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Physically reasonable solutions to steady compressible Navier-Stokes equations in 3D-exterior domains ($v_\infty = 0$)

By

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1. Introduction

In this paper we study the asymptotic properties of the kinetic and density fields of a compressible viscous Navier-Stokes fluid, filling a three dimensional domain exterior to a compact reagion Ω_c, when the prescribed velocity at infinity is zero. As far as the authors know, this is the first contribution in the subject. (The existence and uniqueness to this problem was studied in several papers of Matsumura and Nishida, Novotny and Padula, Novotny, Padula, see [14], [15], [19], [16], [21].)

The same problem, in the simpler case of incompressible Navier-Stokes fluids attracted mathematicians since the paper of Leray [12], who constructed (for the arbitrary size of external data) a solution of problem:

$$\Delta u + \nabla \Pi = -u \cdot \nabla u + f, \qquad \operatorname{div} u = 0\,, \qquad u|_{\partial\Omega_c} = 0 \tag{1.1}$$

(here u denotes the velocity and p the pressure), with the finite Dirichlet integral for the velocity (so called Leray solution). In 1965, Finn [4] proved (for small external forces) existence of solutions with the spatial decay of rate $|x|^{-1}$ for the velocity (so called physically reasonable solutions)[1]. He also proved, in [4], [5], [6] that any physically reasonable solution (if it exists) possesses the decay:

$$u \sim |x|^{-1},\ \nabla u \sim |x|^{-2} \lg |x|,\ \Pi \sim |x|^{-2} \lg |x|\,. \tag{1.2}$$

This statement was (for small data) in a certain sense improved by Borchers and Miyakawa [2]. They have proved existence of solutions in weak Lebesgue spaces $L^{3/2}_w(\Omega)$[2]. More precisely

$$u \sim |x|^{-1}\,, \qquad \nabla u \in L^{3/2}_w(\Omega)\,. \tag{1.3}$$

Similar result was derived independently by Galdi and Simader [10]:

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[1] Only recently, in 1991, Galdi [7] proved that the Leray solution was a physically reasonable one, provided external data are "small".

[2] Recall the definition of $L^{3/2}_w(\Omega)$, $1 < t < \infty$. It is a Bnach space of functions φ with the finite norm $\|\varphi\|_{t,w} = \sup_E \left[(\text{meas } E)^{-1+1/t} \int_E |\varphi|\, dx \right]$. (The supremum is taken over all bounded measurable subset of Ω.)

$$u \sim |x|^{-1}, \ \nabla u, \ \Pi \in L^q(\Omega) \qquad \text{for any } q > \frac{3}{2}. \tag{1.4}$$

Only recently, Novotny and Padula [20] obtained (again only for small data) existence of solutions with the decay

$$u \sim |x|^{-1}, \qquad \nabla u \sim |x|^{-2}, \qquad \Pi \sim |x|^{-2}. \tag{1.5}$$

This result is optimal in the sense that the decay (1.5) is precisely the same as the decay of the fundamental solution to the Stokes operator.

The goal of the present paper is to prove similar result for the (compressible) Poisson-Stokes equations.[3]

More precisely, we investigate the asymptotic structure of the isothermal compressible flow. We prove, for the small external data, existence and uniqueness of solutions in the class of functions with the following properties at infinity:

- The velocity decays to 0 with the rate $|x|^{-1}$ as $|x| \to \infty$.
- The gradient of velocity decays to 0 with the rate $|x|^{-2}$ as $|x| \to \infty$.
- The density tends to a constant with the rate $|x|^{-2}$ as $|x| \to \infty$.

Thus, we get, also for the compressible fluids, the solutions with the same decay at infinity as that one of the fundamental Stokes tensor. Moreover, we show, that the compressible part of the velocity field decays more rapidly than the incompressible one. For the main results see Theorems 5.1, 5.2, 5.3 and 5.4.

The proofs rely on the following techniques

(i) The method of decomposition introduced in [19] which splits the linearized Poisson-Stokes system into a Stokes-type equation (governing the incompressible part of the velocity), a Neumann problem (governing the compressible part of the velocity field), and a transport equation (governing the density).

(ii) Known results for the above auxiliary linear problems in the exterior domains.

(iii) Integral representation formulas for the Laplace and Stokes operators, due to Finn [4], Chang and Finn [3].

(iv) An estimate for weakly singular integrals of certain particular structure, see Section 3.

(v) Some standard estimates of the decay of weakly singular integrals, see e.g. Smirnov [28].

(vi) Estimates of the singular integrals of Calderon-Zygmund type in weighted Sobolev spaces with the polynomial weights, due to Stein [29].

[3] The first existence theorems for small external data in this situation were proved by Matsumura, Nishida [14]–[15] and Novotny, Padula [19]. The velocity and its gradient possess the decay $|x|^{-1}$. From the point of view of the above observation, this classes of functions are not optimal.

2. Equations and heuristic approach

We consider a steady isothermal motion[4] of a viscous compressible fluid in a 3-D exterior domain Ω. The motion is governed by the classical Poisson-Stokes equations for the unknown functions $\rho \geq 0$ (the density) and $v = (v_1, v_2, v_3)$ (the velocity):

$$-\mu_1 \Delta v - (\mu_1 + \mu_2)\nabla \operatorname{div} v + \nabla \rho = \rho f - \operatorname{div}(\rho v \otimes v), \qquad x \in \Omega,$$
$$\operatorname{div}(\rho v) = 0, \qquad x \in \Omega. \tag{2.1}$$

Here μ_1, μ_2 are the constant viscosities satisfying

$$\mu_1 > 0, \qquad \mu_2 \geq -\tfrac{2}{3}\mu_1 \tag{2.2}$$

and f is the external force. The boundary conditions and the conditions at infinity are

$$v|_{\partial\Omega} = 0, \qquad v(x) \to 0, \qquad \rho(x) \to 1 \qquad \text{as } |x| \to \infty. \tag{2.3}$$

The equations for the perturbations (v, σ) where $\rho = 1 + \sigma$, read

$$-\mu_1 \Delta v - (\mu_1 + \mu_2)\nabla \operatorname{div} v + \nabla \sigma = F(\sigma, v), \qquad x \in \Omega,$$
$$\operatorname{div} v + \operatorname{div}(\sigma v) = 0, \qquad x \in \Omega,$$
$$v|_{\partial\Omega} = 0,$$
$$v(x) \to 0,\ \sigma(x) \to 0 \qquad \text{as } |x| \to \infty \tag{2.4}$$

with

$$F(\sigma, v) = -\operatorname{div}((1 + \sigma) v \otimes v) + (1 + \sigma) f \tag{2.5}$$

As in [19] we firstly solve the linearized system

$$-\mu_1 \Delta v - (\mu_1 + \mu_2)\nabla \operatorname{div} v + \nabla \sigma = \mathscr{F} \qquad x \in \Omega,$$
$$\operatorname{div} v + \operatorname{div}(\sigma w) = 0, \qquad x \in \Omega \tag{2.6}$$

with the boundary conditions and the conditions at infinity

$$v|_{\partial\Omega} = 0,$$
$$v(x) \to 0, \qquad \sigma(x) \to 0 \qquad \text{as } |x| \to \infty \tag{2.7}$$

for the unknown functions (σ, v) (and w, $\mathscr{F}$ given).

The solution of (2.6), (2.7) is found as follows. We define a (linear) operator

$$\mathscr{L}: \xi \to \phi \tag{2.8}$$

in the following way:

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[4] The results are valid also for the barotropic (i.e. in particular for the isentropic) case. Then $\nabla \rho$ in equation (1.1) is replaced by $\nabla \pi(\rho)$, where the pressure π is a scalar function, a restriction on $\mathbf{R}^1_+$ (the positive real axe) of an analytic function on $\mathbf{C}_+$ (a complex half-plane containing $\mathbf{R}^1_+$).

i) For a given ξ, we find (Π, u) by solving the Stokes problem

$$-\mu_1 \Delta u + \nabla \Pi = \mathscr{F}, \qquad x \in \Omega,$$
$$\operatorname{div} u = 0, \qquad x \in \Omega,$$
$$u|_{\partial\Omega} = -\nabla\xi|_{\partial\Omega},$$
$$u(x) \to 0 \qquad \text{as } |x| \to \infty. \tag{2.9}$$

ii) When Π is known, we find σ as a solution of the transport equation (with w given)

$$\sigma + (2\mu_1 + \mu_2)\operatorname{div}(\sigma w) = \Pi, \qquad x \in \Omega,$$
$$\sigma(x) \to 0 \qquad \text{as } x \to \infty. \tag{2.10}$$

iii) Once σ is known, ϕ is found as a solution of the Neumann problem

$$\Delta\phi = -\operatorname{div}(\sigma w), \qquad x \in \Omega,$$
$$\left.\frac{\partial\phi}{\partial\nu}\right|_{\partial\Omega} = 0,$$
$$\nabla\phi(x) \to 0 \qquad \text{as } |x| \to \infty. \tag{2.11}$$

The reader easy verifies that the fixed point ϕ of $\mathscr{L}$ (if it exists) and the corresponding (Π, σ, u), satisfy the system of equations

$$-\mu_1 \Delta u + \nabla \Pi = \mathscr{F}, \qquad x \in \Omega,$$
$$\operatorname{div} u = 0, \qquad x \in \Omega,$$
$$u|_{\partial\Omega} = -\nabla\phi|_{\partial\Omega},$$
$$u(x) \to 0 \qquad \text{as } |x| \to \infty, \tag{2.12}$$
$$\sigma + (2\mu_1 + \mu_2)\operatorname{div}(\sigma w) = \Pi, \qquad x \in \Omega,$$
$$\sigma(x) \to 0 \qquad \text{as } x \to \infty, \tag{2.13}$$
$$\Delta\phi = -\operatorname{div}(\sigma w), \qquad x \in \Omega,$$
$$\left.\frac{\partial\phi}{\partial\nu}\right|_{\partial\Omega} = 0,$$
$$\nabla\phi(x) \to 0 \qquad \text{as } |x| \to \infty \tag{2.14}$$

or equivalently

$$\sigma \qquad \text{and} \qquad v = u + \nabla\phi$$

satisfy the linearized problem (2.6), (2.7).

If the system (2.12)–(2.14) is solved, we find the solution of the nonlinear problem (2.4), (2.5) in the form

$$\sigma\,, \qquad v = \nabla\phi + u$$

where (σ, ϕ, u) is a fixed point of the nonlinear operator

$$\mathcal{N}: (\tau, \xi, z) \to (\sigma, \phi, u)\,. \tag{2.15}$$

In (2.15), (σ, ϕ, u) is a solution of the problem (2.12)–(2.14) corresponding to $\mathcal{F} = F(\tau, \nabla\xi + z)$ and $w = \nabla\xi + z$.

It is worth noting, that the fixed point (σ, ϕ, u) of $\mathcal{N}$ and the corresponding Π, solve the system

$$\begin{aligned} -\mu_1 \Delta u + \nabla\Pi &= F(\sigma, u + \nabla\phi)\,, \qquad x \in \Omega\,,\\ \operatorname{div} u &= 0\,, \qquad x \in \Omega\,,\\ u|_{\partial\Omega} &= -\nabla\phi|_{\partial\Omega}\,,\\ u(x) &\to 0 \qquad \text{as } |x| \to \infty\,, \end{aligned} \tag{2.16}$$

$$\begin{aligned} \Delta\phi &= -\operatorname{div}(\sigma(u + \nabla\phi))\,, \qquad x \in \Omega\,,\\ \left.\frac{\partial\phi}{\partial\nu}\right|_{\partial\Omega} &= 0\,,\\ \nabla\phi(x) &\to 0 \qquad \text{as } |x| \to \infty \end{aligned} \tag{2.17}$$

$$\begin{aligned} \sigma + (2\mu_1 + \mu_2)\operatorname{div}(\sigma(u + \nabla\phi)) &= \Pi\,, \qquad x \in \Omega\,,\\ \sigma(x) &\to 0 \qquad \text{as } x \to \infty\,, \end{aligned} \tag{2.18}$$

which is formally equivalent to system (2.4), (2.5).

3. Fundamental solutions and related estimates

3.1. Fundamental solutions. In connection with our problem, we consider two linear operators

$$\mathcal{L}_1\mathcal{E} = \Delta\mathcal{E}, \;\; \mathcal{L}_2(\mathcal{U}, \mathcal{P}) = \mu_1\Delta\mathcal{U} + \nabla\mathcal{P}\,. \tag{3.1}$$

The fundamental solutions to these equations are well known

$$\mathcal{E}(x) = -\frac{1}{4\pi|x|}\,,$$

$$\mathcal{U}_{ij}(x) = -\frac{1}{8\pi\mu_1}\left(\frac{\delta_{ij}}{|x|} + \frac{x_i x_j}{|x|^3}\right), \;\; \mathcal{P}_i(x) = \frac{\partial\mathcal{E}}{\partial x_i}(x) = \frac{x_i}{4\pi|x|^3}\,. \tag{3.2}$$

They solve, in the sense of distributions, the following equations[5,6]

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[5] If not stated explicitly otherwise, we use the Einstein summation convention over the repeated indeces.

[6] In order to avoid the ambiguities, we explain our notation of differentiation of particular composite functions. Let $f: \mathbf{R}^3 \to \mathbf{R}^1$. Then $\nabla f(x - y)$ is a gradient of f calculated in the point $x - y$; $\nabla_x f(x - y) = \nabla f(x - y)$, $\nabla_y f(x - y) = -\nabla f(x - y)$.

$$\Delta_x \mathscr{E}(x-y) = \delta(x-y)\,,$$

$$\mu_1 \Delta_x \mathscr{U}_{ij}(x-y) + \frac{\partial \mathscr{P}_i}{\partial x_j}(x-y) = \delta_{ij}\delta(x-y)\,,$$

$$\frac{\partial \mathscr{U}_{ij}}{\partial x_j}(x-y) = 0\,. \tag{3.3}$$

Here δ_{ij} is the Kronecker delta and δ is the Dirac distribution. Recall the asymptotic properties $\mathscr{E}$, $\mathscr{P}$ and $\mathscr{U}$:

$$|\nabla^\alpha \mathscr{E}(x)| \le \frac{c}{|x|^{\alpha+1}}\,,$$

$$|\nabla^\alpha \mathscr{P}(x)| \le \frac{c}{|x|^{\alpha+2}}\,, \qquad |\nabla^\alpha \mathscr{U}(x)| \le \frac{c}{|x|^{\alpha+1}}\,, \tag{3.4}$$

here $c > 0$, $\alpha = 0, 1, \dots$ and $x \neq 0$. Moreover

$$\nabla^2 \mathscr{U}(x) = \frac{\Sigma_1\left(\frac{x}{|x|}\right)}{|x|^3}\,, \qquad \nabla^2 \mathscr{E}(x) = \nabla \mathscr{P}(x) = \frac{\Sigma_2\left(\frac{x}{|x|}\right)}{|x|^3} \tag{3.5}$$

with $\Sigma_i \in L^\infty(S_1)$, $\int_{S_1} \Sigma_i dS = 0$ $(i = 1, 2)$ where S_1 is a unit sphere, are the singular kernels of the Calderon-Zygmund type, cf. e.g. Stein [30].

3.2. Some estimates of weakly singular and singular integrals. First we prove a lemma, which (besides the decomposition) plays the main tool for the proof of the decay rate of the solutions (see also Novotny, Padula [20]). It concerns with the decay of the weakly singular integral of the type

$$\mathscr{I}(x) = \int_\Omega \mathscr{Q}(x-y)\,\mathrm{div}\, g(y)\, dy\,, \tag{3.6}$$

where Ω is an exterior domain (we suppose, without loss of generality that the unit sphere of center zero is contained in $\mathbf{R}^3 \backslash \Omega$) and $\mathscr{Q}$ is a smooth function on $\mathbf{R}^3 \backslash \{0\}$ with the decay properties

$$|\mathscr{Q}(x)| \le \frac{c}{|x|^2}\,, \qquad |\nabla \mathscr{Q}(x)| \le \frac{c}{|x|^3}\,. \tag{3.7}$$

Lemma 3.1. *Let* $g \in W^{1,\infty}(\Omega)$ *such that* $|x|^2 g \in L^\infty(\Omega)$ *and* $|x|^3\,\mathrm{div}\, g \in L^\infty(\Omega)$. *Let* $\mathscr{Q}$ *be function satisfying* (3.7). *Then* $\mathscr{I}$ *is defined a.e. in* Ω,

$$|x|^2 \mathscr{I} \in L^\infty(\Omega)$$

and we have the estimate

$$\| |x|^2 \mathscr{I} \|_{0,\infty} \le c(\| |x|^2 g \|_{0,\infty} + \| |x|^3\,\mathrm{div}\, g \|_{0,\infty})\,. \tag{3.8}$$

Proof. Fix $x \in \Omega$ and put $|x| = R$. For R sufficiently great, we decompose the exterior domain into five parts (we denote by $B_R(x)$ the sphere of center x and radius R, $B^R(x) = \mathbf{R}^3 \backslash B_R(x)$, $\Omega_R = \Omega \cap B_R(0)$):

$$\mathscr{G}_I = \Omega_{R/2}, \quad \mathscr{G}_{II} = B_1(x), \quad \mathscr{G}_{III} = B_{R/2}(x)\backslash B_1(x),$$
$$\mathscr{G}_{IV} = B_{3R/2}(0)\backslash(B_{R/2}(0) \cup B_{R/2}(x)), \quad \mathscr{G}_V = B^{3R/2}(0). \tag{3.9}$$

Let us estimate integrals $\mathscr{I}_I, \ldots, \mathscr{I}_V$ defined as follows

$$\mathscr{I}(x) = \sum_{i=I}^{V} \int_{\mathscr{G}_i} \mathscr{Q}(x-y) \operatorname{div} g(y)\, dy = \sum_{i=I}^{V} \mathscr{I}_i(x). \tag{3.10}$$

1. Estimates of $\mathscr{I}_I$: We have the identity

$$\mathscr{I}_I(x) = -\int_{\Omega_{R/2}} \nabla_y \mathscr{Q}(x-y)\cdot g(y)\, dy + \int_{\partial\Omega} \mathscr{Q}(x-y) g(y)\cdot \nu\, dS_y$$
$$+ \int_{\partial B_{R/2}(0)} \mathscr{Q}(x-y) g(y)\cdot \nu\, dS_y = \mathscr{I}_{I1}(x) + \mathscr{I}_{I2}(x) + \mathscr{I}_{I3}(x). \tag{3.11}$$

Now, we estimate the integrals at the r.h.s. separately as follows $\left(\text{notice that for } y \in \mathscr{G}_I,\ |x-y| \geq \frac{R}{2}\right)$:

$$\mathscr{I}_{I1} \leq cR^{-3} \int_{\Omega_{R/2}} (|y|^2 |g(y)|)|y|^{-2}\, dy \leq cR^{-3} \int_1^{R/2} \int_S (|y|^2 |g(y)|)\, d|y|\, dS$$
$$\leq cR^{-2} \| |x|^2 g \|_{0,\infty,\Omega},$$
$$|\mathscr{I}_{I2}| \leq cR^{-2} \| |x|^2 g \|_{0,\infty,\Omega},$$
$$|\mathscr{I}_{I3}| \leq cR^{-2} \| |x|^2 g \|_{0,\infty,\Omega}. \tag{3.12}$$

2. Estimates of $\mathscr{I}_{II}$: Notice that in $\mathscr{G}_{II}$, $|y| \geq \frac{R}{2}$. Therefore

$$\mathscr{I}_{II} \leq cR^{-3} \int_{\mathscr{G}_{II}} |\mathscr{Q}(x-y)| |y|^3 |\operatorname{div} g(y)|\, dy$$
$$\leq cR^{-3} \| |x|^3 \operatorname{div} g \|_{0,\infty,\Omega} \int_0^1 \int_{S_1} d|z|\, dS \leq cR^{-3} \| |x|^3 \operatorname{div} g \|_{0,\infty,\Omega}. \tag{3.13}$$

3. Estimates of $\mathscr{I}_{III}$: Notice that in $\mathscr{G}_{III}$, $|x-y| \geq 1$ and $|y| \geq \frac{R}{2}$. Therefore

$$\mathscr{I}_{III} \leq cR^{-3} \int_{\mathscr{G}_{III}} |x-y|^{-2} |y|^3 |\operatorname{div} g(y)|\, dy$$
$$\leq cR^{-3} \| |x|^3 \operatorname{div} g \|_{0,\infty,\Omega} \int_1^R \int_{S_1} d|y|\, dS \leq cR^{-2} \| |x|^3 \operatorname{div} g \|_{0,\infty,\Omega}. \tag{3.14}$$

4. Estimates of $\mathscr{I}_{IV}$: Notice that in $\mathscr{G}_{IV}$, $|x-y| \geq \dfrac{R}{2}$ and $|y| \geq \dfrac{R}{2}$. Hence

$$\mathscr{I}_{IV} \leq cR^{-5} \int_{\mathscr{G}_{IV}} |y|^3 |\operatorname{div} g(y)|\, dy \leq cR^{-5} \| |x|^3 \operatorname{div} g\|_{0,\infty,\Omega} \int_{R/2}^{3R/2} \int_{S_1} |y|^2\, d|y|\, dS$$

$$\leq cR^{-2} \| |x|^3 \operatorname{div} g\|_{0,\infty,\Omega} \,. \tag{3.15}$$

5. Estimates of $\mathscr{I}_V$: Notice that in $\mathscr{G}_V$, $|x-y| \geq |y| - |x| \geq \frac{1}{3}|y|$ and $|y| \geq \dfrac{3R}{2}$. Therefore

$$\mathscr{I}_V \leq \| |x|^3 \operatorname{div} g\|_{0,\infty,\Omega} \int_{3R/2}^{\infty} \int_{S_1} |y|^{-3}\, d|y|\, dS \leq cR^{-2} \| |x|^3 \operatorname{div} g\|_{0,\infty,\Omega} \,. \tag{3.16}$$

The decomposition (3.10) and the estimates (3.11)–(3.16) imply the statement. Lemma 3.1 is thus proved.

Further, we recall a classical lemma about the decay of the integrals

$$I_1(x) = \int_{\Omega} \frac{g(y)}{|x-y|^2}\, dy \,. \tag{3.17}$$

The proof can be found e.g. in Smirnov [28].

Lemma 3.2. (a) *Let* $|x|g \in L^q(\Omega) \cap L^p(\Omega)$ *with* $\frac{3}{2} < q < 3 < p$. *Then* $|x|I_1 \in L^\infty(\Omega)$ *and we have the estimate*

$$\| |x| I_1 \|_{0,\infty} \leq c(\| |x| g\|_{0,q} + \| |x| g\|_{0,p}) \,. \tag{3.18}$$

(b) *Let* $|x|^2 g \in L^\infty(\Omega)$. *Then* $|x|I_1 \in L^\infty(\Omega)$ *and we have the estimate*

$$\| |x| I_1 \|_{0,\infty} \leq c \| |x|^2 g\|_{0,\infty} \,. \tag{3.19}$$

Finally, we need a lemma about the decay of the singular integrals of the Calderon-Zygmund type, i.e.

$$I_2(x) = \int_{\Omega} \frac{\mathscr{W}\left(\dfrac{x-y}{|x-y|}\right)}{|x-y|^3} g(y)\, dy \tag{3.20}$$

where $\mathscr{W}$ is Hölder continuous on the unit sphere S_1 and $\int_{S_1} \mathscr{W}\, dS = 0$. The following lemma is due to Stein [29].

Lemma 3.3. *Let* $1 < t < \infty$, $-\dfrac{3}{t} < \alpha < \dfrac{3}{t'}$ $\left(t' = \dfrac{t}{t-1}\right)$. *Let* $|x|^\alpha g \in L^t(\Omega)$. *Then* $|x|^\alpha I_2 \in L^t(\Omega)$ *and*

$$\| |x|^\alpha I_2 \|_{0,t} \leq c \| |x|^\alpha g\|_{0,t} \,. \tag{3.21}$$

4. Function spaces

Let $\Omega \subset \mathbf{R}^3$ be an exterior domain such that $\Omega_c = \mathbf{R}^3 \backslash \Omega$ is a compact set which contains 0 and its boundary $\partial\Omega$ is of class $\mathscr{C}^{k+3}$, $k = 0, 1, \ldots$; the outer normal to $\partial\Omega$ is denoted by ν. By $B_R(x)$, we denote a ball in $\mathbf{R}^3$ with the radius R and the center in x, $\Omega_R = B_R(0) \backslash \Omega_c$ (provided $\Omega_c \subset B_R(0)$), $\Omega^R = \mathbf{R}^3 \backslash B_R(0)$. Suppose (without loss of generality) that $B_1(0) \subset \Omega_c$.

We use the following function spaces:

- $\mathscr{C}_0^\infty(\Omega)$ is the set of all infinitely differentiable functions with the compact support in Ω.
- $\mathscr{C}^k(\bar{\Omega})$ is a Banach space of continuously differentiable functions up to the order k up to the boundary with the finite norm $|b|_{\mathscr{C}^k} = \sup_{x\in\Omega} (\sum_{m=0}^k |\nabla^m b(x)|)$.
- $L^t(\Omega) = W^{0,t}(\Omega)$, $1 \le t \le \infty$ is the usual Lebesgue space with norm $\|\cdot\|_{0,t}$ and $W^{k,t}(\Omega)$ (resp. $W_0^{k,t}(\Omega)$, $1 < t < \infty$), $k = 1, 2, \ldots$ are usual Sobolev spaces with the norms $\|\cdot\| = \sum_{m=0}^k \|\nabla^m \cdot\|_{0,t}$. (Index zero denotes the zero traces.)
- $\hat{H}_0^{1,t}(\Omega) = \overline{\mathscr{C}_0^\infty(\Omega)}^{|\cdot|_{1,t}}$, $1 < t < \infty$, is a Banach space with the norm $|\cdot|_{1,t} = \|\nabla\cdot\|_{0,t}$. Here and in the sequel, the superposed bar with a norm denotes the completion in this norm. If $1 < t < 3$ then the elements u of $\hat{H}_0^{1,t}(\Omega)$ are such that $u \in L^{3t/(3-t)}(\Omega)$, $\nabla u \in L^t(\Omega)$, $u|_{\partial\Omega} = 0$. Moreover, we have the classical estimates

$$\|u\|_{0,3t/(3-t)} \le c\|\nabla u\|_{0,t}, \qquad \left\|\frac{u}{|x|}\right\|_{0,t} \le c\|\nabla u\|_{0,t}. \tag{4.1}$$

  If $t \ge 3$ then an u, $u \in L^t_{loc}(\Omega)$, $\nabla u \in L^t(\Omega)$, belongs to $\hat{H}_0^{1,t}(\Omega)$ if and only if $u|_{\partial\Omega} = 0$.
- The dual space to $\hat{H}_0^{1,t'}(\Omega)$, $t^{-1} + t'^{-1} = 1$ is denoted by $\hat{H}^{-1,t}(\Omega)$ and its norm is $|\cdot|_{-1,t}$. It is worth noting that for $1 < t < 3$, $\mathscr{C}_0^\infty(\Omega) \subset \hat{H}^{-1,t}(\Omega)$ and the imbedding is dense. The first inequality in (4.1) yields, for $u \in L^{3t/(3+t)}(\Omega)$, $t \in (\frac{3}{2}, \infty)$,

$$|u|_{-1,t} \le \|u\|_{0,3t/(3+t)}. \tag{4.2}$$

- $\hat{H}_\infty^{1,t}(\Omega) = \overline{\mathscr{C}_0^\infty(\bar{\Omega})}^{|\cdot|_{1,t}}$, $1 < t < \infty$, is a Banach space with the norm $|\cdot|_{1,t} = \|\nabla\cdot\|_{0,t}$. If $1 < t < 3$ then the elements u of $\hat{H}_\infty^{1,t}(\Omega)$ are such that $u \in L^{3t/(3-t)}(\Omega)$, $\nabla u \in L^t(\Omega)$. Moreover, we have the classical estimates

$$\|u\|_{0,3t/(3-t)} \le c\|\nabla u\|_{0,t}. \tag{4.3}$$

  If $t \ge 3$, then the elements of $\hat{H}_\infty^{1,t}(\Omega)$ are the equivalence classes $\{u + c\}$, where $u \in L^t_{loc}(\Omega)$, $\nabla u \in L^t(\Omega)$ and $c \in \mathbf{R}^1$.
- The dual space to $\hat{H}_\infty^{1,t'}(\Omega)$, $t^{-1} + t'^{-1} = 1$ is denoted by $(\hat{H}^{1,t'}(\Omega))^*$ and its norm is $|\cdot|_{*,t}$.

- We emphasize, that we do not distinguish in the notation between the spaces of the scalar and the vector valued functions, e.g. $W^{k,t}(\Omega)$, means either $W^{k,t}(\Omega, \mathbf{R}^1)$ or $W^{k,t}(\Omega, \mathbf{R}^3)$. The difference is always clear from the context.

For the detailed description and properties of spaces $W^{k,t}(\Omega)$, $\hat{H}_0^{1,t}(\Omega)$ and $\hat{H}^{-1,t}(\Omega)$ see Adams [1], Simader, Sohr [26], Simader [25], Galdi [8].

Recall some useful technical estimates which are the consequences of the interpolation and imbeddings and which will be currently used without an explicit reference:

(i) Let $1 < q < t < p < \infty$. Then $L^q(\Omega) \cap L^p(\Omega) \subset L^t(\Omega)$ and

$$\|z\|_{0,t} \le \|z\|_{0,q}^{(1-a)t} \|z\|_{0,p}^{at}, \qquad a = \frac{p(t-q)}{t(p-q)}. \tag{4.4}$$

(ii) For $t > 3$, $W^{1,t}(\Omega) \subset \mathscr{C}^0(\bar{\Omega})$ and

$$|z|_{\mathscr{C}^0} \le c\|z\|_{1,t}. \tag{4.5}$$

(iii) Let $1 < q < 3 < p$ and $z \in W^{1,q}(\Omega)$, $\nabla z \in L^p(\Omega)$. Then

$$\nabla z \in L^s(\Omega), \qquad q \le s \le p, \quad |z|_{1,s} \le c(|z|_{1,q} + |z|_{1,p}),$$

$$z \in L^s(\Omega) \cap \mathscr{C}^0(\bar{\Omega}), \quad q \le s \le \infty,$$

$$\|z\|_{0,s} \le c(\|z\|_{1,q} + \|z\|_{1,p}). \tag{4.6}$$

(iv) Let $1 < q < 3 < p$ and $z \in \hat{H}_0^{1,t}(\Omega) \cap \hat{H}_0^{1,t}(\Omega)$, $\nabla z \in W^{1,q}(\Omega) \cap W^{1,p}(\Omega)$. Then

$$\nabla^2 z \in L^s(\Omega), \qquad q \le s \le p,$$

$$\|\nabla^2 z\|_{0,s} \le c(\|\nabla^2 z\|_{0,q} + \|\nabla^2 z\|_{0,p});$$

$$\nabla z \in L^s(\Omega) \cap \mathscr{C}^0(\bar{\Omega}), \qquad q \le s \le \infty,$$

$$\|\nabla z\|_{0,s} \le c(\|\nabla z\|_{1,q} + \|\nabla z\|_{1,p});$$

$$z \in L^s(\Omega) \cap \mathscr{C}^0(\bar{\Omega}), \qquad \frac{3q}{(3-q)} \le s \le \infty,$$

$$\|z\|_{0,s} \le c(\|\nabla z\|_{1,q} + \|\nabla z\|_{1,p}). \tag{4.7}$$

Let $l = 0, 1, \ldots k$, $k = 0, 1, \ldots$, $\frac{3}{2} < q < 3$, $3 < p < \infty$. To investigate the existence of solutions to system (2.12)–(2.14) (or to system (2.6)–(2.7)), and consequently to the fully nonlinear systems (2.4), (2.5) (or (2.16)–(2.18)), it is convenient to introduce the following functional spaces:

(1) *Space for L^p-estimates (see Theorem* 5.1). We set

$$G := \{\sigma : \sigma \in W^{l+1,q}(\Omega) \cap W^{k+1,p}(\Omega)\};$$

$$\Phi := \{\phi\colon \phi \in \hat{H}^{1,q}_{\infty}(\Omega) \cap \hat{H}^{1,p}_{\infty}(\Omega),\ \nabla\phi \in W^{l+2,q}(\Omega) \cap W^{k+2,p}(\Omega)\}\ ;$$

$$U := \{u\colon u \in \hat{H}^{1,q}_{\infty}(\Omega) \cap \hat{H}^{1,p}_{\infty}(\Omega),\ \nabla u \in W^{l+1,q}(\Omega) \cap W^{k+1,p}(\Omega)\}\ ;$$

$$V := U\ ; \qquad \Phi U := \{(\phi, u)\colon \phi \in \Phi,\ u \in U\}\ ;$$

$$G\Phi U := \{(\sigma, \phi, u)\colon \sigma \in G,\ \phi \in \Phi,\ u \in U\} \tag{4.8}$$

which become the Banach spaces when equipped with the norms

$$\|\cdot\|_G = \|\cdot\|_{l+1,q} + \|\cdot\|_{k+1,p}\ ;$$

$$\|\cdot\|_\Phi = \|\nabla\cdot\|_{l+2,q} + \|\nabla\cdot\|_{k+2,p}\ ;$$

$$\|\cdot\|_U = \|\nabla\cdot\|_{l+1,q} + \|\nabla\cdot\|_{k+1,p}\ ;$$

$$\|\cdot\|_V = \|\cdot\|_U\,, \qquad \|(\phi, u)\|_{\Phi U} = \|\phi\|_\Phi + \|u\|_U\ ;$$

$$\|(\sigma, \phi, u)\|_{G\Phi U} = \|\sigma\|_G + \|\phi\|_\Phi + \|u\|_U\ .$$

If $w|_{\partial\Omega} = 0$ and $\operatorname{div} w|_{\partial\Omega} = 0$ then the equation $(2.6)_2$ yields formally $\operatorname{div} v|_{\partial\Omega} = 0$. Taking into account this observation and the boundary conditions for u, ϕ, v, it is convenient to introduce some auxiliary subsets:

$$\Phi U_* := \left\{(\phi, u)\colon \phi \in \Phi,\ u \in U,\ \operatorname{div} u = 0,\ \frac{\partial\phi}{\partial\nu}\bigg|_{\partial\Omega} = 0,\right.$$

$$\left.\Delta\phi|_{\partial\Omega} = 0\,, \qquad (u + \nabla\phi)|_{\partial\Omega} = 0\right\} \tag{4.9}$$

(it is a subspace of ΦU);

$$V_* := \{v\colon v \in V,\ \operatorname{div} v|_{\partial\Omega} = 0,\ v|_{\partial\Omega} = 0\} \tag{4.10}$$

(it is a subspace of V);

$$\Phi_* := \left\{\phi\colon \phi \in \Phi,\ \Delta\phi|_{\partial\Omega} = 0,\ \frac{\partial\phi}{\partial\nu}\bigg|_{\partial\Omega} = 0\right\} \tag{4.11}$$

(it is a subspace of Φ) and

$$G\Phi U_* := \{(\sigma, \phi, u)\colon \sigma \in G,\ (\phi, u) \in \Phi U_*\} \tag{4.12}$$

(it is a subspace of $G\Phi U$). Notice that if $(\phi, u) \in \Phi U_*$ then $v = u + \nabla\phi \in V_*$.

(2) *Spaces for estimating the decay* (*see Theorems* 5.2, 5.3). Let $0 < \gamma \le \gamma' < 1$, $3 < r < \infty$. We set

$$\mathbf{G} := \{\sigma\colon \sigma \in G,\ |x|^2\sigma \in L^\infty(\Omega),\ |x|\nabla\sigma \in L^\infty(\Omega),\ |x|^2\nabla\sigma \in L^r(\Omega)\}\ ;$$

$$\mathbf{\Phi} := \{\phi\colon \phi \in \Phi,\ |x|\nabla\phi \in L^\infty(\Omega),\ |x|^2\nabla^2\phi \in W^{1,r}(\Omega)\}\ ;$$

$$\mathbf{U} := \{u\colon u \in U,\ |x|u \in L^\infty(\Omega),\ |x|^2\nabla u \in L^\infty(\Omega)\}\ ;$$

$$\mathbf{V} := \mathbf{U}, \ \mathbf{\Phi U} := \{(\phi, u) \colon \phi \in \mathbf{\Phi}, u \in \mathbf{U}\} ;$$

$$\mathbf{G\Phi U} := \{(\sigma, \phi, u) \colon \sigma \in \mathbf{G}, \phi \in \mathbf{\Phi}, u \in \mathbf{U}\} . \tag{4.13}$$

These spaces are the Banach spaces with the norms

$$\|\sigma\|_{\mathbf{G}} = \|\sigma\|_{G} + \||x|^2\sigma\|_{0,\infty} + \||x|\nabla\sigma\|_{0,\infty} + \||x|^2\nabla\sigma\|_{0,r} ;$$

$$\|\phi\|_{\mathbf{\Phi}} = \|\phi\|_{\Phi} + \||x|\nabla\phi\|_{0,\infty} + \||x|^2\nabla^2\phi\|_{1,r} ;$$

$$\|u\|_{\mathbf{U}} = \|u\|_{U} + \||x|u\|_{0,\infty} + \||x|^2\nabla u\|_{0,\infty} ;$$

$$\|\cdot\|_{\mathbf{V}} = \|\cdot\|_{\mathbf{U}} ; \qquad \|(\phi, u)\|_{\mathbf{\Phi U}} = \|\phi\|_{\mathbf{\Phi}} + \|u\|_{\mathbf{U}} ;$$

$$\|(\sigma, \phi, u)\|_{\mathbf{G\Phi U}} = \|\sigma\|_{\mathbf{G}} + \|\phi\|_{\mathbf{\Phi}} + \|u\|_{\mathbf{U}} .$$

Taking into account the boundary conditions, we are led to define the following auxiliary spaces and subsets:

$$\mathbf{\Phi U}_* := \left\{(\phi, u) \colon \phi \in \mathbf{\Phi}, u \in \mathbf{U}, \operatorname{div} u = 0, \right.$$

$$\left. \frac{\partial\phi}{\partial\nu}\bigg|_{\partial\Omega} = 0, \Delta\phi|_{\partial\Omega} = 0, (u + \nabla\phi)|_{\partial\Omega} = 0\right\} \tag{4.14}$$

(it is a subspace of $\mathbf{\Phi U}$);

$$\mathbf{V}_* := \{v \colon v \in \mathbf{V}, \operatorname{div} v|_{\partial\Omega} = 0, v|_{\partial\Omega} = 0\} \tag{4.15}$$

(it is a subspace of $\mathbf{V}$);

$$\mathbf{\Phi}_* := \left\{\phi \colon \phi \in \mathbf{\Phi}, \Delta\phi|_{\partial\Omega} = 0, \frac{\partial\phi}{\partial\nu}\bigg|_{\partial\Omega} = 0\right\} \tag{4.16}$$

(it is a subspace of $\mathbf{\Phi}$) and

$$\mathbf{G\Phi U}_* := \{(\sigma, \phi, u) \colon \sigma \in \mathbf{G}, (\phi, u) \in \mathbf{\Phi U}_*\} \tag{4.17}$$

(it is a subspace of $\mathbf{G\Phi U}$). Notice that if $(\phi, u) \in \mathbf{\Phi U}_*$ then $v = u + \nabla\phi \in \mathbf{V}_*$.

(3) *Spaces for description of the right hand sides.* We always have the right hand sides of a particular structure which is suggested by the nature of the nonlinearity, see (2.5), namely

$$\mathscr{F} = \mathscr{F}^0 + \operatorname{div} \mathscr{F}^1$$

The terms in $\mathscr{F}^0$ have their origin in the external force, while $\mathscr{F}^1$ is coming from the nonlinear convective term. Since the goal of the present paper is, essentially, to estimate the contributions to the decay rate, of the nonlinear term, we suppose, without loss of generality, that $\mathscr{F}^0$ has a compact support. We denote

$$L_0 = \{g \colon g \in L^{3q/(3+q)}(\Omega) \cap W^{l,q}(\Omega) \cap W^{k,p}(\Omega), \operatorname{supp} g \in B_{R_0}(0)\} \tag{4.18}$$

a set of admissible $\mathscr{F}^0$'s. (Notice that it is not a Banach space). Further denote

$$\tilde{L} := \hat{H}^{-1,q}(\Omega) \cap W^{l,q}(\Omega) \cap W^{k,p}(\Omega) \tag{4.19}$$

a Banach space with the norm

$$\|\cdot\|_{\tilde{L}} := |\cdot|_{-1,q} + \|\cdot\|_{l,q} + \|\cdot\|_{k,p}$$

and

$$L := W^{l+1,q}(\Omega) \cap W^{k+1,p}(\Omega) \tag{4.20}$$

a Banach space with the norm

$$\|\cdot\|_L := \|\cdot\|_{l+1,q} + \|\cdot\|_{k+1,p} .$$

Finally, we introduce

$$\begin{aligned} \mathbf{L} := \{g\colon g \in L, |x|^3 \operatorname{div} g \in L^\infty(\Omega), |x|^2 g \in L^\infty(\Omega), \\ |x| \nabla \operatorname{div} g \in L^q(\Omega) \cap L^p(\Omega)\} . \end{aligned} \tag{4.21}$$

This is a Banach space with the norm

$$\begin{aligned} \|g\|_{\mathbf{L}} = \|g\|_L + \||x|^3 \operatorname{div} g\|_{0,\infty} + \||x|^2 g\|_{0,\infty} \\ + \||x|\nabla \operatorname{div} g\|_{0,q} + \||x| \nabla \operatorname{div} g\|_{0,p} . \end{aligned}$$

(4) *The coefficients in estimates.* In the sequel, c, c', c_i, c_i' $(i = 1, 2, \ldots)$ are positive constants dependent only of k, l, p, q, r, R_0, $\partial\Omega$ and μ_1, μ_2; they are in particular independent of f. The constants c, c' can have different values even in the same formulas.

5. Main theorems

Here we present the main results: Theorems 5.1 and 5.2 deal with the linearized problem (2.6)–(2.7), or equivalently with system (2.12)–(2.14). Theorem 5.1 contains a fundamental statement about the existence and estimates of solutions to this problem in Sobolev-type spaces (similar proof as presented here can be found in Novotny, Padula [19]), while Theorem 5.2 is an existence result in the spaces with the convenient pointwise decay. Theorem 5.3 deals with the fully nonlinear system (2.4), (2.5), or equivalently with (2.16)–(2.18). Its proof is based on a fixed point argument and on Theorem 5.2. Theorem 5.4 shows, that the compressible part of the velocity field of the fully nonlinear system decays more rapidly than the incompressible one.

Theorem 5.1 (the linearized system—solutions in L^p-spaces): *Let*

$$l = 0, 1, \ldots, k, \qquad k = 0, 1, \ldots, \qquad \tfrac{3}{2} < q < 3, \qquad p > 3, \qquad \Omega \in \mathscr{C}^{k+3} .$$

Let

$$w \in V_* := \{w\colon w \in \hat{H}^{1,q}_\infty(\Omega) \cap \hat{H}^{1,p}_\infty(\Omega)\,,$$
$$\nabla w \in W^{l+1,q}(\Omega) \cap W^{k+1,p}(\Omega),\ \operatorname{div} w|_{\partial\Omega} = 0,\ w|_{\partial\Omega} = 0\}$$

and

$$\mathscr{F} \in \tilde{L} = \hat{H}^{-1,q}(\Omega) \cap W^{l,q}(\Omega) \cap W^{k,q}(\Omega)\,.$$

Then there exists $\gamma' > 0$ *such that if*

$$\|w\|_V < \gamma'$$

then there exists a unique solution (Π, σ, ϕ, u) *of the system* (2.12)–(2.14)

$$\Pi \in G\,, \qquad (\sigma, \phi, u) \in G\Phi U_*\,, \qquad \operatorname{div}(\sigma w) \in G\,,$$

i.e.

$$(\Pi, \sigma, \operatorname{div}(\sigma w))) \in W^{l+1,q}(\Omega) \cap W^{k+1,p}(\Omega)\,,$$
$$u \in \hat{H}^{1,q}_\infty(\Omega) \cap \hat{H}^{1,p}_\infty(\Omega)\,, \qquad \nabla u \in W^{l+1,q}(\Omega) \cap W^{k+1,p}(\Omega)\,,$$
$$\phi \in \hat{H}^{1,q}_\infty(\Omega) \cap \hat{H}^{1,p}_\infty(\Omega)\,, \qquad \nabla \phi \in W^{l+2,q}(\Omega) \cap W^{k+2,p}(\Omega)\,,$$
$$\operatorname{div} u = 0, \qquad \left.\frac{\partial\phi}{\partial\nu}\right|_{\partial\Omega} = 0\,, \qquad \Delta\phi|_{\partial\Omega} = 0\,, \qquad (u + \nabla\phi)|_{\partial\Omega} = 0\,. \tag{5.1}$$

It satisfies the estimates

$$\|\Pi\|_G + \|(\sigma, \phi, u)\|_{G\Phi U} \le c_1 \|\mathscr{F}\|_{\tilde{L}}\,,$$
$$\|\operatorname{div}(\sigma w)\|_G \le c_1 \|\mathscr{F}\|_{\tilde{L}}\,. \tag{5.2}$$

Consequence 5.1 (of Theorem 5.1). *Put* $v = u + \nabla\phi$. *Then* (σ, v) *is a (unique) solution of the problem* (2.6)–(2.7) *and satisfies the estimate*

$$\|\sigma\|_G + \|v\|_U \le c_1 \|\mathscr{F}\|_{\tilde{L}}. \tag{5.3}$$

Theorem 5.2 (the linearized system—solutions with the decay). *Let*

$$l = 0,\ 1,\ \ldots,\ k\,, \qquad k = 2,\ 3,\ \ldots\,, \qquad \tfrac{3}{2} < q < 3\,,$$
$$p > 3\,, \qquad \Omega \in \mathscr{C}^{k+3}\,, \qquad r > 3\,, \qquad R_0 > 0$$

and

$$w \in \mathbf{V}_* := \{w \in V_*,\ |x| w \in L^\infty(\Omega),\ |x|^2 \nabla w \in L^\infty(\Omega)\}$$

(*for the definition of* V_*, *see Theorem* 5.1). *Let*

$$\mathscr{F} = \mathscr{F}^0 + \operatorname{div} \mathscr{F}^1\,,$$

where

$$\mathscr{F}^0 \in L_0 := \{g\colon g \in L^{3q/(3+q)}(\Omega) \cap W^{l,q}(\Omega) \cap W^{k,p}(\Omega),\ \operatorname{supp} g \in B_{R_0}(0)\}\,,$$

$$\mathscr{F}^1 \in \mathbf{L} := \{g: g \in L, |x|^3 \operatorname{div} g \in L^\infty(\Omega), |x|^2 g \in L^\infty(\Omega),$$
$$|x| \nabla \operatorname{div} g \in L^q(\Omega) \cap L^p(\Omega)\}.$$

Then there exists $\gamma' > 0$ *such that if*

$$\|w\|_{\mathbf{V}} < \gamma'$$

then there exists a unique solution (Π, σ, ϕ, u) *of the system* (2.12)–(2.14)

$$\Pi \in \mathbf{G}, \qquad (\sigma, \phi, u) \in \mathbf{G\Phi U}_*,$$

i.e. Π, σ, ϕ, u *satisfy, besides* (5.1), *also the following decay properties*

$$|x|^2(\sigma, \Pi, \operatorname{div}(\sigma w)) \in L^\infty(\Omega), \qquad |x|\nabla(\sigma, \Pi, \operatorname{div}(\sigma w)) \in L^\infty(\Omega),$$
$$|x|^2 \nabla(\sigma, \Pi, \operatorname{div}(\sigma w)) \in L^r(\Omega),$$
$$|x|\nabla\phi \in L^\infty(\Omega), \qquad |x|^2 \nabla^2 \phi \in W^{1,r}(\Omega),$$
$$|x| u \in L^\infty(\Omega), \qquad |x|^2 \nabla u \in L^\infty(\Omega). \tag{5.4}$$

Moreover, we have the estimate

$$\|\Pi\|_{\mathbf{G}} + \|(\sigma, \phi, u)\|_{\mathbf{G\Phi U}} \le c_2(\|\mathscr{F}^0\|_{\tilde{L}} + \|\mathscr{F}^1\|_{\mathbf{L}}). \tag{5.5}$$

Consequence 5.2 (of Theorem 5.2). *Put* $v = u + \nabla\phi$. *Then* (σ, v) *is a (unique) solution of the problem* (2.6)–(2.7) *and satisfies estimate*

$$\|\sigma\|_{\mathbf{G}} + \|v\|_{\mathbf{U}} \le c_2(\|\mathscr{F}^0\|_{\tilde{L}} + \|\mathscr{F}^1\|_{\mathbf{L}}). \tag{5.6}$$

Theorem 5.3 (the fully nonlinear system—solutions with the decay). *Let*

$$l = 1, \ldots, k, \qquad k = 2, 3, \ldots, \qquad \tfrac{3}{2} < q < 3,$$
$$p > 3, \qquad r > 3, \qquad \Omega \in \mathscr{C}^{k+3}, \qquad R_0 > 0$$

and

$$f \in L_0 (\subset \tilde{L}),$$

i.e.

$$f \in \tilde{L} = L^{3q/(3+q)}(\Omega) \cap W^{l,q}(\Omega) \cap W^{k,p}(\Omega) \qquad \text{and } \operatorname{supp} f \in B_{R_0}(0).$$

Then there exist $\gamma_0 > 0$ *and* $\gamma_1 > 0$ *(dependent of* l, k, p, r, $\partial\Omega$ *and* R_0*), such that if*

$$\|f\|_{\tilde{L}} < \gamma_1$$

then in the set

$$\mathbf{B}_{\gamma_0} = \{(\sigma, \phi, u) : (\sigma, \phi, u) \in \mathbf{G\Phi U}_*, \|(\sigma, \phi, u)\|_{\mathbf{G\Phi U}} \le \gamma_0\}$$

(see (5.1) *and* (5.4) *or* (4.8), (4.13), (4.17)*), there exists a unique triplet* (σ, ϕ, u) *such that*

$$\sigma, \qquad v = u + \nabla\phi$$

satisfy the nonlinear system (2.4)–(2.5).[7] *Moreover, we have the estimate*

$$\|(\sigma, \phi, u)\|_{\mathbf{G\Phi U}} \leq c_3 \|f\|_{\tilde{L}} \,. \tag{5.7}$$

Here $c_3 > 0$ *is a constant dependent of* R_0, l, k, q, p, r and $\partial\Omega$.

Theorem 5.4 (further decay of ϕ). *Let* $\varepsilon \in (0, 1)$. *Let* $\Pi \in G$, $(\sigma, \phi, u) \in \mathbf{G\Phi U}_*$ *with* $r > 3/\varepsilon$, $p > 3$, $3/2 < q < 3$, $k = 2, 3, \ldots$, $l = 1, \ldots, k$ *be a solution of the problem* (2.16)–(2.18). *Then*

$$|x|^{2-\varepsilon}\nabla\phi \in L^\infty(\Omega)\,,$$

$$|x|^{3-\varepsilon}\Delta\phi = |x|^{3-\varepsilon} \operatorname{div} v \in L^\infty(\Omega)\,. \tag{5.8}$$

6. Representation formulas

In this section we list the representation formulas for the Laplace and Stokes operators in the form convenient for the further applications. We refer the reader to [8], Vol. I, Ch. 5 and Vol. II, Ch. 9, for more details.

1) *Representation formulas for u and Π*

From the Stokes problem (2.12) we have for any locally smooth weak solution (Π, u) (weak solution means here a solution in the sense of distributions such that $\Pi \in L^q(\Omega)$ and $\nabla u \in L^q(\Omega)$ with some $q \in (2/3, 3)$) and for the r.h.s. $\mathscr{F}$ satisfying the hypothesis of Theorem 5.2 about the decay, the following formulas (cf. Finn [4] or Galdi [8], Vol. I, Ch. 5 and Vol. II, Ch. 9):

Representation formulas for u

$$\begin{aligned} u(x) = &-\int_\Omega \mathscr{U}(x-y)\cdot\mathscr{F}(y)\,dy \\ &+ \int_{\partial\Omega} \{\mu_1 v\cdot[\nabla_y\mathscr{U}(x-y)\cdot u(y) - \nabla_y u(y)\cdot\mathscr{U}(x-y)] \\ &+ \Pi(y)\mathscr{U}(x-y)\cdot v - \mathscr{P}(x-y)u(y)\cdot v\}\,dS_y\,. \end{aligned} \tag{6.1}$$

Considering $\mathscr{F}$ in the form

$$\mathscr{F} = \mathscr{F}^0 + \operatorname{div}\mathscr{F}^1 \tag{6.2}$$

we have in particular from (6.1), (6.2)

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[7] $(\sigma, \phi, u) \in \mathbf{B}_{\gamma_0}$ and $\Pi = \sigma + (2\mu_1 + \mu_2)\operatorname{div}(\sigma v)(\in \mathbf{G})$ obviously satisfy the nonlinear system (2.16)–(2.18) and we have, besides (5.7), also the estimate

$$\|\Pi\|_{\mathbf{G}} \leq c_3\|f\|_{\tilde{L}}\,.$$

$$u(x) = -\int_\Omega \mathscr{U}(x-y)\cdot\mathscr{F}^0(y)\,dy + \int_\Omega \mathscr{F}^1(y):\nabla_y\mathscr{U}(x-y)\,dy$$
$$+ \int_{\partial\Omega} \{\mu_1 \nu\cdot[\nabla_y\mathscr{U}(x-y)\cdot u(y) - \nabla_y u(y)\cdot\mathscr{U}(x-y)]$$
$$+ \Pi(y)\mathscr{U}(x-y)\cdot\nu - \mathscr{P}(x-y)u(y)\cdot\nu - \nu\cdot\mathscr{F}^1(y)\cdot\mathscr{U}(x-y)\}\,dS_y\,. \quad (6.3)$$

We recall that here and in the sequel, we use the Einstein summation convention over the repeated indices. We have denoted by $\mathscr{U}_j$ the vector field $(\mathscr{U}_{j1}, \mathscr{U}_{j2}, \mathscr{U}_{j3})$ and by $\mathscr{F}^1(x):\nabla_y\mathscr{U}(x-y)$ the product $\mathscr{F}^1_{ij}\dfrac{\partial\mathscr{U}_j(x-y)}{\partial y_i}$. Differentiating (6.1) and taking into account (6.2), we get

$$\nabla_x u(x) = -\int_\Omega \nabla_x\mathscr{U}(x-y)\cdot\mathscr{F}^0(y)\,dy + \int_\Omega \nabla_y\mathscr{U}(x-y)\cdot\operatorname{div}\mathscr{F}^1(y)\,dy$$
$$+ \int_{\partial\Omega} \{\mu_1 \nu\cdot[\nabla_x\nabla_y\mathscr{U}(x-y)\cdot u(y) - \nabla_y u_i(y)\nabla_x\mathscr{U}_i(x-y)]$$
$$+ \Pi(y)\nabla_x\mathscr{U}(x-y)\cdot\nu - \nabla_x\mathscr{P}(x-y)u(y)\cdot\nu\}\,dS_y\,. \quad (6.4)$$

We thus have by (6.2), (6.4) the following result

$$u(x) = \mathscr{M}_u^0(x) + \mathscr{N}_u^0(x) + \mathscr{E}_u^0(x)\,, \qquad x\in\Omega\,,$$
$$\nabla_x u(x) = \mathscr{M}_u^1(x) + \mathscr{N}_u^1(x) + \mathscr{E}_u^1(x)\,, \qquad x\in\Omega\,, \quad (6.5)$$

where

$$\mathscr{M}_u^\alpha(x) = -\int_\Omega \nabla_x^\alpha\mathscr{U}(x-y)\cdot\mathscr{F}^0(y)\,dy\,, \qquad \alpha = 0,\ 1\,, \quad (6.6)$$

$$\mathscr{N}_u^0(x) = \int_\Omega \mathscr{F}^1(y):\nabla_y\mathscr{U}(x-y)\,dy\,,$$
$$\mathscr{N}_u^1(x) = \int_\Omega \nabla_y\mathscr{U}(x-y)\cdot\operatorname{div}\mathscr{F}^1(y)\,dy\,, \quad (6.7)$$

and

$$\mathscr{E}_u^0(x) = \int_{\partial\Omega} \{\mu_1 \nu\cdot[\nabla_y\mathscr{U}(x-y)\cdot u(y) - \nabla_y u(y)\cdot\mathscr{U}(x-y)]$$
$$+ \Pi(y)\mathscr{U}(x-y)\cdot\nu - \mathscr{P}(x-y)u(y)\cdot\nu - \nu\cdot\mathscr{F}^1(y)\cdot\mathscr{U}(x-y)\}\,dS_y\,,$$
$$\mathscr{E}_u^1(x) = \int_{\partial\Omega} \{\mu_1 \nu\cdot[\nabla_x\nabla_y\mathscr{U}(x-y)\cdot u(y) - \nabla_y u_i(y)\nabla_x\mathscr{U}_i(x-y)]$$
$$+ \Pi(y)\nabla_x\mathscr{U}(x-y)\cdot\nu - \nabla_x\mathscr{P}(x-y)u(y)\cdot\nu\}\,dS_y\,. \quad (6.8)$$

Representation formulas for Π

Put

$$W(x) = -\int_\Omega \mathscr{U}(x-y)\cdot\mathscr{F}(y)\,dy\,, \qquad P(x) = \int_\Omega \mathscr{P}(x-y)\cdot\mathscr{F}(y)\,dy\,.$$

Then

$$-\mu_1 \Delta W + \nabla P = \mathscr{F}.$$

Therefore (2.12) and (6.1) yield (recall $(3.3)_2$, i.e. $\mu_1 \Delta_x \mathscr{U}(x-y) = -\nabla_x \mathscr{P}(x-y)$ for $x \neq y$ and $(3.2)_2$, i.e. $\mathscr{P}(x-y) = \nabla_x \mathscr{E}(x-y)$ for $x \neq y$):

$$\nabla_x \Pi(x) = \nabla_x P(x) - \int_{\partial\Omega} \nabla_x \{\mu_1 \nu\cdot[\nabla_y \mathscr{P}(x-y)\cdot u(y) - \nabla_y u(y)\cdot \mathscr{P}(x-y)]$$
$$+ \Pi(y)\mathscr{P}(x-y)\cdot\nu\}\,dS_y\,.$$

We thus get, for $\mathscr{F}$ in the form (6.2), in particular

$$\begin{aligned} \Pi(x) &= \mathscr{M}^0_\Pi(x) + \mathscr{N}^0_\Pi(x) + \mathscr{E}^0_\Pi(x)\,, && x\in\Omega\,,\\ \nabla_x \Pi(x) &= \mathscr{M}^1_\Pi(x) + \mathscr{N}^1_\Pi(x) + \mathscr{E}^1_\Pi(x)\,, && x\in\Omega\,,\\ \nabla_x \Pi(x) &= \mathscr{M}^1_\Pi(x) + \mathscr{N}'^1_\Pi(x) + \mathscr{E}'^1_\Pi(x)\,, && x\in\Omega\,. \end{aligned} \tag{6.9}$$

Here

$$\begin{aligned} \mathscr{M}^\alpha_\Pi(x) &= -\int_\Omega \nabla^\alpha_x \mathscr{P}(x-y)\cdot\mathscr{F}^0(y)\,dy\,, \qquad \alpha = 0,\ 1\,,\\ \mathscr{N}^0_\Pi(x) &= \int_\Omega \mathscr{P}(x-y)\cdot \operatorname{div}\mathscr{F}^1(y)\,dy\,,\\ \mathscr{N}^1_\Pi(x) &= \int_\Omega \mathscr{P}_i(x-y)\nabla_y \operatorname{div}\mathscr{F}_i^1(y)\,dy\,,\\ \mathscr{N}'^1_\Pi(x) &= \int_\Omega \nabla_x \mathscr{P}(x-y)\cdot\operatorname{div}\mathscr{F}^1(y)\,dy\,, \end{aligned} \tag{6.10}$$

(in $(6.10)_2$, we have denoted by $\mathscr{F}_i^1$the vector field $(\mathscr{F}^1_{1i}, \mathscr{F}^1_{2i}, \mathscr{F}^1_{3i})$) and

$$\mathscr{E}^0_\Pi(x) = -\int_{\partial\Omega} \{\mu_1 \nu\cdot[\nabla_y \mathscr{P}(x-y)\cdot u(y) - \nabla_y u(y)\cdot\mathscr{P}(x-y)]$$
$$+ \Pi(y)\mathscr{P}(x-y)\cdot\nu\}\,dS_y\,,$$

$$\mathscr{E}^1_\Pi(x) = -\int_{\partial\Omega} \{\mu_1 \nu\cdot[\nabla_x\nabla_y \mathscr{P}(x-y)\cdot u(y) - \nabla_y u_i(y)\nabla_x \mathscr{P}_i(x-y)]$$
$$+ \Pi(y)\nabla_x \mathscr{P}(x-y)\cdot\nu + \nu\mathscr{P}(x-y)\cdot\operatorname{div}\mathscr{F}^1(y)\}\,dS_y\,,$$

$$\mathscr{E}'^{1}_{II}(x) = -\int_{\partial\Omega} \{\mu_1 \nu \cdot [\nabla_x \nabla_y \mathscr{P}(x-y) \cdot u(y) - \nabla_y u_i(y) \nabla_x \mathscr{P}_i(x-y)] + \Pi(y) \nabla_x \mathscr{P}(x-y) \cdot \nu\} \, dS_y . \tag{6.11}$$

2) *Representation formulas for* ϕ

For a locally smooth weak solution (weak means here a solution satisfying the variational formulation and being such that $\nabla\phi \in L^q(\Omega)$ for some $q \in (3/2, 3)$) of the problem

$$\Delta z = g = \operatorname{div} g_0 , \qquad g_0 \in \mathscr{C}_0^\infty(\bar{\Omega}) ,$$

we can write

$$z(x) = \int_\Omega \mathscr{E}(x-y) g(y) \, dy + \int_{\partial\Omega} \nu \cdot [z(y) \nabla_y \mathscr{E}(x-y) - \nabla_y z(y) \mathscr{E}(x-y)] \, dS_y ,$$

and for its gradients

$$\nabla_x z(x) = \int_\Omega \nabla_x \mathscr{E}(x-y) g(y) \, dy + \int_{\partial\Omega} \nu \cdot [z(y) \nabla_y \nabla_x \mathscr{E}(x-y) - \nabla_y z(y) \nabla_x \mathscr{E}(x-y)] \, dS_y;$$

$$\nabla_x^2 z(x) = \int_\Omega \nabla_x^2 \mathscr{E}(x-y) g(y) \, dy + \int_{\partial\Omega} \nu \cdot [z(y) \nabla_y \nabla_x^2 \mathscr{E}(x-y) - \nabla_y z(y) \nabla_x^2 \mathscr{E}(x-y)] \, dS_y;$$

$$\nabla_x^3 z(x) = \int_\Omega \nabla_x^2 \mathscr{E}(x-y) \nabla_y g(y) \, dy + \int_{\partial\Omega} \{\nu \cdot [z(y) \nabla_y \nabla_x^3 \mathscr{E}(x-y) - \nabla_y z(y) \nabla_x^3 \mathscr{E}(x-y)] - \nu \nabla_x^2 \mathscr{E}(x-y) g(y)\} \, dS_y . \tag{6.12}$$

The formulas (6.12) still hold for g_0 "sufficiently regular" and having "sufficient" decay at infinity; the decay $|x|^2 g_0 \in L^\infty(\Omega)$, $|x|^2 \operatorname{div} g_0 \in L^\infty(\Omega)$, $|x|^2 \nabla \operatorname{div} g_0 \in L^\infty(\Omega)$, which we use in this paper, largely satisfies this requirement. Applying formulas (6.12) to problem (2.14), we get

$$\nabla_x \phi(x) = \mathscr{N}_\phi^1(x) + \mathscr{E}_\phi^1(x) , \qquad x \in \Omega ,$$

$$\nabla_x^2 \phi(x) = \mathscr{N}_\phi^2(x) + \mathscr{E}_\phi^2(x) , \qquad x \in \Omega ,$$

$$\nabla_x^3 \phi(x) = \mathscr{N}_\phi^3(x) + \mathscr{E}_\phi^3(x) , \qquad x \in \Omega \tag{6.13}$$

where

$$\mathcal{N}_\phi^1(x) = -\int_\Omega \nabla_x \mathscr{E}(x-y)\operatorname{div}(\sigma w)(y)\,dy\,,$$

$$\mathcal{N}_\phi^2(x) = -\int_\Omega \nabla_x^2 \mathscr{E}(x-y)\operatorname{div}(\sigma w)(y)\,dy\,,$$

$$\mathcal{N}_\phi^3(x) = -\int_\Omega \nabla_x^2 \mathscr{E}(x-y)\nabla_y \operatorname{div}(\sigma w)(y)\,dy \tag{6.14}$$

and

$$\mathscr{E}_\phi^1(x) = \int_{\partial\Omega} v\cdot[\phi(y)\nabla_y\nabla_x\mathscr{E}(x-y) - \nabla_y\phi(y)\nabla_x\mathscr{E}(x-y)]\,dS_y\,;$$

$$\mathscr{E}_\phi^2(x) = \int_{\partial\Omega} v\cdot[\phi(y)\nabla_y\nabla_x^2\mathscr{E}(x-y) - \nabla_y\phi(y)\nabla_x^2\mathscr{E}(x-y)]\,dS_y\,;$$

$$\mathscr{E}_\phi^3(x) = \int_{\partial\Omega} v\cdot[\phi(y)\nabla_y\nabla_x^3\mathscr{E}(x-y) - \nabla_y\phi(y)\nabla_x^3\mathscr{E}(x-y))$$
$$+ v\nabla_x^2\mathscr{E}(x-y)\operatorname{div}(\sigma w)(y)]\,dS_y\,. \tag{6.15}$$

7. Auxiliary linear problems

In this section we recall mostly well-known theorems concerning the solvability of the Neumann and Dirichlet problems for the Laplace operator, the Stokes problem and the transport equation in exterior domains. These results are used in Section 8, in the proof of the existence of solutions of the linearized systems (2.6)–(2.7) and (2.12)–(2.14).

Consider in Ω a Neumann problem

$$\Delta\phi = g \qquad \text{in } \Omega\,,$$
$$\left.\frac{\partial\phi}{\partial v}\right|_{\partial\Omega} = \psi\,, \qquad \nabla\phi \to 0 \qquad \text{as } |x|\to\infty\,. \tag{7.1}$$

This problem, in Sobolev spaces, was studied e.g. by Simader [25], Simader and Sohr [27]. For our purpose, we need the following version of existence statement:

Lemma 7.1 (Neumann problem for the Laplace operator).
(a) *Let* $1 < t < \infty$, $\Omega \in \mathscr{C}^2$ *and*

$$\psi = 0\,, \qquad g \in (\hat{H}_\infty^{1,t'}(\Omega))^* \cap L^t(\Omega) \qquad \left(t' = \frac{t}{t-1}\right).$$

Then there exists a unique solution of the problem (7.1)

$$\phi \in \hat{H}_\infty^{1,t}(\Omega)\,, \qquad \nabla\phi \in W^{1,t}(\Omega)$$

satisfying the estimate

$$\|\nabla\phi\|_{1,t} \le c(|g|_{*,t} + \|g\|_{0,t})\,. \tag{7.2}$$

(b) *Let* $\frac{3}{2} < q < 3$, $3 < p < \infty$, $l = 0, 1, \dots k$, $k = 0, 1, \dots$, $\Omega \in \mathscr{C}^{k+2}$ *and*

$$g \in (\hat{H}_{\infty}^{1,q'}(\Omega))^* \cap (\hat{H}_{\infty}^{1,p'}(\Omega))^* \cap W^{l,q}(\Omega) \cap W^{k,p}(\Omega),$$

$$\psi \in W^{l+1-(1/q),q}(\partial\Omega) \cap W^{k+1-(1/p),p}(\partial\Omega). \tag{7.3}$$

Then there exists just one solution ϕ *of the problem* (7.1)

$$\phi \in \hat{H}_{\infty}^{1,q}(\Omega) \cap \hat{H}_{\infty}^{1,p}(\Omega), \qquad \nabla\phi \in W^{l+1,q}(\Omega) \cap W^{k+1,p}(\Omega) \tag{7.4}$$

which satisfies the estimate

$$\|\nabla\phi\|_{l+1,q} + \|\nabla\phi\|_{k+1,p} \le c(|g|_{*,q} + |g|_{*,p} + \|g\|_{l,q} + \|g\|_{k,p} + \|\psi\|_{l+1-(1/q),q,\partial\Omega} + \|\psi\|_{k+1-(1/p),p,\partial\Omega}). \tag{7.5}$$

Next consider the homogeneous Dirichlet problem for the Laplacian:

$$\Delta\Theta = g, \qquad \Theta|_{\partial\Omega} = 0, \qquad \Theta \to 0 \quad \text{as } |x| \to \infty. \tag{7.6}$$

For our purpose, we need a theorem about the regularity of solutions

Lemma 7.2 (Dirichlet problem for the Laplace operator). *Let* $\frac{3}{2} < q < 3$, $3 < p < \infty$, $l = 0, \dots k$, $k = 0, \dots$, $\Omega \in \mathscr{C}^{k+2}$ *and*

$$g \in \hat{H}^{-1,q}(\Omega) \cap \hat{H}^{-1,p}(\Omega) \cap W^{l,q}(\Omega) \cap W^{k,p}(\Omega). \tag{7.7}$$

Then there exists a unique solution Θ *of the problem* (7.6)

$$\Theta \in \hat{H}_0^{1,q}(\Omega) \cap \hat{H}_0^{1,p}(\Omega), \qquad \nabla\Theta \in W^{l+1,q}(\Omega) \cap W^{k+1,p}(\Omega) \tag{7.8}$$

which satisfies the estimate

$$\|\nabla\Theta\|_{l+1,q} + \|\nabla\Theta\|_{k+1,p} \le c(|g|_{-1,q} + |g|_{-1,p} + \|g\|_{l,q} + \|g\|_{k,p}). \tag{7.9}$$

Next problem to be investigated is the Stokes problem:

$$-\mu_1\Delta u + \nabla\Pi = g \qquad \text{in } \Omega,$$

$$\operatorname{div} u = h \qquad \text{in } \Omega,$$

$$u|_{\partial\Omega} = \psi, \qquad u \to 0, \qquad \Pi \to 0 \qquad \text{as } |x| \to \infty. \tag{7.10}$$

In exterior domains, it was studied by many authors, recall e.g. Maremonti and Solonnikov [13], Galdi and Simader [9], and the exhausting monograph of Galdi [8]. We have in particular

Lemma 7.3 (Dirichlet problem for the Stokes operator).

(a) *Let* $\frac{3}{2} < t < 3$, $\Omega \in \mathscr{C}^2$ *and*

$$\psi \in W^{1-(1/t),t}(\partial\Omega), \qquad g \in \hat{H}^{-1,t}(\Omega), \qquad h \in L^t(\Omega).$$

Then there exists a unique solution of the problem (7.10)

$$u \in \hat{H}_{\infty}^{1,t}(\Omega), \qquad \Pi \in L^t(\Omega)$$

satisfying the estimate

$$\|\nabla u\|_{0,t} + \|\Pi\|_{0,t} \le c(|g|_{-1,t} + \|\psi\|_{1-(1/t),t,\partial\Omega} + \|h\|_{0,t}) . \tag{7.11}$$

(b) *Let* $\frac{3}{2} < q < 3$, $3 < p < \infty$, $l = 0, \dots k$, $k = 0, \dots,$ $\Omega \in \mathscr{C}^{k+2}$ *and*

$$h = 0 , \qquad g \in \hat{H}^{-1,q}(\Omega) \cap W^{l,q}(\Omega) \cap W^{k,p}(\Omega) ,$$

$$\psi \in W^{l+2-(1/q),q}(\partial\Omega) \cap W^{k+2-(1/p),p}(\partial\Omega) .$$

Then there exists a unique solution (Π, u) *of the problem* (7.10)

$$u \in \hat{H}_0^{1,q}(\Omega) \cap \hat{H}_0^{1,p}(\Omega) , \qquad \nabla u \in W^{l+1,q}(\Omega) \cap W^{k+1,p}(\Omega) ,$$

$$\Pi \in W^{l+1,q}(\Omega) \cap W^{k+1,p}(\Omega) \tag{7.12}$$

which satisfies the estimate

$$\|\nabla u\|_{l+1,q} + \|\nabla u\|_{k+1,p} + \|\Pi\|_{l+1,q} + \|\Pi\|_{k+1,p}$$

$$\le c(|g|_{-1,q} + |g|_{-1,p} + \|g\|_{l,q} + \|g\|_{k,p}$$

$$+ \|\psi\|_{l+2-(1/q),q,\partial\Omega} + \|\psi\|_{k+2-(1/p),p,\partial\Omega}) . \tag{7.13}$$

The last axuiliary problem to be considered is the transport equation

$$\omega + \operatorname{div}(w'\omega) = g \qquad \text{in } \Omega , \qquad (w' \cdot v|_{\partial\Omega} = 0) \tag{7.14}$$

We have, cf. Novotny [17], Theorems 5.6, and 7.1 and 7.2:

Lemma 7.4 (transport equation).

(a) *Let* $1 < q < 3$, $3 < p < \infty$, $l = 0, \dots k$, $k = 0, \dots,$ $\Omega \in \mathscr{C}^{k+1}$ *and*

$$w' \in V = \{w' \in \hat{H}_\infty^{1,q}(\Omega) \cap \hat{H}_\infty^{1,p}(\Omega), \nabla w' \in W^{l+1,q}(\Omega) \cap W^{l+1,p}(\Omega)\} ,$$

$$w' \cdot v|_{\partial\Omega} = 0 ,$$

$$g \in W^{l+1,q}(\Omega) \cap W^{k+1,p}(\Omega) . \tag{7.15}$$

Then there exists $\gamma > 0$ *such that if*

$$\|w'\|_V \le \gamma$$

then there exists a unique solution ω *of* (7.14)

$$\omega \in W^{l+1,q}(\Omega) \cap W^{k+1,p}(\Omega) ,$$

$$\operatorname{div}(w'\omega) \in W^{l+1,q}(\Omega) \cap W^{k+1,p}(\Omega)$$

which satisfies the estimate

$$\|\omega\|_{l+1,q} + \|\omega\|_{k+1,p} + \|\operatorname{div}(w'\omega)\|_{l+1,q} + \|\operatorname{div}(w'\omega)\|_{k+1,p}$$

$$\le c(\|g\|_{l+1,q} + \|g\|_{k+1,p}) . \tag{7.16}$$

(b) *Furthermore, we have*

$$\Delta\omega \in \hat{H}^{-1,q}(\Omega) \cap \hat{H}^{-1,p}(\Omega)$$

and the validity of the estimates

$$|\Delta\omega|_{-1,s} \le c(|\Delta g|_{-1,s} + \|w'\|_V \|\omega\|_{1,s}),$$

$$|\Delta \operatorname{div}(w'\omega)|_{-1,s} \le c(|\Delta g|_{-1,s} + \|w'\|_V \|\omega\|_{1,s}), \qquad s = q,\ p. \quad (7.17)$$

If $l \ge 1$ *and* $k \ge 1$, *then moreover it holds*

$$\Delta\omega \in W^{l-1,q}(\Omega) \cap W^{k-1,p}(\Omega)$$

and

$$\|\Delta\omega\|_{l-1,q} + \|\Delta\omega\|_{k-1,p}$$
$$\le c\{\|\Delta g\|_{l-1,q} + \|\Delta g\|_{k-1,p} + \|w'\|_V(\|\omega\|_{l+1,q} + \|\omega\|_{k+1,p})\},$$
$$\|\Delta \operatorname{div}(w'\omega)\|_{l-1,q} + \|\Delta \operatorname{div}(w'\omega)\|_{k-1,p}$$
$$\le c\{\|\Delta g\|_{l-1,q} + \|\Delta g\|_{k-1,p} + \|w'\|_V(\|\omega\|_{l+1,q} + \|\omega\|_{k+1,p})\}. \quad (7.18)$$

Lemma 7.5 (transport equation—the decay). *Let* $1 < r < \infty$, $1 < q < 3$, $3 < p < \infty$, $l = 0, \dots k$, $k = 1, \dots$, $\Omega \in \mathscr{C}^{k+2}$ *and* $w' \in \mathbf{V}_*$, *i.e. it satisfies the assumptions* $(7.15)_{1-2}$ *of the previous lemma and the further assumptions*

$$|x|w' \in L^\infty(\Omega), \qquad |x|^2\nabla w' \in L^\infty(\Omega).$$

Let g *satisfy assumptions of* $(7.15)_3$ *of the previous lemma and*

$$|x|^2 g \in L^\infty(\Omega), \qquad |x|\nabla g \in L^\infty(\Omega), \qquad |x|^2\nabla g \in L^r(\Omega).$$

Then there exists $\gamma > 0$ *such that if*

$$\|w'\|_{\mathbf{V}} \le \gamma$$

then the solution ω *of problem* (7.14), *which is guaranteed by Theorem* 7.4, *i.e.*

$$\omega \in W^{l+1,q}(\Omega) \cap W^{k+1,p}(\Omega),$$
$$\operatorname{div}(w'\omega) \in W^{l+1,q}(\Omega) \cap W^{k+1,p}(\Omega)$$

satisfies, besides (7.16)–(7.18), *also the estimates*

$$\||x|^2\omega\|_{0,\infty} + \||x|\nabla\omega\|_{0,\infty} + \||x|^2\nabla\omega\|_{0,r}$$
$$\le c(\||x|^2 g\|_{0,\infty} + \||x|\nabla g\|_{0,\infty} + \||x|^2\nabla g\|_{0,r}),$$
$$\||x|^2 \operatorname{div}(w'\omega)\|_{0,\infty} + \||x|\nabla \operatorname{div}(w'\omega)\|_{0,\infty} + \||x|^2\nabla \operatorname{div}(w'\omega)\|_{0,r}$$
$$\le c(\||x|^2 g\|_{0,\infty} + \||x|\nabla g\|_{0,\infty} + \||x|^2\nabla g\|_{0,r}). \quad (7.19)$$

Proof. Take ω the solution guaranteed by Lemma 7.4. Consider (7.14) and ∇ (7.14) in the form

$$\omega = -\nabla\omega\cdot w' - \omega\nabla\cdot w' + g\,,$$

$$\nabla\omega = -\nabla^2\omega\cdot w' - \omega\nabla\cdot w' - \nabla w'\cdot\nabla\omega - \nabla\omega\nabla\cdot w' - \omega\nabla\nabla\cdot w' + \nabla g\,. \tag{7.20}$$

Multiplying the first equation by $|x|^2$ and the second by $|x|$, we get estimates

$$\||x|^2\omega\|_{0,\infty} \le c\{\||x|\nabla\omega\|_{0,\infty}\||x|w'\|_{0,\infty} + \||x|^2\omega\|_{0,\infty}|w'|_{\mathscr{C}^1} + \||x|^2 g\|_{0,\infty}\}\,,$$

$$\||x|\nabla\omega\|_{0,\infty} \le c\{|\nabla^2\omega|_{\mathscr{C}^0}\||x|w'\|_{0,\infty} + \||x|\nabla\omega\|_{0,\infty}\||x|\nabla w'\|_{0,\infty} + \||x|\nabla g\|_{0,\infty}\}\,.$$

The last two inequalities, when added, yield for $\|w\|_V$ "small enough"

$$\||x|\nabla\omega\|_{0,\infty} + \||x|^2\omega\|_{0,\infty} \le c\{\||x|\nabla g\|_{0,\infty} + \||x|^2 g\|_{0,\infty}\} \tag{7.21}$$

and consequently

$$\||x|\nabla\,\mathrm{div}(w'\omega)\|_{0,\infty} + \||x|^2\,\mathrm{div}\,(w'\omega)\|_{0,\infty} \le c\{\||x|\nabla g\|_{0,\infty} + \||x|^2 g\|_{0,\infty}\}\,. \tag{7.22}$$

Multiplying $(7.20)_2$ scalarly by $|\nabla\omega|^{r-2}\nabla\omega|x|^{2r}$ and integrating over Ω, we obtain

$$\begin{aligned}
\||x|^2\nabla\omega\|_{0,r}^r \le & -\frac{1}{r}\int_\Omega w'\cdot\nabla|\nabla\omega|^r|x|^{2r}\,dx \\
& - \int_\Omega (\nabla w'\cdot\nabla\omega - \nabla\omega\nabla\cdot w')\cdot|\nabla\omega|^{r-2}\nabla\omega|x|^{2r}\,dx \\
& - \int_\Omega \omega\nabla\nabla\cdot w'\cdot|\nabla\omega|^{r-2}\nabla\omega|x|^{2r}\,dx + \int_\Omega \nabla g\cdot|\nabla\omega|^{r-2}\nabla\omega|x|^{2r}\,dx \\
= & \frac{1}{r}\int_\Omega \mathrm{div}\,w'|\nabla\omega|^r|x|^{2r}\,dx + \int_\Omega w'\cdot\frac{x}{|x|}|\nabla\omega|^r|x|^{2r-1}\,dx \\
& - \int_\Omega (\nabla w'\cdot\nabla\omega - \nabla\omega\nabla\cdot w')\cdot|\nabla\omega|^{r-2}\nabla\omega|x|^{2r}\,dx \\
& - \int_\Omega \omega\nabla\nabla\cdot w'\cdot|\nabla\omega|^{r-2}\nabla\omega|x|^{2r}\,dx + \int_\Omega \nabla g\cdot|\nabla\omega|^{r-2}\nabla\omega|x|^{2r}\,dx \\
\le & c\{|w'|_{\mathscr{C}^2}[\||x|^2\nabla\omega\|_{0,r}^r + \||x|^2\omega\|_{0,\infty}\||x|^2\nabla\omega'\|_{0,r}^{r-1}] + \||x|^2\nabla g\|_{0,r}^r\}\,.
\end{aligned}$$

This estimate and estimate (7.22) yield, for $\|w'\|_V$ sufficiently small, the estimate (7.19). Lemma 7.5 is thus proved.

8. Proof of Theorems 5.1 and 5.2—the linearized system

One version of proof of Theorem 5.1 was given in Novotny, Padula [19]. We repeat it here briefly only for the sake of completeness. The proof of Theorem 5.2 is based on Theorem 5.1 and on the representation formulas (see Section 6). For the sake of simplicity, we use the abbreviated notion for the function spaces (G, Φ, U, L, $\mathbf{G}$, $\boldsymbol{\Phi}$, $\mathbf{U}$, $\mathbf{L}$, ...) introduced in Section 4, formulas (4.8)–(4.21).

Proof of Theorem 5.1. We show that the map $\mathscr{L}$ (see (2.8)) is a contraction in the space Φ for arbitrary $w \in V_*$ "sufficiently small".

Lemma 7.3 (b) applied to problem (2.9) yields the estimate

$$\|u\|_V + \|\Pi\|_G \leq c\{\|\mathscr{F}\|_{\tilde{L}} + \|\nabla\xi\|_{k+2-(1/p),p,\partial\Omega}\} \leq c\{\|\mathscr{F}\|_{\tilde{L}} + \|\xi\|_\Phi\}. \tag{8.1}$$

(Here we have used the Gagliardo inequality about the traces and the inequality $\|\nabla\xi\|_{l+2-(1/q),q,\partial\Omega} \leq \|\nabla\xi\|_{k+2-(1/p),p,\partial\Omega}$ which holds due to the compactness of $\partial\Omega$ and the relations $l \leq k$, $q < p$.) Taking the div of the equation $(2.9)_1$, we get

$$\Delta\Pi = \operatorname{div}\mathscr{F}$$

hence we find

$$]\Pi[:= |\Delta\Pi|_{-1,q} + |\Delta\Pi|_{-1,p} + \kappa_l\|\Delta\Pi\|_{l-1,q} + \kappa_k\|\Delta\Pi\|_{k-1,q} \leq c\|\mathscr{F}\|_{\tilde{L}}. \tag{8.2}$$

(Here $\kappa_s = 0$ if $s = 0$ and $\kappa_s = 1$ otherwise.)

Lemma 7.4 (a) together with (8.1), applied to (2.10), furnishes

$$\|\sigma\|_G + \|\operatorname{div}(\sigma w)\|_G \leq c_1'\{\|\mathscr{F}\|_{\tilde{L}} + \|\xi\|_\Phi\} \tag{8.3}$$

and Lemma 7.4 (b) together with (8.2) yields, in particular,

$$]\operatorname{div}(\sigma w)[\leq c\{\|\mathscr{F}\|_{\tilde{L}} + \|w\|_V\|\sigma\|_G\}. \tag{8.4}$$

Finally, Lemma 7.1 applied to (2.11) gives

$$\|\phi\|_\Phi \leq c\{|\operatorname{div}(\sigma w)|_{*,q} + |\operatorname{div}(\sigma w)|_{*,p} + \|\operatorname{div}(\sigma w)\|_{0,q} + \|\operatorname{div}(\sigma w)\|_{0,p}$$
$$+ \|\nabla\operatorname{div}(\sigma w)\|_{l,q} + \|\nabla\operatorname{div}(\sigma w)\|_{k,p}.\} \tag{8.5}$$

Estimating the first four terms as the nonlinear quadratic terms (with the help of the definition of $|\cdot|_{*,t}$ norm, by the Holder inequality and the Sobolev imbedding theorems), and the last two terms by Lemma 7.2 (recall that $\operatorname{div}(\sigma w)|_{\partial\Omega} = 0$, i.e. it is enough to put in Lemma 7.2 $\Theta = \operatorname{div}(\sigma w)$), we obtain

$$\|\phi\|_\Phi \leq c\{]\operatorname{div}(\sigma w)[+ \|\sigma\|_G\|w\|_V\} \leq c_2'\{\|\mathscr{F}\|_{\tilde{L}} + \|\sigma\|_G\|w\|_V\}. \tag{8.6}$$

Estimates (8.3), (8.4) and (8.6) imply

$$\|\phi\|_\Phi \leq c_3'\{\|\mathscr{F}\|_{\tilde{L}} + \|\xi\|_\Phi\|w\|_V\}. \tag{8.7}$$

Which yields (recall that $\mathscr{L}$ is a linear operator) a contraction in Φ provided $c_3'\|w\|_V < 1$; consequently, there exists a fixed point $\xi = \phi$ of $\mathscr{L}$.

Estimates (5.2) and (5.3) follow from (8.7), (8.1), (8.3) written in the fixed point. Proof of Theorem 5.1 is thus complete.

Proof of Theorem 5.2. We take a solution guaranteed by Theorem 5.1 and estimate the decay at infinity by using the representation formulas (see Section 6). The proof is devided into several steps. In the first step, we derive estimates for (Π, u, ϕ) at finite distances. In the second step, we estimate the decay of the integrals with the compact support, i.e. those integrals in the representation formulas which contain $\mathscr{F}^0$ and the integrals over the boundary. Third step is devoted to the decay properties containing contributions of $\mathscr{F}^1$; this part is divided into four substeps: (3a) estimates of u and ∇u, (3b) estimates of Π, $\nabla\Pi$, (3c) estimates of σ, $\nabla\sigma$, (3d) estimates of ϕ, $\nabla\phi$, $\nabla^2\phi$, $\nabla^3\phi$, (3e) conclusions.

(1) *Estimates at the finite distances*

For a fixed $R \geq 1$, sufficiently large, we have

$$\||x|u\|_{0,\infty,\Omega_R} + \||x|^2\nabla u\|_{0,\infty,\Omega_R} \leq cR^2(\|u\|_{0,\infty,\Omega} + \|\nabla u\|_{0,\infty,\Omega}) \leq c'\|\mathscr{F}\|_{\tilde{L}},$$

$$\||x|^2\Pi\|_{0,\infty,\Omega_R} + \||x|\nabla\Pi\|_{0,\infty,\Omega_R} + \||x|^2\nabla\Pi\|_{0,r,\Omega_R}$$
$$\leq cR^2(1 + (\text{meas}\,\Omega_R)^{1/r})|\Pi|_{\mathscr{C}^1} \leq c'\|\mathscr{F}\|_{\tilde{L}},$$

$$\||x|\nabla\phi\|_{0,\infty,\Omega_R} + \||x|^2\nabla^2\phi\|_{1,r,\Omega_R} \leq cR^2(1 + (\text{meas}\,\Omega_R)^{1/r})|\phi|_{\mathscr{C}^2} \leq c'\|\mathscr{F}\|_{\tilde{L}}. \tag{8.8}$$

(In (8.8), we have used estimates (5.2) and Sobolev imbeddings of the type $W^{s,t}(\Omega) \subset \mathscr{C}^j(\bar{\Omega})$, provided $(s-j)t > 3$; this requires to take $k \geq 2$.)

(2) *Estimates of the terms containing* $\mathscr{F}^0$ *and the estimates of the boundary integrals.* Set $m = \text{meas}\,(\text{supp}\,\mathscr{F}^0)$. We find for R sufficiently large (e.g. such that $\text{supp}\,\mathscr{F}^0 \subset B_{R/2}$) and $r > 3$:

$$\||x|\mathscr{M}_u^0\|_{0,\infty,\Omega^R} + \||x|^2\mathscr{M}_u^1\|_{0,\infty,\Omega^R} \leq cm|\mathscr{F}^0|_{\mathscr{C}^0} \leq c'\|\mathscr{F}^0\|_{\tilde{L}},$$

$$\||x|^2\mathscr{M}_\Pi^0\|_{0,\infty,\Omega^R} + \||x|\mathscr{M}_\Pi^1\|_{0,r,\Omega^R} + \||x|^2\mathscr{M}_\Pi^1\|_{0,r,\Omega^R}$$
$$\leq c\left(m + \left(\int_{\Omega_R} |x|^{-r}\,dx\right)^{1/r}\right)|\mathscr{F}^0|_{\mathscr{C}^0} \leq c'\|\mathscr{F}^0\|_{\tilde{L}}. \tag{8.9}$$

As far as the boundary integrals are concerned, we have (see (6.8), (6.11), (6.15)):

$$\||x|\mathscr{E}_u^0\|_{0,\infty,\Omega^R} + \||x|^2\mathscr{E}_u^1\|_{0,\infty,\Omega^R} \leq c(|u|_{\mathscr{C}^1} + |\Pi|_{\mathscr{C}^0} + |\mathscr{F}^1|_{\mathscr{C}^0})$$
$$\leq c'(\|\mathscr{F}^0\|_{\tilde{L}} + \|\mathscr{F}^1\|_L),$$

$$\||x|^2\mathscr{E}_\Pi^0\|_{0,\infty,\Omega^R} + \||x|\mathscr{E}_\Pi^1\|_{0,\infty,\Omega^R} + \||x|\mathscr{E}_\Pi'^1\|_{0,r,\Omega^R} \leq c(|u|_{\mathscr{C}^1} + |\Pi|_{\mathscr{C}^0} + |\mathscr{F}^1|_{\mathscr{C}^1})$$
$$\leq c'(\|\mathscr{F}^0\|_{\tilde{L}} + \|\mathscr{F}^1\|_L),$$

$$\||x|\mathscr{E}_\phi^1\|_{0,\infty,\Omega^R} + \||x|^2\mathscr{E}_\phi^2\|_{0,r,\Omega^R} + \||x|^2\mathscr{E}_\phi^3\|_{0,r,\Omega^R}$$
$$\leq c(|\phi|_{\mathscr{C}^1} + |\text{div}\,(\sigma w)|_{\mathscr{C}^0}) \leq c'(\|\mathscr{F}^0\|_{\tilde{L}} + \|\mathscr{F}^1\|_L). \tag{8.10}$$

(3) *Estimates of the integrals containing* $\mathscr{F}^1$

(3a) *Estimates of* u, ∇u

We have by Lemma 3.2 (see (6.7))

$$\||x|\mathscr{N}_u^0\|_{0,\infty} \leq \||x|^2\mathscr{F}^1\|_{0,\infty} \tag{8.11}$$

and by Lemma 3.1

$$\| |x|^2 \mathcal{N}_u^1 \|_{0,\infty} \le c(\| |x|^3 \operatorname{div} \mathcal{F}^1 \|_{0,\infty} + \| |x|^2 \mathcal{F}^1 \|_{0,\infty}) . \tag{8.12}$$

Representation (6.5) together with $(8.8)_1$, $(8.9)_1$, $(8.10)_1$, (8.11) and (8.12), gives

$$\| |x| u \|_{0,\infty} + \| |x|^2 \nabla u \|_{0,\infty} \le c(\| \mathcal{F}^0 \|_{\tilde{L}} + \| \mathcal{F}^1 \|_{\mathrm{L}}) . \tag{8.13}$$

(3b) *Estimates of Π, $\nabla \Pi$*

We have by Lemma 3.1 (see (6.10))

$$\| |x|^2 \mathcal{N}_\Pi^0 \|_{0,\infty} \le c(\| |x|^2 \mathcal{F}^1 \|_{0,\infty} + \| |x|^3 \operatorname{div} \mathcal{F}^1 \|_{0,\infty}) ; \tag{8.14}$$

by Lemma 3.2

$$\| |x| \mathcal{N}_\Pi^1 \|_{0,\infty} \le c(\| |x| \nabla \operatorname{div} \mathcal{F}^1 \|_{0,q} + \| |x| \nabla \operatorname{div} \mathcal{F}^1 \|_{0,p}) \tag{8.15}$$

and by Lemma 3.3

$$\| |x|^2 \mathcal{N}_\Pi'^1 \|_{0,r} \le c \| |x|^2 \operatorname{div} \mathcal{F}^1 \|_{0,r} , \qquad r > 3 . \tag{8.16}$$

Representation (6.9) together with $(8.8)_2$, $(8.9)_2$, $(8.10)_2$ and (8.14)–(8.16), gives

$$\| |x|^2 \Pi \|_{0,\infty} + \| |x| \nabla \Pi \|_{0,\infty} + \| |x|^2 \nabla \Pi \|_{0,r} \le c(\| \mathcal{F}^0 \|_{\tilde{L}} + \| \mathcal{F}^1 \|_{\mathrm{L}}) . \tag{8.17}$$

(3c) *Estimates of σ, $\nabla \sigma$*

Applying Lemma 7.5 to problem (2.10), one has

$$\| |x|^2 \sigma \|_{0,\infty} + \| |x| \nabla \sigma \|_{0,\infty} + \| |x|^2 \nabla \sigma \|_{0,r} \le c(\| \mathcal{F}^0 \|_{\tilde{L}} + \| \mathcal{F}^1 \|_{\mathrm{L}}) \tag{8.18}$$

and consequently

$$\begin{aligned} &\| |x|^2 \operatorname{div} (\sigma w) \|_{0,\infty} + \| |x| \nabla \operatorname{div} (\sigma w) \|_{0,\infty} + \| |x|^2 \nabla \operatorname{div} (\sigma w) \|_{0,r} \\ &\quad \le c(\| \mathcal{F}^0 \|_{\tilde{L}} + \| \mathcal{F}^1 \|_{\mathrm{L}}) \end{aligned} \tag{8.19}$$

provided $\|w\|_{\mathrm{V}}$ is "sufficiently small".

(3d) *Estimates of $\nabla \phi$, $\nabla^2 \phi$, $\nabla^3 \phi$*

We have by Lemma 3.2 applied to $(6.14)_1$ (notice that $\| |x|^2 \operatorname{div} (\sigma w) \|_{0,\infty} \le \| |x|^2 \sigma \|_{0,\infty} |\nabla w|_{\mathscr{C}^0} + \| |x| w \|_{0,\infty} \| |x| \nabla \sigma \|_{0,\infty}$),

$$\| |x| \mathcal{N}_\phi^1 \|_{0,\infty} \le c \|w\|_{\mathrm{V}} \| (\sigma, \phi, u) \|_{\mathbf{G}\boldsymbol{\Phi}\mathbf{U}} ; \tag{8.20}$$

by Lemma 3.3 (see $(6.14)_2$) (notice that $\| |x|^2 \operatorname{div} (\sigma w) \|_{0,r} \le \| |x|^2 \sigma \|_{0,\infty} \| \nabla w \|_{0,r} + |w|_{\mathscr{C}^0} \| |x|^2 \nabla \sigma \|_{0,r}$, $\| \nabla w \|_{0,r} \le \| |x|^2 \nabla w \|_{0,\infty} \| |x|^{-2} \|_{0,r} \le c \| |x|^2 \nabla w \|_{0,\infty}$), we obtain

$$\| |x|^2 \mathcal{N}_\phi^2 \|_{0,r} \le c \|w\|_{\mathrm{V}} \| (\sigma, \phi, u) \|_{\mathbf{G}\boldsymbol{\Phi}\mathbf{U}} ; \tag{8.21}$$

by Lemma 3.3, see $(6.14)_3$ and (8.19) (i.e. $\| |x|^2 \nabla \operatorname{div} (\sigma w) \|_{0,r}$ is bounded),

$$\| |x|^2 \mathcal{N}_\phi^3 \|_{0,r} \le c(\| \mathcal{F}^0 \|_{\tilde{L}} + \| \mathcal{F}^1 \|_{\mathrm{L}}) . \tag{8.22}$$

Representation formulas (6.13) and (8.20)–(8.22) together with $(8.8)_3$ and $(8.10)_3$ yield

$$\| |x| \nabla \phi \|_{0,\infty} + \| |x|^2 \nabla^2 \phi \|_{1,r} \leq c(\|\mathscr{F}^0\|_{\tilde{L}} + \|\mathscr{F}^1\|_{\mathbf{L}} + \|w\|_{\mathbf{V}} \|(\sigma, \phi, u)\|_{\mathbf{G\Phi U}}) \tag{8.23}$$

and, in particular, by the imbedding $W^{1,r}(\Omega) \subset \mathscr{C}^0(\bar{\Omega})$, also,

$$\| |x|^2 \nabla^2 \phi \|_{0,\infty} \leq c(\|\mathscr{F}^0\|_{\tilde{L}} + \|\mathscr{F}^1\|_{\mathbf{L}} + \|w\|_{\mathbf{V}} \|(\sigma, \phi, u)\|_{\mathbf{G\Phi U}}) \tag{8.24}$$

(3e) *Conclusion*

Estimates (5.2), (8.13), (8.17), (8.18) and (8.23) yield

$$\|\Pi\|_{\mathbf{G}} + \|(\sigma, \phi, u)\|_{\mathbf{G\Phi U}} \leq c(\|\mathscr{F}^0\|_{\tilde{L}} + \|\mathscr{F}^1\|_{\mathbf{L}} + \|w\|_{\mathbf{V}} \|(\sigma, \phi, u)\|_{\mathbf{G\Phi U}}) \tag{8.25}$$

which yields estimate (5.5) (and as a consequence (5.6)), for $\|w\|_{\mathbf{V}}$ sufficiently small. Theorem 5.2 is thus proved.

9. Proof of Theorem 5.3—the fully nonlinear system

We show that the nonlinear operator $\mathscr{N}$, formally defined by (2.15) has the following properties:

(i) It is well defined on a closed ball

$$\mathbf{B}_{\gamma_0} := \{(\sigma, \phi, u) \in \mathbf{G\Phi U}_*, \|(\sigma, \phi, u)\|_{\mathbf{G\Phi U}} \leq \gamma_0\} \tag{9.1}$$

of the space $\mathbf{G\Phi U}$, provided γ_0 is sufficiently small.

(ii) It maps $\mathbf{B}_{\gamma_0}$ into itself provided γ_0 and γ_1 are sufficiently small (recall that γ_1 is the bound for $\|f\|_{\tilde{L}}$, i.e. $\|f\|_{\tilde{L}} \leq \gamma_1$).

(iii) Denote by G', Φ', U', $\tilde{L}'$ the Banach space G, Φ, U, $\tilde{L}$, respectively with $l-1$ and $k-1$ instead of l, k ($l \geq 1, k \geq 1$). Then put

$$\mathbf{X} := \{(\sigma, \phi, u) : \sigma \in G', \phi \in \Phi', u \in U'\}, \tag{9.2}$$

It is a Banach space with norm

$$\|(\sigma, \phi, u)\|_{\mathbf{X}} = \|\sigma\|_{G'} + \|\phi\|_{\Phi'} + \|u\|_{U'} .$$

Obviously $\mathbf{B}_{\gamma_0} \subset \mathbf{X}$ is a closed subset in $\mathbf{X}$. The operator $\mathscr{N}$ is a contraction in $\mathbf{B}_{\gamma_0}$ in the topology of $\mathbf{X}$, provided γ_0 and γ_1 are sufficiently small.

As a consequence, $\mathscr{N}$ possesses a fixed point, say (σ, ϕ, u). The corresponding σ and $v = u + \nabla\phi$ satisfy the fully nonlinear system (2.4), (2.5).

Proof of (i), (ii). Set

$$F = F^0 + \operatorname{div} F^1 , \qquad w = z + \nabla\xi ,$$
$$F^0 = (1+\tau)f , \qquad F^1 = -(1+\tau) w \otimes w . \tag{9.3}$$

We easily find, for $(\tau, \xi, z) \in \mathbf{B}_1$:

$$\|F\|_{\tilde{L}} \leq c(\|f\|_{\tilde{L}} + \|(\tau, \xi, z)\|^2_{\mathbf{G\Phi U}}) ,$$
$$\|F^0\|_{\tilde{L}} \leq c\|f\|_{\tilde{L}} , \qquad \|F^1\|_{\mathbf{L}} \leq c\|(\tau, \xi, z)\|^2_{\mathbf{G\Phi U}} \tag{9.4}$$

with $r > 3$, $\frac{3}{2} < q < 3$, $p > 3$ and $l = 1, \ldots, k$, $k = 1, 2, \ldots$.[8] Theorem 5.2 applied to system (2.12)–(2.14) with $F = F(\tau, w)$ yields estimate

$$\|(\sigma, \phi, u)\|_{\mathbf{G\Phi U}} \leq c(\|f\|_{\tilde{L}} + \|(\tau, \xi, z)\|^2_{\mathbf{G\Phi U}}) . \tag{9.5}$$

This yields existence of γ_1 and γ_0 such that

$$\mathcal{N}\mathbf{B}_{\gamma_0} \subset \mathbf{B}_{\gamma_0} \qquad \text{provided } \|f\|_{\tilde{L}} \leq \gamma_1 .$$

Proof of (iii). Let (σ_1, ϕ_1, u_1), (σ, ϕ, u) be solutions of problem (2.12)–(2.14) corresponding to $\mathscr{F} = F(\tau_1, w_1 = z_1 + \nabla\xi_1)$ and $\mathscr{F} = F(\tau, w = z + \nabla\xi)$, respectively, where (τ_1, ξ_1, z_1), $(\tau, \xi, z) \in \mathbf{B}_{\gamma_0}$. We want to prove that $\mathcal{N}$ is a contraction, i.e.

$$\|\mathcal{N}(\tau_1, \xi_1, z_1) - \mathcal{N}(\tau, \xi, z)\|_{\mathbf{X}} < \hbar\|(\tau_1, \xi_1, z_1) - (\tau, \xi, z)\|_{\mathbf{X}} \tag{9.6}$$

with a $0 < \hbar < 1$ (provided γ_0 and γ_1 are sufficiently small).

Denote $\tilde{\tau} = \tau - \tau_1$, $\tilde{\sigma} = \sigma - \sigma_1$, $\tilde{\xi} = \xi - \xi_1$, $\tilde{\phi} = \phi - \phi_1$, $\tilde{z} = z - z_1$, $\tilde{u} = u - u_1$, $\tilde{F} = F(\tau_1, w_1) - F(\tau, w)$, $w_1 = z_1 + \nabla\xi_1$, $w = z + \nabla\xi$. We calculate

$$\|\tilde{F}\|_{\tilde{L}'} \leq c\{\|f\|_{\tilde{L}'} + (\|(\tau, \xi, z)\|_{\mathbf{G\Phi U}} + \|(\tau_1, \xi_1, z_1)\|_{\mathbf{G\Phi U}})\|(\tilde{\tau}, \tilde{\xi}, \tilde{z})\|_{\mathbf{X}}\} . \tag{9.7}$$

Writing system (2.12)–(2.14) for the differences, we get

$$\begin{gathered} -\mu_1 \Delta\tilde{u} + \nabla\tilde{\Pi} = \tilde{F} , \\ \operatorname{div} \tilde{u} = 0 , \\ \tilde{u}|_{\partial\Omega} = \nabla\tilde{\phi}|_{\partial\Omega}, \qquad \tilde{u}(x),\ \tilde{\Pi}(x) \to 0 \qquad \text{as } |x| \to \infty , \end{gathered} \tag{9.8}$$

$$\begin{gathered} \tilde{\sigma} + (2\mu_1 + \mu_2) \operatorname{div}(\tilde{\sigma}w) = \tilde{\Pi} - (2\mu_1 + \mu_2) \operatorname{div}(\sigma_1\tilde{w}) , \\ \tilde{\sigma} \to 0 \qquad \text{as } |x| \to \infty , \end{gathered} \tag{9.9}$$

$$\begin{gathered} \Delta\tilde{\phi} = -\operatorname{div}(\tilde{\sigma}w) - \operatorname{div}(\sigma_1\tilde{w}) , \\ \left.\frac{\partial\tilde{\phi}}{\partial\nu}\right|_{\partial\Omega} = 0 , \qquad \tilde{\phi} \to 0 \qquad \text{as } |x| \to \infty . \end{gathered} \tag{9.10}$$

Applying to equation (9.8) Lemma 7.3, we get estimate

$$\|\tilde{u}\|_{U'} + \|\tilde{\Pi}\|_{G'} \leq c(\|\tilde{F}\|_{\tilde{L}'} + \|\tilde{\phi}\|_{\Phi'}) . \tag{9.11}$$

Taking the divergence of equation (9.8), we get

$$\Delta\tilde{\Pi} = \operatorname{div} \tilde{F}$$

hence

$$]\tilde{\Pi}[' := |\Delta\tilde{\Pi}|_{-1,q} + |\Delta\tilde{\Pi}|_{-1,p} + \kappa_{l-1}\|\Delta\tilde{\Pi}\|_{l-2,q} + \kappa_{k-1}\|\Delta\tilde{\Pi}\|_{k-2,p} \leq c\|\tilde{F}\|_{\tilde{L}'}$$

(for the definition of κ_s, see (8.2)). Lemma 7.4 (a), (b) together with (9.11) and

[8] Condition $l \geq 1$ is required when estimating $\||x|\nabla \operatorname{div} F^1\|_{0,q}$.

the previous estimate, applied to (9.9) yields

$$\|\tilde{\sigma}\|_{G'} + \|\mathrm{div}\,(\tilde{\sigma}w)\|_{G'} \leq c(\|\tilde{F}\|_{\tilde{L}'} + \|\tilde{\phi}\|_{\Phi'} + \|\sigma_1\|_{G}(\|\tilde{\xi}\|_{\Phi'} + \|\tilde{z}\|_{\Phi'})) \tag{9.12}$$

and

$$]\Delta\,\mathrm{div}\,(\tilde{\sigma}w)[' \leq c(\|\tilde{F}\|_{\tilde{L}'} + \|\tilde{w}\|_{V'}\|\sigma_1\|_{G} + \|\tilde{\sigma}\|_{G'}\|w\|_{V})\,. \tag{9.13}$$

After this, Lemma 7.1 applied to (9.10) furnishes by the similar reasoning as that one in formulas (8.5)–(8.7)

$$\|\tilde{\phi}\|_{\Phi'} \leq c(\|\tilde{F}\|_{\tilde{L}'} + \|\tilde{w}\|_{V'}\|\sigma_1\|_{G} + \|\tilde{\sigma}\|_{G'}\|w\|_{V})\,. \tag{9.14}$$

Estimates (9.11), (9.12), (9.14) yield finally (for γ_0, γ_1 "sufficiently small")

$$\|\tilde{u}\|_{U'} + \|\tilde{\phi}\|_{\Phi'} + \|\tilde{\sigma}\|_{G'} \leq c_4'(\gamma_0 + \gamma_1)(\|\tilde{z}\|_{U'} + \|\tilde{\xi}\|_{\Phi'} + \|\tilde{\tau}\|_{G'})\,. \tag{9.15}$$

The last inequality yields the contraction in B_{γ_0} for the operator $\mathcal{N}$, provided

$$c_4'(\gamma_0 + \gamma_1) < 1\,.$$

The existence of a (unique) fixed point of $\mathcal{N}$ in $\mathbf{B}_{\gamma_0}$ thus follows from the standard Banach contraction principle. This fixed point (say (σ, ϕ, u)) determines the solution (σ, v) of the nonlinear problem (2.4), (2.5)

$$\sigma\,, \qquad v = u + \nabla\phi\,.$$

The estimate (5.7) follows from (9.5) written in the fixed point. Theorem 5.3 is thus proved.

10. Proof of Theorem 5.4—further decay of $\nabla\phi$

Let us set in the representation formulas of Section 6

$$\mathcal{F}^0 = (1+\sigma)f\,, \qquad \mathcal{F}^1 = -(1+\sigma)(v\otimes v)\,,$$

$w = v$, $\tau = \sigma$, where $(\sigma, v = u + \nabla\phi) \in \mathbf{G}\times\mathbf{V}$ is a solution of problem (2.4), (2.5) guaranteed by Theorem 5.3. Integrating by parts in $(6.14)_1$, we get

$$\mathcal{N}_\phi^1(x) = -\int_\Omega \nabla_x^2\mathcal{E}(x-y)\cdot(\sigma v)(y)\,dy - \int_{\partial\Omega}\nabla_x\mathcal{E}(x-y)(\sigma v\cdot\nu)(y)\,dS_y$$

which yields by $(6.13)_1$

$$\nabla\phi(x) = \overline{\mathcal{N}}_\phi^1(x) + \overline{\mathcal{C}}_\phi^1(x)$$

where

$$\overline{\mathcal{N}}_\phi^1(x) = -\int_\Omega (\sigma v)(y)\cdot\nabla_x^2\mathcal{E}(x-y)\,dy$$

$$\bar{\mathcal{E}}_\phi^1(x) = -\int_{\partial\Omega}\nabla_x\mathcal{E}(x-y)(\sigma v\cdot\nu)(y)\,dS_y + \mathcal{E}_\phi^1\,.$$

We therefore find (see Lemma 3.3)

$$|x|^{2-\varepsilon}\nabla\phi \in L^r(\Omega)\,, \qquad r > 3/\varepsilon\,.$$

Recall that (see Theorem 5.3)

$$|x|^{2-\varepsilon}\nabla^2\phi \in L^r(\Omega)\,, \qquad r > 3/\varepsilon\,.$$

The assertion $(5.8)_1$ follows thus by the imbedding $W^{1,r} \subset \mathscr{C}^0(\bar{\Omega})$ applied to the function $|x|^{2-\varepsilon}\nabla\phi$.

The proof of $(5.8)_2$ is not so straightforward. First, we have to calculate the decay of $\nabla^2\Pi$, $\nabla^4\phi$ and $\nabla^2 u$, $\nabla^3 u$. We have, by differentiating $(6.5)_2$, $(6.9)_2$ and (6.13)

$$\begin{aligned}
\nabla^2\Pi(x) &= \mathscr{M}^2_\Pi(x) + \mathscr{N}^2_\Pi(x) + \mathscr{E}^2_\Pi(x)\,, \qquad x \in \Omega - \operatorname{supp}\mathscr{F}^0\,,\\
\nabla^2 u(x) &= \mathscr{M}^2_u(x) + \mathscr{N}^2_u(x) + \mathscr{E}^2_u(x)\,, \qquad x \in \Omega\,,\\
\nabla^3 u(x) &= \mathscr{M}^3_u(x) + \mathscr{N}^3_u(x) + \mathscr{E}^3_u(x)\,, \qquad x \in \Omega - \operatorname{supp}\mathscr{F}^0\,,\\
\nabla^4\phi &= \mathscr{N}^4_\phi(x) + \mathscr{E}^4_\phi(x)\,, \qquad x \in \Omega\,,
\end{aligned} \tag{10.1}$$

where

$$\begin{aligned}
\mathscr{M}^2_\Pi(x) &= \int_\Omega \nabla^2_x\mathscr{P}(x-y)\cdot\mathscr{F}^0(y)\,dy\,, \qquad x \in \Omega - \operatorname{supp}\mathscr{F}^0\,,\\
\mathscr{M}^2_u(x) &= -\int_\Omega \nabla^2_x\mathscr{U}(x-y)\cdot\mathscr{F}^0(y)\,dy\,, \qquad x \in \Omega\,,\\
\mathscr{M}^3_u(x) &= -\int_\Omega \nabla^3_x\mathscr{U}(x-y)\cdot\mathscr{F}^0(y)\,dy\,, \qquad x \in \Omega - \operatorname{supp}\mathscr{F}^0\,,\\
\mathscr{N}^2_\Pi(x) &= \int_\Omega \nabla_x\mathscr{P}_i(x-y)\nabla_y \operatorname{div}\mathscr{F}^1_i(y)\,dy\,,\\
\mathscr{N}^2_u(x) &= -\int_\Omega \nabla^2_x\mathscr{U}(x-y)\cdot\operatorname{div}\mathscr{F}^1(y)\,dy\\
\mathscr{N}^3_u(x) &= -\int_\Omega \nabla^2_x\mathscr{U}_i(x-y)\nabla_y \operatorname{div}(\mathscr{F}^1)_i(y)\,dy\\
\mathscr{N}^4_\phi(x) &= -\int_\Omega \nabla^2_x\mathscr{E}(x-y)\nabla^2_y \operatorname{div}(\sigma v)(y)\,dy\\
\mathscr{E}^2_\Pi &= \nabla_x\mathscr{E}^1_\Pi\,, \qquad \mathscr{E}^2_u = \nabla_x\mathscr{E}^1_u\,,\\
\mathscr{E}^3_u &= \nabla_x\mathscr{E}^2_u + \int_{\partial\Omega} \nu\nabla^2_x\mathscr{U}(x-y)\cdot\operatorname{div}\mathscr{F}^1(y)\,dS_y\,,\\
\mathscr{E}^4_\phi &= \nabla_x\mathscr{E}^3_\phi + \int_{\partial\Omega} \nu\nabla^2_x\mathscr{E}(x-y)\nabla_y \operatorname{div}(\sigma v)(y)\,dS_y\,.
\end{aligned} \tag{10.2}$$

Formulas (10.1) and (10.2) yield, for

$$(\sigma, \phi, u) \in \mathbf{B}_{\gamma_0} \subset \mathbf{G\Phi U}\,,$$

the following estimates:

$$|x|^{2-\varepsilon}\nabla^2 u \in L^r(\Omega)\,, \qquad r > 3/\varepsilon\,.$$

Recall that (see Theorem 5.3)

$$|x|^{2-\varepsilon}\nabla^3 \phi \in L^r(\Omega)\,, \qquad r > 3/\varepsilon\,,$$

therefore

$$|x|^{2-\varepsilon}\nabla^2 v \in L^r(\Omega)\,, \qquad r > 3/\varepsilon\,.$$

Now, $|x|^{2-\varepsilon}\nabla \operatorname{div} F^1 \in L^r(\Omega)$. Therefore

$$|x|^{2-\varepsilon}\nabla^2 \Pi \in L^r(\Omega)\,, \qquad r > 3/\varepsilon\,.$$

Next we get from the transport equation (2.18)[9]

$$|x|^{2-\varepsilon}\nabla^2 \sigma \in L^r(\Omega)\,, \qquad r > 3/\varepsilon\,,$$

$$|x|^{2-\varepsilon}\nabla^2 \operatorname{div}(\sigma v) \in L^r(\Omega)\,, \qquad r > 3/\varepsilon\,.$$

After this, we deduce from $(10.1)_3$ (the reader verifies that $|x|^{3-\varepsilon}\nabla \operatorname{div} F^1 \in L^r(\Omega)$)

$$|x|^{3-\varepsilon}\nabla^3 u \in L^r(\Omega)\,, \qquad r > 3/\varepsilon$$

and by $(10.1)_1$ also

$$|x|^{3-\varepsilon}\nabla^2 \Pi \in L^r(\Omega)\,, \qquad r > 3/\varepsilon\,.$$

From $(10.1)_4$

$$|x|^{2-\varepsilon}\nabla^4 \phi \in L^r(\Omega)\,, \qquad r > 3/\varepsilon$$

(see Lemma 3.3) and as a consequence

$$|x|^{2-\varepsilon}\nabla^3 v \in L^r(\Omega)\,, \qquad r > 3/\varepsilon\,.$$

Now we find, again from the transport equation (2.18).[9]

$$|x|^{3-\varepsilon}\nabla^2 \sigma \in L^r(\Omega)\,, \qquad r > 3/\varepsilon\,.$$

After this, we obtain

$$|x|^{3-\varepsilon} \operatorname{div}(\sigma v) \in W^{1,r}(\Omega)\,, \qquad r > 3/\varepsilon\,.$$

[9] This requires some explication: Differentiating $(7.20)_2$ (with $w' = v$, $\omega = \sigma$, we obtain an transport equation for $\nabla^2\sigma$ (we put all terms containing derivatives of σ in the order less then 2 to the r.h.s. of it). One easily verifies that $|x|^{a-\varepsilon}$ (r.h.s.) (where a states firstly for 2 and then for 3) belongs to $L^r(\Omega)$. Therefore, multiplying this equation scalarly by $|x|^{r(a-\varepsilon)}|\nabla^2\sigma|^{r-2}\nabla^2\sigma$ and integrating over Ω, we obtain, after some standard manipulations, the desired result. See Novotny [17], Theorems 5.7 and 5.8 for details.

The last estimate yields by (2.17) and the well known Sobolev imbedding theorems, the estimates $(5.8)_2$. Theorem 5.4 is thus proved.

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