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Singular principal normality in the Cauchy problem

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1. Introduction

It is well known now, after the work of Hörmander [H] and Alinhac [A], that the two relevant concepts for the uniqueness in the Cauchy problem are principal normality and pseudo-convexity. However there are simple examples of operators for which uniqueness holds but principal normality fails on the initial surface. The purpose of this work, whose starting point has been the understanding of the recent example given in [CD], is to show that, in some cases, one can relax the notion of principal normality. Indeed, for operators with simple characteristics we introduce a notion of singular principal normality and we prove that it ensures compact uniqueness. In a second part we show that this condition is relevant in proving that if it is violated in a strong sense then non uniqueness holds for a zero$^{\text{th}}$ order perturbation of the operator.

The uniqueness result uses Carleman estimates with singular weights, which are proved by the method introduced by Lerner [L] in the standard case. The main difficulty is then that the proof requires a Fefferman-Phong inequality for pseudo-differential operators with symbols in a non temperate class in the sense of Hörmander [H]. However this inequality has been established in a recent paper by the authors [CDZ]. The proof of the non uniqueness result uses the method developed in earlier works (see [A], [Z]) with new difficulties related to the singularities.

2. Statement of the results

Let P be a homogeneous differential operator of order $m \geq 1$, with complex valued C^∞ coefficients, in a neighborhood V of a point x_0 in $\mathbf{R}^n$, and symbol p. Let S be a C^∞ hypersurface through x_0, given in V by $S \cap V = \{x \in V\colon \varphi(x) = 0\}$, with $\varphi \in C^\infty$ and $\varphi'(x) \neq 0$ in $S \cap V$. We shall set $V^+ = \{x \in V\colon \varphi(x) > 0\}$.

The symbol p will be assumed to have simple complex roots

$$\text{(H.1)} \quad \begin{cases} \xi \in \mathbf{R}^n,\ \tau \in \mathbf{R} \quad \text{and} \quad p(x_0, \xi + i\tau\varphi'(x_0)) = \{p, \varphi\}(x_0, \xi + i\tau\varphi'(x_0)) = 0 \\ \text{imply } \xi = \tau = 0\ . \end{cases}$$

Here $\{,\}$ is as usual the Poisson bracket.

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The second assumption is an extension of the usual principal normality condition (see [H]).

$$\text{(H.2)}\quad \begin{cases} \text{We can write } \dfrac{1}{2i}\{\bar{p}, p\} = q + r \text{ where } q,\ r \text{ are two functions such that} \\ \text{i) there exists } C>0 \text{ such that } |q(x,\xi)| \le C|p(x,\xi)|\cdot|\xi|^{m-1} \text{ in } V^+ \times \mathbf{R}^n \\ \text{ii) there exist } \varepsilon \in]0,1[\text{ and } \mu \in L^\infty(V^+ \times \mathbf{R}^n) \text{ with } \mu(x,\xi) \le 1-\varepsilon \\ \quad\ \text{and } \varphi(x) r(x,\xi) = \mu(x,\xi)(\operatorname{Im} \bar{p}\{p,\varphi\})(x,\xi) \text{ in } V^+ \times \mathbf{R}^n . \end{cases}$$

Then we can state

Theorem 2.1. *Assume that the operator P satisfies* (H.1) *and* (H.2). *Then if u is a C^∞ function in V such that*

$$\text{(2.1)}\qquad |Pu(x)| \le C \sum_{|\alpha|\le m-1} \frac{|D^\alpha u(x)|}{(\varphi(x))^{m-|\alpha|}} \text{ in } V^+ ,$$

$$\text{(2.2)}\qquad \operatorname{supp} u \subset \{x \in V : \varphi(x) \ge 0\} \quad \textit{and} \quad \operatorname{supp} u \cap S \textit{ is compact},$$

there exists a neighborhood W of x_0 in which u vanishes.

Remarks 2.2. a) In fact Theorem A still holds under the slightly more general condition

$$\text{(H.2) ii)}'\quad \begin{cases} \text{there exist } \varepsilon \in]0,1[,\ \mu \in L^\infty(V^+ \times \mathbf{R}^n) \quad \text{and } \nu > 0 \\ \text{such that } \mu(x,\xi) + \nu \le 1 - \varepsilon \quad \text{and} \\ \varphi(x) r(x,\xi) = \mu(x,\xi) \operatorname{Im} \bar{p}\{p,\varphi\}(x,\xi) + s(x,\xi) \\ \text{with } |s(x,\xi)| \le \nu |p(x,\xi)|\cdot|\{p,\varphi\}(x,\xi)| . \end{cases}$$

b) If $n = 2$ one can remove in Theorem 2.1 the assumption that $\operatorname{supp} u \cap S$ is compact which means that we have true uniqueness. This follows from the fact that, according to Remark a), the condition (H.2) is invariant under the singular change of coordinates $t = T(\delta - |X|^2)$, $x = X$ (when S is given by $\{t = 0\}$) which convexify the support of u. The invariance mentioned above is a consequence of the inequality $\left|\operatorname{Im} \bar{p} \dfrac{\partial p}{\partial \xi}\right| \le C|p| \left|\dfrac{\partial p}{\partial \tau}\right|$ which holds for any non characteristic symbol in two dimensions.

c) We note that the condition (H.2) depends on the surface S but not on the function φ.

d) In the case of the operator considered in [CD],

$$P = D_t^2 - D_1^2 - D_2^2 + i((D_1 + tD_2)^2 + \alpha t^2 D_2^2)$$

and $S = \{t = 0\}$, the condition (H.2) ii) is equivalent to $\alpha > \frac{1}{3}$ (and we take $q \equiv 0$). Another example for which Theorem 2.1 applies is $P = D_t + it^k D_x$, $k \in \mathbf{N}$.

e) If there are tangent bicharacteristics, the condition (H.1) should be replaced by a pseudo-convexity condition. However the method used here does not allow such an extension.

We show now that the condition (H.2) is in some sense necessary for uniqueness.

Let us consider in a neighborhood of the origin in $\mathbf{R}_x^{n-1} \times \mathbf{R}_t^+$ a differential operator $P = D_t^m + \sum_{\substack{|\alpha|+j\leq m \\ j<m}} a_{\alpha j}(x, t) D_x^\alpha D_t^j$ with principal symbol p.

According to the hypothesis (H.1) we shall assume that p has a smooth simple characteristic which means that in a conic neighborhood $V \times \Gamma$ of a point $(0, 0; \xi_0, \tau_0)$ we can write

$$p(x, t; \xi, \tau) = (\tau - \lambda(x, t, \xi)) q(x, t; \xi, \tau)$$

where λ, q are C^∞ functions and $q(x, t; \xi, \tau) \neq 0$ in $V \times \Gamma$. We shall set $\lambda = \lambda_1 + i\lambda_2$ with λ_1 and λ_2 real.

Then we can state

Theorem 2.3. *Let U be a neighborhood of the origin in $\mathbf{R}^{n-1}$ and $\delta_0 \in]0, 1[$. Assume that one can find a function ξ in $C^\infty(U \times [0, \delta_0], \mathbf{R})$ with $\nabla_x \xi(0, 0) = \xi_0$, such that*

$$\text{(C.1)} \qquad \begin{cases} \text{there exist } \varepsilon > 0 \text{ and } h \in L^\infty(U \times [0, \delta_0]) \text{ such that} \\ h(x, t) \geq 1 + \varepsilon \text{ and, with } \zeta = (x, t, \nabla_x \xi(x, t), \lambda_1(x, t; \nabla_x \xi(x, t)), \\ \dfrac{t}{2i}\{\bar{p}, p\}(\zeta) = h(x, t)(\operatorname{Im} \bar{p}\{p, t\})(\zeta) . \end{cases}$$

We assume moreover that there exists an integer $M \geq 1$ such that, for $|\alpha| + j \leq M - 1$ and (x, t) in $U \times [0, \delta_0]$, one has

$$\text{(C.2)} \qquad (\partial_\xi^\alpha \partial_t^j \lambda_2)(x, t; \nabla_x \xi(x, t)) = t^{M-|\alpha|-j} g_{\alpha j}(x, t) ,$$

where $g_{\alpha j}$ are C^∞ and $g_{00}(x, t) \neq 0$.

Then there exist two C^∞ functions u and a, vanishing for $t \leq 0$, such that $Pu + au = 0$ near the origin and $(0, 0) \in \operatorname{supp} u$.

Remark 2.4. This result can be applied to the example given in Remark 2.2 d) if $\alpha \in [0, \frac{1}{3}[$ by taking $\xi(x, t) = tx_1 - \dfrac{2x_2}{3(1 + \alpha)}$.

3. Proof of Theorem 2.1

As usual one can assume that $x_0 = 0$, $S \cap V = \{(x, t) \in \mathbf{R}^{n-1} \times \mathbf{R}: t = 0\}$. The uniqueness will follow from a Carleman estimate. We shall prove

Theorem 3.1. *There exist positive constants C, γ_0 and a neighborhood W of x_0 such that*

$$\text{(3.1)} \qquad \sum_{|\alpha|\leq m-1} \gamma^{m-|\alpha|-1/2} \| t^{-\gamma-1/2+|\alpha|} D^\alpha u \|_{L^2} \leq C \| t^{-\gamma-1/2+m} Pu \|_{L^2} ,$$

for $\gamma \geq \gamma_0$ *and* $u \in C_0^\infty(W)$, $\operatorname{supp} u \subset \{(x, t): t \geq 0\}$.

Proof of Theorem 3.1. We set $u = t^\gamma v$. Then

$$P(x, t; D_x, D_t)u = t^\gamma P(x, t; D_x, D_t - i\gamma t^{-1})v .$$

Since P is homogeneous we get

$$t^m P(x, t; D_x, D_t) = t^\gamma P(x, t; tD_x, tD_t - i\gamma) + t^\gamma R_{m-1}(x, t; tD_x, tD_t - i\gamma) ,$$

where R_{m-1} is a symbol of order $\leq m - 1$.

In it easy to see that (3.1) will follow from

$$\sum_{|\alpha| \leq m-1} \gamma^{m-|\alpha|-1/2} \| t^{-1/2+|\alpha|} D^\alpha v \|_{L^2} \leq C \| t^{-1/2} P(x, t; tD_x, tD_t - i\gamma)v \|_{L^2} . \tag{3.2}$$

In the L^2 norms which appear in (3.2) we set $t = e^y$. This is a diffeomorphism from $]0, 1[$ to $]-\infty, e[$. Let us set $w(x, y) = v(x, e^y)$. Then w is rapidly decreasing when $y \to -\infty$. A straightforward computation shows that (3.2) will follow from

$$\sum_{|\alpha|+k \leq m-1} \gamma^{m-|\alpha|-k-1/2} \| (e^y D_x)^\alpha D_y^k w \|_{L^2} \leq C \| P(x, e^y; e^y D_x, D_y - i\gamma)w \|_{L^2} \tag{3.3}$$

if γ is large enough.

We set

$$P_\gamma = P(x, e^y; e^y D_x, D_y - i\gamma) = \sum_{|\beta|+k=m} a_{\beta k}(x, e^y) e^{y|\beta|} D_x^\beta (D_y - i\gamma)^k .$$

The usual symbol of P_γ is $p(x, e^y; e^y\xi, \eta - i\gamma)$. Following Lerner [L] we shall denote by R_γ the operator whose Weyl symbol is $p(x, e^y; e^y\xi, \eta - i\gamma)$. Then $R_\gamma - P_\gamma$ is a differential operator whose usual symbol is

$$b(x, y, \xi, \eta, \gamma) = \sum_{\substack{|\alpha| \leq m \\ \alpha \neq 0}} \frac{1}{\alpha!} \left(\frac{i}{2}\right)^{|\alpha|} D_X^\alpha D_\Xi^\alpha [p(x, e^y; e^y\xi, \eta - i\gamma)]$$

where $X = (x, y)$, $\Xi = (\xi, \eta)$.

If follows from Proposition A.1 in the Appendix that

$$\| (P_\gamma - R_\gamma)w \|_{L^2} \leq C \sum_{|\beta|+k \leq m-1} \gamma^{m-1-|\beta|-k} \| (e^y D_x)^\beta D_y^k w \|_{L^2} . \tag{3.4}$$

It follows that (3.3) will be a consequence of

$$\sum_{|\beta|+k \leq m-1} \gamma^{m-|\beta|-k-1/2} \| (e^y D_x)^\beta D_y^k w \|_{L^2} \leq C \| R_\gamma w \|_{L^2} . \tag{3.5}$$

Now $\| R_\gamma w \|^2 = (R_\gamma^* R_\gamma w, w)$ and the advantage to work with R_γ instead of P_γ is that the Weyl symbol of R_γ^* is simply $\overline{p(x, e^y, e^y\xi, \eta - i\gamma)}$.

To compute the Weyl symbol of $R_\gamma^* R_\gamma$ let us recall the following formula

$$\sigma^w(A \circ B) = \sum_{\alpha, \beta} \frac{1}{\alpha!} \frac{1}{\beta!} \left(\frac{1}{2}\right)^{|\alpha|} \left(-\frac{1}{2}\right)^{|\beta|} D_\xi^\alpha \partial_x^\beta a D_\xi^\beta \partial_x^\alpha b \tag{3.6}$$

if a and b are the Weyl symbols of the differential operators A and B. Here we take $a = \overline{p(\zeta)}$, $b = p(\zeta)$ where $\zeta = (x, e^y, e^y\xi, \eta - i\gamma)$. The term in (3.6) with $\alpha = \beta = 0$ is equal to $|p(\zeta)|^2$ and that with $|\alpha| + |\beta| = 1$ can be written as

$$\frac{e^y}{2i}\sum_j\left(\overline{\frac{\partial p}{\partial \xi_j}(\zeta)}\frac{\partial p}{\partial x_j}(\zeta) - \overline{\frac{\partial p}{\partial x_j}(\zeta)}\frac{\partial p}{\partial \xi_j}(\zeta)\right) + \frac{e^y}{2i}\left(\overline{\frac{\partial p}{\partial \tau}(\zeta)}\frac{\partial p}{\partial t}(\zeta) - \overline{\frac{\partial p}{\partial t}(\zeta)}\frac{\partial p}{\partial \tau}(\zeta)\right)$$
$$+ \frac{1}{2i}\underbrace{\left(\overline{\frac{\partial p}{\partial \tau}(\zeta)}\sum_j e^y\xi_j\frac{\partial p}{\partial \xi_j}(\zeta) - \frac{\partial p}{\partial \tau}(\zeta)\sum_j e^y\xi_j\overline{\frac{\partial p}{\partial \xi_j}(\zeta)}\right)}_{(1)}.$$

Now since p is homogeneous of degree m we have

$$\sum_j e^y\xi_j\frac{\partial p}{\partial \xi_j}(\zeta) + (\eta - i\gamma)\frac{\partial p}{\partial \tau}(\zeta) = mp(\zeta).$$

It follows that

$$(1) = \frac{1}{2i}\left[\frac{\overline{\partial p}}{\partial \tau}mp - \frac{\overline{\partial p}}{\partial \tau}(\eta - i\gamma)\frac{\partial p}{\partial \tau} - \frac{\partial p}{\partial \tau}m\bar{p} + \frac{\partial p}{\partial \tau}(\eta + i\gamma)\frac{\overline{\partial p}}{\partial \tau}\right](\zeta).$$

$$(1) = \frac{m}{2i}\left(p\cdot\frac{\overline{\partial p}}{\partial \tau} - \bar{p}\cdot\frac{\partial p}{\partial \tau}\right)(\zeta) + \gamma\left|\frac{\partial p}{\partial \tau}(\zeta)\right|^2.$$

Then if we set $X = (x, t)$, $\Xi = (\xi, \tau)$ we find that the Weyl symbol of $R_\gamma^* R_\gamma$ is equal to

$$\sigma^w(R_\gamma^* R_\gamma) = |p(\zeta)|^2 + e^y \operatorname{Im}\overline{\frac{\partial p}{\partial \Xi}(\zeta)}\cdot\frac{\partial p}{\partial X}(\zeta) + \gamma\left|\frac{\partial p}{\partial \tau}(\zeta)\right|^2 + m \operatorname{Im}\overline{\frac{\partial p}{\partial \tau}(\zeta)}\cdot p(\zeta) + S \tag{3.7}$$

where

$$S = \sum_{|\alpha|+|\beta|\geq 2}\frac{1}{\alpha!}\frac{1}{\beta!}\left(\frac{1}{2}\right)^{\alpha}\left(-\frac{1}{2}\right)^{|\beta|} D^{\alpha}_{(\xi,\eta)}\partial^{\beta}_{(x,y)}[\overline{p(\zeta)}]D^{\beta}_{(\xi,\eta)}\partial^{\alpha}_{(x,y)}[p(\zeta)]. \tag{3.8}$$

(Note that this is a finite sum.)

The main part of the proof of Theorem 3.1 is then:

Lemma 3.2. *Assume that p satisfies* (H.1) *and* (H.2). *Then there exists a positive constant c_0 such that for $|x| + e^y \leq c_0$ and $\gamma \geq e^{c_0}$*

$$\sigma^w(R_\gamma^* R_\gamma) \geq c_0\gamma(e^y|\xi| + |\eta| + \gamma)^{2m-2}, \qquad \forall(\xi, \eta) \in \mathbf{R}^{n-1} \times \mathbf{R}. \tag{3.9}$$

Proof. It is first easy to see that one can find a positive contact C independent of γ such that if $\gamma \geq 1$

$$|S| \leq C(e^y|\xi| + |\eta| + \gamma)^{2m-2},\ \forall y \in]-\infty, e[,\ \forall(\xi, \eta) \in \mathbf{R}^{n-1} \times \mathbf{R}. \tag{3.10}$$

It follows from (3.7) that (3.9) will be a consequence of

$$|p(\zeta)|^2 + e^y \operatorname{Im} \overline{\frac{\partial p}{\partial \Xi}(\zeta)} \frac{\partial p}{\partial X}(\zeta) + \gamma \left| \frac{\partial p}{\partial \tau}(\zeta) \right|^2 + m \operatorname{Im} \overline{\frac{\partial p}{\partial \tau}(\zeta)} \cdot p(\zeta) \tag{3.11}$$

$$\geq c_1 \gamma (e^y |\xi| + |\eta| + \gamma)^{2m-2}$$

if γ is large enough. Recall that $\zeta = (x, e^y; e^y \xi, \eta - i\gamma)$, $\Xi = (\xi, \tau)$ and $X = (x, t)$. Since $e^y \xi$ appears in ζ and in the right hand side of (3.11), this inequality will follow from the following claim:

there exists c_0 such that for $|x| + t \leq c_0$, $\gamma \geq e^{c_0}$, $(\xi, \eta) \in \mathbf{R}^{n-1} \times \mathbf{R}$,

$$|p(Z)|^2 + t \operatorname{Im} \overline{\frac{\partial p}{\partial \Xi}(Z)} \frac{\partial p}{\partial X}(Z) + \gamma \left| \frac{\partial p}{\partial \tau}(Z) \right|^2 + m \operatorname{Im} \overline{\frac{\partial p}{\partial \tau}(Z)} p(Z) \tag{3.12}$$

$$\geq c_0 \gamma (|\xi|^2 + |\eta|^2 + \gamma^2)^{m-1}$$

where $Z = (x, t; \xi, \eta - i\gamma)$.

Let us set $A^2 = |\xi|^2 + |\eta|^2 + \gamma^2$ and

$$\Gamma = \frac{\gamma}{A}, \ \tilde{\xi} = \frac{\xi}{A}, \ \tilde{\eta} = \frac{\eta}{A}, \ \tilde{Z} = (x, t, \tilde{\xi}, \tilde{\eta} - i\Gamma) . \tag{3.13}$$

Dividing both sides of (3.12) by $A^{2(m-1)}$ and using the homogeneity, we find that (3.12) is equivalent to

$$\gamma \Gamma^{-2} |p(\tilde{Z})|^2 + t\Gamma^{-1} \operatorname{Im} \overline{p'_\Xi(\tilde{Z})} p'_X(\tilde{Z}) + \left| \frac{\partial p}{\partial \tau}(\tilde{Z}) \right|^2 + m\Gamma^{-1} \operatorname{Im} \overline{p'_\tau(\tilde{Z})} p(\tilde{Z}) \geq c_0 . \tag{3.14}$$

We prove (3.14) by contradiction. If this claim is not true one can find sequences (x_k), (t_k), $(\tilde{\xi}_k)$, $(\tilde{\eta}_k)$, (γ_k), (Γ_k) such that

$$|x_k| + t_k \leq \frac{1}{k}, \ |\tilde{\xi}_k|^2 + |\tilde{\eta}_k|^2 + \Gamma_k^2 = 1, \ \gamma_k \geq e^k \tag{3.15}$$

$$\gamma_k \Gamma_k^{-2} \underbrace{|p(\tilde{Z}_k)|^2}_{(1)} + \underbrace{t_k \Gamma_k^{-1} \operatorname{Im} \overline{p'_\Xi(\tilde{Z}_k)} p'_X(\tilde{Z}_k)}_{(2)} + \underbrace{\left| \frac{\partial p}{\partial \tau}(\tilde{Z}_k) \right|^2}_{(3)} \tag{3.16}$$

$$\underbrace{+ m\Gamma_k^{-1} \operatorname{Im} \overline{p'_\tau(\tilde{Z}_k)} p(\tilde{Z}_k) \leq \frac{1}{k}}_{(4)} ,$$

where $\tilde{Z}_k = (x_k, t_k, \tilde{\xi}_k, \tilde{\eta}_k - i\Gamma_k)$.

Taking subsequences one may assume that

$$\tilde{\xi}_k \to \xi_0, \ \eta_k \to \eta_0, \ \Gamma_k \to \Gamma_0 \quad \text{as } k \to +\infty, \quad \text{with } |\xi_0|^2 + \eta_0^2 + \Gamma_0^2 = 1 . \tag{3.17}$$

Let us note that for every continuous function F of $(x, t, \xi, \eta, \gamma)$ the quantity $F(\tilde{Z}_k)$ is bounded by a constant independent of k (by (3.15)). We shall write $F(\tilde{Z}_k) = O(1)$.

Case 1: $\Gamma_0 \neq 0$

Then $t_k\Gamma_k^{-1}$ Im $\overline{p'_\Xi(\tilde{Z}_k)}p'_X(\tilde{Z}_k) = t_k\Gamma_k^{-1}O(1) \to 0$ as $k \to +\infty$. Moreover

$$|m\Gamma_k^{-1} \operatorname{Im} \overline{p'_\tau(\tilde{Z}_k)}p(\tilde{Z}_k)| \leq \frac{1}{4}\gamma_k\Gamma_k^{-2}|p(\tilde{Z}_k)|^2 + c\gamma_k^{-1}\left|\frac{\partial p}{\partial \tau}(\tilde{Z}_k)\right|^2 . \tag{3.18}$$

It follows from (3.16) that

$$\frac{1}{2}\gamma_k\Gamma_k^{-2}|p(\tilde{Z}_k)|^2 + (1 - c\gamma_k^{-1})\left|\frac{\partial p}{\partial \tau}(\tilde{Z}_k)\right|^2 \leq \frac{1}{k} + t_k\Gamma_k^{-1}O(1) .$$

Then $|p(\tilde{Z}_k)|$ and $\left|\frac{\partial p}{\partial \tau}(\tilde{Z}_k)\right|$ tend to zero. By (3.17) we get $p(0, 0; \xi_0, \eta_0 - i\Gamma_0) = \frac{\partial p}{\partial \tau}(0, 0; \xi_0, \eta_0 - i\Gamma_0) = 0$ with $\Gamma_0 \neq 0$. This contradicts (H.1).

Case 2: $\Gamma_0 = 0$

We can write in (3.16), $(2) = t_k\Gamma_k^{-1}F(\Gamma_k)$ with

$$F(\Gamma_k) = \operatorname{Im} \overline{p'_\Xi(\tilde{Z}_k)}p'_X(\tilde{Z}_k) = F(0) + \Gamma_k O(1) .$$

Now $F(0) = \frac{1}{2i}\{\bar{p}, p\}(m_k)$ where $m_k = (x_k, t_k, \tilde{\xi}_k, \tilde{\eta}_k)$. It follows that

$$(2) = t_k\Gamma_k^{-1}\frac{1}{2i}\{\bar{p}, p\}(m_k) + t_k O(1) .$$

Thanks to (H.2) we can write

$$(2) = \underbrace{t_k\Gamma_k^{-1}q(m_k)}_{(5)} + \underbrace{t_k\Gamma_k^{-1}r(m_k)}_{(6)} + t_k O(1) \tag{3.19}$$

and

$$|(5)| \leq ct_k\Gamma_k^{-1}|p(m_k)| .$$

Since $p(m_k) = p(\tilde{Z}_k) + \Gamma_k O(1)$ we get

$$|(5)| \leq \frac{1}{4}\Gamma_k^{-2}\gamma_k|p(\tilde{Z}_k)|^2 + c'\gamma_k^{-1}t_k + t_k^2 O(1) . \tag{3.20}$$

Let us consider the term (6) in (3.19). By (H.2) ii) we have

$$(6) = \Gamma_k^{-1}\mu(m_k) \operatorname{Im} \bar{p}(m_k)\frac{\partial p}{\partial \tau}(m_k) .$$

Now $p(m_k) = p(\tilde{Z}_k) + i\Gamma_k\frac{\partial p}{\partial \tau}(\tilde{Z}_k) + \Gamma_k^2 O(1)$, and $\frac{\partial p}{\partial \tau}(m_k) = \frac{\partial p}{\partial \tau}(\tilde{Z}_k) + \Gamma_k O(1)$. Therefore

$$(3.21)\qquad \begin{cases} (6) = -\mu(m_k)\left|\dfrac{\partial p}{\partial \tau}(\tilde{Z}_k)\right|^2 + (7) \\ |(7)| \le \dfrac{1}{4}\Gamma_k^{-2}\gamma_k|p(\tilde{Z}_k)|^2 + \Gamma_k O(1) + c\gamma_k^{-1}O(1)\,. \end{cases}$$

It follows from (3.19), (3.20) and (3.21) that

$$(3.22)\qquad (2) = -\mu(m_k)\left|\frac{\partial p}{\partial \tau}(\tilde{Z}_k)\right|^2 + \frac{1}{2}\Gamma_k^{-2}\gamma_k|p(\tilde{Z}_k)|^2 + \varepsilon_k,\ \varepsilon_k \to 0\,.$$

Using (3.16), (3.18), (3.22) and (H.2) ii) we get

$$\frac{1}{4}\gamma_k\Gamma_k^{-2}|p(\tilde{Z}_k)|^2 + \varepsilon\left|\frac{\partial p}{\partial \tau}(\tilde{Z}_k)\right|^2 \le \varepsilon_k',\ \varepsilon_k' \to 0\,.$$

Letting k go to $+\infty$ we get

$$p(0, 0; \xi_0, \eta_0) = \frac{\partial p}{\partial \tau}(0, 0; \xi_0, \eta_0) = 0$$

with, by (3.17), $|\xi_0|^2 + \eta_0^2 = 1$. This contradicts (H.1). The proof is complete.

End of the proof of Theorem 3.1. Let $\theta \in C^\infty(\mathbf{R})$ be such that, with c_0 defined in Lemma 3.2,

$$(3.23)\qquad \theta(y) = \begin{cases} e^y & \text{if } e^y \le c_0 \\ 2c_0 & \text{if } e^y \ge 2c_0 \end{cases},\qquad 0 < \theta(y) \le 2c_0\,.$$

Let $\chi \in C^\infty$ be such that, $0 \le \chi \le 1$ and

$$(3.24)\qquad \chi(x, y) = \begin{cases} 1 & \text{if } |x| + e^y \le \dfrac{c_0}{2} \\ 0 & \text{if } |x| + e^y \ge c_0\,. \end{cases}$$

Let us set $\Phi^2 = \theta^2(y)|\xi|^2 + |\eta|^2 + \gamma^2$. It is easy to see that

$$(3.25)\qquad |\sigma^w(R_\gamma^* R_\gamma)| \le M\Phi^{2m},\ |x| + e^y \le c_0,\ \gamma \ge e^{c_0}\,.$$

Let us consider the following symbol in $\mathbf{R}^{n-1} \times \mathbf{R} \times \mathbf{R}^{n-1} \times \mathbf{R}$,

$$(3.26)\qquad q = \chi\sigma^w(R_\gamma^* R_\gamma) + (1 - \chi)M(\theta^2|\xi|^2 + \eta^2 + \gamma^2)^m\,.$$

We shall show that, with the notation in Appendix

$$(3.27)\qquad \begin{cases} q \in S(\Phi^{2m}, g) \\ q \ge c\gamma\Phi^{2m-2} \end{cases} \quad \text{for all } (x, y, \xi, \eta) \text{ in } \mathbf{R}^{n-1} \times \mathbf{R} \times \mathbf{R}^{n-1} \times \mathbf{R},\ \gamma \ge e^{c_0}\,.$$

Indeed if $|x| + e^y \le c_0$ we write

$$q = \sigma^w(R_\gamma^* R_\gamma) + (1 - \chi)[M(\theta^2|\xi|^2 + \eta^2 + \gamma^2)^m - \sigma^w(R_\gamma^* R_\gamma)]\,,$$

therefore, by Lemma 3.2 and (3.25) we have $q \geq c_0 \gamma \Phi^{2m-2}$ in this region. If $|x| + e^y > c_0$ then $\chi = 0$ so $q = M(\theta^2 |\xi|^2 + \eta^2 + \gamma^2)^m \geq M\gamma\Phi^{2m-2}$. This proves (3.27).

It follows from (3.27) and Proposition A.1 in the Appendix that $(q_\psi^w u, u) \geq c\gamma \|u\|_{m-1}^2$ for every u in $C_0^\infty(\mathbf{R}^{n-1} \times \mathbf{R})$. Now, since q^w is a differential operator, we have $q_\psi^w = q^w$ and if u has its support in $|x| + e^y \leq \frac{1}{2}c_0$, where $\chi = 1$, we get $\|R_\gamma u\|_{L_2}^2 \geq c\gamma \|u\|_{m-1}^2$. This proves (3.5) and completes the proof of Theorem 3.1.

4. Proof of Theorem 2.3

The proof of Theorem 2.3 is inspired by that one of Theorem 1 in [A].

First of all we show that, without any restriction, we can always suppose that for all $(x, t) \in U \times [0, \delta_0]$

$$g_{0,0}(x, t) \leq -c' < 0, \tag{4.1}$$

for some $c' > 0$. In fact if $g_{0,0}(x, t) \geq c > 0$ we consider the point $(0, 0, -\xi_0, -\tau_0)$ and we obtain (4.1) together with the conditions (C.1) and (C.2) for $\tilde{\lambda}$ and $\tilde{\xi}$, where $\tilde{\lambda}(x, t; \xi) = -\lambda(x, t; -\xi)$ and $\tilde{\xi}(x, t) = -\xi(x, t)$. Moreover we remark that, eventually taking smaller U and δ_0, there exists $c'' > 0$ such that, for all $(x, t) \in U \times [0, \delta_0]$,

$$\lambda_2(x, t; \nabla_x \xi(x, t)) + t(\partial_t \lambda_2 + \{\lambda_2, \lambda_1\})(x, t; \nabla_x \xi(x, t)) \geq t^M c''. \tag{4.2}$$

In fact

$$\frac{1}{2i}\{\bar{p}, p\} = (-\partial_t \lambda_2 - \{\lambda_2, \lambda_1\})|q|^2 + (\tau - \lambda_1)g_1 + \lambda_2 g_2$$

where g_1 and g_2 are C^∞ functions, and

$$\operatorname{Im}(\bar{p}\{p, t\}) = \lambda_2 |q|^2 + |\tau - \lambda|^2 \operatorname{Im}(\bar{q}\{q, t\}).$$

Setting $\zeta = (x, t; \nabla_x \xi(x, t), \lambda_1(x, t, \nabla_x \xi(x, t))$, by the condition (C.1) we have

$$\begin{aligned} &t(-\partial_t \lambda_2 - \{\lambda_2, \lambda_1\})(x, t, \nabla_x \xi(x, t)) \\ &\quad = \lambda_2(x, t, \nabla_x \xi(x, t)) h(x, t) \left[1 + \frac{(\lambda_2 \operatorname{Im}(\bar{q}\{q, t\}))}{|q|^2}(\zeta) - t\left(\frac{g_2}{|q|^2}\right)(\zeta) \right], \end{aligned}$$

and consequently

$$(\lambda_2 + t(\partial_t \lambda_2 + \{\lambda_2, \lambda_1\}))(x, t; \nabla_x \xi(x, t)) = -\lambda_2(x, t; \nabla_x \xi(x, t)) \cdot l(x, t)$$

where $l \in L^\infty$ and $l(x, t) \geq \frac{\varepsilon}{2}$ in $U' \times [0, \delta_0']$ with $U' \subseteq U$ and $\delta_0' \in]0, \delta_0]$. Using now (C.2) and (4.1) we obtain (4.2).

Construction of the phase φ

Lemma 4.1. *There exist a neighborhood U of the origin in $\mathbf{R}^{n-1}$, two positive constants s_0, $\delta_0 < 1$, and a function φ in $C^\infty(U \times [-s_0, s_0] \times [0, \delta_0], \mathbf{C})$ such that*

$$p\left(x, \delta + s\delta, \nabla_x\xi(x, \delta) + \frac{\delta}{i}\nabla_x\varphi(x, s, \delta), \frac{1}{i}\partial_s\varphi(x, s, \delta)\right) = O(s^\infty) \tag{4.3}$$

where $O(s^\infty)$ means C^∞ function vanishing together with all its derivatives for $s = 0$. Moreover

$$\varphi(x, 0, \delta) = 0 \quad \text{and} \quad \partial_x\varphi(x, 0, \delta) = i\lambda(x, \delta; \nabla_x\xi(x, \delta))\,; \tag{4.4}$$

$$\operatorname{Re}\varphi(x, s, \delta) = \delta^M\left[A(x, \delta)s + B(x, \delta)\frac{s^2}{2} + R(x, s, \delta)s^3\right] \tag{4.5}$$

where A, B, $R \in C^\infty$, and for all $(x, \delta) \in U \times [0, \delta_0]$,

$$A(x, \delta) \geq c' > 0,\ A(x, \delta) + B(x, \delta) \leq -\frac{c''}{2} < 0\,. \tag{4.6}$$

Proof. We want to solve formally the following problem

$$\begin{cases}\partial_s\psi(x, s, \delta) = \lambda(x, \delta + s\delta; \nabla_x\xi(x, \delta) + \delta\nabla_x\psi(x, s, \delta)),\ |s| \leq s_0\\ \psi(x, 0, \delta) = 0\,.\end{cases} \tag{4.7}$$

As a necessary condition we determine all the derivatives in s of ψ for $s = 0$, using (4.7). We have

$$\partial_s\psi(x, 0, \delta) = \lambda(x, \delta; \nabla_x\xi(x, \delta))$$

and

(4.8)
$$\partial_s^2\psi(x, 0, \delta) = \delta\partial_t\lambda(x, \delta; \nabla_x\xi(x, \delta))$$
$$+ \delta\sum_{j=1}^{n-1}\partial_{\xi_j}\lambda(x, \delta; \nabla_x\xi(x, \delta))\left[\partial_{x_j}\lambda + \sum_{k=1}^{n-1}\partial_{x_k}\lambda\partial_{x_j}\partial_{\xi_k}\xi\right](x, \delta; \nabla_x\xi(x, \delta))\,.$$

We prove now that we have

$$\begin{cases}\partial_s^k\psi(x, 0, \delta) = \delta^{k-1}G_k(x, \delta), & k \geq 1\,,\\ \partial_s^k \operatorname{Im}\psi(x, 0, \delta) = \delta^M H_k(x, \delta), & k \geq 1\,,\end{cases} \tag{4.9}$$

where G_k and H_k are smooth functions.

Indeed setting $\theta = \delta\psi$ we show easily by induction on k, using (4.7), that

$$\partial_s^k\theta(x, s, \delta) = \delta^k M_k(x, \delta, s\delta, (\partial_x^\alpha\theta)_{|\alpha| \leq k})$$

$$\partial_s^k \operatorname{Im}\theta(x, s, \delta) = \delta^k\sum_{|\alpha| \leq k-1} a_{\alpha k}(x, s\delta, (\partial_x^\beta\theta)_{|\beta| \leq k})\partial_{(t,\xi)}^\alpha\lambda_2(x, \delta + s\delta, \nabla_x\xi(x, \delta) + \nabla_x\theta)$$

where M_k, $a_{\alpha k}$ are smooth functions. Now (4.9) follows from the hypothesis (C.2).

From (4.9) we deduce that there exist two sequences of real valued C^∞ functions $(h_{jk})_k$, $j = 1, 2$, such that

$$\partial_s^k \psi(x, 0, \delta) = \delta^{k-1}(h_{1k}(x, \delta) + i\delta^{\max(0, M-k+1)} h_{2k}(x, \delta)).$$

We construct now a solution of (4.7) as usual. We take a function $\chi \in C_0^\infty(\mathbf{R})$, $0 \le \chi \le 1$, $\chi(u) = 1$ for $|u| \le 1$, and we set

$$\tilde{\psi}(x, s, \delta) = \sum_{k=1}^{+\infty} [\delta^{k-1} h_{1k}(x, \delta) + i\delta^{\max(0, M-k+1)} h_{2k}(x, \delta)] \chi(\lambda_k s) \frac{s^k}{k!}$$

where (λ_k) is an appropriate positive divergent real sequence. Then $\tilde{\psi}$ is a C^∞ solution of (4.7). Therefore $\varphi(x, s, \delta) = i\tilde{\psi}(x, s, \delta)$ is a solution of (4.3), (4.4) which satisfies (4.5), by (4.9).

Let us prove (4.6). Condition (C.2) gives

$$\delta^M A(x, \delta) = \partial_s(\operatorname{Re} \varphi)(x, 0, \delta) = -\lambda_2(x, \delta; \nabla_x \xi(x, \delta)) = -\delta^M g_{0,0}(x, \delta),$$

so that, by (4.1), $A(x, \delta) = -g_{0,0}(x, \delta) \ge c' > 0$.

From (4.8) we have

$$\delta^M B(x, \delta) = -\delta \left[\partial_t \lambda_2 + \sum_{j=1}^{n-1} \partial_{\xi_j} \lambda_2 \partial_{x_j} \lambda_1 + \partial_{\xi_j} \lambda_1 \partial_{x_j} \lambda_2 \right.$$
$$\left. + 2 \sum_{j,k=1}^{n-1} \partial_{\xi_j} \lambda_1 \partial_{\xi_k} \lambda_2 \partial_{x_j} \partial_{x_k} \xi \right] (x, \delta; \nabla_x \xi(x, \delta)).$$

Recalling that $\lambda_2(x, \delta; \nabla_x \xi(x, \delta)) = \delta^M g_{0,0}(x, \delta)$ we deduce

$$\partial_{x_j} \lambda_2(x, \delta; \nabla_x \xi(x, \delta)) + \sum_{k=1}^{n-1} \partial_{\xi_k} \lambda_2(x, \delta; \nabla_x \xi(x, \delta)) \partial_{x_j} \partial_{x_k} \xi(x, \delta) = \delta^M \partial_{x_j} q_{0,0}(x, \delta);$$

consequently

$$\left(\sum_{j,k=1}^{n-1} \partial_{\xi_j} \lambda_1 \partial_{\xi_k} \lambda_2 \partial_{x_j} \partial_{x_k} \xi \right) (x, \delta; \nabla_x \xi(x, \delta))$$
$$= \left(-\sum_{j=1}^{n-1} \partial_{\xi_j} \lambda_1 \partial_{x_j} \lambda_2 + \delta^M \sum_{j=1}^{n-1} \partial_{\xi_j} \lambda_1 \partial_{x_j} g_{0,0} \right) (x, \delta; \nabla_x \xi(x, \delta)).$$

So that

$$\delta^M B(x, \delta) = -\delta(\partial_t \lambda_2 + \{\lambda_2, \lambda_1\})(x, \delta; \nabla_x \xi(x, \delta))$$
$$- 2\delta^{M+1} \sum_{j=1}^{n-1} \partial_{\xi_j} \lambda_1(x, \delta; \nabla_x \xi(x, \delta)) \partial_{x_j} g_{0,0}(x, \delta)$$

and

$$A(x, \delta) + B(x, \delta) = \delta^{-M}[-\lambda_2 - \delta(\partial_t \lambda_2 + \{\lambda_2, \lambda_1\})](x, \delta; \nabla_x \xi(x, \delta))$$
$$- 2\delta \sum_{j=1}^{n-1} \partial_{\xi_j} \lambda_1(x, \delta; \nabla_x \xi(x, \delta)) \partial_{x_j} g_{0,0}(x, \delta),$$

and the condition (4.2) gives the conclusion if δ_0 is sufficiently small. This ends the proof of Lemma 4.1.

Construction of the function γ. We define the function $h\colon U \times [-s_0, s_0] \times]0, \delta_0] \to \mathbf{C}$ in the following way

$$h(x, s, \delta) = e^{-\gamma(x,\delta)} e^{i\nu\delta^{-M-1}\xi(x,\delta)} e^{\nu\delta^{-M}\varphi(x,s,\delta)} w(x, s, \delta)$$

where ν is a parameter, γ, ξ, φ, w are functions, with γ and w to be determined. We set $\delta = \delta_k = k^{-\rho}$, $\nu = \nu_k = k^{\sigma}$, with $\rho, \sigma > 0$. For $(x, t) \in U \times [\delta_{k+1}, \delta_{k-1}]$ we define

$$h_k(x, t) = e^{-\gamma(x,\delta_k)} e^{i\nu_k\delta_k^{-M-1}\xi(x,\delta_k)} e^{\nu_k\delta_k^{-M}\varphi(x,(t-\delta_k)/\delta_k,\delta_k)} w\left(x, \frac{t-\delta_k}{\delta_k}, \delta_k\right).$$

We construct now the function γ. For $t \in [\delta_{k+1}, \delta_{k-1}]$ we define

$$G_k(x, t) = \nu_k\delta_k^{-M}\operatorname{Re}\varphi\left(x, \frac{t-\delta_k}{\delta_k}, \delta_k\right) - \nu_{k+1}\delta_{k+1}^{-M}\operatorname{Re}\varphi\left(x, \frac{t-\delta_{k+1}}{\delta_{k+1}}, \delta_{k+1}\right).$$

Lemma 4.2. *We set* $m_k = \dfrac{\delta_k + \delta_{k+1}}{2}$ *and* $I_k(x) = G_k(x, m_k)$. *For* $k \to +\infty$ *we have*

$$I_k(x) = -\rho k^{\sigma-1} A(x, 0)(1 + o(1)).$$

Proof.

$$I_k(x) = \nu_k\left[\delta_k^{-M}\operatorname{Re}\varphi\left(x, \frac{\delta_{k+1}-\delta_k}{2\delta_k}, \delta_k\right)\right] - \nu_{k+1}\left[\delta_{k+1}^{-M}\operatorname{Re}\varphi\left(x, \frac{\delta_k-\delta_{k+1}}{2\delta_{k+1}}, \delta_{k+1}\right)\right]$$

$$= \nu_k A(x, \delta_k)\frac{\delta_{k+1}-\delta_k}{2\delta_k} + \nu_{k+1}A(x, \delta_{k+1})\frac{\delta_{k+1}-\delta_k}{2\delta_{k+1}} + O(k^{\sigma-2}).$$

But

$$\nu_k\frac{\delta_{k+1}-\delta_k}{2\delta_k} = \nu_{k+1}\frac{\delta_{k+1}-\delta_k}{2\delta_{k+1}}(1 + O(k^{-1})) = -\frac{1}{2}\rho k^{\sigma-1}(1 + O(k^{-1}))$$

and $A(x, \delta_k) = A(x, 0)(1 + O(k^{-\rho}))$.

Then $I_k(x) = -\rho k^{\sigma-1}A(x, 0)(1 + o(1))$. The proof is complete.

We set $\gamma_k(x) = -\sum_{j=k_0}^{k-1} I_j(x)$ and consequently, for $k \to +\infty$

$$\gamma_k(x) = \frac{\rho}{\sigma}k^{\sigma}A(x, 0)(1 + o(1)).$$

We choose a function $\tilde{\gamma} \in C^\infty$ such that $\delta_k^{-\sigma/\rho}\tilde{\gamma}(x, \delta_k) = \gamma_k(x)$ and we set $\gamma(x, \delta) = \delta^{-\sigma/\rho}\tilde{\gamma}(x, \delta)$.

Transport equation. We will call B^∞ the set of the functions which are C^∞ in x, s and in a positive rational power of δ ($\delta \in [0, \delta_0]$). By using a simple recurrence argument it is possible to show the following

Lemma 4.3. *Let $q(x, t; D_x, D_t)$ be a homogeneous operator of order k with C^∞ coefficients and*

$$\Phi(x, s, \delta) = -\gamma(x, \delta) + iv\delta^{-M-1}\xi(x, \delta) + v\delta^{-M}\varphi(x, s, \delta)\,.$$

Assume that $0 < \sigma < \rho$. Then there exists a function $g \in B^\infty$ such that

$$q\left(x, \delta + s\delta; \frac{\delta^{M+1}}{v} D_x, \frac{\delta^M}{v} D_s\right)e^\Phi = \left[q\left(x, \delta + s\delta; \frac{\delta^{M+1}}{v}\frac{1}{i}\nabla_x\Phi, \frac{\delta^M}{v} D_s\Phi\right) + \frac{\delta^M}{v} g\right]e^\Phi\,.$$

Remarking that $q\left(x, \delta + s\delta, \frac{\delta^{M+1}}{v}\frac{1}{i}\nabla_x\Phi, \frac{\delta^M}{v}D_s\Phi\right)$ is a B^∞ function, we deduce from Lemma 4.3 that $e^{-\Phi}q\left(x, \delta + s\delta; \frac{\delta^{M+1}}{v}D_x, \frac{\delta^M}{v}D_s\right)e^\Phi$ is a B^∞ function. Consequently if $w(x, s, \delta)$ is a B^∞ function, then by Leibniz formula $q\left(x, \delta + s\delta; \frac{\delta^{M+1}}{v}D_x, \frac{\delta^M}{v}D_s\right)e^\Phi w$ is a B^∞ function, in fact

$$(4.10)\quad q\left(x, \delta + s\delta, \frac{\delta^{M+1}}{v}D_x, \frac{\delta^M}{v}D_s\right)e^\Phi w$$
$$= \sum_{\alpha, j}\frac{1}{\alpha!}\frac{1}{j!}\frac{\delta^{(M+1)|\alpha|+Mj}}{v^{|\alpha|+j}}\left[q^{(\alpha,j)}\left(x, \delta + s\delta; \frac{\delta^{M+1}}{v}D_x, \frac{\delta^M}{v}D_s\right)e^\Phi\right]D_x^\alpha D_s^j w$$

where $q^{(\alpha,j)}(x, t; \xi, \tau) = \partial_\xi^\alpha\partial_\tau^j q(x, t; \xi, \tau)$.

Lemma 4.4. *Assume $0 < \sigma < \rho$. There exist $H(x, s)$, $K(x, s)$ C^∞ functions with $H(0, 0) \neq 0$, an operator $\tilde{Q}(x, s, \delta; D_x, D_s)$ with B^∞ coefficients, a positive rational number $\tilde{r} > 0$ and a function $F(x, s, \delta)$ which is $O(s^\infty)$* such that

$$e^{-\Phi}\left(\frac{\delta^{M+1}}{v}\right)^m P(x, \delta + s\delta, D_x, \delta^{-1}D_s)(e^\Phi w)$$
$$= \frac{\delta^M}{v}\left[H(x, s)D_s w + K(x, s)w + \delta^{\tilde{r}}\tilde{Q}(x, s, \delta; D_x, D_s)w\right] + F(x, s, \delta)w\,.$$

Proof. We have

$$\left(\frac{\delta^{M+1}}{v}\right)^m P(x, \delta + s\delta, D_x, \delta^{-1}D_s)$$
$$= p\left(x, \delta + s\delta, \frac{\delta^{M+1}}{v}D_x, \frac{\delta^M}{v}D_s\right)$$
$$+ \frac{\delta^M}{v}\left[\delta\cdot\sum_{k=1}^m\left(\frac{\delta^{M+1}}{v}\right)^{k-1}p_{m-k}\left(x, \delta + s\delta, \frac{\delta^{M+1}}{v}D_x, \frac{\delta^M}{v}D_s\right)\right].$$

By using (4.10) the terms coming from $\sum_{k=1}^{m}\left(\frac{\delta^{M+1}}{\nu}\right)^{k-1} p_{m-k}$ will be absorbed in $\tilde{Q}$. Again using (4.10) we have

$$
\begin{aligned}
p\left(x, \delta + s\delta; \frac{\delta^{M+1}}{\nu} D_x, \frac{\delta^M}{\nu} D_s\right)(e^{\Phi} w) = &\left[p\left(x, \delta + s\delta; \frac{\delta^{M+1}}{\nu} D_x, \frac{\delta^M}{\nu} D_s\right) e^{\Phi}\right] w \\
&+ \frac{\delta^M}{\nu}\left[\partial_\tau p\left(x, \delta + s\delta, \frac{\delta^{M+1}}{\nu} D_x, \frac{\delta^M}{\nu} D_s\right) e^{\Phi}\right] D_s w \\
&+ \frac{\delta^{M+1}}{\nu} \sum_{j=1}^{n-1} [\partial_{\xi_j} p(\ldots) e^{\Phi}] D_{x_j} w \\
&+ \sum_{|\alpha|+j \geq 2} \frac{\delta^{(M+1)|\alpha|+Mj}}{\nu^{|\alpha|+j}} [\partial_\xi^\alpha \partial_\tau p(\ldots) e^{\Phi}] D_x^\alpha D_s^j w\,.
\end{aligned}
$$

Then by Lemma 4.3 we deduce

$$
\begin{aligned}
p\left(x, \delta + s\delta, \frac{\delta^{M+1}}{\nu} D_x, \frac{\delta^M}{\nu} D_s\right)(e^{\Phi} w) = &\left[p\left(x, \delta + s\delta; \frac{\delta^{M+1}}{\nu} \frac{1}{i} \nabla_x \Phi, \frac{\delta^M}{\nu} D_s \Phi\right) w + \frac{\delta^M}{\nu} g w\right. \\
&+ \frac{\delta^M}{\nu} \partial_\tau p\left(x, \delta + s\delta; \frac{\delta^{M+1}}{\nu} \frac{1}{i} \nabla_x \Phi, \frac{\delta^M}{\nu} D_s \Phi\right) D_s w \\
&\left.+ \frac{\delta^{M+1}}{\nu} \tilde{Q}(x, s, \delta; D_x, D_s) w\right] e^{\Phi}\,,
\end{aligned}
$$

where $g \in B^\infty$. From Lemma 4.1 and the choice of γ we have

$$
\begin{aligned}
p\left(x, \delta + s\delta; \frac{\delta^{M+1}}{\nu} \frac{1}{i} \nabla_x \Phi, \frac{\delta^M}{\nu} D_s \Phi\right) &= p\left(x, \delta + s\delta, \frac{\delta^{M+1}}{\nu} i\nabla_x \gamma + \nabla_x \xi + \frac{1}{i} \nabla_x \varphi, \frac{1}{i} \partial_s \varphi\right) \\
&= O(s^\infty) + \frac{\delta^{M+1-(\sigma/\rho)}}{\nu} g
\end{aligned}
$$

where $g \in B^\infty$. Using a similar argument for $\partial_\tau p$ we obtain

$$
\begin{aligned}
&e^{-\Phi} P\left(x, \delta + s\delta; \frac{\delta^{M+1}}{\nu} D_x, \frac{\delta^M}{\nu} D_s\right)(e^{\Phi} w) \\
&= \frac{\delta^M}{\nu}\left[\partial_\tau p\left(x, \delta + s\delta; \nabla_x \xi + \frac{1}{i} \delta \nabla_x \varphi, \frac{1}{i} \partial_s \varphi\right) D_s w\right. \\
&\qquad \left.+ g(x, s, \delta) w + \delta^{\tilde{r}} \tilde{Q}(x, s, \delta; D_x, D_s) w\right] + F(x, s, \delta) w
\end{aligned}
$$

where g, $F \in B^\infty$, and F is $O(s^\infty)$. We conclude the proof of the Lemma with Taylor expansions for the functions $\partial_\tau p$ and g with respect to δ, recalling moreover that $\partial_\tau p(0, 0; \xi_0, \tau_0) \neq 0$.

We construct a sequence $(w_j(x, s, \delta))_j$ of B^∞ functions solving

$$\begin{cases} H(x, s)D_s w_0(x, s) + K(x, s)w_0(x, s) = 0 \\ w_0(x, 0) = 1 \end{cases}$$

and for $j \geq 1$

$$\begin{cases} H(x, s)D_s w_j(x, s, \delta) + K(x, s)w_j(x, s, \delta) = \tilde{Q}(x, s, \delta; D_x, D_s)w_{j-1}(x, s, \delta) \\ w_j(x, 0, \delta) = 0\,. \end{cases}$$

We consider then a function $z(x, s, y, \delta)$, C^∞ in x, s, y and in a positive fractional power of δ, such that for all $\alpha \in \mathbf{N}^{n-1}$, j, $N \in \mathbf{N}$, there exists $C_{\alpha,j,N} > 0$ such that

$$\left| D_x^\alpha D_s^j \left(z(x, s, y, \delta) - \sum_{k=0}^{N} w_k(x, s, \delta) y^k \right) \right| \leq C_{\alpha,j,N} |y|^{N+1}\,.$$

Finally we define

$$w(x, s, \delta) = z(x, s, \delta^{\bar{r}}, \delta)\,.$$

As a consequence of this definition we have that for all $\alpha \in \mathbf{N}^{n-1}$, j, $N \in \mathbf{N}$, there exists $C_{\alpha,j,N} > 0$ such that

$$|D_x^\alpha D_s^j((HD_s + K + \delta^{\bar{r}}\tilde{Q})w)| \leq C_{\alpha,j,N}\delta^N\,. \tag{4.11}$$

By using (4.11) and Lemma 4.4 it is possible to prove the following

Lemma 4.5. *For* $(x, t) \in U \times [\delta_{k+1}, \delta_{k-1}]$ *let us define*

$$r_k(x, t) = \frac{P(x, t; D_x, D_t)h_k(x, t)}{h_k(x, t)}\,.$$

Then there exists $k_0 \in \mathbf{N}$ *such that for all* $\alpha \in \mathbf{N}^{n-1}$, j, $N \in \mathbf{N}$ *there exists* $C_{\alpha,j,N} > 0$ *such that*

$$|D_x^\alpha D_t^j r_k(x, t)| \leq C_{\alpha,j,N} k^{-N} \tag{4.12}$$

for all $k \geq k_0$ *and for all* $(x, t) \in U \times [\delta_{k+1}, \delta_{k-1}]$.

The set where $|h_k| = |h_{k+1}|$. For $(x, t) \in U \times [\delta_{k+1}, \delta_k]$ we define

$$F_k(x, t) = \operatorname{Log} \frac{|h_k(x, t)|}{|h_{k+1}(x, t)|}\,.$$

We have

$$\partial_t F_k(x, t) = v_k \delta_k^{-M-1} \partial_s \operatorname{Re} \varphi\left(x, \frac{t - \delta_k}{\delta_k}, \delta_k\right) - v_{k+1} \delta^{-M-1} \partial_s \operatorname{Re} \varphi\left(x, \frac{t - \delta_{k+1}}{\delta_{k+1}}, \delta_{k+1}\right)$$

$$+ \frac{1}{\delta_k} \frac{\partial_s w\left(x, \dfrac{t - \delta_k}{\delta_k}, \delta_k\right)}{w\left(x, \dfrac{t - \delta_k}{\delta_k}, \delta_k\right)} - \frac{1}{\delta_{k+1}} \frac{\partial_s w\left(x, \dfrac{t - \delta_{k+1}}{\delta_{k+1}}, \delta_{k+1}\right)}{w\left(x, \dfrac{t - \delta_{k+1}}{\delta_{k+1}}, \delta_{k+1}\right)}\,.$$

Lemma 4.6. *Let $\sigma > 1$ and assume that for all $x \in U$*

$$(\sigma + \rho)A(x, 0) + \rho B(x, 0) \leq -c < 0 .$$

Then there exists $k_0 \in \mathbf{N}$ such that for all $k \geq k_0$ and for all $(x, t) \in U \times [\delta_{k+1}, \delta_k]$

$$\partial_t F_k(x, t) \geq \frac{c}{2} k^{\sigma+\rho-1} .$$

Proof. Using (4.5) we can write

$$\begin{aligned}
&\partial_t F_k(x, t)\\
&= v_k \delta_k^{-1}\left[A(x, \delta_k) + B(x, \delta_k)\frac{t-\delta_k}{\delta_k} + R'\left(x, \frac{t-\delta_k}{\delta_k}, \delta_k\right)\left(\frac{t-\delta_k}{\delta_k}\right)^2\right]\\
&\quad - v_{k+1}\delta_{k+1}^{-1}\left[A(x, \delta_{k+1}) + B(x, \delta_{k+1})\frac{t-\delta_{k+1}}{\delta_{k+1}} + R'\left(x, \frac{t-\delta_{k+1}}{\delta_{k+1}}, \delta_{k+1}\right)\left(\frac{t-\delta_{k+1}}{\delta_{k+1}}\right)^2\right]\\
&\quad + O(k^\sigma)\\
&= v_k \delta_k^{-1}\left[A(x, \delta_k) + B(x, \delta_k)\frac{t-\delta_k}{\delta_k}\right] - v_{k+1}\delta_{k+1}^{-1}\left[A(x, \delta_{k+1}) + B(x, \delta_{k+1})\frac{t-\delta_{k+1}}{\delta_{k+1}}\right]\\
&\quad + O(k^{\sigma+\rho-2}) + O(k^\rho).\\
&= A(x, \delta_k)[v_k\delta_k^{-1} - v_{k+1}\delta_{k+1}^{-1}] + B(x, \delta_k)\left[v_k\frac{t-\delta_k}{\delta_k^2} - v_{k+1}\frac{t-\delta_{k+1}}{\delta_{k+1}^2}\right]\\
&\quad + O(k^{\sigma+\rho-2}) + O(k^\rho).
\end{aligned}$$

And, as $\sigma > 1$, we deduce that for $k \to +\infty$

$$\partial_t F_k(x, t) = -k^{\sigma+\rho-1}[(\rho + \sigma)A(x, 0) + \rho B(x, 0)](1 + o(1)) .$$

The proof is complete.

Recalling the definition of γ and m_k we have

$$F_k(x, m_k) = \operatorname{Log}\frac{\left|w\left(x, \dfrac{\delta_{k+1}-\delta_k}{2\delta_k}, \delta_k\right)\right|}{\left|w\left(x, \dfrac{\delta_k-\delta_{k+1}}{2\delta_{k+1}}, \delta_{k+1}\right)\right|} = O(1) .$$

If $\sigma > 2$ then, for all $x \in U$, $F_k(x, \delta_k) - F_k(x, \delta_{k+1}) \to +\infty$ for $k \to +\infty$. Consequently we can find a C^∞ function $m_k(x)$ such that $F_k(x, m_k(x)) = 0$. Moreover defining $e_k(x) = m_k(x) - m_k$ we have

$$e_k(x) = O(k^{-(\sigma+\rho-1)}) .$$

Choice of σ and ρ. We set $\sigma = 3$ and, recalling (4.6), we fix $\rho > \sigma$ in such a way that

$$\left(1 + \frac{\sigma}{\rho}\right)A(x, 0) + B(x, 0) \leq -\frac{c''}{4} < 0 .$$

End of the proof. From this point on the proof is standard. We give only a sketch, refering to [Z] for the details.

Using Whitney's Theorem we construct a sequence of B^∞ functions $(y_k(x, t))_k$ such that, setting

$$u_k(x, t) = h_k(x, t)(1 + y_k(x, t)) \qquad \text{and} \qquad \tilde{r}_k(x, t) = \frac{Pu_k(x, t)}{u_k(x, t)}$$

we obtain that $\tilde{r}_k$ satisfies (4.12) and it is flat on the surfaces $t = m_k(x)$ and $t = m_{k-1}(x)$.

After that we consider a real function $\chi \in C^\infty(\mathbf{R})$, $0 \leq \chi \leq 1$, with $\chi(u) = 1$ for $|u| \leq \frac{3}{4}$ and $\chi(u) = 0$ for $|u| \geq 1$. We define

$$\chi_k(t) = \chi\left(\frac{t - \delta_k}{\delta_k - \delta_{k+1}}\right) \qquad \text{and} \qquad u(x, t) = \sum_{k \geq k_0} \chi_k(t) u_k(x, t)\,.$$

It is a routine computation to verify that $a(x, t) = -\dfrac{Pu(x, t)}{u(x, t)}$ is a C^∞ function, flat on $t = 0$. Moreover, as

$$\gamma(x, \delta_k) = \frac{\rho}{\sigma} A(x, 0) k^\sigma (1 + o(1)) \qquad \text{and} \qquad \left| \nu_k \delta_k^{-M} \operatorname{Re} \varphi\left(x, \frac{t - \delta_k}{\delta_k}, \delta_k\right)\right| \leq C k^{\sigma - 1}\,,$$

we have that u is flat on $t = 0$. Finally remarking that u can be zero only on $t = m_k(x)$, we deduce that $(0, 0) \in \operatorname{supp} u$ and this achieves the proof of the Theorem.

Appendix

We collect in this Appendix some results used in the proof of Theorem 2.1. These results, which include the Fefferman-Phong inequality, are particular cases of those in [CDZ] to which we refer.

We consider the metric in $\mathbf{R}^{n-1} \times \mathbf{R} \times \mathbf{R}^{n-1} \times \mathbf{R} = W$

$$g_{x,y,\xi,\eta} = dx^2 + dy^2 + \frac{d\xi^2 + d\eta^2}{\Phi^2(x, y, \xi, \eta)}\,, \tag{A.1}$$

where $\Phi^2(x, y, \xi, \eta) = \theta^2(y)|\xi|^2 + \eta^2 + \gamma^2$, $\gamma \geq 1$, and θ is a C^∞ function in $\mathbf{R}$ such that $\theta(y) = e^y$ if $e^y \leq c_0$, $\theta(y) = 2c_0$ if $e^y \geq 2c_0$ and $0 < \theta \leq 2c_0$.

We shall denote by G by Euclidian metric in $\mathbf{R}^{n-1} \times \mathbf{R}$. It is then easy to see that

$$\begin{cases} g^\sigma_{x,y,\xi,\eta} = \Phi^2(x, y, \xi, \eta)(dx^2 + dy^2) + d\xi^2 + d\eta^2\,, \\ G_{x,y} \leq g_{x,y,\xi,\eta} \qquad \text{for every } (x, y, \xi, \eta) \text{ in } W\,, \\ \dfrac{g_{x,y,\xi,\eta}}{g^\sigma_{x,y,\xi,\eta}} = \Phi^{-2}(x, y, \xi, \eta) \leq 1\,. \end{cases} \tag{A.2}$$

It is also easy to see that g is slowly varying but not temperate in the sense of Hörmander [H]. However g is locally temperate in the sense of Dencker [D] as we shall show. We have to prove that one can find positive constants c, C and $N \in \mathbf{N}$ such that $G_X(X - X') \leq c$ implies $g_{X\Xi} \leq C g_{X'\Xi'}(1 + g^{\sigma}_{X\Xi}((X, \Xi) - (X', \Xi')))^N$ for every $X = (x, y)$, $X' = (x', y')$, $\Xi = (\xi, \eta)$, $\Xi' = (\xi', \eta')$.

A straightforward computation shows that this will be implied by

$$\frac{\gamma^2 + \theta^2(y')|\xi'|^2 + \eta'^2}{\gamma^2 + \theta^2(y)|\xi|^2 + \eta^2} \leq c(1 + |\xi - \xi'|^2 + |\eta - \eta'|^2)^N \tag{A.3}$$

if $|y - y'| \leq c$.

We have $\gamma^2 + |\eta'|^2 \leq 2(\gamma^2 + \eta^2 + |\eta - \eta'|^2)$. Since $\gamma \geq 1$ we get

$$\frac{\gamma^2 + \eta'^2}{\gamma^2 + \eta^2 + \theta^2(y)|\xi|^2} \leq 2 + |\eta - \eta'|^2 .$$

Now, since $\theta \leq 2c_0$, we have $\theta^2(y')|\xi'|^2 \leq 2\theta^2(y')|\xi|^2 + 8c_0^2|\xi - \xi'|^2$. Therefore

$$\frac{\theta^2(y')|\xi'|^2}{\theta^2(y)|\xi|^2 + \eta^2 + \gamma^2} \leq 2\frac{\theta^2(y')}{\theta^2(y)} + 8c_0^2|\xi - \xi'|^2 .$$

Assume $|y - y'| \leq \operatorname{Log} 2$. Then if $y \leq \operatorname{Log}\frac{c_0}{2}$ we have $y' \leq \operatorname{Log} c_0$. Therefore $\theta(y) = e^y$, $\theta(y') = e^{y'}$ and

$$2\frac{\theta^2(y')}{\theta^2(y)} = 2e^{2(y'-y)} \leq 8 .$$

Finally if $y \in \left[\operatorname{Log}\frac{c_0}{2}, +\infty\right[$ then $\theta(y)$ is bounded below by a fixed constant c_1 and $\theta(y') \leq 2c_0$. Therefore

$$2\frac{\theta^2(y')}{\theta^2(y)} \leq \frac{8c_0^2}{c_1^2} .$$

The above considerations show that we can apply, for these metrics, the results of [D] and [CDZ].

We introduce now the Sobolev spaces for $m \in \mathbf{N}$,

$$\mathscr{H}^m = \left\{u\colon \|u\|^2 = \sum_{|\alpha|+k\leq m} \gamma^{2(m-|\alpha|-k)}\|(\theta(y)D_x)^{\alpha}D_y^k u\|^2_{L^2} < +\infty\right\} .$$

Then we can state

Proposition A.1. 1) *If $a \in S(\Phi^{-k}, g)$, $k \in \mathbf{N}$ then a_χ^w is continuous from L^2 to $\mathscr{H}^k$.*

2) *If $a \in S(\Phi^m, g)$, $m \in \mathbf{N}$, then a_χ^w is continuous from $\mathscr{H}^m$ to L^2.*

3) *Let $a \in S(\Phi^{2m}, g)$. Assume that $a \geq c_0\gamma\Phi^{2m-2}$ in W. Then there exists $c_1 > 0$ such that $(a_\chi^w u, u) \geq c_1\gamma\|u\|^2_{m-1}$, $u \in C_0^\infty(W)$.*

Proof. 1) This is true if $k=0$, by the L^2 – continuity of operators with symbols in $S(1,g)$ proved by Dencker [D]. By induction let $a \in S(\Phi^{-k-1}, g)$. Then $\|a_\chi^w u\|_{k+1} \leq \gamma \|a_\chi^w u\|_k + \|\theta(y) D_x a_\chi^w u\|_k + \|D_y a_\chi^w u\|_k$. Now if $r \in S(\Phi, g)$ the symbolic calculus in [D] shows that $r_\chi^w a_\chi^w = (ra)_\psi^w + s_\psi^w$ with ra and s in $S(\Phi^{-k}, g)$. Since γ, $\theta(y)\xi$ and η are in $S(\Phi, g)$ the result follows by induction.

2) Taking χ real and symmetric gives Φ_χ^w self adjoint. Then

$$\|\Phi_\chi^w u\|_{L^2}^2 = (\Phi_\chi^w \Phi_\chi^w u, u) = ((\Phi^2)_\psi^w u, u) + (r_\psi^w u, u)$$

where $r \in S(\Phi, g)$. Since Φ^2 is a polynomial in (ξ, η) we have $(\Phi^2)_\psi^w = (\Phi^2)^w = \gamma^2 - \theta^2(y)\Delta_x - \partial_y^2$. Therefore for u in C_0^∞ we get

$$\|\Phi_\chi^w u\|_{L^2}^2 = \gamma^2 \|u\|_{L^2}^2 + \|\theta(y) D_x u\|_{L^2}^2 + \|D_y u\|_{L^2}^2 + O(\|u\|_{L^2}^2)\,. \tag{A.4}$$

Now we use an induction on m. Let $a \in S(\Phi^m, g)$ we can write

$$a_\chi^w u = a_\chi^w \circ (\Phi^{-1})_\chi^w \circ \Phi_\chi^w + a_\chi^w r_\psi^w,\ r \in S(\Phi^{-1}, g)\,.$$

It follows that $a_\chi^w = (a\Phi^{-1})_\psi^w \Phi_\chi^w + s^w$ with $s \in S(\Phi^{m-1}, g)$. Then

$$\|a_\chi^w u\|_{L^2} \leq \|(a \circ \Phi^{-1})_\psi^w \Phi_\chi^w u\|_{L^2} + \|s_\varphi^w u\|_{L^2} \leq C(\|\Phi_\chi^w u\|_{m-1} + \|u\|_{m-1})$$

by the induction. We have

$$\|\Phi_\chi^w u\|_{m-1} = \sum_{|\alpha|+k \leq m-1} \gamma^{2(m-1-|\alpha|-k)} \|(\theta(y)D_x)^\alpha D_y^k \Phi_\chi^w u\|_{L^2}\,.$$

Then we commute Φ_χ^w with the operators in front of it (which belongs to $S(\Phi^{|\alpha|}, g)$) and we use (A.4) to get the result.

3) Let us set $u = (\Phi^{1-m})_\chi^w v$. It follows that

$$(a_\chi^w u, u) = ((\Phi^{1-m})_\chi^w a_\chi^w (\Phi^{1-m})_\chi^w v, v)\,.$$

But, by the symbolic calculus, we have

$$(\Phi^{1-m})_\chi^w a_\chi^w (\Phi^{1-m})_\chi^w = (\Phi^{2-2m} a)_\psi^w + r_\psi^w$$

where $r \in S(1, g)$. It follows that

$$(a_\chi^w u, u) = ((\Phi^{2-2m} a - c_0\gamma)_\psi^w v, v) + c_0 \gamma \|v\|_{L_2}^2 + (r_\psi^w v, v)\,.$$

Now $\Phi^{2-2m} a - c_0 \gamma \in S(\Phi^2, g) = S(h^{-2}, g)$ and $\Phi^{2-2m} a - c_0 \gamma \geq 0$. We can apply the Fefferman-Phong inequality proved in [CDZ] to get

$$(a_\chi^w u, u) \geq c_0 \gamma \|v\|_{L_2}^2 - C\|v\|_{L_2}^2 - C'\|v\|_{L_2}^2 \geq \tfrac{1}{2} c_0 \gamma \|v\|_{L^2}^2$$

if γ is large enough, since the constants C and C' depend only on the semi norms of $\Phi^{2-2m} a - c_0\gamma$ in $S(\Phi^2, g)$ and therefore are independent of γ.

Now by 1, $\|u\|_{m-1}^2 = \|(\Phi^{1-m})_\chi^w v\|_{m-1}^2 \leq C\|v\|_{L_2}^2$. The proof is complete.

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