Lambda D^2)$.

By putting $\varphi(x) \equiv 1$ in the formula (5.1), we also obtain

$$\int_M \Gamma_x^{N+1}(y)\,d\mu(y) = 1.$$

Therefore we can define a function $R_x$ by solving the equation

$$\Delta R_x = \Gamma_x^{N+1} - \frac{1}{V} \tag{5.3}$$

under the condition

$$\int_M R_x(y)\,d\mu(y) = 0.$$

The elliptic regularity theorem (Theorem 1.2) and Proposition 4.4 imply that $R_x$ is of $C^2$-class.

Putting (5.3) into (5.1), we obtain

$$\begin{aligned}\varphi(x) &= \int_M K_x(y)\,\Delta\varphi(y)\,d\mu(y) + \int_M \Delta R_x(y)\,\varphi(y)\,d\mu(y)\\ &\quad + \frac{1}{V}\int_M \varphi(y)\,d\mu(y)\\ &= \int_M K_x(y)\,\Delta\varphi(y)\,d\mu(y) + \int_M R_x(y)\,\Delta\varphi(y)\,d\mu(y)\\ &\quad + \frac{1}{V}\int_M \varphi(y)\,d\mu(y)\end{aligned}$$

i.e., $\Delta(K_x+R_x) = \delta_x - V^{-1}$. Since $\int_M \{K_x(y)+R_x(y)\}\,d\mu(y) = \int_M K_x(y)\,d\mu(y)$, we have

$$G_x(y) = K_x(y) + R_x(y) - \frac{1}{V}\int_M K_x(y)\,d\mu(y). \tag{5.4}$$

We can now estimate the Green function near the singularity.

**Theorem 5.1.** *There exist contants $C_1$ and $C_2$, depending only on $n$, $p$, $\Lambda D^2$, and $V$, such that*

$$|G_x(y)| \leq C_1 d(x, y)^{2-n} \quad \textit{for } d(x, y) \leq C_2 i_0.$$

*Proof.* Since we have already estimated $K_x(y)$ and

$$\left|\frac{1}{V}\int_M K_x(y)\,d\mu(y)\right| \leq \frac{Cr_0^2}{V} \leq \frac{CD^n}{V} r_0^{2-n},$$

it remains to estimate $R_x(y)$. By (5.1), we have

$$\begin{aligned}R_x(z) &= \int_M K_z(y)\,\Delta R_x(y)\,d\mu(y) + \int_M \Gamma_z^{N+1}(y)\,R_x(y)\,d\mu(y)\\ &= \int_M K_z(y)\,\Gamma_x^{N+1}(y)\,d\mu(y) - \frac{1}{V}\int_M K_z(y)\,d\mu(y) + \int_M \Gamma_z^{N+1}(y)\,R_x(y)\,d\mu(y).\end{aligned}$$

Therefore we obtain

$$|R_x(z)| \leq \|\Gamma_x^{N+1}\|_\infty \int_M |K_z| d\mu + \frac{1}{V}\int_M |K_z| d\mu + \| \Gamma_z^{N+1} \|_{\frac{2n}{n+2}} \| R_x \|_{\frac{2n}{n-2}}. \quad (5.5)$$

Applying Sobolev's inequality (1.3) and Hölder's inequality, we have

$$\| R_x \|^2_{\frac{2n}{n-2}} \leq C_S^2 \| \nabla R_x \|_2^2 = C_S^2 \int_M R_x \Delta R_x d\mu = C_S^2 \int_M R_x \Gamma_x^{N+1} d\mu - \frac{1}{V}\int_M R_x d\mu$$
$$= C_S^2 \int_M R_x \Gamma_x^{N+1} d\mu \leq C_S^2 \| R_x \|_{\frac{2n}{n-2}} \| \Gamma_x^{N+1} \|_{\frac{2n}{n+2}}$$

and hence

$$\| R_x \|_{\frac{2n}{n-2}} \leq C_S^2 \| \Gamma_x^{N+1} \|_{\frac{2n}{n+2}}.$$

From Proposition 4.4, we have

$$\| \Gamma_x^{N+1} \|_\infty \leq C r_0^{-n}$$

and

$$\| \Gamma_x^{N+1} \|_{\frac{2n}{n+2}} \leq C r_0^{\frac{2-n}{2}}$$

for some constant $C = C(n, p, \Lambda D^2)$. Then putting these inequalities and (5.2) into (5.5), we obtain

$$\| R_x \|_\infty \leq C r_0^{2-n}, \quad (5.6)$$

where $C$ is a constant that depends only on $n$, $p$, $\Lambda D^2$, and $D^n/V$. The proof has been completed.

We turn to the first derivative of the Green function.

**Theorem 5.2** *There exist constants $C_1$ and $C_2$, depending only on $n$, $p$, $\Lambda D^2$, and $D^n/V$, such that*

$$| \nabla G_x(y) | \leq C_1 d(x, y)^{1-n} \quad \text{for } d(x, y) \leq C_2 i_0.$$

*Proof.* Differentiating (5.4), we have

$$\nabla G_x(y) = \nabla K_x(y) + \nabla R_x(y).$$

By the argument similar to [5, Lemma 4.1], the formula

$$\nabla K_x(y) = \nabla H_x(y) + \int_M \sum_{k=1}^N \Gamma_x^k(z) \nabla H_z(y) d\mu(z)$$

is justified for $y \neq x$. Then Propositions 3.1, 4.1, and 4.2 imply that

$$| \nabla K_x(y) | \leq C d(x, y)^{1-n}$$

for some constant $C = C(n, p, \Lambda D^2)$ and that $\nabla K_x(y) = 0$ for $d(x, y) \geq 2(N+1) t_0$. Hence we have

$$\int_M | \nabla K_x | d\mu \leq C r_0$$

for some $C = C(n, p, \Lambda D^2)$.

In order to estimate $\nabla R_x$, we approximate $R_x$ with smooth functions $\{\varphi_k\}_{k=1}^{\infty}$ in the $C^2$-topology.

From Propositions 4.1 and 4.2, we see that the leading part of $K_x(y)$ is $H_x(y)$ and we deduce that

$$K_x(y) \geq -ar_0^{2-n}$$

for some constant $a$ depending only on $n$, $p$, and $\Lambda D^2$. Then we have

$$|\nabla \varphi_k|^2(y) = \int_M K_y \Delta |\nabla \varphi_k|^2 d\mu + \int_M \Gamma_y^{N+1} |\nabla \varphi_k|^2 d\mu$$
$$= \int_M (K_y + ar_0^{2-n}) \Delta |\nabla \varphi_k|^2 d\mu + \int_M \Gamma_y^{N+1} |\nabla \varphi_k|^2 d\mu.$$

Using Weizenböck's formula, we have

$$\Delta |\nabla \varphi_k|^2 = 2\langle \nabla \Delta \varphi_k, \nabla \varphi_k\rangle - 2|\nabla^2 \varphi_k|^2 - 2\mathrm{Ric}(\nabla \varphi_k, \nabla \varphi_k)$$
$$\leq 2\langle \nabla \Delta \varphi_k, \nabla \varphi_k\rangle + 2(n-1)\Lambda |\nabla \varphi_k|^2.$$

Since $K_y + ar_0^{2-n}$ is non-negative (by the definition of $a$),

$$\int_M (K_y + ar_0^{2-n}) \Delta |\nabla \varphi_k|^2 d\mu \leq 2\int_M (K_y + ar_0^{2-n}) \langle \nabla \varphi_k, \nabla \Delta \varphi_k + (n-1)\Lambda \nabla \varphi_k\rangle d\mu$$
$$= 2\int_M (K_y + ar_0^{2-n}) \Delta \varphi_x \{\Delta \varphi_k + (n-1)\Lambda \varphi_k\} d\mu$$
$$-2\int_M \{\Delta \varphi_k + (n-1)\Lambda \varphi_k\} \langle \nabla K_y, \nabla \varphi_k\rangle d\mu.$$

Passing to the limit, we obtain

$$|\nabla R_x|^2(y) \leq \int_M K_y |\Delta R_x|^2 d\mu + ar_0^{2-n} \int_M |\Delta R_x|^2 d\mu + (n-1)\Lambda \int_M K_y R_x \Delta R_x d\mu$$
$$+ a(n-1)\Lambda r_0^{2-n} \int_M |\nabla R_x|^2 d\mu - 2\int_M \Delta R_x \langle \nabla K_y, \nabla R_x\rangle d\mu$$
$$-2(n-1)\Lambda \int_M R_x \langle \nabla K_y, \nabla R_x\rangle d\mu + \int_M \Gamma_y^{N+1} |\nabla R_x|^2 d\mu. \qquad (5.7)$$

The right-hand side of (5.7) is estimated with a constant $C = C(n, p, \Lambda D^2, D^n/V)$ as follows:

$$\int_M \Gamma_y^{N+1} |\nabla R_x|^2 d\mu \leq \| \Gamma_y^{N+1} \|_\infty \int_M |\nabla R_x|^2 d\mu = Cr_0^{-n} \int_M R_x \Delta R_x d\mu$$
$$\leq Cr_0^{-n} \int_M R_x \Gamma_x^{N+1} d\mu \leq Cr_0^{2-2n},$$
$$a(n-1)\Lambda r_0^{2-n} \int_M |\nabla R_x|^2 d\mu \leq aC\Lambda r_0^{4-2n} \leq aC\Lambda D^2 r_0^{2-2n},$$

$$\int_M K_y|\Delta R_x|^2 d\mu = \int_M K_y \left|\Gamma_x^{N+1} - \frac{1}{V}\right|^2 d\mu \leq 2\left(\|\Gamma_x^{N+1}\|_\infty^2 + \frac{1}{V^2}\right)\int_M |K_y| d\mu$$
$$\leq C\left(r_0^{2-2n} + \frac{r_0^2}{V^2}\right) \leq C\left(1 + \frac{D^{2n}}{V^2}\right) r_0^{2-2n},$$

$$a r_0^{2-n}\int_M |\Delta R_x|^2 d\mu \leq 2a r_0^{2-n}\left(\int |\Gamma_x^{N+1}|^2 d\mu + \frac{1}{V}\right)$$
$$\leq 2a r_0^{2-n}\left(C r_0^{-n} + \frac{1}{V}\right) \leq 2a\left(C + \frac{D^n}{V}\right) r_0^{2-2n},$$

$$(n-1)\Lambda\int_M K_y R_x \Delta R_x d\mu \leq (n-1)\Lambda\|R_x\|_\infty\left(\|\Gamma_x^{N+1}\|_\infty + \frac{1}{V}\right)\int_M |K_y| d\mu$$
$$\leq C\Lambda r_0^{4-n}\left(r_0^{-n} + \frac{1}{V}\right) \leq C\Lambda D^2\left(1 + \frac{D^n}{V}\right) r_0^{2-2n},$$

$$-2\int_M \Delta R_x\langle \nabla K_y, \nabla R_x\rangle d\mu \leq 2\left(\|\Gamma_x^{N+1}\|_\infty + \frac{1}{V}\right)\|\nabla R_x\|_\infty \int_M |\nabla K_y| d\mu$$
$$\leq C r_0\left(r_0^{-n} + \frac{1}{V}\right)\|\nabla R_x\|_\infty$$
$$\leq C\left(1 + \frac{D^n}{V}\right) r_0^{1-n}\|\nabla R_x\|_\infty,$$

$$-2(n-1)\Lambda\int_M R_x\langle \nabla K_y, \nabla R_x\rangle d\mu \leq 2(n-1)\Lambda\|R_x\|_\infty\|\nabla R_x\|_\infty\int_M |\nabla K_y| d\mu$$
$$\leq C\Lambda r_0^{3-n}\|\nabla R_x\|_\infty \leq C\Lambda D^2 r_0^{1-n}\|\nabla R_x\|_\infty.$$

Hence we obtain

$$\|\nabla R_x\|_\infty^2 \leq C_1 r_0^{1-n}\|\nabla R_x\|_\infty + C_2 r_0^{2-2n}$$

for some constants $C_1$ and $C_2$ depending only on $n$, $p$, $\Lambda D^2$ and $D^n/V$. This implies

$$\|\nabla R_x\|_\infty \leq C r_0^{1-n}$$

for some constant $C = C(n, p, \Lambda D^2, D^n/V)$ and the theorem follows.

**Remark 5.3.** Using the estimate of the heat kernel, one can estimate $G_x$ and $\nabla G_x$ globally in terms of $n$, $\Lambda D^2$, $D^n/V$. See [7] .

## 6. $L^p$-estimate for the Laplace operator

Let us show Calderon-Zygmund type inequality for $G_x$ in this section. We first fix some notations. Let $E_1$ and $E_2$ be vector bundles over $M$ with norms. We use the same symbol $|\cdot|$ for the norms on $E_1$ and $E_2$. For a section $s$ of $E_1$ or $E_2$, we denote by $\mu(s; a)$ the volume of the subset $\{x \in M : |s(x)| > a\}$. Notice that

$$\mu(s; a) \leq a^{-q}\int_{|s|>a} |s|^q d\mu \leq \frac{\|s\|_q^q}{a^q}.$$

We denote by $L^q(E_1)$ (resp. $L^q(E_2)$) the space of the sections whose $L^q$-norm is finite.

Let us introduce the following basic interpolation theorem which is repeatedly used in this section. For the proof, see [5, Theorem 9.8] .

**Theorem 6.1 (Marcinkiewicz's interpolation inequality).** *Let $A$ be a linear operator from $L^{q_1}(E_1)\cap L^{q_2}(E_1)$ to $L^{q_1}(E_2)\cap L^{q_2}(E_2)$ with $1\le q_1<q_2<\infty$ satisfying*

$$\mu(As;a)\le\frac{C_1\|s\|_{q_1}^{q_1}}{a^{q_1}}\quad \text{and}\quad \mu(As;a)\le\frac{C_2\|s\|_{q_2}^{q_2}}{a^{q_2}}$$

*for some constants $C_1$ and $C_2$. Then $A$ can be extended to a linear bounded operator on $L^q(E_1)$ for $q_1<q<q_2$ and*

$$\|As\|_q\le 2\left\{\frac{q}{q-q_1}+\frac{q}{q_2-q}\right\}^{1/q}C_1^{\eta}C_2^{1-\eta}\|s\|_q$$

*for $\eta=q_1(q_2-q)/q(q_2-q_1)$.*

For a function $f$ on $M$, we put

$$u(x)=\int_M H_y^{\lambda}(x)\chi_{\lambda}(y)f(y)\,d\mu(y).$$

By Green's formula, we have

$$\chi_{\lambda}(y)\varphi(y)=\int_M H_y^{\lambda}(x)\chi_{\lambda}(y)\Delta\varphi(x)\,d\mu(x)-\int_M \Delta H_y^{\lambda}(x)\chi_{\lambda}(y)\varphi(x)\,d\mu(x)$$

for any smooth function $\varphi$. Therefore

$$\int_M f(x)\chi_{\lambda}(y)\varphi(x)\,d\mu(x)=\int_M u(x)\Delta\varphi(x)\,d\mu(x)-\int_M\left\{\int_M \Delta H_y^{\lambda}(x)\chi_{\lambda}(y)f(y)\,d\mu(y)\right\}\varphi(x)\,d\mu(x)$$

and we obtain

$$f(x)\chi_{\lambda}(x)=\Delta u(x)-\int_M \Delta H_y^{\lambda}(x)\chi_{\lambda}(y)f(y)\,d\mu(y).\tag{6.1}$$

We first show the following proposition.

**Proposition 6.2** *Let $1<q\le p$ and*

$$u(x)=\int_M H_y^{\lambda}(x)\chi_{\lambda}(y)f(y)\,d\mu(y).$$

*There exists a constant $C$, depending only on $q$, $n$, $p$, $\Lambda D^2$, and $D/i_0$, such that*

$$\|\nabla^2 u\|_q\le C\|f\|_q.$$

*Proof.* We carry out the proof in nine steps. We always calculate in the the coordinate $F_\lambda$ and denote the metric tensor by $g_{ij}$ and the Christoffel symbols by $\Gamma_{ij}^k$. Notice that there holds $\nabla_{ij}^2 u = \partial_{ij}^2 u - \Gamma_{ij}^k \partial_k u$.

**Step 1.** First we prove this proposition for $q = 2$. By Weitzenböck's formula, we have

$$\|\nabla^2 u\|_2^2 \leq \|\Delta u\|_2^2 + (n-1)\Lambda \|\nabla u\|_2^2.$$

From (6.1) and Hölder's inequality, we obtain

$$|\Delta u(x)|^2 \leq 2|f(x)|^2 + 2\left\{\int_M |\Delta H_y^\lambda(x)|\, d\mu(y)\right\}\left\{\int_M |\Delta H_y^\lambda(x)|\,|f(y)|^2 d\mu(y)\right\}.$$

From the estimate of $\Delta h_\xi^\lambda(\xi)$, we can estimate the integrals $\int_M |\Delta H_y^\lambda(x)|d\mu(y)$ and $\int_M |\Delta H_y^\lambda(x)|d\mu(x)$ with some constant $C_1$ that depends only on $n$, $p$, and $\Lambda D^2$. Hence we have

$$\|\Delta u\|_2^2 \leq 2(1+C_1^2)\|f\|_2^2.$$

Similarly we have

$$\begin{aligned} |\nabla u(x)| &\leq \int_M |\nabla H_y^\lambda(x)||f(y)|d\mu(y) \\ &\leq \left\{\int_M |\nabla H_y^\lambda(x)|d\mu(y)\right\}^{1/2}\left\{\int_M |\nabla H_y^\lambda(x)||f(y)|^2 d\mu(y)\right\}^{1/2}. \end{aligned}$$

The estimate of $\partial h_\xi^\lambda(\xi)$ implies that there is a constant $C_2$, depending only on $n$, $p$, and $\Lambda D^2$, such that

$$\int_M |\nabla H_y^\lambda(x)|d\mu(y) \leq C_2 D; \quad \int_M |\nabla H_y^\lambda(x)|d\mu(x) \leq C_2 D.$$

Hence we have

$$\|\nabla u\|_2^2 \leq C_2^2 D^2 \|f\|_2^2.$$

Therefore we obtain

$$\|\nabla^2 u\|_2^2 \leq 2(1 + C_1^2)\|f\|_2^2 + (n-1)C_2^2 \Lambda D^2 \|f\|_2^2. \tag{6.2}$$

**Step 2.** We denote by $S^2T^*M$ the bundle of symmetric bilinear forms. We apply Theorem 6.1 to the operator $f \mapsto \nabla^2 u$. By the result of Step 1, we have

$$\mu(\nabla^2 u; a) \leq \frac{\|\nabla^2 u\|_2^2}{a^2} \leq \frac{C\|f\|_2^2}{a^2}. \tag{6.3}$$

**Step 3.** In Steps 3 and 4, we will prove that there is a constant $C$ depending only on $n$, $p$, $\Lambda D^2$, and $D/i_0$, and satisfying

$$\mu(\nabla^2 u; a) \leq \frac{C\|f\|_1}{a} \tag{6.4}$$

for any function $f \in L^1(M)$.

For simplicity, we denote the volume of a subset $S \subset M$ by$|S|$. We have

$$\frac{1}{|B_x(t_0)|}\int_{B_x(t_0)}|f|d\mu \leq \frac{V}{|B_x(t_0)|}\frac{\|f\|_1}{V}$$

$$\leq \frac{\gamma D^n}{t_0^n}\frac{\|f\|_1}{V} = \frac{c\|f\|_1}{V}, \tag{6.5}$$

where $c$ is a constant that depends only on $n$, $p$, $\Lambda D^2$, and $D/i_0$. Here we have used Bishop-Gromov's volume comparison theorem, which says that for $0 < r < R$ we have

$$\frac{|B_x(R)|}{|B_x(r)|} \leq \frac{\gamma R^n}{r^n}.$$

Notice that we may assume $\|f\|_1 \leq aV/c$. Otherwise, (6.4) is valid because $\mu(\nabla^2 u; a) \leq V \leq c\|f\|_1/a$. Hence

$$\frac{1}{|B_x(t_0)|}\int_{B_x(t_0)}|f|d\mu \leq a$$

for any $x \in M$.

Set $E_0 = \{x \in M: |f(x)| \leq a\}$ and define a sequence $\{t_k\}_{k=1}^{\infty}$ by $t_k = 2^{-k}t_0$. For $k \leq 1$, we put

$$\widetilde{E}_k = \left\{x \in E: \frac{1}{|B_x(t_k)|}\int_{B_x(t_k)}|f|d\mu > a\right\}$$

and $E = \cup_{k=1}^{\infty} \widetilde{E}_k$. Then the set $M \setminus (E \cup E_0)$ has measure 0, because

$$\lim_{k\to\infty} \frac{1}{|B_x(t_k)|}\int_{B_x(t_k)}|f|d\mu = |f(x)|$$

for a.e. $x \in M$.

We now define a family of subsets $\{E_k\}_{k\geq 1}$ inductively by $E_1 = \widetilde{E}_1$ and $E_k = \widetilde{E}_k \setminus \widetilde{E}_{k-1}$ for $k > 1$. Notice that for $x$ contained in the closure of $E_k$ we have

$$\frac{1}{|B_x(t_k)|}\int_{B_x(t_k)}|f|d\mu \geq a;$$
$$\frac{1}{|B_x(2t_k)|}\int_{B_x(2t_k)}|f|d\mu \leq a. \tag{6.6}$$

We can choose a finite subset $N_1$ of the closure of $E_1$ such that the geodesic balls $\{B_x(t_1)\}_{x\in N_1}$ are mutually disjoint and the geodesic balls $\{B_x(2t_1)\}_{x\in N_1}$ cover the closure of $E_1$. Inductively we choose a finite subset $N_k$ of the closure of $E_k \setminus \cup_{j=1}^{k-1} \cup_{x\in N_j} B_x(2t_j)$ such that the geodesic balls $\{B_x(t_k)\}_{x\in N_k}$ are mutually disjoint and the geodesic balls $\{B_x(2t_k)\}_{x\in N_k}$ cover the closure of $E_k \setminus \cup_{j=1}^{k-1} \cup_{x\in N_j} B_x(2t_j)$. In this way, we obtain a set of pairs $\{(x_k, \rho_k): x_k \in M, \rho_k > 0\}_{k\geq 1} = \{(x, t_j): x \in N_j, j = 1,2,\cdots\}$ such that the geodesic balls $\{B_{x_k}(2\rho_k)\}_{k\geq 1}$ cover

$E$ and the geodesic balls $\{B_{x_k}(\rho_k)\}_{k\geq 1}$ are mutually disjoint. Moreover, (6.6) implies that

$$\frac{1}{|B_{x_k}(\rho_k)|}\int_{B_{x_k}(\rho_k)}|f|d\mu \geq a$$

and

$$\frac{1}{|B_{x_k}(2\rho_k)|}\int_{B_{x_k}(2\rho_k)}|f|d\mu \leq a.$$

We define a family of mutually disjoint subsets $\{D_k\}_{k\geq 1}$ inductively by

$$D_1 = B_{x_1}(2\rho_1)\setminus \bigcup_{j\geq 2} B_{x_j}(\rho_j);$$

$$D_k = B_{x_k}(2\rho_k)\setminus\Big[\big(\bigcup_{1\leq i\leq k-1} D_i\big) \cup \big(\bigcup_{j>k} B_{x_j}(\rho_j)\big)\Big] \quad \text{for } k > 1.$$

Obviously $B_{x_k}(\rho_k) \subset D_k \subset B_{x_k}(2\rho_k)$ and $\cup_{k\geq 1} D_k = \cup_{k\geq 1} B_{x_k}(2\rho_k) \supset E$. From Bishop-Gromov's volume comparison theorem, we have

$$\begin{aligned}\frac{1}{|D_k|}\int_{D_k}|f|d\mu &\leq \frac{1}{|B_{x_k}(\rho_k)|}\int_{B_{x_k}(2\rho_k)}|f|d\mu\\ &\leq \frac{2^n\gamma}{|B_{x_k}(2\rho_k)|}\int_{B_{x_k}(2\rho_k)}|f|d\mu\\ &\leq 2^n\gamma a\end{aligned}$$

and

$$a \leq \frac{1}{|B_{x_k}(\rho_k)|}\int_{B_{x_k}(\rho_k)}|f|d\mu \leq \frac{4^n\gamma}{|B_{x_k}(4\rho_k)|}\int_{D_k}|f|d\mu.$$

Therefore the volume of the subset $\cup_{k\geq 1} B_{x_k}(16\rho_k)$ is equal to or less than

$$\sum_{k\geq 1}|B_{x_k}(16\rho_k)| \leq \sum_{k\geq 1}\frac{16^n\gamma}{a}\int_{D_k}|f|d\mu \leq \frac{4^n\gamma\|f\|_1}{a}. \tag{6.7}$$

Using the defining function $\phi_k$ of $D_k$, we decompose $f$ as follows:

$$f = f_0 + \sum_{k\geq 1} f_k,$$

where

$$f_k = \phi_k f - \frac{\phi_k}{|D_k|}\int_{D_k} f d\mu.$$

Then the function $f_0$ satisfies $|f_0| \leq 2^n\gamma a$ for a.e. $x \in M$ and $\|f_0\|_1 \leq \|f\|_1$. The functions $\{f_k\}_{k\geq 1}$ satisfy $\int_M f_k d\mu = 0$.

**Step 4.** Set $u_k(x) = \int_M H_y^\lambda(x)\chi_\lambda(y) f_k(y)\, d\mu(y)$ for $k \geq 0$. From the result of Step 1, we have

$$\mu(\nabla^2 u_0; a/2) \leq \frac{4\|\nabla^2 u_0\|_2^2}{a^2} \leq \frac{C\|f_0\|_2^2}{a^2}$$

$$\leq \frac{C\|f_0\|_\infty\|f_0\|_1}{a^2} \leq \frac{2^n\gamma C\|f\|_1}{a} \tag{6.8}$$

for some constant $C = C(n, p, \Lambda D^2)$.

Next we analyze $u_k$ for $k \geq 1$. Recall that the support of $\chi_\lambda$ is contained in $F_\lambda(B_0(t_0))$ and $H_y^\lambda \equiv 0$ outside $F_\lambda(B_0(2t_0))$. If $D_k \cap F_\lambda(B_0(t_0)) \neq \emptyset$, then $B_{x_k}(t_0) \cap B_{F_\lambda(0)}(2t_0) \neq \emptyset$. Since $8t_0 \leq r_0$, we have

$$D_k \subset B_{x_k}(t_0) \subset B_{F_\lambda(0)}(4t_0) \subset F_\lambda(B_0(r_0))$$

and

$$F_\lambda^{-1}(D_k) \subset F_\lambda^{-1}(B_{x_k}(2\rho_k)).$$

Therefore we can analyze $u_k$ in the $p$-harmonic coordinate $F_\lambda$, i.e.,

$$u_k(\xi) = \int_{|\zeta-\xi_k|\leq 4\rho_k} h_\zeta^\lambda(\xi)\chi_\lambda(\zeta)f_k(\zeta)\sqrt{\det g_{ij}(\zeta)}\,d\zeta.$$

Here we have put $\xi = F_\lambda^{-1}(x)$, $\xi_k = F_\lambda^{-1}(x_k)$, and $\zeta = F_\lambda^{-1}(y)$. Recall that $g_{ij}$ satisfies $4^{-1}\delta_{ij} \leq g_{ij} \leq 4\delta_{ij}$ as symmetric bilinear forms in the coordinate $F_\lambda$. If $x \notin B_{x_k}(16\rho_k)$, then $|\xi-\xi_k| \geq 8\rho_k$ and $|\xi-\zeta| \geq 4\rho_k$. Hence there exists a constant $C = C(n)$ such that

$$\begin{aligned}
&\int_{M\setminus B_{x_k}(16\rho_k)} |\nabla^2 u_k(x)|\,d\mu(x) \\
&\leq C\Big[\int_{|\xi-\xi_k|\geq 8\rho_k} |\partial_{ij}^2 u_k(\xi)|\sqrt{\det g_{ij}(\xi)}\,d\xi \\
&\quad + \int_{|\xi-\xi_k|\geq 8\rho_k} |\Gamma_{ij}^l(\xi)\,\partial_l u_k(\xi)|\sqrt{\det g_{ij}(\xi)}\,d\xi\Big].
\end{aligned} \tag{6.9}$$

In the first integral of the right hand side, we can interchange the order of integration and differentiation:

$$\begin{aligned}
\partial_{ij}^2 u_k(\xi) &= \int_{|\zeta-\xi_k|\leq 4\rho_k} \partial_{ij}^2 h_\zeta^\lambda(\xi)\chi_\lambda(\zeta)f_k(\zeta)\sqrt{\det g_{ij}(\zeta)}\,d\zeta \\
&= \int_{|\zeta-\xi_k|\leq 4\rho_k} \{\partial_{ij}^2 h_\zeta^\lambda(\xi)\chi_\lambda(\zeta) - \partial_{ij}^2 h_{\xi_k}^\lambda(\xi)\chi_\lambda(\xi_k)\}f_k(\zeta)\sqrt{\det g_{ij}(\zeta)}\,d\zeta.
\end{aligned}$$

The last equality holds because $\int_M f_k d\mu = 0$. From Lemma 3.1 and (2.2), we observe that

$$\begin{aligned}
&|\partial^2 h_\zeta^\lambda(\xi)\chi_\lambda(\zeta) - \partial^2 h_{\xi_k}^\lambda(\xi)\chi_\lambda(\xi_k)| \\
&\qquad \leq C\{r_0^{-\alpha}|\zeta-\xi_k|^\alpha|\xi-\xi_k|^{-n} + |\zeta-\xi_k||\xi-\xi_k|^{-n-1}\}
\end{aligned}$$

for some constant $C = C(n, p, \Lambda D^2)$. Since $\sqrt{\det g_{ij}(\xi)} \leq 2^n$, we obtain

$$\int_{|\xi-\xi_k|\geq 8\rho_k} |\partial^2 u_k(\xi)|\sqrt{\det g_{ij}(\xi)}\,d\xi$$

$$\leq 2^n C\int_{8\rho_k\leq|\xi-\xi_k|\leq r_0}\left\{\left(\frac{4\rho_k}{r_0}\right)^{\alpha}|\xi-\xi_k|^{-n}+4\rho_k|\xi-\xi_k|^{-n-1}\right\}d\xi$$
$$\times\int_{|\zeta-\xi_k|\leq 4\rho_k}|f_k(\zeta)|\sqrt{\det g_{ij}(\zeta)}\,d\zeta$$
$$\leq 2^n C\omega\int_{8\rho_k}^{r_0}\left\{\left(\frac{4\rho_k}{r_0}\right)^{\alpha}r^{-1}+4\rho_k r^{-2}\right\}dr\cdot\|f_k\|_1$$
$$=2^n C\omega\left\{\left(\frac{4\rho_k}{r_0}\right)^{\alpha}\log\frac{r_0}{8\rho_k}+\frac{1}{2}\right\}\|f_k\|_1$$
$$\leq 2^n C\omega\left(\frac{1}{2^{\alpha}\alpha e}+1\right)\|f_k\|_1.$$

As to the second integral, we use the estimates

$$\int_{B_0(r_0)}|\partial_l h_\xi^\lambda(\xi)|^{p/(p-1)}\sqrt{\det g_{ij}(\xi)}\,d\xi$$
$$\leq C\int_0^{2r_0}r^{-(n-1)/(p-1)}dr$$
$$=\frac{C(p-1)}{p-n}(2r_0)^{(p-n)/(p-1)} \tag{6.10}$$

and

$$\|\Gamma_{ij}^l\|_{p,r_0}\leq C\|\partial g\|_{p,r_0}\leq Cr_0^{(n-p)/p} \tag{6.11}$$

for some constant $C=C(n)$. These inequalities imply

$$\int_{|\xi-\xi_k|\geq 8\rho_k}|\Gamma_{ij}^l(\xi)\,\partial_l u_k(\xi)|\sqrt{\det g_{ij}(\xi)}\,d\xi$$
$$\leq\int_{|\zeta-\xi_k|\leq 4\rho_k}\left\{\int_{|\xi-\xi_k|\geq 8\rho_k}|\Gamma_{ij}^l(\xi)\,\partial_l h_\xi^\lambda(\xi)|\sqrt{\det g_{ij}(\xi)}\,d\xi\right\}$$
$$\times|f_k(\zeta)|\sqrt{\det g_{ij}(\zeta)}\,d\zeta$$
$$\leq\|\Gamma_{ij}^l\|_{p,r_0}\int_{|\zeta-\xi_k|\leq 4\rho_k}\left\{\int_{B_0(r_0)}|\partial_l h_\xi^\lambda(\xi)|^{p/(p-1)}\sqrt{\det g_{ij}(\xi)}\,d\xi\right\}^{(p-1)/p}$$
$$\times|f_k(\zeta)|\sqrt{\det g_{ij}(\zeta)}\,d\zeta$$
$$\leq C\|f_k\|_1$$

for some constant $C=C(n)$. Thus we can find a constant $C$, depending only on $n$, $p$, $\Lambda D^2$, and $D^n/V$, such that

$$\int_{M\setminus B_{x_k}(16\rho_k)}|\nabla^2 u_k|d\mu\leq C\|f_k\|_1.$$

Hence the volume of the subset $\left\{x\in M\setminus\cup_{k\geq 1}B_{x_k}(16\rho_k):|\nabla^2(u-u_0)(x)|\geq a/2\right\}$ is equal to or less than

$$\frac{2}{a}\int_{M\setminus\cup_{k\geq1}B_{x_k}(16\rho_k)}|\nabla^2(u-u_0)|d\mu \leq \frac{2}{a}\sum_{k\geq1}\int_{M\setminus B_{x_k}(16\rho_k)}|\nabla^2 u_k|d\mu$$
$$\leq\frac{2C}{a}\sum_{k\geq1}\|f_k\|_1$$
$$\leq\frac{2C}{a}\|f-f_0\|_1$$
$$\leq\frac{4C}{a}\|f\|_1. \tag{6.12}$$

Combining (6.7) and (6.12), we obtain

$$\mu(\nabla^2(u-u_0);a/2) \leq \frac{C\|f\|_1}{a} \tag{6.13}$$

for some constant $C = C(n, p, \Lambda D^2, D^n/V)$. Now (6.4) follows from (6.8), (6.13), and

$$\mu(\nabla^2 u;a) \leq \mu(\nabla^2(u-u_0);a/2)+\mu(\nabla^2 u_0;a/2).$$

**Step 5.** From (6.3) and (6.4), we obtain Proposition 6.2 for the case $1 < q < 2$ by applying Marcinkiewicz's interpolation inequality.

**Step 6.** In the case $2 < q \leq p$, we need to consider the adjoint operator. Let $b$ be a section of symmetric 2-tensor $S^2TM$ and define a function $\nabla^{*2}b$ in the sense of distribution, that is, it satisfies

$$\int_M\phi\nabla^{*2}bd\mu = \int_M\nabla^2_{ij}\phi b^{ij}d\mu$$

for any smooth funtion $\phi$ on $M$. Let $2 \leq q < p$ and set $q' = q/(q-1)$, $p' = p/(p-1)$. We define a function $v$ by

$$v(x) = \int_M H^\lambda_x(y)\chi_\lambda(x)\nabla^{*2}b(y)d\mu(y).$$

Then for the function $v(x) = \int_M H^\lambda_y(x)\chi_\lambda(y)f(y)d\mu(y)$, we see that

$$\int_M\nabla^2_{ij}u b^{ij}d\mu = \int_M fvd\mu.$$

By duality, it suffices to show the existence of a constant $C = C(n, p, q, \Lambda D^2, D/i_0)$ satisfying

$$\|v\|_{q'} \leq C\|b\|_{q'}.$$

Notice that, from (6.2) and by duality, we already have

$$\|v\|_2 \leq C\|b\|_2 \tag{6.14}$$

for some constant $C = C(n, p, \Lambda D^2)$.

**Step 7.** We define a function $w$ by

$$w(x) = v(x) + \int_{F_\lambda(B_0(r_0))} \Gamma^l_{ij}(y)\, \partial_l H^\lambda_x(y)\, \chi_\lambda(x)\, b^{ij}(y)\, d\mu(y).$$

By Hölder's inequality, we see that

$$\begin{aligned} |w(x) - v(x)| \leq & \left\{ \int_{F_\lambda(B_0(r_0))} |\Gamma^l_{ij}(y)|^p d\mu(y) \right\}^{1/p} \\ & \times \left\{ \int_{F_\lambda(B_0(r_0))} |\partial_l H^\lambda_x(y)|^{p/(p-1)} d\mu(y) \right\}^{1/q - 1/p} \\ & \times \left\{ \int_{F_\lambda(B_0(r_0))} |\partial_l H^\lambda_x(y)|^{p/(p-1)} |b^{ij}(y)|^{q'} d\mu(y) \right\}^{1/q'}. \end{aligned}$$

Then (6.10), (6.11), and the estimate

$$\begin{aligned} & \int_{F_\lambda(B_0(r_0))} \left\{ \int_{F_\lambda(B_0(r_0))} |\partial_l H^\lambda_x(y)|^{p/(p-1)} d\mu(x) \right\} |b^{ij}(y)|^{q'} d\mu(y) \\ & \leq C r_0^{(p-n)/(p-1)} \int_{F_\lambda(B_0(r_0))} |b^{ij}(y)|^{q'} d\mu(y) \end{aligned}$$

imply

$$\begin{aligned} & \int_{F_\lambda(B_0(r_0))} |w(x) - v(x)|^{q'} d\mu(x) \\ & \leq C \int_{F_\lambda(B_0(r_0))} |b^{ij}(y)|^{q'} d\mu(y) \end{aligned}$$

and hence

$$\| w - v \|_{q'} \leq C \| b \|_{q'} \tag{6.15}$$

for some constant $C = C(n, p, q)$.

**Step 8.** In Steps 8 and 9, let us show that, if $1 < q' \leq 2$, we have

$$\| w \|_{q'} \leq C \| b \|_{q'} \tag{6.16}$$

for some constant $C = C(n, p, q, \Lambda D^2, D/i_0)$.

From (6.14) and (6.15), it follows that there is a constant $C = C(n, p, q, \Lambda D^2)$ such that

$$\| w \|_2 \leq \| v \|_2 + \| w - v \|_2 \leq C \| b \|_2,$$

and hence we have

$$\mu(w; a) \leq \frac{C \| b \|_2^2}{a^2}. \tag{6.17}$$

**Step 9.** In view of Theorem 6.1, it remains to show that

$$\mu(w; a) \leq \frac{C \| b \|_1}{a} \tag{6.18}$$

for some constant $C = C(n, p, q, \Lambda D^2, D/i_0)$. We shall decompose $b$ as in Step 3.

We may assume $\int_M |b| d\mu \le aV/c$ for the same constant $c$ of (6.5). We can construct a set of triplets

$$\{(x_k, \rho_k, D_k): x_k \in D_k \subset M, 0 < \rho_k \le t_0/2\}_{k\ge 1}$$

satisfying the following properties:

(1) $\{D_k\}_{k\ge 1}$ is a family of mutually disjoint measurable sets such that $D_k \subset B_{x_k}(2\rho_k)$.

(2) $|b| \le a$ for a.e. $x \in M\setminus \cup_{k\ge 1} D_k$.

(3) $\frac{1}{|D_k|}\int_{D_k} |b| d\mu \le 2^n \gamma a.$

(4) The volume of the subset $\cup_{k\ge 1} B_{x_k}(16\rho_k)$ is equal to or less than $4^n\gamma \| b \|_1/a$.

If $F_\lambda(B_0(2t_0))$ intersects with $D_k$ then $B_{F_\lambda(0)}(4t_0) \cap B_{x_k}(t_0) \ne \emptyset$. Since $12t_0 = r_0$, we have

$$D_k \subset B_{x_k}(t_0) \subset B_{F_\lambda(0)}(6t_0) \subset F_\lambda(B_0(r_0)).$$

Using the coordinate $F_\lambda$, we express $b$ as $b^{ij}\partial_i\partial_j$ by functions $b^{ij}$ on $F_\lambda(B_0(r_0))$. We define sections $\bar{b}_k = \bar{b}_k^{ij}\partial_i\partial_j$ of $S^2TM|_{D_k}$ by setting

$$\bar{b}_k^{ij} = \frac{1}{|D_k|}\int_{D_k} b^{ij} d\mu.$$

For $x, y \in F_\lambda(B_0(r_0))$, the norms of the fibers $S^2T_xM$, $S^2T_yM$ satisfy $|\cdot|_x \le 16 |\cdot|_y$. Therefore, we verify

$$|\bar{b}_k| \le \frac{16}{|D_k|}\int_{D_k} |b| d\mu$$

and

$$\int_{D_k} |\bar{b}_k| d\mu \le 16\int_{D_k} |b| d\mu.$$

Using the defining function $\phi_k$ of $D_k$, we now decompose $b$ into $b_0 + \Sigma_{k\ge 1} b_k$ by setting

$$b_k = \phi_k(b - \bar{b}_k) \quad \text{for } k \ge 1.$$

Then we have

$$\| b_0 \|_1 \le 16 \| b \|_1; \quad | b_0 | \le 2^{n+4}\gamma a \quad \text{for a.e. } x \in M. \tag{6.19}$$

We set

$$v_k(x) = \int_M H_x^\lambda(y)\chi_\lambda(x)\nabla^{*2} b_k(y) d\mu(y)$$

and

$$w_k(x) = v_k(x) + \int_M \Gamma_{ij}^l(y)\partial_l H_x^\lambda(y)\chi_\lambda(x) b_k^{ij}(y) d\mu(y),$$

where $b_k^{ij}$ is the local expression of $b_k$ in the coordinate $F_\lambda$. From (6.14), we have

$$\mu(w_0; a/2) \leq \frac{4\|w_0\|_2^2}{a^2} \leq \frac{C\|b_0\|_2^2}{a^2} \leq \frac{C\|b_0\|_\infty\|b_0\|_1}{a^2}.$$

Then it follows from (6.19) that

$$\mu(w_0; a/2) \leq \frac{C\|b\|_1}{a}$$

for some constant $C = C(n, p, \Lambda D^2)$. For $x \notin B_{x_k}(16\rho_k)$, we have

$$w_k(x) = \int_M \nabla_{ij}^2 H_x^\lambda(y) b_k^{ij}(y) d\mu(y) + \int_M \Gamma_{ij}^l(y) \partial_l H_x^\lambda(y) \chi_\lambda(x) b_k^{ij}(y) d\mu(y).$$
$$= \int_M \chi_\lambda(x) \partial_{ij}^2 H_x^\lambda(y) b_k^{ij}(y) d\mu(y)$$

because the singularity of $H_x$ lies outside the support of $b_k$. Notice that both the domain of integral and the support of $v_k$ are subdomains of $F_\lambda(B_0(r_0))$. By putting $F_\lambda(\zeta) = x$, $F_\lambda(\xi) = y$, and $F_\lambda(\xi_k) = x_k$, we have

$$v_k(\zeta) = \int_{|\xi-\xi_k|\leq 4\rho_k} \chi_\lambda(\zeta) \partial_{ij}^2 h_\zeta^\lambda(\xi) b_k^{ij}(\xi) \sqrt{\det g_{ij}(\xi)}\, d\xi.$$

Since

$$\int_{|\xi-\xi_k|\leq 4\rho_k} b_k^{ij}(\xi) \sqrt{\det g_{ij}(\xi)}\, d\xi = 0,$$

we obtain

$$|w_k(\zeta)| \leq C\int_{B_0(4\rho_k)} |\{\partial_{ij}^2 h_\zeta^\lambda(\xi) - \partial_{ij}^2 h_\zeta^\lambda(\xi_k)\} b_k^{ij}(\xi)| d\xi$$

for some constant $C = C(n)$. From Lemma 3.1, we see that there exists a constant $C = C(n, p)$ such that

$$|\partial_{ij}^2 h_\zeta^\lambda(\xi) - \partial_{ij}^2 h_\zeta^\lambda(\xi_k)| \leq C|\zeta - \xi_k|^{-n-1}|\xi - \xi_k|.$$

Therefore we have

$$|w_k(\xi)| \leq \int_{|\xi-\xi_k|\leq 4\rho_k} |\partial_{ij}^2 h_\zeta^\lambda(\xi) - \partial_{ij}^2 h_\zeta^\lambda(\xi_k)|\, |b_k^{ij}(\xi)| \sqrt{\det g_{ij}(\xi)}\, d\xi$$
$$\leq C\rho_k |\zeta - \xi_k|^{-n-1} \|b_k\|_1,$$

and hence

$$\int_{|\zeta-\xi_k|\geq 8\rho_k} |w_k(\xi)| \sqrt{\det g_{ij}(\zeta)}\, d\zeta$$
$$\leq C\rho_k \int_{|\zeta-\xi_k|\geq 8\rho_k} |\zeta - \xi_k|^{-n-1} d\zeta \cdot \|b_k\|_1$$
$$\leq C\omega\rho_k \int_{8\rho_k}^{2r_0} r^{-2} dr \cdot \|b_k\|_1$$

$$\leq \frac{C\omega}{8} \| b_k \|_1.$$

As in Step 4, we can estimate the volume of the subset

$$\left\{x \in M \setminus \bigcup_{k \geq 1} B_{x_k}(16\rho_k) : | w - w_0(x) | \geq a/2\right\}$$

from above by the quantity $C \| b \|_1 / a$ for some constant $C = C(n, p, \Lambda D^2)$. Then (6.18) follows. Thus we obtain (6.16) by applying Theorem 6.1 again. This completes the proof of Proposition 6.2.

The following corollary is a direct consequence of Proposition 6.2.

**Corollary 6.3.** *Let $1 < q \leq p$. For a function $f$ on $M$, define a function by*

$$u(x) = \int_M H_y(x) f(y) d\mu(y).$$

*Then there is a constant $C$, depending only on $n$, $p$, $q$, $\Lambda D^2$, and $D/i_0$, such that*

$$\| \nabla^2 u \|_q \leq C \| f \|_q.$$

Thus we can estimate the constant that appears in the $L^p$-estimate for the Laplace operator in terms of the diameter, the injectivity radius, and the lower bound of the Ricci tensor.

**Theorem 6.4.** *Let $q > 1$ and $f$ be a function on $M$. Define a function $u$ by*

$$u(x) = \int_M G_y(x) f(y) d\mu(y).$$

*Then there is a constant $C$, depending only on $n$, $q$, $\Lambda D^2$, and $D/i_0$, such that*

$$\| \nabla^2 u \|_q \leq C \| f \|_q.$$

*In particular, by putting $f = \Delta u$, we have*

$$\| \nabla^2 u \|_q \leq C \| \Delta u \|_q.$$

*Proof.* Choose $p$ such that $p > n$ and $p \geq q$. On account of Corollaries 4.3 and 6.3, we have only to show that, for the function $u$ defined by

$$u(x) = \int_M R_y(x) f(y) d\mu(y),$$

there is a constant $C = C(n, p, q, \Lambda D^2, D/i_0)$ such that

$$\| \nabla^2 u \|_q \leq C \| f \|_q.$$

Set $\alpha = 1 - n/p$ and $\beta = \alpha/2$. In every $p$-harmonic coordinate $F : B_0(r_0) \to M$, by the elliptic regularity theorem (Theorem 1.2), we have

$$r_0^2 \| \partial^2 R_x \|_{\infty, r_0/2} \leq C\left\{\| R_x \|_{\infty, r_0} + r_0^2 \left\| \Gamma_x^{N+1} - \frac{1}{V} \right\|_{\infty, r_0} + r_0^{2+\beta} \, [\Gamma_x^{N+1}]_{\beta, r_0}\right\}$$

for some $C = C(n, p)$. Hence, by Proposition 4.4 and (5.6), we obtain

$$\| \partial^2 R_x \|_{\infty, r_0/2} \leq C r_0^{-n}$$

for some constant $C = C(n, p, \Lambda D^2, D^n/V)$. Recall that $i_0^n/V$ is estimated from above in terms of $n$ (cf. [4]). Therefore we can estimate the $L^p$-norm of $\nabla^2_{ij} R_x = \nabla^2_{ij} R_x - \Gamma^k_{ij} \partial_k R_x$:

$$\| \nabla^2 R_x \|_p \leq C i_0^{n(1-p)/p}, \tag{6.20}$$

where $C$ is a constant depending only on $n$, $p$, $\Lambda D^2$, and $D/i_0$. Notice that the ratio $i_0/r_0$ depends only on $n$, $p$, and $\Lambda D^2$. Since $R_x$ is of $C^2$-class, we have

$$\nabla^2 u(y) = \int_M \nabla^2 R_x(y) f(x) d\mu(x).$$

Applying Hölder's inequality, we deduce

$$|\nabla^2 u(y)|^q \leq V^{q-1} \int_M |\nabla^2 R_x(y)|^q |f(x)|^q d\mu(x).$$

Integrating this in $y$ and using (6.20), we obtain

$$\begin{aligned} \| \nabla^2 u \|_q^q &\leq V^{q-1} \int_M \| \nabla^2 R_x \|_q^q |f(x)|^q d\mu(x) \\ &\leq V^{q(p-1)/p} \int_M \| \nabla^2 R_x \|_p^q |f(x)|^q d\mu(x) \\ &\leq C (V/D^n)^{q(p-1)/p} (D/i_0)^{nq(p-1)/p} \| f \|_q^q \end{aligned}$$

for some constant $C = C(n, p, \Lambda D^2, D/i_0)$. This shows the theorem because $V/D^n$ is estimated from above by $n$, $p$, and $\Lambda D^2$.

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